HETEROGENEOUS BELIEFS, RISK AND LEARNING IN A SIMPLE ASSET PRICING MODEL

CARL CHIARELLA AND XUE-ZHONG HE

School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

carl.chiarella@uts.edu.au, tony.he1@uts.edu.au

Abstract. Trade among individuals occurs either because tastes (risk aversion) differ, endowments differ, or beliefs differ. We study a simple discounted present value asset price model where agents have heterogeneous beliefs, different risk attitudes and learning schemes. According to different risk attitudes, agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. By using both bifurcation theory and numerical analysis, we find that the dynamics of asset pricing is affected by the relative risk attitudes of different types of investors and the external noise and learning schemes can significantly affect the dynamics. Compared with the findings of Brock and Hommes [7] on the dynamics caused by changing of the intensity of choice to switch predictors, we find that many of their insights are robust to the generalization we consider, however, the resulting dynamical behavior is considerably enriched and exhibits some significant differences.

Date: First version: Oct. 1998, Latest version: April 16, 1999.

1. Introduction

In recent years a body of evidence on the role of heterogeneous beliefs in financial markets has presented a sharp challenge to the traditional view that assets in financial markets are rationally priced to reflect all publicly available information. It is well known that trade among individuals occurs either because tastes (risk aversion) differ, endowments differ, or beliefs differ. With different groups of traders having different expectations about future prices, asset price fluctuations can be caused by an endogenous mechanism. For instance, Beja and Goldman [3] and Chiarella [9] consider models with interaction of two types of traders, fundamentalists and chartists. The first type base their decision on the deviation of the asset prices from fundamentals and the second type on simple technical trading rules, extrapolation of trends and other patterns observed in past prices. The interaction between these two types may lead to market instability, which is global for Beja and Goldman's linear model and gives rise to a stable limit cycle for Chiarella's nonlinear model. For more related work along this line, we refer to Brock and Hommes [5], Day and Huang [10], Lux [16, 17, 18] and Sethi [22].

Recently, Brock and Hommes [6, 7] have introduced the concept of an *adaptively rational equilibrium*, where agents base decisions upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. A key aspect of these models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. The resulting dynamical system is nonlinear and as Brock and Hommes [7] (henceforth BH) show, capable of generating the entire "zoo" of complex behaviour from local stability to high order cycles and chaos as the intensity of choice to switch predictors increases.

In this paper we focus on some different aspects of the BH mechanism. When different types of investors, say, "smart-money" investors and "noise-traders", are involved in the market, it is believed (see Black [4], Campbell and Kyle [8], DeLong, Shleifer, Summers and Waldmann [11], Fama and French [13], Miller [20], Poterba and Summers [21], Summers [23])) that the *smart-money* investors are more risk averse than *noise-traders*. In other words, different types of investors have different risk attitudes. We explicitly consider how asset price dynamics are influenced by relative risk attitudes of different type of investors. Our aim in particular is to take the framework of Brock and Hommes [7] on asset price dynamics and to relax a number of its assumptions. In so doing we seek to determine the robustness of their findings to changes in some of the key economic characteristics of their model. We focus on a number of aspects of the modeling framework. Firstly we allow agents to have differing attitudes to risk. Secondly we relax the assumption that all agents have the same view of variance of the risky assets of the economy. In our experience this is probably the quantity on which financial analysts will have differing opinions. We allow some of the agents to estimate variance from a time series of past realized returns. Thirdly, the length of the time series window over which agents form their estimates of the mean and variance of the return on the risky asset is also a parameter with which we experiment numerically to determine its effect on the dynamic behaviour of the model. A detailed summary of our findings is presented in the concluding section. Briefly, we find that many of the insights of Brock and Hommes are robust to the generalizations we consider, however there are also some important differences in the dynamic behaviour of the system when agents differ significantly in attitudes to risk, learning schemes and their estimates of the variance of returns on the risky asset.

The plan of the paper is as follows. In Section 2 a generalized version of the Brock and Hommes [7] model is presented which takes account of the fact that different agents have different risk attitudes and opinions on formations of expectations and variances. By allowing some of the agents to estimate expectation and variance from a time series of realized returns, we incorporate the length of the time series window into the model. In Section 3, we focus first on the dynamics of equilibrium asset prices with two belief types, one being fundamentalists and the other one either trend chasers or contrarians. We then consider the market interaction of these three groups. Using both bifurcation theory and numerical analysis, we investigate the effects on the dynamics of the asset price models with relatively different risk aversion coefficients and different learning schemes. In these models, we also gain some numerical insights into the effect of external noise on the system as measured by the intensity of the noise in the dividend process. We find that the resulting dynamical behavior is considerably enriched and has some significant differences compared to the original Brock-Hommes [7] analysis. Proofs of the results in this section are given in an appendix. Summary of the main results of the paper and a brief discussion are included in the last section.

2. Adaptive beliefs system

This section is devoted to a generalization of the simple asset pricing model established by Brock and Hommes [7]. Following the framework of Brock and Hommes [7], we consider an asset pricing model with one risky asset and one risk free asset. It is assumed that the risk free asset is perfectly elastically supplied at gross return $R > 1$. Let p_t be the price (ex dividend) per share of the risky asset at time t and $\{y_t\}$ be the stochastic dividend process of the risky asset. Then investor wealth at $t+1$ is given by

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t, \tag{2.1}$$

where W_t is the wealth at time t and z_t is the number of shares of the risky asset purchased at t.

As in Brock and Hommes [7], we use a Walrasian scenario to derive the demand equation, i.e. each trader is viewed as a price taker (see Brock and Hommes [5] and Grossman [15] for detailed discussion). The market is viewed as finding the price p_t that equates the sum of these demand schedules to the supply. That is, the price p_t at time t is formed by using information available as of time $t-1$ and the expected utility for time $t+1$ (see more details in the following discussion).

Denote by $F_t = \{p_t, p_{t-1}, \cdots ; y_t, y_{t-1}, \cdots\}$ the information set formed at time t. Let E_t, V_t be the conditional expectation and variance, respectively, based on F_t, and E_{ht}, V_{ht} be the "beliefs" of investor type h about the conditional expectation and variance. Denote R_{t+1} the excess return at $t+1$, that is

$$R_{t+1} = p_{t+1} + y_{t+1} - Rp_t. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$\begin{aligned} E_{ht}(W_{t+1}) &= RW_t + E_{ht}(p_{t+1} + y_{t+1} - Rp_t)z_t \\ &= RW_t + E_{ht}(R_{t+1})z_t, \\ V_{ht}(W_{t+1}) &= z_t^2 V_{ht}(p_{t+1} + y_{t+1} - Rp_t) \\ &= z_t^2 V_{ht}(R_{t+1}). \end{aligned} \tag{2.3}$$

Assume each investor type is a myopic mean variance maximizer, but different traders (say, type h) have different attitudes towards risk, characterized by the risk aversion coefficient, say a_h. Then, for type h, the demand for shares z_{ht} solves

$$\max_z \{E_{ht}(W_{t+1}) - \frac{a_h}{2} V_{ht}(W_{t+1})\}, \tag{2.4}$$

i.e.,

$$z_{ht} = \frac{E_{ht}(R_{t+1})}{a_h V_{ht}(R_{t+1})}. \tag{2.5}$$

Let z_{st} denote the supply of (risky) shares and n_{ht} the fraction of investors of type h at t (so that $\sum_h n_{ht} = 1$). Then the equilibrium of demand and supply implies[1]

$$\sum_h n_{h,t-1} z_{ht} = z_{st} \tag{2.6}$$

or (using (2.5))

$$\sum_h n_{h,t-1} \frac{E_{ht}(R_{t+1})}{a_h V_{ht}(R_{t+1})} = z_{st}. \tag{2.7}$$

We now assume zero supply of outside shares, i.e. $z_{st} = 0$, then (2.7) leads to

$$\sum_h n_{h,t-1} \frac{E_{ht}(R_{t+1})}{a_h V_{ht}(R_{t+1})} = 0. \tag{2.8}$$

In order to get a bench mark notion of the rational expectation 'fundamental solution' p_t^*, consider the equation

$$Rp_t^* = E_t\{p_{t+1}^* + y_{t+1}\},$$

where E_t is expectation conditional on the information set F_t. In the case where the dividend process $\{y_t\}$ is IID, $E_t(y_{t+1}) = \bar{y}$ which is a constant. Then the only solution satisfying the "no bubbles" condition ($\lim_{t\to\infty} Ep_t/R^t = 0$) is the constant solution $\bar{p} = \bar{y}/(R+1)$. Let x_t denote the deviation of p_t from the benchmark fundamental p_t^*, that is

$$x_t = p_t - p_t^*.$$

Regarding the different class of beliefs about the deviations from the fundamental solution, we assume

$$\begin{array}{ll} \text{(A1)}: & \text{Heterogeneous beliefs on return} \\ & E_{ht}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}) + f_{ht}; \\ \text{(A2)}: & \text{Heterogeneous beliefs on variance} \\ & V_{ht}(p_{t+1} + y_{t+1}) = V_t(p_{t+1}^* + y_{t+1}) + g_{ht}, \\ & V_t(p_{t+1}^* + y_{t+1}) = \sigma^2, \end{array} \tag{2.9}$$

where

$$f_{h,t} = f_h(x_{t-1}, \cdots, x_{t-L}), \qquad g_{h,t} = g_h(x_{t-1}, \cdots, x_{t-L}) \tag{2.10}$$

are some *deterministic* function and σ^2 is a constant. Following Brock and Hommes [7] our assumption (A1) states that each group of agents base their prediction of the mean into two parts, a fundamental part (on which all agents agree) and an agent specific prediction component f_{ht}. Our assumption (A2) makes a similar statement about each group of agents prediction of the variance. For the current analysis we assume that the fundamental component of variance prediction is constant at σ^2. Brock and Hommes [7] assume that $g_{ht} = 0$ for all h. Our rationale for incorporating non-zero g_{ht} is to capture the fact that differences in opinion about volality is a feature of financial markets. Under (A1), (A2), we have

$$E_{ht}(R_{t+1}) = f_{h,t} - Rx_t, \tag{2.11}$$

$$V_{ht}(R_{t+1}) = V_{ht}(p_{t+1} + y_{t+1}) = \sigma^2 + g_{h,t} \tag{2.12}$$

[1] Note that the typical investor makes these plans at time $(t-1)$, hence the fraction $n_{h,t-1}$. In this decision process p_t is treated as parametric and is revealed to the investor when the Walrasian auctioneer announces the market clearing price.

and hence the equilibrium equation is given by

$$R\Big[\sum_h \frac{n_{h,t-1}}{a_h(\sigma^2+g_{h,t})}\Big]x_t = \sum_h \frac{n_{h,t-1}f_{h,t}}{a_h(\sigma^2+g_{h,t})}. \tag{2.13}$$

To define the 'fitness function', we rewrite the excess return in the following form

$$R_{t+1} = x_{t+1} - Rx_t + \delta_{t+1}, \tag{2.14}$$

where

$$\delta_{t+1} = p^*_{t+1} + y_{t+1} - E_t\{p^*_{t+1} + y_{t+1}\} \tag{2.15}$$

is a Martingale Difference Sequence w.r.t. F_t, i.e. $E\{\delta_{t+1}|F_t\} = 0$ for all t. Let $\pi_{h,t}$ be the 'fitness function' which, according to Brock and Hommes [7], is defined by the realized profits of trader type h:

$$\pi_{h,t} = R_{t+1}z_{ht}, \qquad \text{with} \qquad z_{ht} = \frac{E_{ht}(R_{t+1})}{a_h V_{ht}(R_{t+1})}. \tag{2.16}$$

More generally, one can introduce additional memory into the performance measure, by considering a weighted average of realized profits, as follows:

$$U_{h,t} = \pi_{h,t} + \eta U_{h,t-1}, \tag{2.17}$$

where the parameter η represents the *memory strength*.

Let the updated fractions be formed on the bases of discrete choice probability (see Manski and McFadden [19], Anderson, de Palma and Thisse [1], Brock and Hommes [6, 7]),

$$n_{h,t} = exp[\beta U_{h,t-1}]/Z_t, \qquad Z_t = \sum_h exp[\beta U_{h,t-1}], \tag{2.18}$$

where $\beta(>0)$ is the *intensity of choice* measuring how fast agents switch between different prediction strategies. In particular, $\beta = +\infty$ means the entire mass of traders uses the strategy that has highest fitness; while $\beta = 0$ means that the mass of traders distributes itself evenly across the set of available strategies.

To sum up, the evolutionary dynamics is described by the following adaptive beliefs system

$$\begin{cases} Rx_t = & \sum_h \frac{n_{h,t-1}f_{h,t}}{a_h(\sigma^2+g_{h,t})} / \sum_h \frac{n_{h,t-1}}{a_h(\sigma^2+g_{h,t})}, \\ n_{h,t} = & \exp[\beta U_{h,t-1}]/Z_t, \end{cases} \tag{2.19}$$

where

$$\begin{cases} Z_t = & \sum_h \exp[\beta U_{h,t-1}], \\ U_{h,t} = & \pi_{h,t} + \eta U_{h,t-1}, \\ \pi_{h,t} = & R_{t+1}z_{ht} = [x_{t+1} - Rx_t + \delta_{t+1}]\frac{f_{h,t} - Rx_t}{a_h(\sigma^2+g_{h,t})}. \end{cases} \tag{2.20}$$

The system can be written as a high-order difference equation. Say, for $\eta = 0$, $U_{h,t} = \pi_{h,t}$ is a function of $x_{t+1}, x_t, \cdots$ and

$$n_{h,t} = n_h(x_t, x_{t-1}, \cdots).$$

Therefore the system (2.19)-(2.20) can be written as a high-order difference equation of the form

$$x_t = G(x_{t-1}, x_{t-2}, \cdots).$$

By setting $a_h = a$ and $g_{ht} = 0$ we would obtain the dynamic system studied by Brock and Hommes [7]. In the following section, we will investigate the dynamics of this more generalized asset price model with heterogeneous beliefs, risk and learning schemes.

3. Dynamics of simple belief types

When all the belief types have the same risk aversion and the heterogeneous return function f_{ht} has a single lag (that is $f_{ht} = f_h(x_{t-1})$), the dynamics caused by the intensity of choice to switch predictors has been investigated by Brock and Hommes [7]. They have shown that the irregular fluctuations in asset prices are triggered by a rational choice, based upon realized profits, in prediction strategies, known as *Rational Animal Spirits*.

In this section, we investigate how the dynamics are affected by the different risk attitudes of the various investors, as characterized by the different risk aversion coefficients. We also consider the impact of learning schemes with different lag lengths in the formation of expectations when there are two different types of beliefs.

