INFINITE-HORIZON OPTIMAL HEDGING UNDER CONE CONSTRAINTS

by

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ABSTRACT

We address the issue of hedging in infinite horizon markets under cone constraints on the number of shares of assets. We show that the minimum cost of hedging a liability stream is equal to its largest present value with respect to admissible stochastic discount factors, thus can be determined without finding an optimal hedging strategy. We develop an algorithm by which an optimal portfolio in one date-event can be obtained without finding that in others. We apply the results to a variety of trading restrictions and show how the admissible stochastic discount factors can be characterized.

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1. INTRODUCTION

Often financial institutions are faced with liability streams which the cost of not meeting is large. There are many examples. Lack of means to pay mature debts may involve corporations in costly financial restructuring. Failure to provide for requested withdrawals may put banks in runs. Insurance companies that default on compensatory payments may incur legal expenses. Employers that do not fulfill pension obligations may loss reputations. More striking examples can be found in derivatives and futures markets, which have grown tremendously in recent years. On one side, new instruments have been developed and the volume of transactions within individual markets has skyrocketed. On the other, inability of market makers and securities traders to cover their positions are likely to trigger financial crises.

What can market participants do to reduce the default risks? The answer is, *hedging*. Hedging is a set of transactions in financial markets that generates a dominating stream, one whose payoffs are at least as large as in meeting the underlying liability, therefore offsets the default risks. The standard models of hedging and valuation of contingent claims, which can be traced back to the pioneering option pricing work of Black and Scholes (1973), Merton (1973), and Cox, Ross and Rubinstein (1979), assume the absence of market frictions. However, investors are usually faced with trading restrictions such as no short-sales constraints, non-negative restrictions of portfolio values, margin requirements on stocks and bonds (leverage restrictions in futures markets are typically imposed through margin requirements as well), and target debt to equity ratios. These restrictions, as well as the unrestricted case, are special examples of *cone* constraints on the number of shares of assets. Formally, a cone is a collection of vectors that is stable under addition and multiplication by nonnegative real numbers. In addition to the aforementioned generality in representing various trading restrictions, modeling market frictions by cones has an advantage that arbitrage cannot exist in equilibrium under such constraints, provided that investors' preferences are monotone. In consequence, one can derive the implications of hedging from the absence of arbitrage under cone constraints without making explicit use of utility maximization or market equilibrium.[1]

In this paper we determine analytically the minimum hedging cost and the optimal hedging strategies in infinite horizon markets in the absence of arbitrage under cone constraints on the number of shares of assets. We show that the minimum cost of hedging a liability stream is equal to its largest present value with respect to admissible stochastic discount factors. This in particular implies that the cost can be determined without finding an optimal hedging strategy. We show that an optimal hedging strategy can be obtained through solving a sequence of independent programs. Independence means that an optimal portfolio in one date-event can be obtained without finding that in others. The results hold for arbitrary liability streams, not limited to payoff streams contingent on asset prices or interest rates in the usual sense. We apply the results to a variety of trading restrictions and show how the admissible stochastic discount factors can be characterized. The model presented here nests the standard finite horizon setting as a special case in which the results hold for arbitrary payoff streams.

The work presented in this paper contributes to the literature on hedging with market frictions. Ever since Black and Scholes (1973) and Merton (1973), much has been written on hedging and valuation of contingent claims with transactions costs. Some studies, including Garman and Ohlson (1981) and Jouini and Kallal (1995a), have dealt with the minimum hedging cost, while others, including Bensaid, Lesne, Pages and Scheinkman (1991) and Edirisinghe, Naik and Uppal (1993), have also addressed the optimal hedging strategies, in the presence of proportional transactions costs.[2] The finite-horizon version of our analysis extends these studies since proportional transactions costs, or bid-ask spreads, can be reinterpreted as no short-sales constraints (see, for example, Foley (1970)), which, as pointed out earlier, are a special example of cone constraints. In particular, Jouini and Kallal (1995a) show that the minimum hedging cost equals the largest present value of the underlying payoff stream with respect to some stochastic processes (whose existence is implied by the celebrated Hahn-Banach Theorem or the Riesz Representation Theorem). Our analysis offers a computational advantage in this regard as well; here, the admissible stochastic discount factors are characterized by explicit linear (in)equalities, therefore the minimum hedging cost can be determined by solving a standard linear program.