To investigate the role of heterogeneous belief types, we assume that all beliefs will follow a linear return and a nonlinear variance learning process. More precisely, let

$$\bar{x}_t = \frac{1}{L}\sum_{i=1}^{L} x_{t-i}, \qquad \bar{\sigma}_t^2 = \frac{1}{L}\sum_{i=1}^{L}[x_{t-i} - \bar{x}_t]^2 \tag{3.1}$$

and let

$$v_h(x) = \mu[1 - \frac{1}{(1+x)^{\xi}}], \tag{3.2}$$

where L is a positive integer and $\mu, \xi \geq 0$ are constants. We assume that

$$f_{ht} = d_h \bar{x}_t, \tag{3.3}$$

where d_h is the *trend* of trader type h. As in Brock and Hommes [7], we call agent h a pure *trend chaser* if $d_h > 0$ (strong trend chaser if $d_h > R$) and a *contrarian* if $d_h < 0$ (strong contrarian if $d_h < -R$). The simple predictors (3.3) could be considered as a generalization of the simplest idealization of overreacting securities analysts or overreacting investors. When $d_h = 0$, equation (3.3) reduces to *fundamentalists*, believing that prices return to their fundamental value. When $d_h \neq 0$, equation (3.3) reduces to *chartists*, believing that prices follow the past prices governing by a learning process. On the variance, we assume that

$$g_{ht} = \sigma^2 v_h(\bar{\sigma}_t^2). \tag{3.4}$$

Motivated by Franke and Sethi [14], (3.4) means that although traders increase their variance estimate as the estimate variance from past realized returns increases, but they do not do so without bound. This is to be expected for the fundamentalists since they know the fundamental values in long run.

3.1. Fundamentalists versus Trend Chasers.

In this part, we assume that type 1 agents are fundamentalists, believing that prices return to their fundamental solution $x_t = 0$, whereas type 2 agents believe in a pure trend. Also, the fundamentalists realize the participation of the chartists and they adjust their formulation on the variance correspondingly. We assume following belief schemes for the two groups of agents:

$$
\begin{array}{lll}
\text{Type 1:} & \text{Fundamentalists:} & \\
 & \text{Mean:} & f_{1t}=0, \\
 & \text{Variance:} & g_{1t}=\sigma^2 v_h(\bar{\sigma}_t^2), \\
 & \text{Risk Aversion Coefficient :} & a_1 \\
\text{Type 2:} & \text{Pure trend chaser:} & \\
 & \text{Mean:} & f_{2t}=d\bar{x}_t, \\
 & \text{Variance:} & g_{2t}=0, \\
 & \text{Risk Aversion Coefficient :} & a_2,
\end{array}
\tag{3.5}
$$

where $a_1, a_2, d > 0$ are positive real constants. Thus, the fundamentalists believe that the prices will return to their fundamental values and, based on the observed prices, they adjust their formulation on variance according to the function $v_h(x)$. On the other hand, the second type of traders form their expectations by trying to learn the mean of the price process as measured by the last L realized prices and chase the trend of the prices. Here, we allow $a_i\,(i=1,2)$ to be different to characterize the different risk attitudes of the two types of investors. Typically, as pointed by Campbell and Kyle [8], we would expect the fundamentalists to be more risk averse than the trend chasers.

Adaptive Belief System: Let $\eta = 0, \delta_t = 0$. We have from (2.19), (2.20) and (3.1)-(3.5) that

$$
\left\{
\begin{array}{rl}
x_t = & \frac{d}{R}\frac{a_1(1+v_h(\bar{\sigma}_t^2))}{a_2 n_{1,t-1}+a_1(1+v_h(\bar{\sigma}_t^2))n_{2,t-1}} n_{2,t-1}\bar{x}_t, \\
n_{1,t} = & \exp[\beta(\frac{1}{a_1\sigma^2(1+v_h(\bar{\sigma}_{t-1}^2))} Rx_{t-1}[Rx_{t-1}-x_t]-C)]/Z_t, \\
n_{2,t} = & \exp[\beta(\frac{1}{a_2\sigma^2}[x_t-Rx_{t-1}][d\bar{x}_{t-1}-Rx_{t-1}]/Z_t, \\
Z_t = & \exp[\beta(\frac{1}{a_1\sigma^2(1+v_h(\bar{\sigma}_{t-1}^2))} Rx_{t-1}[Rx_{t-1}-x_t]-C)] \\
 & +\exp[\beta(\frac{1}{a_2\sigma^2}[x_t-Rx_{t-1}][d\bar{x}_{t-1}-Rx_{t-1}],
\end{array}
\right.
\tag{3.6}
$$

where $C \geq 0$ is the cost incurred by the fundamentalists.

Let $a = a_2/a_1$ and

$$m_t = n_{1,t} - n_{2,t}.$$

Then

$$n_{1,t} = \frac{1+m_t}{2}, \qquad n_{2,t} = \frac{1-m_t}{2} \tag{3.7}$$

and hence the system (3.6) can be expressed as

$$
\left\{
\begin{array}{rl}
x_t = & \frac{d}{R}\frac{(1+v_h(\bar{\sigma}_t^2))(1-m_{t-1})}{a(1+m_{t-1})+(1+v_h(\bar{\sigma}_{t-1}^2))(1-m_{t-1})}\bar{x}_t \\
m_t = & \tanh\left[\frac{\beta}{2a_1\sigma^2}(Rx_{t-1}-x_t)\left(\frac{Rx_{t-1}}{1+v_h(\bar{\sigma}_t^2)}+\frac{d\bar{x}_{t-1}-Rx_{t-1}}{a}\right)-\frac{\beta C}{2}\right].
\end{array}
\right.
\tag{3.8}
$$

We divide the following discussion on the dynamics of (3.8) into two cases.

Case 1: $L = 1$

We first consider a special case: $L = 1$. It follows from (3.1) and (3.2) that

$$\bar{x}_t = x_{t-1}, \qquad \bar{\sigma}_t^2 = 0, \qquad v_h(\bar{\sigma}_t^2) = 0.$$

Then we have the following system

$$
\left\{
\begin{array}{rl}
x_t = & \frac{d}{R}\frac{1-m_{t-1}}{a+1+(a-1)m_{t-1}}x_{t-1} \\
m_t = & \tanh\left[\frac{\beta}{2a_1\sigma^2}(Rx_{t-1}-x_t)\left(Rx_{t-1}+\frac{dx_{t-2}-Rx_{t-1}}{a}\right)-\frac{\beta C}{2}\right].
\end{array}
\right.
\tag{3.9}
$$

Lemma 3.1. (**Existence and stability of equilibrium**) *Let* $m^{eq} = \tanh(-\frac{\beta C}{2})$, $m^* = 1 - \frac{2aR}{d+R(a-1)}$ *and* x^* *be the positive solution (if it exists) of*

$$\tanh\left[\frac{\beta}{2\sigma^2 a_1}(R-1)(R+\frac{d-R}{a})(x^*)^2 - \frac{\beta C}{2}\right] = m^*. \tag{3.10}$$

- *For* $0 < d < R$, $E_1 = (0, m^{eq})$ *is the unique, globally stable steady state of (3.9);*
- *For* $R < d < (a+1)R$, *there are two possibilities:*
  - *if* $m^* < m^{eq}$ *then* E_1 *is the unique, globally stable steady state of (3.9);*
  - *if* $m^* > m^{eq}$ *then (3.9) has three steady states* E_1, E_2 *and* E_3*;* E_1 *is unstable;*
- *For* $d > (a+1)R$, *(3.9) has three steady states* $E_1, E_2 = (x^*, m^*)$ *and* $E_3(-x^*, m^*)$*;* E_1 *is unstable.*

Lemma 3.1 indicates that, when the trend chasers extrapolate only weakly ($0 < d < R$), the fundamental steady state E_1 is globally stable, no matter what risk attitude investors have. However, when $d > R$, the stability of the fundamental equilibrium E_1 depends on the ratio a, which measures the relative risk attitude. When the trend chasers extrapolate very strongly ($d > (a+1)R$) the fundamental equilibrium E_1 becomes unstable and bifurcate two additional nonzero steady states E_2 and E_3. In the case of $R < d < (a+1)R$, the fundamental equilibrium is stable when a is large, that is when the trend chasers become more risk averse than the fundamentalists. This point will become more clear in the following discussion.

When $a = 1$, Lemma 3.1 leads to Lemma 2 in Brock and Hommes [7]. Hence, Lemma 3.1 provides a more general result by allowing different risk aversion coefficients. For more detailed discussion of Lemma 3.1, we refer to Brock and Hommes [7]. The relative risk aversion coefficient a has the effect of reducing or expanding the second and third regimes in Lemma 3.1. If $a < 1$ (i.e. fundamentalists are more risk averse) the middle region shrinks whilst the third region expands. The converse holds for $a > 1$. Typically we would expect fundamentalists to be more risk averse than trend chasers, i.e. $a < 1$.

To fully understand the impact of the relative risk aversion coefficient a on the dynamical behaviour of the model we need to study how changes in a affect bifurcations. The following two Lemmas clarify this issue, where we are more interested in the case when $R < d < (a+1)R$. Let a^* satisfy

$$\tanh(-\frac{\beta C}{2}) = 1 - \frac{2a^*R}{d+(a^*-1)R}. \tag{3.11}$$

Then one can verify that $m^* < m^{eq}$ if and only if $a > a^*$. Hence, a pitchfork bifurcation occurs for $a = a^*$. As a decreases further, the system will have a Hopf bifurcation. More precisely, we have the following Lemmas 3.2 and 3.3 (The proof of Lemma 3.2 comes from Lemma 3.1 directly and one can find the proof of Lemma 3.3 in the appendix).

Lemma 3.2. (**Pitchfork bifurcation**) *Assume that* $R < d < (a+1)R$. *Let* a^* *be solution of (3.11). Then, for* $a > a^*$, E_1 *is the unique equilibrium; for* $0 < a < a^*$, *(3.9) has three equilibria* E_1, E_2 *and* E_3. *Therefore the system has a pitchfork bifurcation at* $a = a^*$.

Lemma 3.3. (**Second Bifurcation**) *Let* $E_2 = (x^*, m^*)$ *and* $E_3(-x^*, m^*)$ *be the non-fundamental steady states as in Lemma 3.1. Assume* $R < d < (a+1)R$ *and* $C > 0$ *and let* a^* *be the pitchfork bifurcation value as in Lemma 3.2. There exists* $a^{**} < a^*$ *such that* E_2 *and* E_3 *are stable for* $a \in (a^{**}, a^*)$ *and unstable for* $a < a^{**}$. *For* $a = a^{**}$, E_2 *and* E_3 *exhibit a Hopf bifurcation.*

For $a = 1$, Brock and Hommes [7] obtained a similar dynamics when changing the parameter β. Here instead, we use parameter a to measure the relative attitudes toward the risk for two different types investors.

Numerical simulations

Equations (3.9) is equivalent to a three dimensional first-order system in terms of (x_t, x_{t-1}, m_t). In the following numerical simulations, we choose

$$R = 1.1, d = 1.2, C = 1.0, \beta = 3.5, \sigma^2 = 1.0, a_1 = 1.0$$

with initial value (1.2, 0.7, -0.2) and different $a_2 = a = 0.9, 0.95, 1.0, 1.1, 1.2$ (initial values can be selected so that the solutions stay non-positive, instead of nonnegative as reported here[2]). Obviously, $R < d < (a+1)R$ is satisfied. Then the pitchfork bifurcation parameter, indicated by Lemma 3.2, $a^* = 3.01$. Then Lemma 3.2 implies that the fundamental steady state E_1 is globally stable for $a > 3.01$. Fig. 3.1 (a) shows plots of the attractors in the (x_t, m_t) plane and Fig. 3.1 (b) and (c) plot the solutions of x_t and m_t, respectively. In Fig. 3.1 (a), the orbit converges to the positive equilibrium E_2 for $a = 1.2$ and then to an attracting invariant 'circle' surrounding E_2 for $a = 1.1$, which indicates that the Hopf bifurcation value $a^{**} \in (1.1, 1.2)$. Then as a decreases, the 'circle' break into invariant sets. Fig. 3.1 (b) and (c) show the corresponding time series. For $a = 1$, the prices oscillate about the positive equilibrium E_2 and the market is dominated by the chartists. As a increases, the prices are stabilized to E_2 and as a decreases, the prices are switching between an unstable phase of an upward trend and a stable phase with prices close to the fundamental value. Fig. 3.1(d)-(e) show the corresponding time series of x_t, m_t with noise. The noise comes from a stochastic dividend process $y_t = \bar{y} + \epsilon_t$ with IID noise ϵ_t, uniformly distributed on the interval $[-0.05, 0.05]$, added to the constant dividend process $\bar{y}$. We note that, without noise, the time series have regular patterns and, after adding noise, the patterns become more irregular. We highlight the case of $a = 1.2$. Without noise, the solution converges to E_2; with noise, the solution fluctuates in a quite irregular fashion and exhibits bursts of volatility. This simulation makes the point that nonlinear models of financial markets dynamics (even when stable) "process" external noise in a far more complicated way than is possible with linear models. However this is a theme that we do not develop further here but rather for future research.

Fig. 3.2(a) shows a bifurcation diagram w.r.t. the relative risk ratio a without noise, suggesting periodic and quasi-periodic dynamics after the primary Hopf bifurcation as a decreases. Fig. 3.2(b) shows the corresponding Largest Lyapunov Exponent (LLE) plot without noise. When the fundamentalists are more risk averse than trend chaser (i.e. $a < 1$), decreasing of a leads to weakly chaotic asset prices fluctuations with an irregular switching between close to the EMH fundamental prices and upward and downward trends. On the other hand, when we allow an external dividend process to be stochastic, i.e. $y_t = \bar{y} + \epsilon_t$ with IID noise ϵ_t, uniformly distributed on the interval $[-0.05, 0.05]$. The bifurcation diagram and the largest Lyapunov exponent plot are shown in Fig. 3.2 (c) and (d), respectively. The bifurcation diagram itself does not indicate much difference from the one without noise, however, the largest Lyapunov exponent (λ) plot does indicate a significant affect of the noise on the dynamics. When adding the noise, it is expected that the dynamics become more chaotic (indicated by an increase in λ) if the dynamics

---

[2]From the first equation of the system (3.9), one can see that a positive (negative) initial value produce a positive (negative) solution, even adding noise from dividend process. Hence in the case of fundamentalists versus trend chasers, the fluctuation among the three phases (stable, upward trend and downward trend), reported in Brock and Hommes [7], cannot occur once the initial value is selected.