In contrast to the extensive transactions costs literature, it is not until recently that hedging and valuation with trading restrictions have received a great deal of attention. Naik and Uppal (1994) have first developed an algorithm of backward recursion for finding the minimum hedging cost as well as the optimal hedging strategies, in the presence of margin requirements on stocks and bonds.[3] With this algorithm to determine the minimum hedging cost requires finding an optimal hedging strategy while to find an optimal portfolio in one date-event requires finding that in subsequent ones. Broadie, Cvitanic and Soner (1998) have extended this result to a continuous time setting.[4] The finite-horizon version of our analysis extends this result by incorporating general cone constraints, and by showing that the minimum hedging cost can be determined without finding an optimal hedging strategy while an optimal portfolio in one date-event can be obtained without finding that in others.

Another contribution of our model is attributed to its infinite-horizon feature. The existing studies of hedging of contingent claims have been carried out in the finite horizon setting, i.e., there is a final date by which all assets are liquidated. Yet, markets are of infinite horizon in nature if assets of no maturity date (such as stocks), or if an infinite sequence of assets of finite maturity, are traded. Moreover, there are conceivable situations in which institutional investors may need to hedge payoff streams over an infinite horizon as well. Our model is the first one to analyze the problem of hedging in infinite horizon markets and nevertheless encompasses the standard finite-horizon setting as a special case.[5]

The rest of the paper is organized in the following order. Section 2 describes the model and presents the main results. Section 3 applies the main results to various trading restrictions and characterizes the admissible stochastic discount factors. Section 4 concludes. All proofs are contained in the Appendix.

2. THE MODEL AND MAIN RESULTS

We model dynamic uncertainty by a set Ω of states of the world and an increasing sequence $\{N_t\}_{t=0}^{\infty}$ of finite information partitions with $N_0 = \{\Omega\}$. We map this information structure onto an *event-tree* D, where an information set $s^t \in N_t$ is referred to as a date-event or a node of the event-tree. For each s^t, we denote by s^t_- its unique immediate predecessor if $t \neq 0$, $\{s^t_+\}$ a finite set of its immediate successors, and $D(s^t)$ a subtree with root s^t. With this notation we have $D(s^0) = D$. In each date-event there are a finite number of assets traded on spot markets in exchange for a single consumption goods that is taken as the unit of account. We denote by (q, d) a price-dividend process adapted to $\{N_t\}_{t=0}^{\infty}$. A holder of one share of an asset j traded for a price $q_j(s^t)$ at s^t is entitled to a payoff $R_j(s^{t+1})$ at each $s^{t+1} \in \{s^t_+\}$, where $R_j(s^{t+1}) = q_j(s^{t+1}) + d_j(s^{t+1})$ if the asset continues to be traded for a price $q_j(s^{t+1})$ at s^{t+1} and $R_j(s^{t+1}) = d_j(s^{t+1})$ if the asset is liquidated at s^{t+1}. We denote by $q(s^t)$ a vector of prices for assets traded at $s^t \in D$ and $R(s^t)$ a vector of one-period payoffs for assets traded at s^t_- for $s^t \in D \backslash \{s^0\}$. That is, a holder of one share of each of the assets traded for price $q(s^t)$ at s^t is entitled to payoff $R(s^{t+1})$ at each $s^{t+1} \in \{s^t_+\}$. At each $s^t \in D \backslash \{s^0\}$ new assets can be issued while existing assets can be liquidated, so the dimensions of $R(s^t)$ and $q(s^t)$ can be different. The difference is equal to the number of existing assets liquidated subtracting the number of new assets issued at s^t.