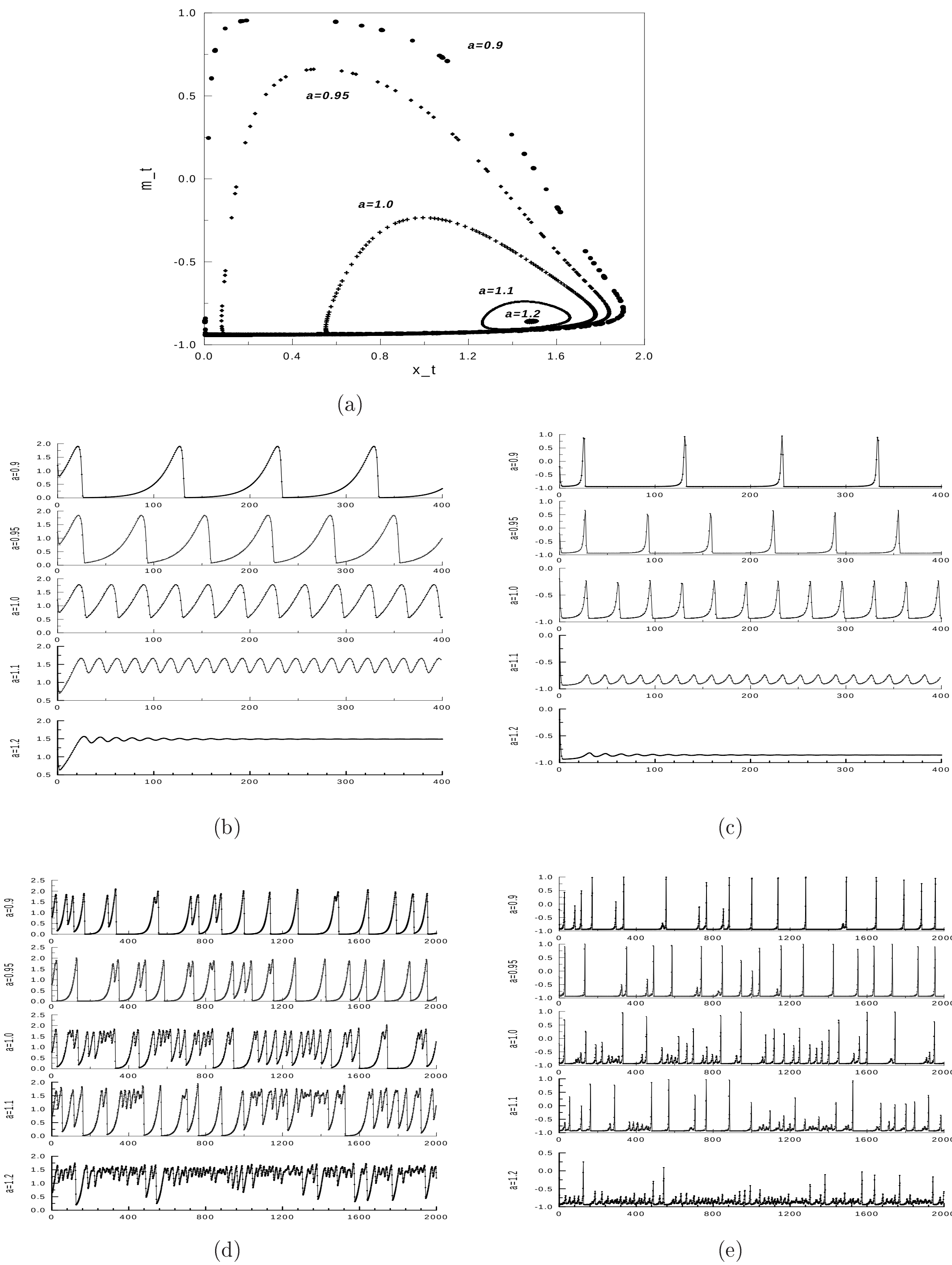


FIGURE 3.1. Trend versus fundamentalists: Phase plot of (x, m) (a) (without noise) and the time series of x_t and m_t without noise (b) and (c) and with noise (d) and (e) for $a = a_2 = 0.9, 0.95, 1.0, 1.1, 1.2$.

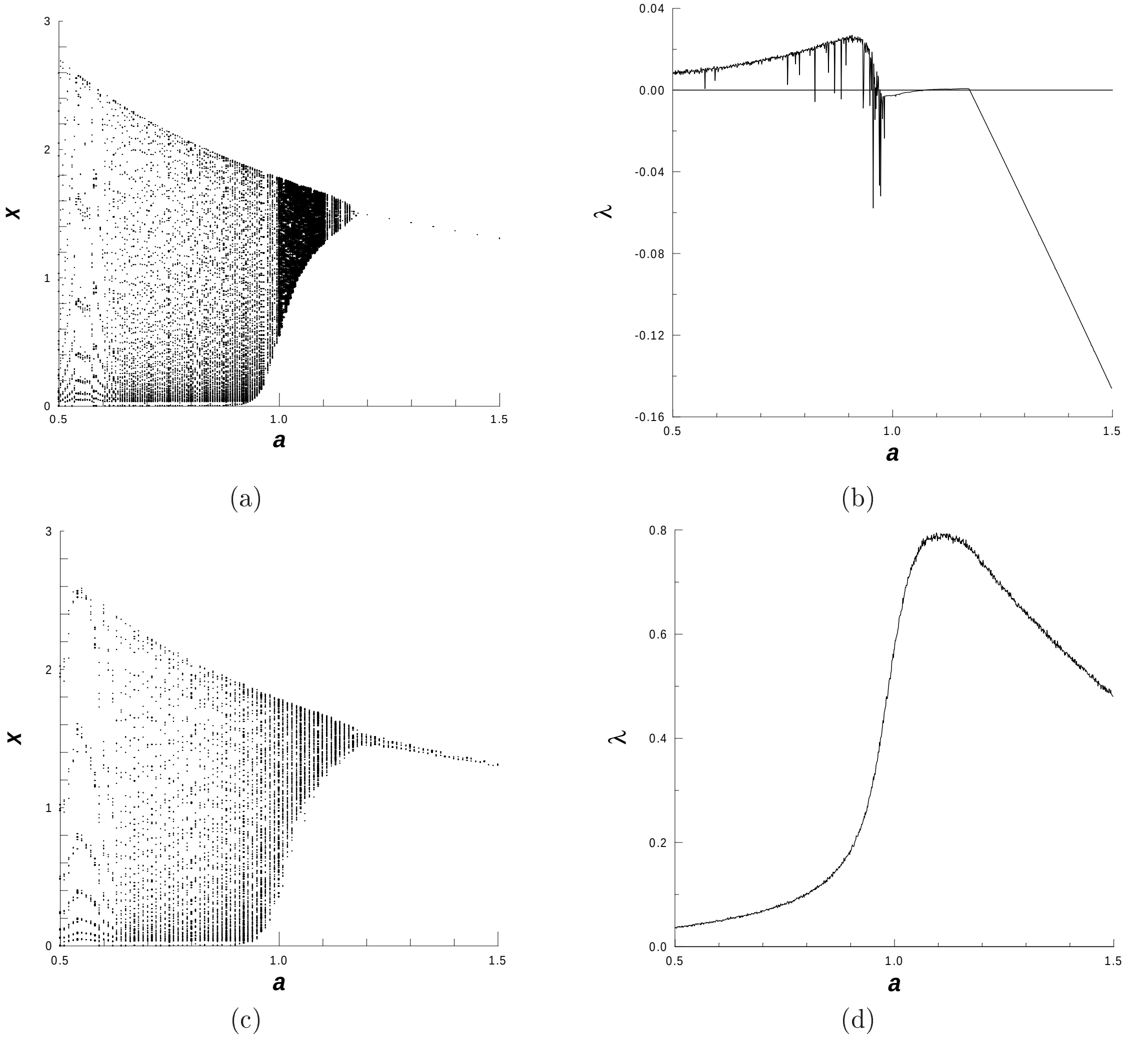


FIGURE 3.2. Trend versus fundamentalists: Bifurcation diagram—(a) without noise (c) with noise; Largest Lyapunov exponent—(b) without noise (d) with noise.

without noise is chaotic. However, for large a, say $a \in [1.2, 1.5]$, the system without noise is stable ($\lambda < 0$). After adding noise, positive λ indicates it becomes chaotic, even though the magnitudes of the oscillations is very small. This phenomena can also be seen from Fig. 3.1(d) for $a = 1.2$. Hence, adding noise can make a system from chaotic to more chaotic, also from stable to chaotic. In other words, noise here has a destabilizing effect.

Fig. 3.3 (a) and (b) report the effect of changing variance on the dynamics of the model where we select $\sigma^2 \in (0.1, 1)$ and $a = 0.9$ in Fig. 3.3(a) and $a = 1.2$ in Fig. 3.3(b) without noise. From Fig. 3.1 we know that $a = 0.9$ and $\sigma^2 = 1$ correspond to chaotic behaviour. As expected, the bifurcation diagram in Fig. 3.3(a) indicates the increasing magnitudes of the fluctuations as variance increases, however the positive Lyapunov exponent reported in 3.3(a) indicates that the system stays in the chaotic regime for $\sigma^2 \in (0.1, 1)$. For $a = 1.2$, Fig 3.3(b) indicates that the system stays in the stable regime for various $\sigma^2 \in (0.1, 1)$. Surprisingly, the Lyapunov exponent plot indicates that the system become more stable as the variance increases. In

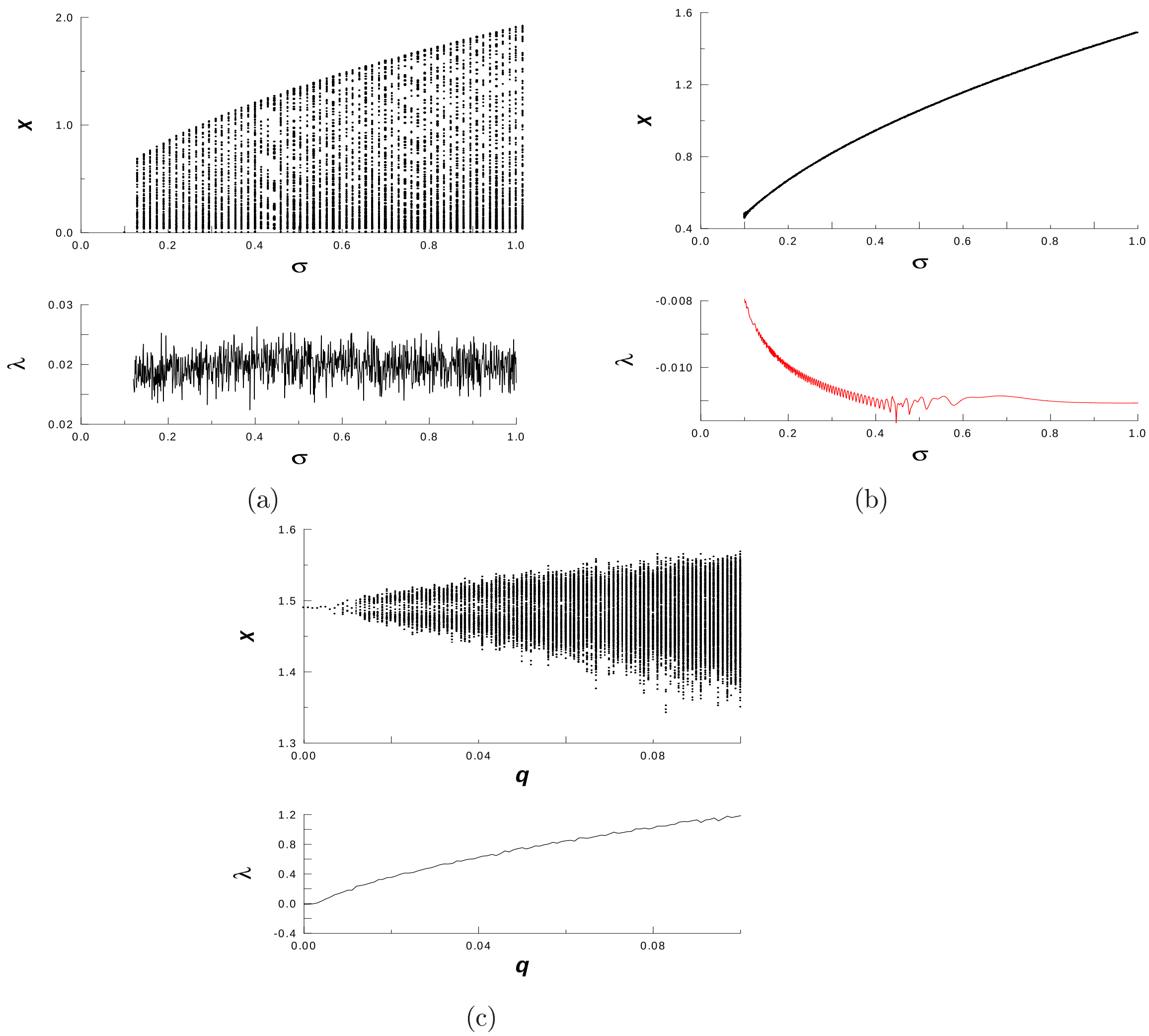


FIGURE 3.3. Trend versus fundamentalists—Bifurcation Diagram and Lyapunov Exponent Plot: (a) a=0.9 and $\sigma^2 \in (0,1)$ without noise; (b) a=1.2 and $\sigma^2 \in (0,1)$ without noise; (c) a=1.2 with uniformly distributed noise on $[-q,q]$ for $q \in [0,0.1]$

both cases, the dynamical behaviour of the system is more affected by the relative risk ratio a, rather than the variance σ^2.

In Fig. 3.3(c), we select $a = 1.2$ and vary the strength of the external noise ϵ_t, uniformly distributed on the interval $[-q,q]$, by varying $q \in [0,0.1]$. As reported earlier, when $a = 1.2$ and without noise ($q = 0$), the system is stable. Fig. 3.3(c) reports the bifurcation diagram and the Lyapunov exponent for $q \in [0,0.1]$. It indicates the system changes from stable to unstable at $q = 0.003$ and then becomes more and more chaotic when q increases. Therefore the intensity of the noise (characterized by the parameter q here) does affect the dynamics of the model and, in general, it destabilizes the dynamics of the system.

The above numerical simulations suggest that:

- When the chartists are more risk averse, the market is dominated by the fundamentalists and the prices converge to the fundamental value. When the fundamentalists are more risk averse, the market become unstable, even chaotic.

- The dynamics of the model is more affected by the relative risk ratio a, rather than the variance σ^2.
- The external noise has significant affect on the dynamical behaviour of the model. It can destabilize an otherwise stable dynamics.

Further numerical simulations (not reported here) confirm that, when $d < R$, the fundamental equilibrium is globally stable, no matter the degree of their risk aversion for both groups. When $d > (a+1)R$, the fundamental equilibrium is locally unstable.

Case 2: $L \geq 2$

We now turn to case when $L > 1$. For $L \geq 2$, one can check that the equilibrium of the system is same as the case when $L = 1$. It then follows from Lemma 3.1 that the system has either one unique equilibrium E_1 or three equilibria E_1, E_2 and E_3. In this more general case, we are more interested in the stability of the fundamental equilibrium. To study the stability of the fundamental equilibrium E_1, we can write the high order difference equation as an equivalent first order difference system. Denote $\mathbf{x} = (x_1, \cdots, x_{L+2})$ to be an $L+2$ dimensional vector and

$$\begin{aligned}
F(\mathbf{x}) &= \tfrac{d}{LR}\tfrac{1}{a(1+m(\mathbf{x}))+(1+v_h(\mathbf{x}))(1-m(\mathbf{x}))}(1+v_h(\mathbf{x})(1-m(\mathbf{x}))[x_1+\cdots+x_L],\\
m(\mathbf{x}) &= \tanh\left[\tfrac{\beta}{2a_1\sigma^2}(Rx_2-x_1)\left(\tfrac{Rx_2}{1+\tilde{h}(\mathbf{x})}+\tfrac{d(x_3+\cdots+x_{L+2})/L-Rx_2}{a}\right)-\tfrac{\beta C}{2}\right],\\
v_h(\mathbf{x}) &= \mu\left[1-\left(1+\tfrac{1}{L}\textstyle\sum_{i=1}^{L}(x_i-\tfrac{x_1+\cdots+x_L}{L})^2\right)^{-\xi}\right] = v_h(x_1,x_2,\cdots,x_L),\\
\tilde{v}_h(\mathbf{x}) &= v_h(x_3,x_4,\cdots,x_{L+2}).
\end{aligned} \tag{3.12}$$

To analyse the local stability and bifurcations of equations (3.8), we write it as the equivalent $L+2$ dimensional system

$$\begin{cases} x_{1,t+1} = & F(\mathbf{x}_t)\\ x_{2,t+1} = & x_{1,t}\\ & \vdots\\ x_{L+2,t+1} = & x_{L+1,t},\end{cases} \tag{3.13}$$

where F and m are defined by (3.12). Then the stability of E_1 is equivalent to the stability of $\mathbf{x} = \mathbf{0}$ of system (3.13). We readily find that the Jacobian matrix of the system (3.13) at $\mathbf{x} = \mathbf{0}$ is given by the $L+2$ square matrix

$$J = \begin{pmatrix} -\gamma & -\gamma & \cdots & -\gamma & -\gamma & 0 & 0\\ 1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 1 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 1 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 1 & 0\end{pmatrix}, \tag{3.14}$$

where

$$\gamma \equiv -\frac{1}{L}\frac{d}{R}\frac{1-m^{eq}}{a+1+(a-1)m^{eq}}. \tag{3.15}$$

Then the characteristic equation is given by

$$D(\lambda) \equiv \lambda^2[\lambda^L + \gamma\lambda^{L-1} + \gamma\lambda^{L-2} + \cdots + \gamma\lambda + \gamma] = 0. \tag{3.16}$$

Obviously, 0 is a double eigenvalue. Using Jury's Test (see appendix), we have following result.

Lemma 3.4. *The zeros of the characteristic polynomial*

$$P(\lambda) \equiv \lambda^L + \gamma\lambda^{L-1} + \gamma\lambda^{L-2} + \cdots + \gamma\lambda + \gamma \tag{3.17}$$

lie inside the unit circle if and only if

$$-\frac{1}{L} \leq \gamma < 1. \tag{3.18}$$

Based on Lemma 3.4 and (3.15), we have the following local stability result.

Theorem 3.5. *Let $d > 0$. The fundamental steady state E_1 of the system (3.13) is locally asymptotically stable if and only if*

$$\frac{d}{R}\frac{1-m^{eq}}{a+1+(a-1)m^{eq}} < 1. \tag{3.19}$$

We note that the condition (3.19) is independent of the lag length L. In particular, when $a_1 = a_2$, the condition can be written as

$$m^* \equiv 1 - 2R/d < m^{eq}, \tag{3.20}$$

which is the same condition derived by Brock and Hommes [7]. Thus, Theorem 3.5 is a generalization of their result to the situation of different risk aversion coefficients. The region of stability of the fundamental steady state in the $(a, d/R)$ plane is shown in Fig. 3.4, where the four regions A, B, C and D are divided by $\frac{d}{R} = 1, 1 + \frac{1+m^{eq}}{1-m^{eq}}a$ and $1 + a$, respectively. Following Theorem 3.5, the fundamental steady state is locally stable in A and B and unstable in C and D. Furthermore, we know from Lemma 3.1 that there exist two other steady states E_2 and E_3 in C and D.

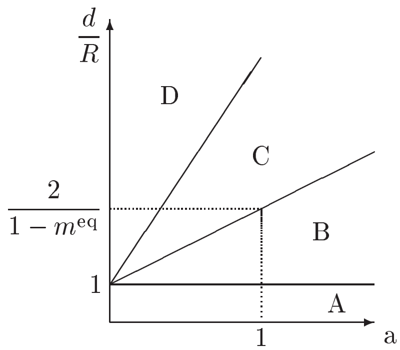


FIGURE 3.4. Stability region of the fundamental steady state E_1: stable in A and B and unstable in C and D

Let a^* be the solution indicated by Lemma 3.2. Then one can verify that it is also the solution of the following equation

$$\frac{d}{R}\frac{1-m^{eq}}{a+1+(a-1)m^{eq}} = 1; \tag{3.21}$$

that is,

$$a^* = [\frac{d}{R} - 1]\frac{1-m^{eq}}{1+m^{eq}}. \tag{3.22}$$

Therefore, E_1 is locally asymptotically stable for $a > a^*$.

Numerical simulations

In the following simulations, we choose

$$R = 1.1, d = 1.2, C = 1.0, \sigma^2 = 1.0, a_1 = 1.0, \beta = 3.8, \gamma = 0 \qquad (3.23)$$

Then, with the selection in (3.23), $a^* = 4.06375$ (defined by (3.22)). For $L = 2$, we choose $a_2 = a = 0.5, 0.8, 0.9, 1.0, 1.1$, respectively. Fig. 3.5(a1) shows the phase plots of (x_t, m_t) plane and Fig. 3.5(a2)-(a3) plot the corresponding time series of $\{x_t\}$ without noise (a2) and with noise (a3). Fig. 3.5(b1) and (c1) show the phase plot (x_t, m_t) for $L = 5$ and $L = 10$, respectively, where we select $a_2 = a = 0.2, 0.5, 0.8, 1.0, 1.2$. The corresponding time series are plotted in Fig. 3.5 (b2) and (c2) without noise and Fig. 3.5 (c3) and (c3) with noise.

The above numerical simulations suggest that:

- Just as the case when $L = 1$, there exists a second bifurcation value $a^{**} \in (1.0, 1.2)$ for $L = 2, 5$ and 10.
- When lag length L increases, the attractors on the (x_t, m_t) plane become more complicated. Say, for $a = 0.5$, when $L = 2$, the prices switch between an unstable phase of an upward trend and a stable phase with prices close to the fundamental value; when $L = 5$ and $L = 10$, the prices fluctuate away from the fundamental value; the periods of upward trend for $L = 10$ is longer than the case of $L = 5$.
- The external noise has a more significant affect on the dynamical behaviour of the model with short lag length (say $L = 2$) than long lag length (say $L = 5$ and 10). Also, it has more affect for high ratio a than for low ratio a. In other words, when the system exhibits chaotic behaviour without noise, adding noise could have no significant affect on the dynamics, however, if the system without noise is stable, adding noise can lead to significant changes on the dynamics of the system.

Bifurcation diagrams and the Lyapunov exponent plots are shown in Fig. 3.6 where (a) is for $L = 2$, (b) is for $L = 5$ and (c) is for $L = 10$. There are some differences, however it is in general not clear whether increasing of lag length L stabilize or destabilizes the dynamics. When $\mu > 0$, further numerical simulations (not reported here) indicate not much difference in dynamical behaviour.

3.2. **Fundamentalists versus Contrarians.**

In this part, we assume that $d < 0$ in (3.8), that is the type 2 investors are contrarians. The adaptive belief system is thus identical to system (3.8) but now with $d < 0$. As in the previous part, we first consider the case $L = 1$ and then move to the case when $L > 1$.

Case 1: $L = 1$

When $L = 1$, our adaptive belief system is also given by (3.9). Since $d < 0$, one can see that the only steady state is the fundamental equilibrium E_1. However now, the system may have a two cycle. More precisely, we have the following result.

Lemma 3.6. *(***Existence of steady state and 2-cycles and their stability***) Let $m^{eq} = \tanh(-\frac{\beta C}{2})$, $\bar{m} = 1 + \frac{2aR}{d - R(a-1)}$ and $\bar{x}$ be the positive solution (if it exists) of*

$$\tanh\left[\frac{\beta}{2\sigma^2 a_1}(R+1)(R - \frac{d+R}{a})(\bar{x})^2 - \frac{\beta C}{2}\right] = \bar{m}. \qquad (3.24)$$

- *The fundamental steady state $E_1 = (0, m^{eq})$ is the unique steady state; it is globally stable for $-R < d < 0$;*
- *For $-(a+1)R < d < -R$ there are two possibilities:*

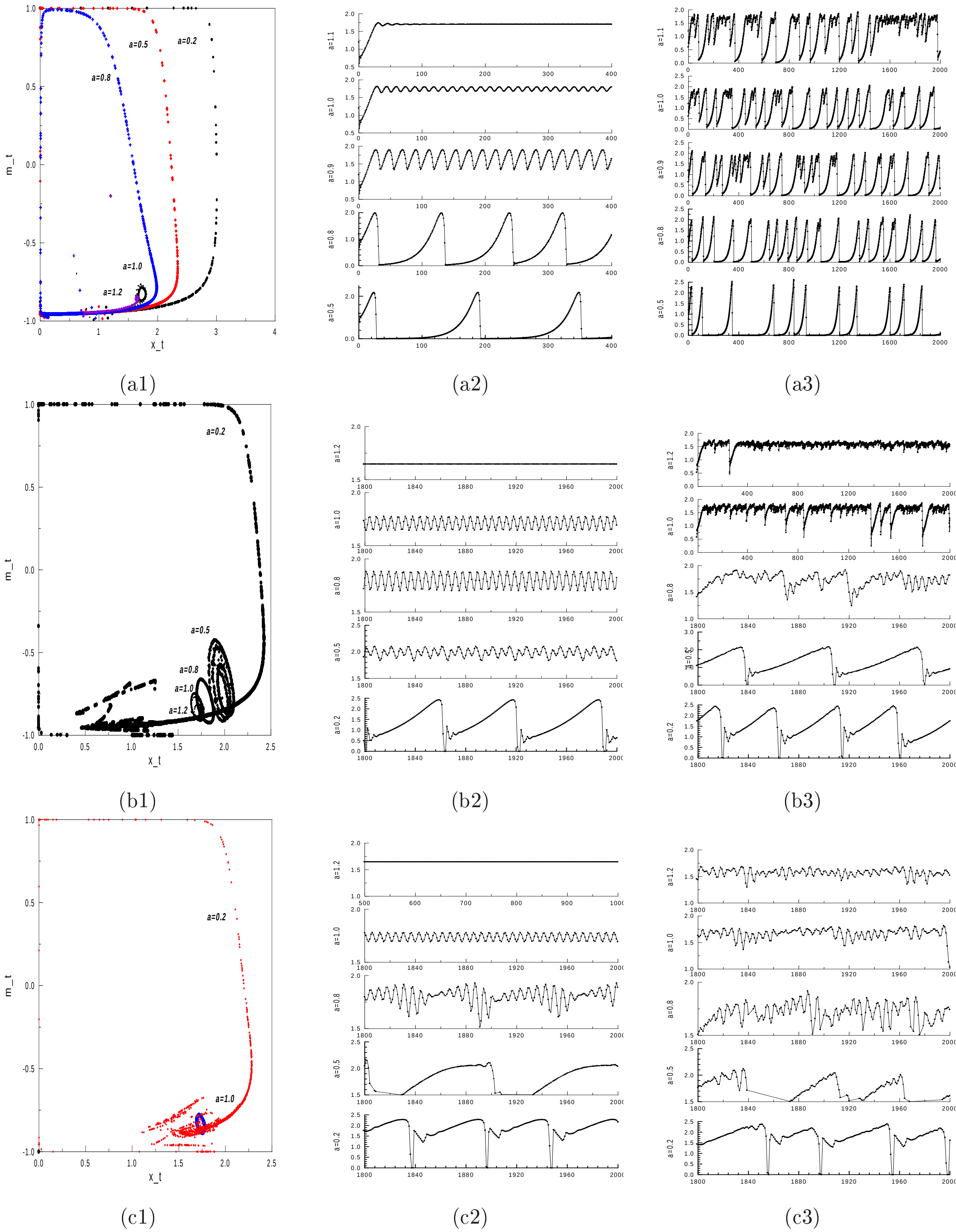
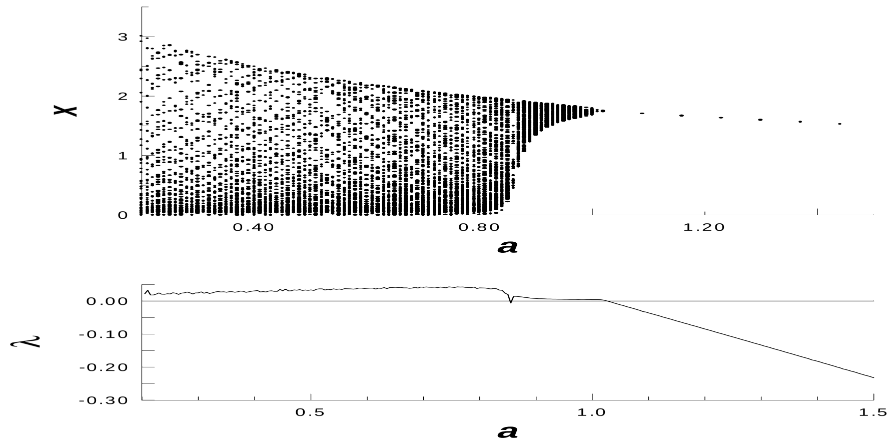
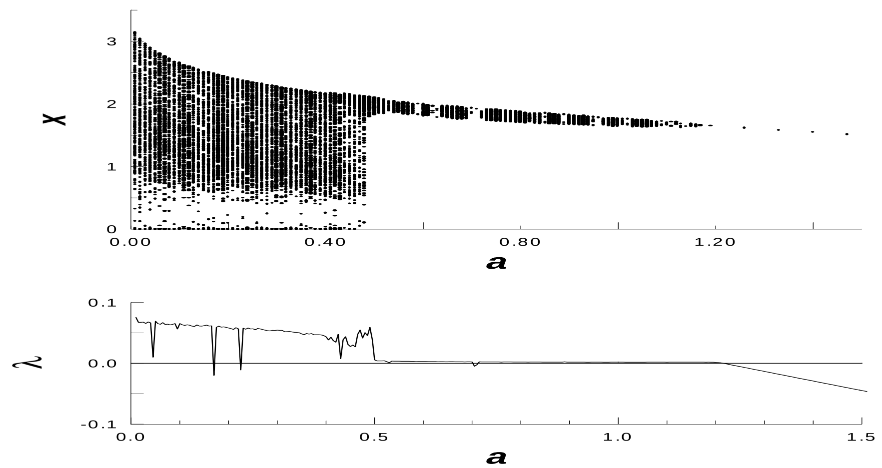


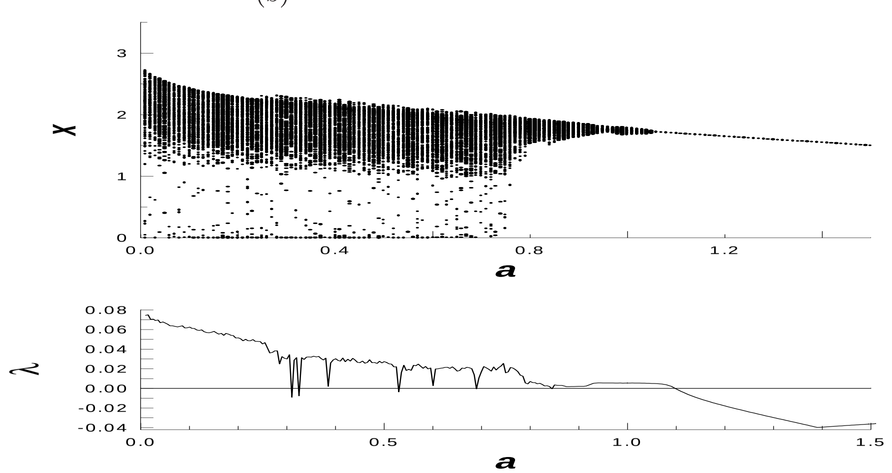
FIGURE 3.5. Trend versus fundamentalists: Phase plots (x, m) for $L = 2$ (a1), $L = 5$ (b1) and $L = 10$ (c1); Time series of x_t without noise (for $L = 2$ (a2), $L = 5$ (b2) and $L = 10$ (c2)) and with noise (for $L = 2$ (a3), $L = 5$ (b3) and $L = 10$ (c3))
.