A portfolio $\theta(s^t)$ specifies the number of shares of assets to be held at the end of trade at s^t. We denote by $\Theta(s^t)$ a set of feasible portfolios at s^t, which is assumed to be a *polyhedral cone*,[6] and Θ the Cartesian product $\prod_{s^t \in D} \Theta(s^t)$. That is, a *portfolio strategy* θ is in Θ if and only if its portfolio component $\theta(s^t)$ is in $\Theta(s^t)$ for each s^t. By z^θ we denote the payoff stream generated by a feasible portfolio strategy θ given by

$$z^\theta(s^t) \equiv R(s^t)'\theta(s^t_-) - q(s^t)'\theta(s^t), \qquad \forall\ s^t \in D \backslash \{s^0\}.$$

An *arbitrage* in Θ is a feasible portfolio strategy θ that generates a positive payoff stream at a nonnegative cost or a nonnegative payoff stream at a negative cost, i.e., such that

$$q_(s^0)'\theta(s^0) \leq 0, \;\; z^\theta(s^t) \geq 0, \;\; \forall\ s^t \in D \backslash \{s^0\},$$

with at least one strict inequality. A feasible *finite arbitrage* is an arbitrage $\theta \in \Theta$

that involves nonzero asset holdings only at finitely many dates, of which a feasible *one-period arbitrage* is an example. A feasible one-period arbitrage at a node s^τ is a finite arbitrage θ such that $\theta(s^t) = 0$ for $s^t \neq s^\tau$ and $\theta(s^\tau) \in \Theta(s^\tau)$. Applying a generalized Farkas lemma to polyhedral cones establishes the equivalence between the absence of one-period arbitrage in $\Theta(s^t)$ and the existence of strictly positive numbers $\{a(s^t),\ a(s^{t+1}),\ s^{t+1} \in \{s^t_+\}\}$ such that

$$\left\{ q(s^t) - \sum_{s^{t+1} \in \{s^t_+\}} \frac{a(s^{t+1})}{a(s^t)} R(s^{t+1}) \right\} \in \Theta(s^t)^*, \tag{1}$$

where $\Theta(s^t)^* \equiv \{\vartheta : \vartheta'\theta \geq 0,\ \forall\ \theta \in \Theta(s^t)\}$ is the polar cone of $\Theta(s^t)$, thus a polyhedral cone as well (see, for example, Ben-Israel (1969), Sposito (1989), Sposito and David (1971, 1972)).[7] These positive numbers are referred to as *admissible stochastic discount factors*. Since only the ratios $\{a(s^{t+1})\backslash a(s^t)\}$ are restricted by (1), the absence of one-period arbitrage in Θ allows one to define a system of admissible stochastic discount factors consistent with (1) at each node. We denote by $A(s^t)$ the set of the systems of admissible stochastic discount factors on subtree $D(s^t)$. To simplify, we denote $A(s^0)$ by A.

We now formulate the optimal hedging problem. Let z be an adapted nonnegative payoff stream such that there is a portfolio strategy $\theta \in \Theta$ with $z^\theta \geq z$. The objective is to determine

$$V(z) \equiv inf\{q(s^0)'\theta(s^0)\ :\ z^\theta \geq z,\ \ \theta \in \Theta\}, \tag{2}$$

and to find a feasible portfolio strategy that achieves $V(z)$ whenever there exists one. Our main results are that the absence of arbitrage in Θ implies that, $V(z)$ is equal to the largest present value of z with respect to the systems of admissible stochastic discount factors and is achieved by a feasible strategy obtained through solving a sequence of independent programs. The following theorem is concerned with the determination of the minimum hedging cost.

THEOREM 1: *Suppose that there is no arbitrage in Θ. Then $A \neq \emptyset$, and*

$$V(z) = \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t). \tag{3}$$

Proof: See the Appendix.

According to (3), the minimum cost of hedging a nonnegative payoff stream is equal to its largest present value with respect to the admissible stochastic discount factors. This in particular implies that the cost can be determined without finding an optimal hedging strategy. The following theorem provides an algorithm for finding an optimal strategy by which an optimal portfolio in one date-event can be obtained without finding that in others.