(a)



(b)



(c)

FIGURE 3.6. Trend versus fundamentalists: Bifurcation diagram and Lyapunov exponent plot—(a) $L = 2$; (b) $L = 5$; (c) $L = 10$

  - *if* $\bar{m} < m^{eq}$ *then* E_1 *is the unique, globally stable steady state.*
  - *if* $\bar{m} > m^{eq}$ *then* E_1 *is unstable and there exists a periodic two cycle* $\{(\bar{x},\bar{m}),(-\bar{x},\bar{m})\}$*;*
- *For* $d < -(a+1)R$*,* E_1 *is unstable and there exists a periodic two cycle* $\{(\bar{x},\bar{m}),(-\bar{x},\bar{m})\}$*.*

The above result has been obtained by Brock and Hommes [7] in the case when $a = 1$ and it can be proved similarly. Furthermore, it leads to the following periodic doubling bifurcation result (its proof follows from Lemma 3.6 and $\bar{m} = m^{eq}$).

Lemma 3.7. *Let* $\{(\bar{x},\bar{m}),\ (-\bar{x},\bar{m})\}$ *be the two cycle as in Lemma 3.6. Assume* $-(a+1)R < d < -R$*,* $C > 0$ *and denote*

$$\bar{a} \equiv \frac{d+R}{R}\frac{m^{eq}-1}{m^{eq}+1}. \tag{3.25}$$

Then $\bar{a}$ *is a period doubling bifurcation value; that is, the system (3.9) has unique fundamental equilibrium* E_1 *for* $a > \bar{a}$ *and has a (locally) stable 2-cycle for* $a < \bar{a}$ *(with* a *near* $\bar{a}$*).*

In the following numerical simulations, we choose

$$R = 1.1, d = -1.5, C = 1.0, \beta = 4.0, \sigma^2 = 1.0, a_1 = 1.0$$

with different $a_2 = a = 0.05, 0.1, 1.0, 1.5, 2.0$. Fig. 3.7(a) and (b) show the time series of x_t and m_t, respectively for different a and Fig. 3.7(e) plots the attractors in the (x_t, m_t) plane. The corresponding time series in Fig. 3.7(a) and (b) with noise added to the dividend process are shown in Fig. 3.7(c) and (d). It suggests that noise does not seem to do much in this case. Fig. 3.7(e) indicates that; when $a_2 = 2.0$ all the solutions converge to the period 2 equilibrium (period doubling bifurcation); when $a = 1.5$, the orbit converges to an attractor consisting of the two invariant 'circles' created after the secondary Hopf bifurcation of the two cycle. As a decreases, the two circles move closer to each other; when $a = 0.1$, the system seems to be already close to a homoclinic orbit. As a decreases further to $a = 0.05$, the attractor becomes a point, which implies that all the orbits converge to the fundamental equilibrium. In fact, the dynamical behaviour is very similar to the chaotic fluctuations in Brock and Hommes [7]. The bifurcation diagram in Fig. 3.9(f) shows periodic doubling bifurcation of the steady state and the breaking of the invariant circle into strange attractors as a decreases. The chaotic region (characterised by positive λ) is interspersed with many stable cycles where $\lambda < 0$.

Numerical simulations (not reported here) show also that, for $d < -R$, the fundamental equilibrium is globally stable for any $a > 0$; while for $d < -(a+1)R$, the fundamental equilibrium is unstable.

In a market with fundamentalists versus contrarians a small ratio of the relative risk aversion coefficient produces asset price dynamics, with irregular fluctuations around the EMH fundamental. This implies that when the fundamentalists become relatively more risk averse than the contrarians, market instability and chaos ensue.

Case 3: $L > 1$

We now turn to the case where $d < 0$ and $L > 1$. The adaptive belief system is identical to the system (3.12) and (3.13). Following the discussion on the system (3.12) and using Lemma 3.4 and (3.15), we have the following local stability result.

Theorem 3.8. *Let* $d < 0$*. The unique steady state* E_1 *of the system (3.13) is locally stable if and only if*

$$0 \le -\frac{d}{R}\frac{1-m^{eq}}{a+1+(a-1)m^{eq}} < L. \tag{3.26}$$

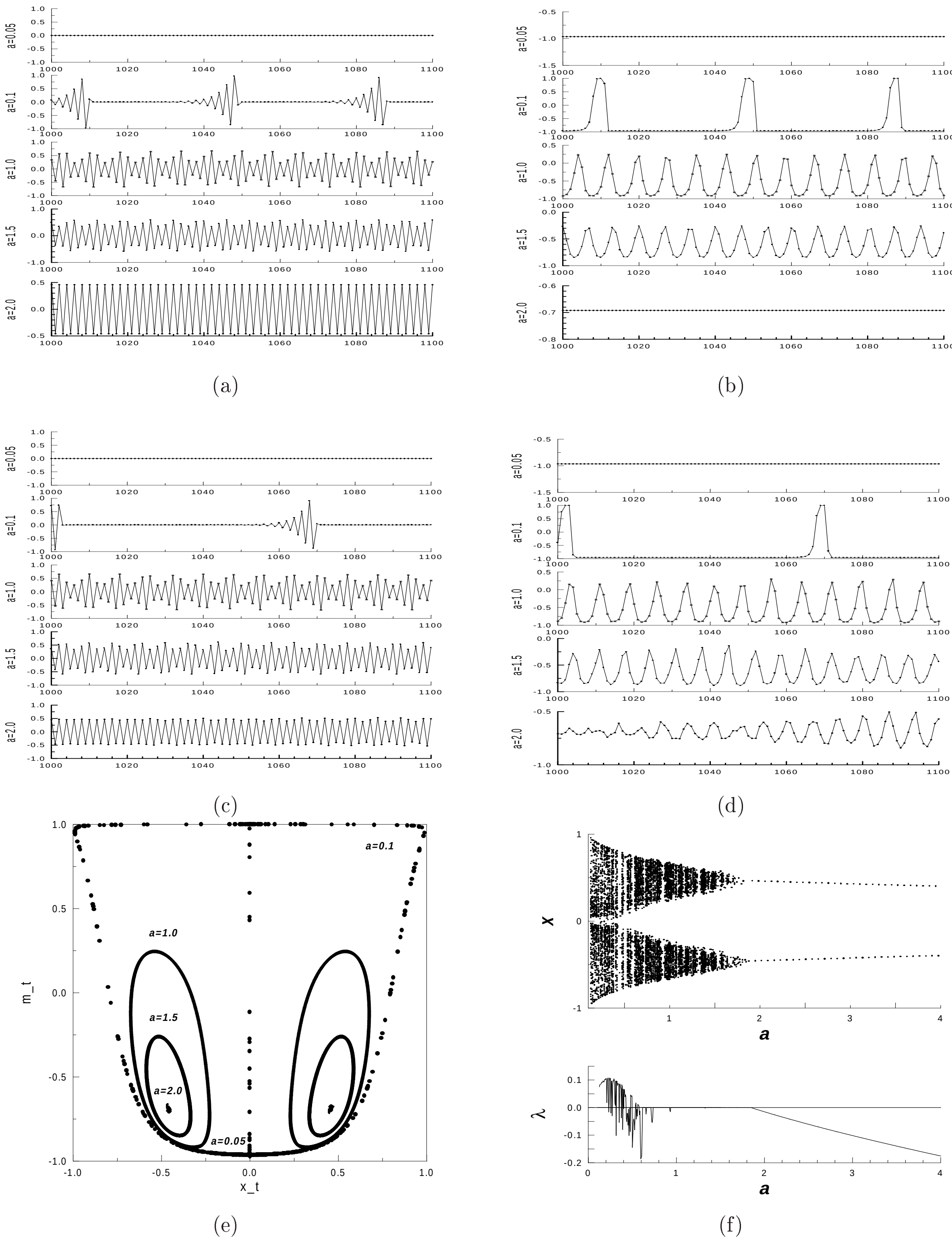


FIGURE 3.7. Contrarian versus fundamentalists: the time series of x_t and m_t without noise(a) and (b) and with noise (c) and (d) and the phase plot of (x, m) (without noise) (e) for $a_2 = 0.05, 0.1, 1.0, 1.5, 2.0$; bifurcation diagram and Lyapunov exponent plot (f);

Different from the case when $d > 0$, the stability condition (3.26) depends on the lag length L is illustrated in Fig. 3.8 where the fundamental steady state is locally stable in the region below the line (labelled by A) and unstable in the region above the line (labelled by B). Also, different from the case $d < 0$ and $L = 1$, the system may have no 2-cycle even if $d < -(a+1)R$. In fact, the condition (3.26) can be written as

$$a > a^* \equiv -(1 + \frac{d}{LR})\frac{1 - m^{eq}}{1 + m^{eq}}. \tag{3.27}$$

So, a necessary condition for the occurrence of bifurcation (from the steady state) is

$$d < -LR. \tag{3.28}$$

Therefore, the fundamental steady state is always locally stable when $0 > d > -LR$ for any $a > 0$. The stability region of the fundamental steady state w.r.t the parameter d is proportional to the lag length L. It is in this sense that increasing of the lag length L can stabilize an otherwise unstable dynamics.

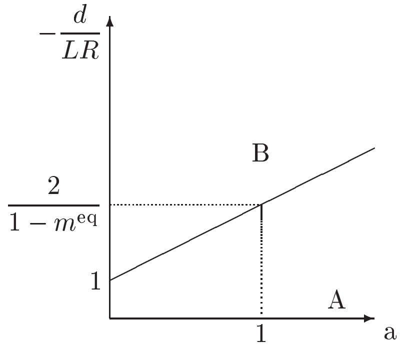


FIGURE 3.8. Stability region of the fundamental steady state E_1: stable in A and unstable in B

Numerical simulations

For $L = 2$, we choose

$$R = 1.1, g = -2.5, \beta = 4.0, a_1 = 1.0, C = 1.0, \sigma^2 = 1.0, \mu = 0.$$

Then $a^* = 7.445$ (defined by (3.27)) implies that the fundamental equilibrium is locally stable for $a > a^* = 7.445$. For $a_2 = a = 0.01, 1, 2, 5, 7, 8$, the time series of x_t and m_t are plotted in Fig. 3.9(a) and (b) without noise and (c) and (d) with noise. The phase plot (without noise) on (x_t, x_{t-1}) is shown in Fig. 3.9(e). Fig. 3.9(a) shows that, when a is around 7, the fundamental solution breaks into periodic circles, in particular, into a 3-cycle. The bifurcation diagram and the Lyapunov exponent plot in Fig. 3.9(f) indicate that, as a decreases, the dynamical behaviour is mainly characterised by periodic cycles. To see closely the Lyapunov exponent plot when $a \in [2, 8]$, we enlarge that part of the plot and, from which, one can see some positive Lyapunov exponent for a is near 2. Noting that when $a < 1$, the solutions exhibit very simple dynamics, characterised by either 3 or 6 cycles. In particular, when $a = 1$, the solution converges to a 3-cycle and when $a = 0.1$ to a 6-cycle. Furthermore, prices oscillate among different phases, including the fundamental equilibrium. Also, we notice that the addition of noise has not much affect when a is large, but it does affect the solution behavior when a is small.

For $L = 5$, we select

$$R = 1.0, g = -5.5, \beta = 4.0, a_1 = 1.0, C = 1.0, \sigma^2 = 1.0, \mu = 0.$$

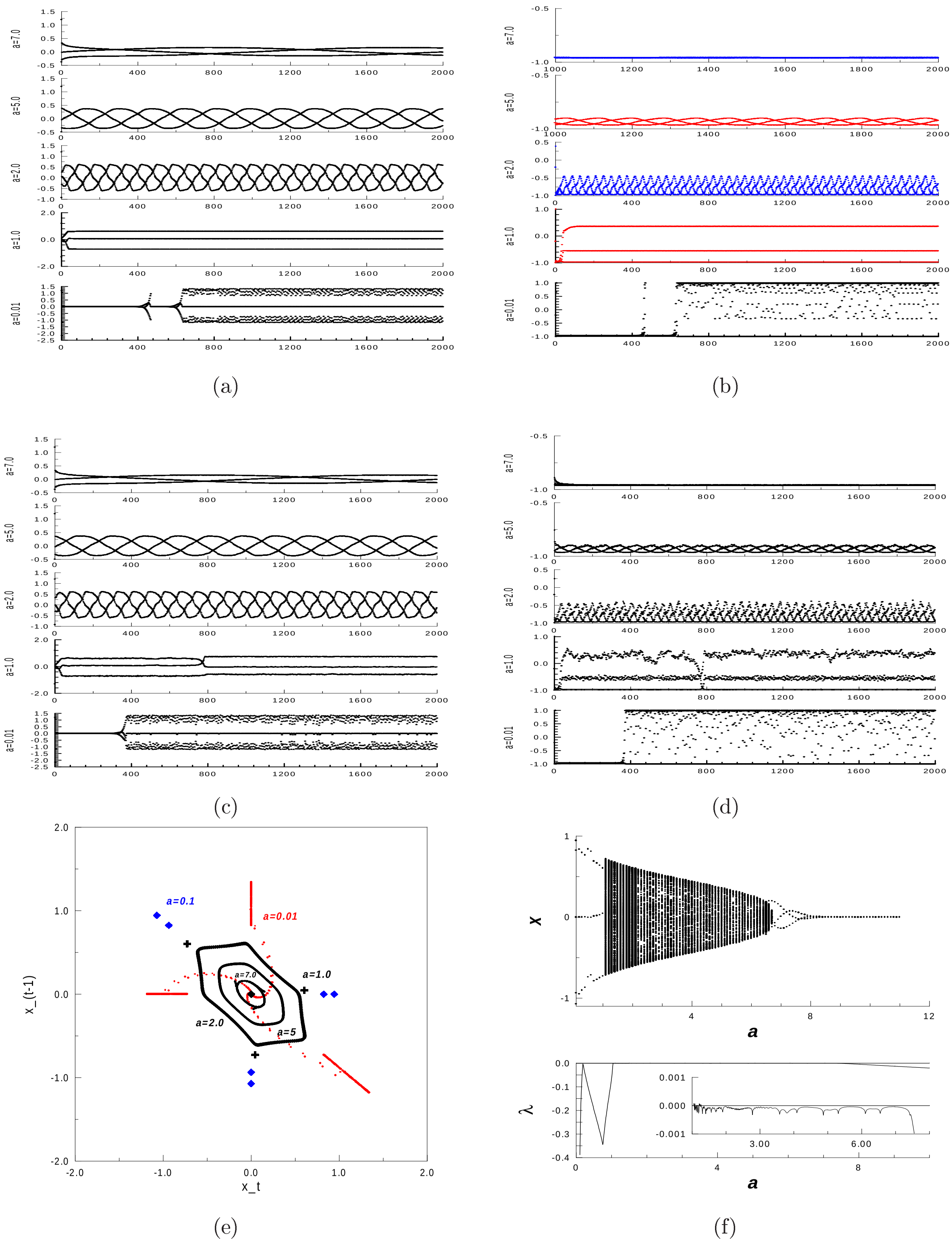


FIGURE 3.9. Contrarian versus fundamentalists–L=2: Time series of x_t and m_t, without noise (a)-(b) and with noise (c)-(d); phase plot of (x_t, x_{t-1}) (e); bifurcation diagram and Lyapunov exponent plot (f)

Then $a^* = 5.46$ (defined by (3.27)) implies that the fundamental equilibrium is locally stable for $a > a^* = 5.46$. For $a_2 = a = 0.05, 1, 3, 5, 7$, the time series of x_t and m_t are plotted in Fig. 3.10(a) and (b) without noise and (c) and (d) with noise. The phase plot on (x_t, x_{t-1}) is shown in Fig. 3.10(e) for $a = 5.0$. Fig. 3.10(a) shows that, as a decreases, the system has more complicated dynamics, in particular, this is indicated by the phase plot in Fig. 3.10(e) with $a = 0.5$. The bifurcation diagram and the Lyapunov exponent plot in Fig. 3.10 illustrate more complicated dynamics as a decreases.

One can see there is much difference in dynamics for different lag length $L = 1, 2$ and 5. When $\mu > 0$, numerical simulations (not reported here) indicate not much difference for both $L = 2$ and $L = 5$.

3.3. **Three beliefs types: Fundamentalists versus Trends.** In this part we consider a combination of three different belief types. As before, type 1 are fundamentalists. Type 2 are trend chasers and type 3 are contrarians. This model can be treated as a combination of the previous two cases. Consequently, we might expect to have some kind of mixture of the dynamics of two beliefs models when the evolution of the state vector is such that the relative importance of the trend chasers or contrarians is diminished. On the other hand at times when all three groups are of roughly equal relative importance we might expect some new dynamic features.

Following the notation in the previous subsections, we assume that

$$f_{1t} = 0, \qquad f_{2t} = d_2 \bar{x}_{t,L_2}, \qquad f_{3t} = -d_3 \bar{x}_{t,L_3} \tag{3.29}$$

and

$$g_{1t} = \sigma^2 v_h(\bar{\sigma}_t^2), \qquad g_{2t} = g_{3t} = 0, \tag{3.30}$$

where $d_2, d_3 > 0$ are constants and $\bar{x}_t, \bar{\sigma}_t^2$ and v_h are defined as before. Let $a_i\,(i = 1, 2, 3)$ be the risk aversion coefficient of type $i\,(i = 1, 23)$ and $\eta = 0, \delta_t = 0$. Then the adaptive belief system (2.19) and (2.20) can be written as follows:

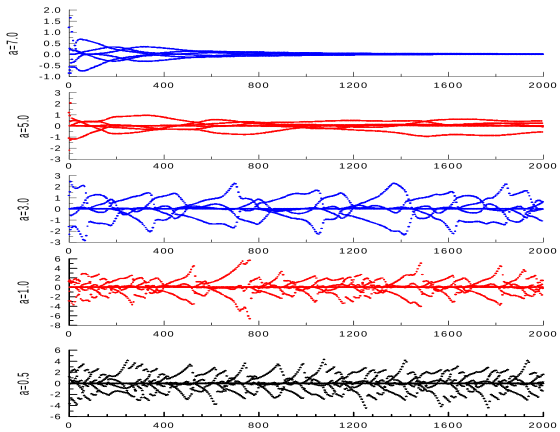
$$\begin{cases} x_t = & \frac{a_1}{R}\frac{(1+h(\bar{\sigma}_t^2))[d_2 a_3 n_{2,t-1}\bar{x}_{t,L_2} - d_3 a_2 n_{3,t-1}\bar{x}_{t,L_3}]}{a_2 a_3 n_{1,t-1} + a_1(1+h(\bar{\sigma}_t^2))[a_3 n_{2,t-1} + a_2 n_{3,t-1}]} \\ n_{1,t} = & \exp[\beta(\frac{1}{a_1\sigma^2(1+h(\bar{\sigma}_{t-1}^2))} R x_{t-1}[R x_{t-1} - x_t] - C)]/Z_t, \\ n_{2,t} = & \exp[\beta(\frac{1}{a_2\sigma^2}[x_t - R x_{t-1}][d_2 \bar{x}_{t-1,L_2} - R x_{t-1}]/Z_t, \\ n_{3,t} = & \exp[\beta(\frac{1}{a_3\sigma^2}[x_t - R x_{t-1}][-d_3 \bar{x}_{t-1,L_3} - R x_{t-1}]/Z_t, \\ Z_t = & \exp[\beta(\frac{1}{a_1\sigma^2(1+h(\bar{\sigma}_{t-1}^2))} R x_{t-1}[R x_{t-1} - x_t] - C)] \\ & + \exp[\beta(\frac{1}{a_2\sigma^2}[x_t - R x_{t-1}][d_2 \bar{x}_{t-1,L_2} - R x_{t-1}] \\ & + \exp[\beta(\frac{1}{a_3\sigma^2}[x_t - R x_{t-1}][-d_3 \bar{x}_{t-1,L_3} - R x_{t-1}] \end{cases} \tag{3.31}$$

As the simplest case, we first assume that $L_2 = L_3 = 1$. Then $\bar{x}_{t,L_2} = \bar{x}_{t,L_3} = x_{t-1}, \bar{\sigma}_t^2 = 0$ and $h(\bar{\sigma}_t^2) = 0$ so that the system (3.31) is reduced to

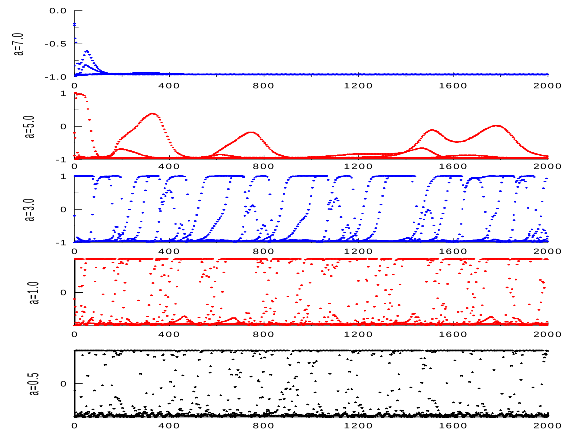
$$\begin{cases} x_t = & \frac{a_1}{R}\frac{d_2 a_3 n_{2,t-1} - d_3 a_2 n_{3,t-1}}{a_2 a_3 n_{1,t-1} + a_1 a_3 n_{2,t-1} + a_1 a_2 n_{3,t-1}} x_{t-1} \\ n_{1,t} = & \exp[\beta(\frac{1}{a_1\sigma^2} R x_{t-1}[R x_{t-1} - x_t] - C)]/Z_t, \\ n_{2,t} = & \exp[\beta(\frac{1}{a_2\sigma^2}[x_t - R x_{t-1}][d_2 x_{t-2} - R x_{t-1}]/Z_t, \\ n_{3,t} = & \exp[\beta(\frac{1}{a_3\sigma^2}[x_t - R x_{t-1}][-d_3 x_{t-2} - R x_{t-1}]/Z_t. \end{cases} \tag{3.32}$$

System (3.32) is equivalent to a third-order difference equation in x_t. Denote

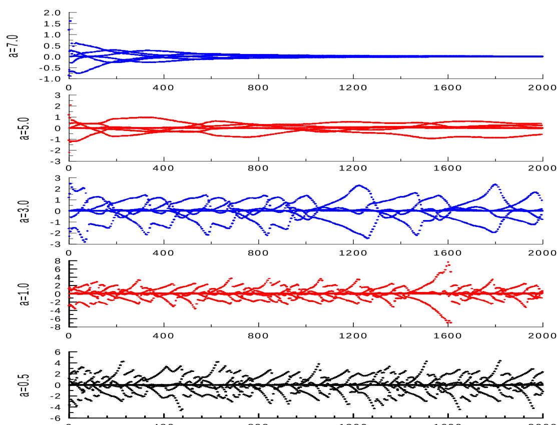
$$A_2 = \frac{a_2}{a_1}, A_3 = \frac{a_3}{a_1}, D_2 = \frac{d_2}{R}, D_3 = \frac{d_3}{R}.$$



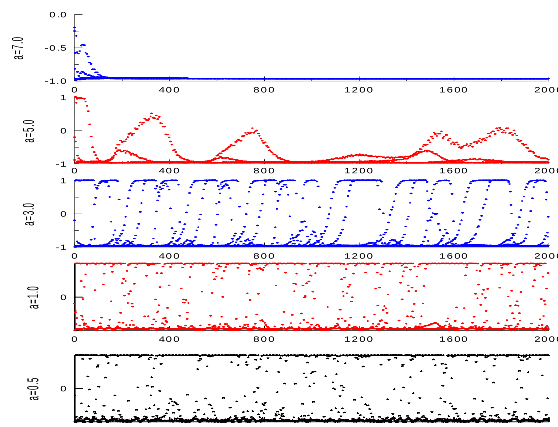
(a)



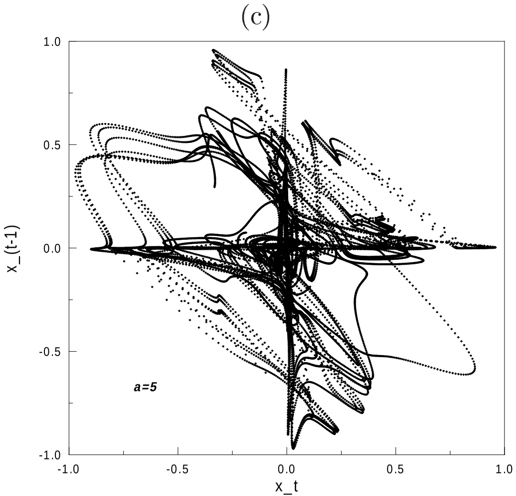
(b)



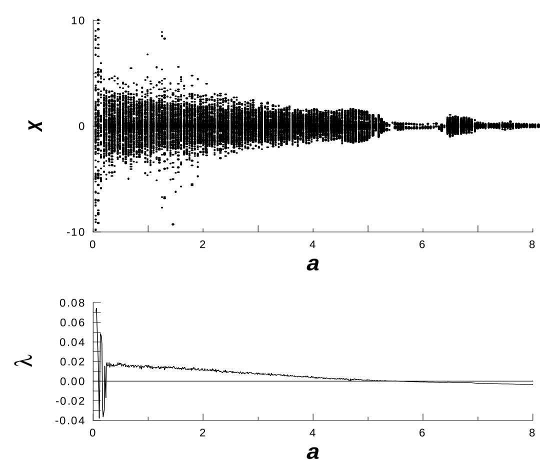
(c)



(d)



(e)



(f)

FIGURE 3.10. Contrarian versus fundamentalists–L=5: Time series of x_t and m_t, without noise (a)-(b) and with noise (c)-(d); phase plot of (x_t, x_{t-1}) for $a = 5$ (e); bifurcation diagram and Lyapunov exponent plot (f)

Then we have

$$\begin{cases} x_{t+1} &= f(x_t, y_t, z_t) \\ y_{t+1} &= x_t \\ z_{t+1} &= y_t, \end{cases} \tag{3.33}$$

where

$$\begin{aligned} f(x,y,z) &= \tfrac{D_2 A_3 g_2(x,y,z) - D_3 A_2 g_3(x,y,z)}{A_2 A_3 g_1(x,y,z) + A_3 g_2(x,y,z) + A_2 g_3(x,y,z)} x \\ g_1(x,y,z) &= exp[\tfrac{\beta R}{a_1 \sigma^2} y(Ry - x) - C] \\ g_2(x,y,z) &= exp[\tfrac{\beta}{a_2 \sigma^2}(x - Ry)(d_2 z - Ry)] \\ g_3(x,y,z) &= exp[\tfrac{\beta}{a_3 \sigma^2}(x - Ry)(-d_3 z - Ry)] \end{aligned} \tag{3.34}$$

The Jacobian matrix of the linearized system at the fundamental equilibrium $(0,0,0)$ is given by

$$J = \begin{pmatrix} \alpha & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.35}$$

where

$$\alpha = \frac{D_2 A_3 - D_3 A_2}{A_2 A_3 e^{-C} + A_2 + A_3}.$$

Obviously, 0 is a double eigenvalue and α is the third eigenvalue. Therefore, the fundamental equilibrium is locally asymptotically stable if and only if

$$|D_2 A_3 - D_3 A_2| < A_2 + A_3 + A_2 A_3 e^{-C}. \tag{3.36}$$

Let us take A_2 as a bifurcation parameter. Then condition (3.36) can be written as

$$\begin{array}{lll} A_2 > \frac{D_2 - 1}{D_3 + 1 + A_3 e^{-C}} A_3 & \text{if} & D_3 < 1 + A_3 e^{-C} \\ \frac{D_2 - 1}{D_3 + 1 + A_3 e^{-C}} A_3 < A_2 < \frac{D_2 + 1}{D_3 - 1 - A_3 e^{-C}} A_3 & \text{if} & D_3 > 1 + A_3 e^{-C}. \end{array} \tag{3.37}$$

The condition (3.37) indicates that:

- if $D_2 < 1$ and $D_3 < 1 + A_3 e^{-C}$, the fundamental equilibrium is locally asymptotically stable for any $A_2 > 0$;
- if $D_2 < 1$ and $D_3 > 1 + A_3 e^{-C}$, the fundamental equilibrium is locally asymptotically stable for $A_2 < \frac{D_2 + 1}{D_3 - 1 - A_3 e^{-C}} A_3$;
- if $D_2 > 1$ and $D_3 < 1 + A_3 e^{-C}$, the fundamental equilibrium is locally asymptotically stable for $A_2 \geq \frac{D_2 - 1}{D_3 + 1 + A_3 e^{-C}} A_3$;
- if $D_2 > 1$ and $D_3 > 1 + A_3 e^{-C}$, the fundamental equilibrium is locally asymptotically stable for $A_2 \in (b_1, b_2)$, where $b_1 = \frac{D_2 - 1}{D_3 + 1 + A_3 e^{-C}} A_3$ and $b_2 = \frac{D_2 + 1}{D_3 - 1 - A_3 e^{-C}} A_3$.