THEOREM 2: *Suppose that there is no arbitrage in Θ. Then $A \neq \emptyset$, and there is a solution to the following program*

$$\min_{\theta(s^t)} \quad q(s^t)'\theta(s^t) \tag{4}$$

$$s.t. \quad R(s^{t+1})'\theta(s^t) \geq \sup_{a \in A(s^{t+1})} \sum_{s^\tau \in D(s^{t+1})} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau) \tag{5}$$

$$s^{t+1} \in \{s^t_+\}, \quad \theta(s^t) \in \Theta(s^t),$$

which is the portfolio component at s^t of a feasible strategy that achieves $V(z)$.
Proof: See the Appendix.

According to theorem 2, the task of finding an optimal hedging strategy reduces to solving a sequence of independent programs (4)-(5). Independence refers to the fact that a solution to the program in one date-event can be obtained without finding that in others. A critical step in solving these programs, as well as in determining the minimum hedging cost as of (3), is calculating the largest present value of the underlying payoff stream, which in turn relies on characterizing the admissible stochastic discount factors. In the following section, we apply theorems 1 and 2 to various trading restrictions and show how the admissible stochastic discount factors can be characterized.

3. APPLICATIONS

In this section we use polyhedral cone constraints on the number of shares of assets to describe market frictions including no short-sales constraints, nonnegative restrictions of portfolio values, margin requirements on stocks and bonds, and target debt to equity ratios. We characterize the admissible stochastic discount factors by a system of linear (in)equalities, thus, reduce the task of calculating the largest present value of the underlying payoff stream to solving a linear program. To help exposition yet not lose generality, we assume that there are two assets in each date-event. To simplify, we assume prices are strictly positive so that the admissible stochastic discount factors can be characterized using rates of returns on traded assets, defined by

$$(r_1(s^t), r_2(s^t)) \equiv \left(\frac{R_1(s^t)}{q_1(s^t_-)}, \frac{R_2(s^t)}{q_2(s^t_-)} \right)$$

for each $s^t \in D \backslash \{s^0\}$. In each of the following subsections, the set $\Theta(s^t)$ of feasible portfolios at s^t is a polyhedral cone for each $s^t \in D$. Consequently, theorems 1 and 2 are applicable.

3.1. *No Short-sales Constraints*

No short-sales constraints can be modeled by taking

$$\Theta(s^t) = \Theta(s^t)^* = I\!R_+^2 \tag{6}$$

for each $s^t \in D$. The set A of the systems of admissible stochastic discount factors is characterized by the following linear inequalities

$$\sum_{s^{t+1}\in\{s_+^t\}} r_1(s^{t+1})\left[\frac{a(s^{t+1})}{a(s^t)}\right] \le 1, \quad \sum_{s^{t+1}\in\{s_+^t\}} r_2(s^{t+1})\left[\frac{a(s^{t+1})}{a(s^t)}\right] \le 1, \tag{7}$$

$$a(s^t) > 0, \;\; a(s^{t+1}) > 0, \;\; s^t \in D, \;\; s^{t+1} \in \{s_+^t\}. \tag{8}$$

Consequently, the largest present value of the underlying payoff stream can be calculated by solving a linear program.

3.2. *Nonnegative Restrictions of Portfolio Values*

Consider a constraint that the end-of-trade portfolio value be nonnegative. That is, any indebtedness held at the beginning of trade must be fully repaid upon the completion of trade. This constraint can be modeled by taking

$$\Theta(s^t) = \{\theta(s^t) \in I\!R^2 : q_1(s^t)\theta_1(s^t) + q_2(s^t)\theta_2(s^t) \ge 0\} \tag{9}$$

for each $s^t \in D$. It follows that

$$\Theta(s^t)^* = \{\vartheta(s^t) \in I\!R_+^2 : -q_2(s^t)\vartheta_1(s^t) + q_1(s^t)\vartheta_2(s^t) = 0\} \tag{10}$$