Numerical simulations on the nonlinear system also show that the fundamental equilibrium is asymptotically stable for $D_2 < 1$ and $D_3 < 1 + A_3 e^{-C}$. We are more interested in the case where trend chasers extrapolate strongly while contrarians extrapolate weakly, that is when $D_2 > 1$ and $D_3 < 1 + A_3 e^{-C}$. To compare with the case of fundamentalists versus trend chasers, we select

$$R = 1.1, C = 1.0, a_1 = 1, \sigma^2 = 1, \beta = 3.5$$

and $d_2 = 1.2, d_3 = 0.1, a_3 = 3.0$. Then $D_2 > 1$, $D_3 < 1 + A_3 e^{-C}$ and the condition (3.37) implies the fundamental equilibrium is locally stable for $A_2 > 0.124$. The bifurcation diagram and Lyapunov exponent plot in Figure 3.11(a) on our nonlinear model indicate that the fundamental equilibrium bifurcates when A_2 is near 0.124. Depending on different initial values, the bifurcation is either a positive equilibrium or a negative one. In some cases, the bifurcating equilibria may switch from a positive one to an negative one as A_2 decreases. Comparing with Figure 3.2(a)

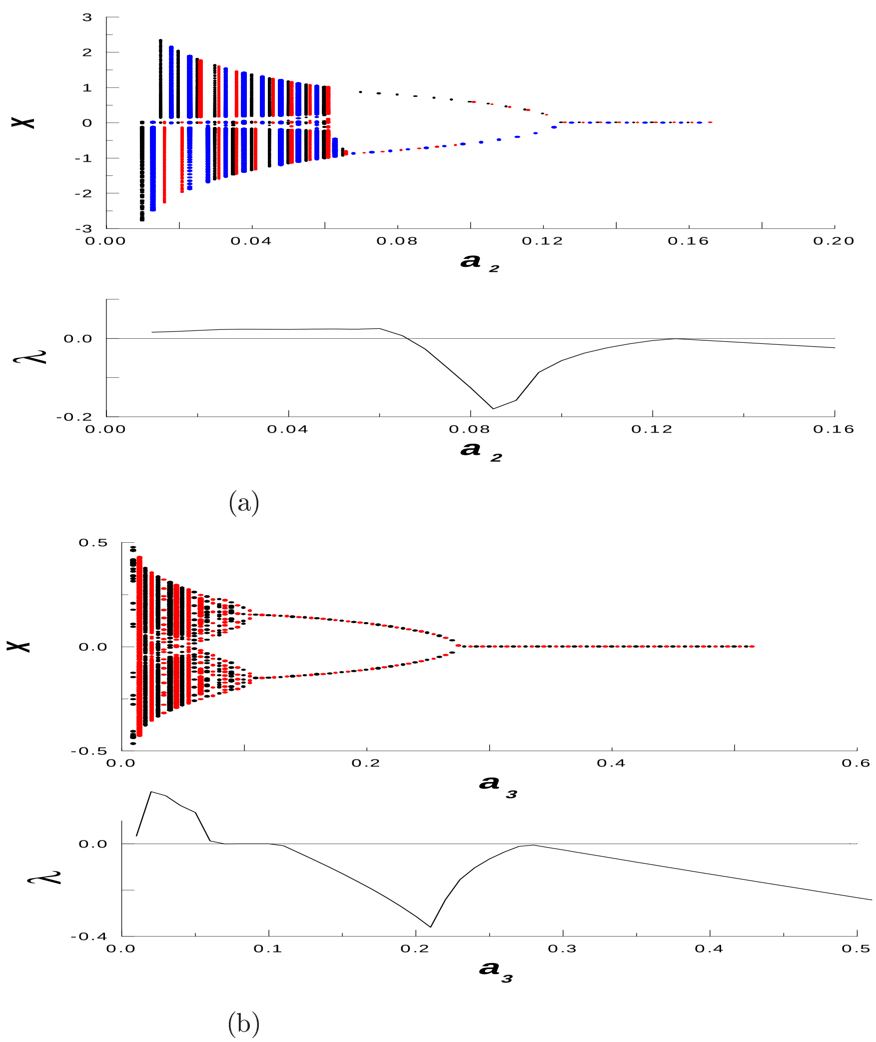


FIGURE 3.11. Three beliefs with $L_1 = L_2 = 1$: Bifurcation diagram and Lyapunov exponent plot

and (b), we can see that adding the third group can stabilize the fundamental equilibrium, even if the contrarians extrapolate only weakly ($d_3 = 0.1$) and are more risk averse (reluctant to be involved in the markets ($A_3 = 3.0$)). We note that $\frac{D_2-1}{D_3+1+A_3e^{-C}}A_3$ is an increasing function of A_3. So, when the contrarians become less risk averse (i.e. A_3 decreases), the equilibrium becomes more stabilized in terms of the risk attitude of trend chasers.

On the other hand, when $D_3 > 1 + A_3e^{-C}$, that is when the contrarians extrapolate strongly, the fundamental equilibrium is locally stable for $A_2 \in (b_1, b_2)$. Let $d_2 = 1.2, d_3 = 2$ Depending on the risk attitude of the contrarians, the size of the interval for A_2 varies. Say, $(b_1, b_2) = (0.0285, 4.64)$ for $a_3 = 1$ and $(b_1, b_2) = (0.015, 1.648)$ for $a_3 = 0.5$. Numerical simulations indicate that adding the contrarians may result in the stabilization of the fundamental equilibrium when they are more risk averse and the dynamic explode when they become less risk averse.

Noting the symmetry of A_2 and A_3, D_2 and D_3 in the condition (3.36), we can also treat A_3 as a parameter. To compare with the case of fundamentalists versus contrarians, we choose

$$R = 1.1, C = 1.0, a_1 = 1.0, \sigma^2 = 1, \beta = 4.$$

When $d_2 = 1, d_3 = 1.5$ and $a_2 = 2.0$, the bifurcation diagram and Lyapunov exponent plot are shown in Figure 3.11(b). In this case, local stability condition is given by $A_3 > 0.275$. Numerical simulations show that

- When both trend chasers and contrarians extrapolate weakly ($D_2 < 1+A_2e^{-C}$ and $D_3 < 1$), the fundamental equilibrium is asymptotically stable;
- When contrarians extrapolate strongly ($D_3 > 1$), the fundamental equilibrium is stabilized when adding trend chasers as a third group, even if they extrapolate weakly ($D_2 < 1 + A_2e^{-C}$) and are more risk averse ($A_2 = 2.0$). Also, the maximum magnitudes of the oscillations are reduced;
- When both trend chasers and contrarians extrapolate strongly ($D_2 > 1 + A_2e^{-C}$ and $D_3 > 1$), by adding trend chasers as a third group, the fundamental equilibrium may be stabilized when they are more risk averse or be destabilized when they become less risk averse (leading to exploding dynamics).

It would be interesting to consider the price dynamics for different combinations of the window lengths L_1 and L_2 in the three beliefs model. We omit this discussion to short the paper.

4. Conclusion

Adopting the framwork developed by Brock and Hommes [6, 7] as a starting point, the present paper has incorporated risk and learning schemes into an asset pricing model with heterogeneous beliefs. Fundamentalists, trend chasers and contrarians as the main trading groups cause various dynamics of price changes. It is well known that different types of investors have different risk attitudes. However, how their risk attitudes affect the dynamics of asset prices has not received a great deal of attention in the literature. The current paper has bought to address this deficiency.

This paper provides an explicit study on how the dynamics of asset pricing is affected by different risk attitudes of different types of investors and different learning schemes. We might summarize our results as follows:

- The dynamics of asset pricing is affected by the relative risk attitudes of different types of investors (measured by a). In general, when the chartists (trend chasers or/and contrarians) are more risk averse, the prices converge either to the fundamental equilibrium (this is the case when the fundamentalists are less risk averse), or to a positive (or negative) equilibrium (in the model of fundamentalists and trend chasers), or to a period doubling cycle (in the model of fundamentalists and contrarians). However, when the fundamentalists are relatively more risk averse, the price dynamics become unstable and lead to complicated dynamics, such as periodic cycles and chaotic behavior.
- Learning schemes affect the models differently. In the model of fundamentalists and trend chasers, increasing the window length L does not affect the (local) bifurcation values (when other parameters are fixed) and leads to the prices fluctuating away from the fundamental value. But in the model of fundamentalists and contrarians, an increase in L leads to an increase in the bifurcation values and the price dynamics become more complicated.
- The price dynamics is more affected by the relatively risk aversion ratio a, rather than the size of variance σ^2.
- The external noise has a significant effect on the dynamical behavior of the model. It can destabilize an otherwise stable dynamics.

- When adding a third belief into two beliefs models, as expected, we have some kind of mixture of the dynamics of two beliefs models when the evolution of the state vector is such that the relative importance of the trend chasers or contrarians is diminished. On the other hand at times when all three groups are of roughly equal relative importance we obtain some new dynamic features. In particular, mixing the fundamentalists with the two other groups can significantly stabilize the price dynamics, even if either of the other two groups extrapolates weakly and is more risk averse.

In summary, we find that the resulting dynamical behavior is considerably enriched and has some significant differences compared to the original Brock-Hommes [7] analysis. However the interaction of external noise with the nonlinear dynamics of the model is a topic that requires more extensive research. The techniques discussed in Arnold [2] may be useful in this regard.

The asset pricing model of this paper is established by using the scenario of the Walrasian auctioneer to obtain the market clearing price. As pointed by Grossman [15], the demand function in the Walrasian scenario specifies a desired level of holdings of the security at each particular price p_t, irrespective of whether or not p_t is a market clearing price. Instead, Grossman assumes that the consumer faces a price that is a real offer of another person, or the outcome of a market process. Hence the fact that a particular price is offered is itself information about what someone else thinks about the future payoff. Therefore to extend the model discussed here to the non-Walrasian scenario to incorporate costs of information is an interesting problem which we leave to future research work.

Appendix A.

Proof of Lemma 3.1: Let $\bar{x}, \bar{m}$ be constants satisfying

$$\begin{aligned} \bar{x} &= \frac{d}{R}\frac{1-\bar{m}}{a+1+(a-1)\bar{m}}\bar{x} \\ \bar{m} &= \tanh\left[\frac{\beta}{2a_1\sigma^2}(R-1)\left(R+\frac{d-R}{a}\right)(\bar{x})^2-\frac{\beta C}{2}\right] \end{aligned} \tag{A.1}$$

The first equation implies $\bar{x}=0$, or $\bar{m}=m^*$. $\bar{x}=0$ implies $\bar{m}=m^{eq}$, which leads to the existence of $E_1(0,m^{eq})$. Regarding to the existence of E_2 and E_3, we consider three different cases.

(i) Assume $d<R$, then $\frac{aR}{aR+(d-R)}>1$, which implies $\bar{m}<-1$. Hence there is no solution from the equation (3.10) and E_1 is the only equilibrium.

(ii). Assume $d>(a+1)R$. Then one can verify that $0<\bar{m}<1$ and the equation (3.10) always has two solutions $\pm x^*$. Therefore, apart from E_1, we have two other equilibria E_2, E_3.

(iii). Assume $R<d<(a+1)R$. Then $-1<\bar{m}<0$ and the equation (3.10) has solutions if and only if $m^*>m^{eq}$.

We next consider the stability of equilibrium E_1. Let $y_t=x_{t-1}$ and $z_t=y_{t-1}=x_{t-2}$ and one can rewrite as the following 3 dimensional difference system

$$\begin{aligned} x_{t+1} &= F(x_t,y_t,z_t) \\ y_{t-1} &= x_t \\ z_{t-1} &= y_t \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} F(x,y,z) &= \frac{d}{R}\frac{1-m}{a+1+(a-1)m}x, \\ m &= \tanh\left[\frac{\beta}{2a_1\sigma^2}(Ry-x)\left(Rz+\frac{dz-Ry}{a}\right)-\frac{\beta C}{2}\right] \end{aligned} \tag{A.3}$$

Calculating the Jacobian matrix of the system at $(x, y, z) = (0, 0, 0)$, we obtain the three eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = F_x(0, 0, 0)$, where

$$F_x(0,0,0) = \frac{d}{R}\frac{1 - m^{eq}}{a + 1 + (a-1)m^{eq}}. \tag{A.4}$$

It is easy to verify that, for $-1 < m^{eq} \le 0$, $\frac{1}{1+a} \le \frac{1-m^{eq}}{a+1+(a-1)m^{eq}} < 1$. Hence $\lambda_3 < 1$ for $d < R$ and $\lambda_3 > 1$ for $d > (a+1)R$. When $R < d < (a+1)R$, $\lambda < 1$ if and only if $m^{eq} < m^*$. Therefore, the local stability of E_1 follows. Furthermore, when E_1 is local stable, it is also global stable. This completes the proof.

Proof of Lemma 3.3: The characteristic equation of the system at (x^*, x^*, x^*) is given by

$$G(\lambda) \equiv \lambda^3 - F_x^*\lambda^2 - F_y^*\lambda - F_z^* = 0, \tag{A.5}$$

where

$$\begin{aligned} F_x^* &= 1 + A\frac{[d+(a-1)R]^2}{a[a+1+(a-1)m^*]} \\ F_y^* &= -A\frac{[d+(a-1)R][d+(a-1)(2R-1)]}{a[a+1+(a-1)m^*]} \\ F_z^* &= -A\frac{d(R-1)[d+R(a-1)]}{a[a+1+(a-1)m^*]} \end{aligned} \tag{A.6}$$

with $A = \frac{\beta(x^*)^2}{2\sigma^2 a_1}\left(1 - tanh^2\left[\frac{\beta}{2a_1\sigma^2}(R-1)\left(R + \frac{d-R}{a}\right)(x^*)^2 - \frac{\beta C}{2}\right]\right)$. It is know that $|\lambda_i| < 1 (i = 1, 2, 3)$ if and only if

$$|F_x^* + F_z^*| < 1 - F_y^*, \qquad |F_y^* - F_z^* F_x^*| < 1 - (F_z^*)^2. \tag{A.7}$$

From Lemma 3.2, we know that, for $a = a^*$, $x^* = 0$ so that the equation has $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$. Following the argument used in the proof of Lemma 3 in Brock and Hommes, one can prove that there exists $a^{**} < a^*$ such that the system has a Hopf bifurcation at a^{**}.

Jury's Test

We first introduce concepts of the *inners* of a matrix and the *positive innerwise* matrix, which can be found from the book by Elaydi [12] (pages 180–181).

Let $B = (b_{ij})_{n\times n}$ be a matrix. The *inners* of the matrix B are the matrix itself and all the matrices obtained by omitting successively the first and last rows and the first and last columns. A matrix B is said to be *positive innerwise* if the determinants of all its inners are positive.

We now consider the kth order scalar equation

$$x_{n+k} + p_1 x_{n+k-1} + p_2 x_{n+k-2} + \cdots + p_k x_n = 0, \tag{A.8}$$

where the p_i's are real numbers. Obviously, the characteristic equation of the equation (A.8) is given by

$$p(\lambda) = \lambda^k + p_1\lambda^{k-1} + \cdots + p_k. \tag{A.9}$$

The Schue-Cohn criterion defines the conditions for the characteristic roots of equation (A.9) to fall inside the unit circle. More precisely, the following Jury's test will be used in our proof to Lemma 3.4.

Lemma A.1. (**Jury's test**) *The zeros of the characteristic polynomial (A.9) lie inside the unit circle if and only if the following hold:*

- $p(1) > 0$
- $(-1)^k p(-1) > 0,$

- *the* $(k-1)\times(k-1)$ *matrices*

$$B_{k-1}^{\pm}=\begin{pmatrix}1&0&\cdots&0&0\\p_1&1&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&0\\p_{k-3}&p_{k-4}&\cdots&1&0\\p_{k-2}&p_{k-3}&\cdots&p_1&1\end{pmatrix}\pm\begin{pmatrix}0&0&\cdots&0&p_k\\0&0&\cdots&p_k&p_{k-1}\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&p_k&\cdots&p_4&p_3\\p_k&p_{k-1}&\cdots&p_3&p_2\end{pmatrix}$$

are positive innerwise.

Proof of Lemma 3.4: Let $D_L(\lambda)=D(\lambda)/\lambda^2$, where $D(\lambda)$ is defined by (3.16). What we need to show is that, apart from the double zero eigenvalue of (3.16), all the zeros of the characteristic polynomial $D_L(\lambda)$ lie inside of the unit circle if and only if $0\le\gamma<1$ or $-\frac{1}{L}<\gamma\le 0$.

From $\gamma>0$,it is easy to see that $D_L(1)=1+(L-1)\gamma>0$ and $(-1)^L D_L(-1)=1-\gamma$ if L is odd and $(-1)^L D_L(-1)=1$ if L is even. Hence the first two conditions of Theorem A.1 hold if and only if $0\le\gamma<1$ or $-\frac{1}{L}<\gamma\le 0$. To show the third condition is satisfied, it is enough to show that, for $k=1,2,\cdots,L-1$, the matrix $B_k^{\pm}$ with $p_1=p_2=\cdots=p_L=\gamma$ are positive if and only if $0\le\gamma<1$ or $-\frac{1}{L}<\gamma\le 0$.

Let $k=2m$ be even. Then we have

$$B_k^{+}=\begin{pmatrix}1&0&\cdots&0&0&0&0&\cdots&0&\gamma\\\gamma&1&\cdots&0&0&0&0&\cdots&\gamma&\gamma\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&⋰&\vdots&\vdots\\\gamma&\gamma&\cdots&1&0&0&\gamma&\cdots&\gamma&\gamma\\\gamma&\gamma&\cdots&\gamma&1&\gamma&\gamma&\cdots&\gamma&\gamma\\\gamma&\gamma&\cdots&\gamma&2\gamma&1+\gamma&\gamma&\cdots&\gamma&\gamma\\\gamma&\gamma&\cdots&2\gamma&2\gamma&2\gamma&1+\gamma&\cdots&\gamma&\gamma\\\vdots&\vdots&⋰&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\\gamma&2\gamma&\cdots&2\gamma&2\gamma&2\gamma&2\gamma&\cdots&1+\gamma&\gamma\\2\gamma&2\gamma&\cdots&2\gamma&2\gamma&2\gamma&2\gamma&\cdots&2\gamma&1+\gamma\end{pmatrix}\tag{A.10}$$

To evaluate the determinate of B_k^+, we use (-1) to multiply the i-th columns and add to the $2m-(i-1)$-th columns, respectively, for $i=1,\cdots,m$. We then have

$$|B_k^+| = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma-1 \\ \gamma & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \gamma-1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ \gamma & \gamma & \cdots & 1 & 0 & 0 & \gamma-1 & \cdots & 0 & 0 \\ \gamma & \gamma & \cdots & \gamma & 1 & \gamma-1 & 0 & \cdots & 0 & 0 \\ \gamma & \gamma & \cdots & \gamma & 2\gamma & 1-\gamma & 0 & \cdots & 0 & 0 \\ \gamma & \gamma & \cdots & 2\gamma & 2\gamma & 0 & 1-\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma & 2\gamma & \cdots & 2\gamma & 2\gamma & 0 & 0 & \cdots & 1-\gamma & 0 \\ 2\gamma & 2\gamma & \cdots & 2\gamma & 2\gamma & 0 & 0 & \cdots & 0 & 1-\gamma \end{vmatrix}$$

$$= (1-\gamma)^m \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ \gamma & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ \gamma & \gamma & \cdots & 1 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \gamma & \gamma & \cdots & \gamma & 1 & -1 & 0 & \cdots & 0 & 0 \\ \gamma & \gamma & \cdots & \gamma & 2\gamma & 1 & 0 & \cdots & 0 & 0 \\ \gamma & \gamma & \cdots & 2\gamma & 2\gamma & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma & 2\gamma & \cdots & 2\gamma & 2\gamma & 0 & 0 & \cdots & 1 & 0 \\ 2\gamma & 2\gamma & \cdots & 2\gamma & 2\gamma & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \tag{A.11}$$

Now for $i=1,2,\cdots,m$, we first add the $2m-(i-1)$-the columns to the i-the columns, respectively. Then, multiply γ to the $2m-(i-1)$-th column and add to the all the first $m-1$ columns. as a result, the upper left block matrix become a zero matrix and the down left block matrix has 2γ as non-diagonal elements and $2\gamma+1$ as diagonal elements. Correspondingly,

$$|B_k^+| = (-1)^m(1-\gamma)^m \begin{vmatrix} 2\gamma & \cdots & 2\gamma+1 \\ \vdots & ⋰ & \vdots \\ 2\gamma+1 & \cdots & 2\gamma \end{vmatrix} \tag{A.12}$$

We use -1 to time the first column and add to all the rest columns. Then, use -1 to multiply the columns 2 to k and add them to the first column. As as result, we have a low triangle matrix with $(1,1,\cdots,1,2m\gamma+1)$. Therefore,

$$\det(B_k^+) = (1-\gamma)^m(L\gamma+1). \tag{A.13}$$

Similarly,

$$B_k^- = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\gamma \\ \gamma & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\gamma & -\gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ \gamma & \gamma & \cdots & 1 & 0 & 0 & -\gamma & \cdots & -\gamma & -\gamma \\ \gamma & \gamma & \cdots & \gamma & 1 & -\gamma & -\gamma & \cdots & -\gamma & -\gamma \\ \gamma & \gamma & \cdots & \gamma & 0 & 1-\gamma & -\gamma & \cdots & -\gamma & -\gamma \\ \gamma & \gamma & \cdots & 0 & 0 & 0 & 1-\gamma & \cdots & -\gamma & -\gamma \\ \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1-\gamma & -\gamma \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1-\gamma \end{pmatrix} \tag{A.14}$$

To find the $\det(B_k^-)$, we expand it first by the last row and then by the first row and these lead to $\det(B_k^-) = (1-\gamma)\det(B_{k-2}^-)$. Since $k = 2m$, it follows from the formula $\det(B_{2m}^-) = (1-\gamma)\det(B_{2(m-1)}^-)$ that

$$\det(B_k^-) = (1-\gamma)^m. \tag{A.15}$$

In conclusion, we have for $k = 2m$,

$$\det(B_k^+) = (1-\gamma)^m(2m\gamma+1), \qquad \det(B_k^-) = (1-\gamma)^m. \tag{A.16}$$

Next we assume that $k = 2m+1$. Then

$$B_k^- = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & -\gamma \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \gamma & \cdots & 1 & 0 & -\gamma & \cdots & -\gamma \\ \gamma & \cdots & \gamma & 1-\gamma & -\gamma & \cdots & -\gamma \\ \gamma & \cdots & 0 & 0 & 1-\gamma & \cdots & -\gamma \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1-\gamma \end{pmatrix} \tag{A.17}$$

It is easy to see that $\det(B_k^-) = (1-\gamma)\det(B_{2m}^-)$. Using (A.16), we have

$$\det(B_k^-) = (1-\gamma)^{m+1}. \tag{A.18}$$

On the other hand,

$$B_k^+ = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & \gamma \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \gamma & \cdots & 1 & 0 & \gamma & \cdots & \gamma \\ \gamma & \cdots & \gamma & 1+\gamma & \gamma & \cdots & \gamma \\ \gamma & \cdots & 2\gamma & 2\gamma & 1+\gamma & \cdots & \gamma \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\gamma & \cdots & 2\gamma & 2\gamma & 2\gamma & \cdots & 1+\gamma \end{pmatrix} \tag{A.19}$$

To find the $\det(B_k^-)$, we multiply the i-th column by -1 and add to the $2m-(i-1)$-the column, respectively, for $i = 1, \cdots, m$.

$$\det(B_k^+) = (1-\gamma)^m \begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \gamma & \cdots & 1 & 0 & -1 & \cdots & 0 \\ \gamma & \cdots & \gamma & 1+\gamma & 0 & \cdots & 0 \\ \gamma & \cdots & 2\gamma & 2\gamma & 1 & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\gamma & \cdots & 2\gamma & 2\gamma & 0 & \cdots & 1 \end{vmatrix}. \tag{A.20}$$

Similarly, one can use row operations to reduce the upper left $m \times m$ matrix to a zero matrix and correspondingly,

$$\det(B_k^+) = (-1)^{m+1}(1-\gamma)^m \begin{vmatrix} \gamma & \cdots & \gamma & 1+\gamma \\ 2\gamma & \cdots & 2\gamma+1 & 2\gamma \\ \vdots & \iddots & \vdots & \vdots \\ 2\gamma+1 & \cdots & 2\gamma & 2\gamma \end{vmatrix}. \tag{A.21}$$

Multiply the first column by -1 and add all the rest of the columns of $\det(B_k^-)$ and then, multiply the last column by $-\gamma$ and add to the first column, multiply -2γ to the columns $2, 3, \cdots, m$ and add to the first column. We then add up with

$$\det(B_k^+) = (-1)^{m+1}(1-\gamma)^m \begin{vmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ \gamma(2m+1)+1 & -1 & \cdots & -1 \end{vmatrix}. \tag{A.22}$$

Therefore

$$\det(B_k^+) = (\gamma k + 1)(1-\gamma)^m. \tag{A.23}$$

Then from (A.18) and (A.23), for $k = 2m+1$,

$$\det(B_k^+) = (\gamma k + 1)(1-\gamma)^m, \qquad \det(B_k^-) = (1-\gamma)^{m+1}. \tag{A.24}$$

Finally, it follows from (A.16) and (A.24) that $B_k^{\pm}$ are positive if and only if $0 \leq \gamma < 1$ or $-\frac{1}{L} < \gamma \leq 0$ and this completes the proof.

References

[1] S. Anderson, A. de Palma, and J. Thisse. *Discrete Choice Theory of Product Differentiation*. MIT Press, 1993.
[2] L. Arnold. *Random Dynamical Systems*. Springer-Verlag, 1998.
[3] A. Beja and M.B. Goldman. On the dynamic behavior of prices in disequilibrium. *Journal of Finance*, 35:235–247, 1980.
[4] F. Black. Noise. *Journal of Finance*, 41:529–543, 1986.
[5] W. Brock and C. Hommes. *Models of Complixity in Economics and Finance*, chapter 1, pages 3–44. Wiley, New York, 1997. in *Systems Dynamics in Economic and Finance Models*, Eds. Heij, C., Schumacher, J.M., Hanzon, B., Praagman, C.
[6] W. Brock and C. Hommes. A rational route to randomness. *Econometrica*, 65:1059–1095, 1997.
[7] W. Brock and C. Hommes. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, 22:1235–1274, 1998.
[8] J Campbell and A. Kyle. Smart money, noise trading and stock price behaviour. *Review of Economic Studies*, 60:1–34, 1993.
[9] C. Chiarella. The dynamics of speculative behaviour. *Annals of Operations Research*, 37:101–123, 1992.
[10] R. Day and W. Huang. Bulls, bears and market sheep. *Journal of Economic Behavior and Organization*, 14:299–329, 1990.
[11] J.B. DeLong, A. Shleifer, Summers L.H., and R.J. Waldmann. Noise trader risk in financial markets. *Journal of Political Economy*, 98:703–738, 1990.
[12] S. Elaydi. *An Introduction to Difference Equations*. Springer, New York, 1996.
[13] E.F. Fama and K.R. French. Permanent and transitory components of stock prices. *Journal of Political Economy*, 98:246–273, 1988.
[14] R. Franke and R. Sethi. Cautious trend-seeking and complex asset price dynamics. *Research in Economics*, 52:61–79, 1998.
[15] S. Grossman. *The Informational Role of Prices*. MIT Press, Cambridge, MA, 1989.
[16] T. Lux. Herd behaviour, bubbles and crashes. *Economic Journal*, 105:881–896, 1995.
[17] T. Lux. Time variation of second moments from a noise trader/infection model. *Journal of Economic Dynamics and Control*, 22:1–38, 1997.
[18] T. Lux. The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions. *Journal of Economic Behavior and Organization*, 33:143–165, 1998.
[19] C. Manski and D. McFadden. *Structural Analysis of Discrete Data with Econometric Applications*. MIT Press, 1981.
[20] E.M. Miller. Risk, uncertainity, and divergence of opinion. *Journal of Finance*, 32:1151–1168, 1977.
[21] J.M. Poterba and L.H. Summers. Mean reversion in stock prices: Evidence and implications. *Journal of Financial Economics*, 22:27–60, 1988.
[22] R. Sethi. Endogenous regime switching in speculative markets. *Structural Change and economic dynamics*, 7:99–118, 1996.
[23] L.H. Summers. Does the stock market rationally reflect fundamental values. *Journal of Finance*, 41:591–601, 1986.