for each $s^t \in D$. Therefore, A can be characterized by the following linear (in)equalities

$$\sum_{s^{t+1}\in\{s_+^t\}} r_1(s^{t+1})\left[\frac{a(s^{t+1})}{a(s^t)}\right] = \sum_{s^{t+1}\in\{s_+^t\}} r_2(s^{t+1})\left[\frac{a(s^{t+1})}{a(s^t)}\right] \le 1, \tag{11}$$

$$a(s^t) > 0, \;\; a(s^{t+1}) > 0, \;\; s^t \in D, \;\; s^{t+1} \in \{s_+^t\}. \tag{12}$$

Note that A characterized by (11)-(12) is a subset of that characterized by (7)-(8).

3.3. *Margin Requirements on Stocks and Bonds*

Investors who need to hedge a payoff stream in securities markets are often faced with margin requirements on stocks and bonds, which capture their ability to increase short-sales or borrowing as a function of their creditworthiness. For the purpose of illustration, assume that one traded asset in each date-event is an one-period bond while the other a stock, that is,

$$R_1(s^t) = d_1(s^t), \;\; R_2(s^t) = q_2(s^t) + d_2(s^t), \;\; s^t \in D \backslash \{s^0\}.$$

Margin requirements can be modeled by taking

$$\begin{aligned} \Theta(s^t) = \{\theta(s^t) \in I\!R^2 : q_1(s^t)\theta_1(s^t) &\geq -m_1(s^t)[q_1(s^t)\theta_1(s^t) + q_2(s^t)\theta_2(s^t)], \\ q_2(s^t)\theta_2(s^t) &\geq -m_2(s^t)[q_1(s^t)\theta_1(s^t) + q_2(s^t)\theta_2(s^t)]\} \end{aligned} \tag{13}$$

for each $s^t \in D$, where $m_1(s^t)$ and $m_2(s^t)$ are nonnegative numbers representing margin requirements on the bond and stock, respectively. The margin requirements described by (13) implies nonnegative end-of-trade portfolio values. It follows that

$$\begin{aligned} \Theta(s^t)^* = \{\vartheta(s^t) \in I\!R^2 : [1 + m_1(s^t)]q_1(s^t)\vartheta_2(s^t) &\geq m_1(s^t)q_2(s^t)\vartheta_1(s^t), \\ [1 + m_2(s^t)]q_2(s^t)\vartheta_1(s^t) &\geq m_2(s^t)q_1(s^t)\vartheta_2(s^t)\} \end{aligned} \tag{14}$$

for each $s^t \in D$. It is worth pointing out that (14) implies $\Theta(s^t)^* \subseteq I\!R^2_+$. Therefore, A can be characterized by the following linear inequalities

$$\sum_{s^{t+1} \in \{s^t_+\}} \left\{[1 + m_2(s^t)]r_1(s^{t+1}) - m_2(s^t)r_2(s^{t+1})\right\} \left[\frac{a(s^{t+1})}{a(s^t)}\right] \leq 1, \tag{15}$$

$$\sum_{s^{t+1} \in \{s^t_+\}} \left\{[1 + m_1(s^t)]r_2(s^{t+1}) - m_1(s^t)r_1(s^{t+1})\right\} \left[\frac{a(s^{t+1})}{a(s^t)}\right] \leq 1, \tag{16}$$

$$a(s^t) > 0, \;\; a(s^{t+1}) > 0, \;\; s^t \in D, \;\; s^{t+1} \in \{s^t_+\}. \tag{17}$$

In the case when $m_1(s^t) = m_2(s^t) = 0$, corresponding to no borrowing on bond and no short-selling in stock, (15)-(17) reduce to (7)-(8).

3.4. *Target Debt to Equity Ratios*

Financial managers are often required to maintain certain debt to equity ratios while hedging a payoff stream. Assuming as in 3.3 that one traded asset in each date-event is an one-period bond and the other a stock, we can model this leverage requirement by taking

$$\Theta(s^t) = \{\theta(s^t) \in \mathbb{R}^2_+ : \alpha(s^t)q_2(s^t)\theta_2(s^t) \leq q_1(s^t)\theta_1(s^t) \leq \beta(s^t)q_2(s^t)\theta_2(s^t)\} \quad (18)$$

where $0 < \alpha(s^t) \leq \beta(s^t)$ for each $s^t \in D$. The interval $[\alpha(s^t), \beta(s^t)]$ specifies the range of feasible debt to equity ratios in date-event s^t (the restriction that $\theta(s^t)$ be nonnegative in (18) is redundant in the case when $\alpha(s^t) < \beta(s^t)$). It follows that

$$\begin{aligned} \Theta(s^t)^* = \{\vartheta(s^t) \in \mathbb{R}^2 : \alpha(s^t)q_2(s^t)\vartheta_1(s^t) + q_1(s^t)\vartheta_2(s^t) &\geq 0, \\ \beta(s^t)q_2(s^t)\vartheta_1(s^t) + q_1(s^t)\vartheta_2(s^t) &\geq 0\} \end{aligned} \quad (19)$$

for each $s^t \in D$. Therefore, A can be characterized by the following linear inequalities

$$\sum_{s^{t+1} \in \{s^t_+\}} \left[\frac{\alpha(s^t)r_1(s^{t+1}) + r_2(s^{t+1})}{\alpha(s^t) + 1}\right] \left[\frac{a(s^{t+1})}{a(s^t)}\right] \leq 1, \quad (20)$$

$$\sum_{s^{t+1} \in \{s^t_+\}} \left[\frac{\beta(s^t)r_1(s^{t+1}) + r_2(s^{t+1})}{\beta(s^t) + 1}\right] \left[\frac{a(s^{t+1})}{a(s^t)}\right] \leq 1, \quad (21)$$

$$a(s^t) > 0, \;\; a(s^{t+1}) > 0, \;\; s^t \in D, \;\; s^{t+1} \in \{s^t_+\}. \quad (22)$$

In the degenerate case when $\alpha(s^t) = \beta(s^t)$, the value of debt versus that of equity in date-event s^t must be kept at a single ratio, and (20) and (21) are identical.

4. CONCLUDING REMARKS

We have addressed in this paper the issue of hedging an arbitrary liability stream in the presence of polyhedral cone constraints on the number of shares of assets. We have derived a representation for the minimum hedging cost in terms of the largest present value of the underlying liability stream with respect to the admissible stochastic discount factors. This in particular implies that the cost can be determined without finding an optimal hedging strategy. We have shown that an optimal portfolio in one date-event can be obtained without finding that in others. We have applied the results to trading restrictions often proposed and characterized the admissible stochastic discount factors by linear (in)equalities.

Our analysis has gone beyond the standard finite horizon paradigm and nests it as a special case. This can be seen by taking $\Theta(s^t)$ to be a singleton set of null asset holdings for each $t \geq T$ and some finite T. In this case, theorems 1 and 2 hold for arbitrary payoff streams.

Applications of our results have been illustrated with two assets, but are readily extended to account for arbitrary (yet finite) number of securities. Such extension is trivial for no short-sales constraints and nonnegative restrictions of portfolio values, and straightforward for margin requirements and target debt to equity ratios. For instance, a margin requirement can be imposed on each of a finite number of assets traded by investors, while target debt to equity ratios can be imposed through a restriction on the ratio of portfolio value of bonds to that of stocks held by mutual fund managers.

APPENDIX

Proof of theorems 1 and 2: Under the assumption that there is no arbitrage in Θ, one can apply a generalized Farkas lemma to $\Theta(s^t)$ for each s^t to establish $A \neq \emptyset$. The following inequality, which holds for any feasible strategy θ that hedges z, is useful in establishing (3):

$$q(s^t)'\theta(s^t) \geq 0, \qquad \forall\ s^t \in D. \tag{23}$$

To prove (23) suppose, by contradiction, that there is some s^t at which $q(s^t)'\theta(s^t) < 0$. Then the strategy $\bar{\theta}$ such that, $\bar{\theta}(s^\tau)$ coincides with $\theta(s^\tau)$ if $s^\tau \in D(s^t)$ and with null asset holdings otherwise, is an arbitrage in Θ. A contradiction. So (23) must hold. We now establish

$$V(z) \geq \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t). \tag{24}$$

Let θ be a portfolio strategy in Θ that hedges z, and choose an arbitrary $a \in A$. By definition of polar cones, the inner product of the left hand side of (1) and portfolio $\theta(s^t)$ is nonnegative. Using this and $z^\theta \geq z$ repeatedly, we obtain for any $\tau \geq 1$,

$$a(s^0)q(s^0)\theta(s^0) \geq \sum_{t=1}^{\tau} \sum_{s^t \in N_t} a(s^t)z(s^t) + \sum_{s^\tau \in N_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau) \geq \sum_{t=1}^{\tau} \sum_{s^t \in N_t} a(s^t)z(s^t),$$

where the second inequality follows from (23). Taking $\tau \to \infty$ on the right-most side of the above inequalities leads to

$$a(s^0)q(s^0)'\theta(s^0) \geq \sum_{t=1}^{\infty} \sum_{s^t \in N_t} a(s^t)z(s^t) \equiv \sum_{s^t \in D \setminus \{s^0\}} a(s^t)z(s^t).$$

That a is arbitrarily chosen implies

$$q(s^0)'\theta(s^0) \geq \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).$$

That θ is an arbitrary strategy in Θ that hedges z implies

$$V(z) \geq \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t),$$

which establishes (24).

We now use a duality technique of convex programming and inequality (23) to establish

$$V(z) \leq \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t). \tag{25}$$

Note that (25) is non-trivial only if the right-hand side is finite, so we assume this is the case. Consider the following dual of the program (4)-(5),

$$\max_{\substack{\alpha(s^{t+1}) \\ s^{t+1} \in \{s^t_+\}}} \quad \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1}) [\sup_{a \in A(s^{t+1})} \sum_{s^\tau \in D(s^{t+1})} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau)] \tag{26}$$

$$s.t. \quad \left\{ q(s^t) - \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1}) R(s^{t+1}) \right\} \in \Theta(s^t)^* \tag{27}$$

$$\alpha(s^{t+1}) \geq 0, \quad s^{t+1} \in \{s^t_+\},$$

where $\Theta(s^t)^*$ is the polar cone of $\Theta(s^t)$. We claim that both (5) and (27) have feasible solutions. That (27) has a feasible solution simply follows from the existence of a system of admissible stochastic discount factors. We now prove that any feasible strategy θ that hedges z, induces a portfolio $\theta(s^t)$ at s^t that is a feasible solution to (5). To proceed we use relations (1), (23), $z^\theta \geq z$ and definition of polar cones to obtain, for each $s^{t+1} \in \{s^t_+\}$, an arbitrary system of discount factors $a \in A(s^{t+1})$, and any $r \geq t+1$,

$$\begin{aligned} a(s^{t+1}) R(s^{t+1})' \theta(s^t) \quad &\geq \quad \sum_{\tau=t+1}^{r} \sum_{s^\tau \in D(s^{t+1}) \cap N_\tau} a(s^\tau) z(s^\tau) + \sum_{s^r \in D(s^{t+1}) \cap N_r} a(s^r) q(s^r)' \theta(s^r) \\ &\geq \quad \sum_{\tau=t+1}^{r} \sum_{s^\tau \in D(s^{t+1}) \cap N_\tau} a(s^\tau) z(s^\tau), \end{aligned}$$

where the second inequality follows from (23). Taking $r \to \infty$ on the right-most side of above inequalities leads to

$$a(s^{t+1})R(s^{t+1})'\theta(s^t) \geq \sum_{s^\tau \in D(s^{t+1})} a(s^\tau)z(s^\tau).$$

That a is arbitrarily chosen from $A(s^{t+1})$ implies

$$R(s^{t+1})'\theta(s^t) \geq \sup_{a \in A(s^{t+1})} \sum_{s^\tau \in D(s^{t+1})} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau).$$

Thus, $\theta(s^t)$ is a feasible solution to (5).

By the duality theorem of convex programming (see, for example, Sposito (1989)), both the primal and dual problems have finite optimal solutions, and the values of their optimal objectives (4) and (26) are equal. Since $A \neq \emptyset$, $\Theta(s^t)^*$ is a cone, and the objective (26) is continuous in $\alpha(s^{t+1})$ for $s^{t+1} \in \{s^t_+\}$, the dual problem (26)-(27) can be re-written as

$$\sup_{\substack{\alpha(s^{t+1}) \\ s^{t+1} \in \{s^t_+\}}} \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1})[\sup_{a \in A(s^{t+1})} \sum_{s^\tau \in D(s^{t+1})} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau)] \tag{28}$$

$$s.t \quad \left\{ q(s^t) - \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1})R(s^{t+1}) \right\} \in \Theta(s^t)^* \tag{29}$$
$$\alpha(s^{t+1}) > 0, \quad s^{t+1} \in \{s^t_+\}.$$

The value of the optimal objective of the problem (28)-(29) is equal to

$$\sup \sum_{s^{t+1} \in \{s^t_+\}} \frac{a(s^{t+1})}{a(s^t)} [\sup_{a \in A(s^{t+1})} \sum_{s^\tau \in D(s^{t+1})} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau)] \tag{30}$$

where the outer supremum is taken over the admissible stochastic discount factors $\{a(s^{t+1}) \backslash a(s^t)\}$ given by relation (1). By a dynamic programming argument, (30) is equal to

$$\sup_{a \in A(s^t)} \sum_{s^\tau \in D(s^t) \backslash \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau). \tag{31}$$

Repeating the above procedure for every node of the event-tree shows that there is a feasible portfolio strategy $\hat{\theta}$ such that, $\hat{\theta}(s^t)$ is an optimal solution to the primal problem (4)-(5) for each s^t. It follows that

$$
\begin{aligned}
q(s^t)'\hat{\theta}(s^t) &= \sup_{a \in A(s^t)} \sum_{s^\tau \in D(s^t)\setminus\{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \geq 0, \quad s^t \in D, && (32)\\
R(s^t)'\hat{\theta}(s^t_-) &\geq \sup_{a \in A(s^t)} \sum_{s^\tau \in D(s^t)} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \geq 0, \quad s^t \in D\setminus\{s^0\}. && (33)
\end{aligned}
$$

Relations (32) and (33) imply

$$
\begin{aligned}
z^{\hat{\theta}}(s^t) &\equiv R(s^t)'\hat{\theta}(s^t_-) - q(s^t)'\hat{\theta}(s^t) \\
&\geq \sup_{a \in A(s^t)} [z(s^t) + \sum_{s^\tau \in D(s^t)\setminus\{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau)] - \sup_{a \in A(s^t)} \sum_{s^\tau \in D(s^t)\setminus\{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \\
&= z(s^t)
\end{aligned}
$$

for each $s^t \in D\setminus\{s^0\}$. Therefore, $\hat{\theta}$ generates a payoff stream $z^{\hat{\theta}} \geq z$ at a date-0 cost equal to

$$
q(s^0)'\hat{\theta}(s^0) = \sup_{a \in A} \sum_{s^t \in D\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t). \qquad (34)
$$

This establishes (25) and, combined with (24), gives rise to (3). This proves theorem 1. Equation (3) together with the above calculations shows that, $\hat{\theta}(s^t)$, an optimal solution to program (4)-(5), is the portfolio component at s^t of a feasible strategy that achieves $V(z)$. This completes the proof of theorem 2. $\square$

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Endnotes

1. When addressing the issue of derivatives pricing or financial innovations, one should carefully distinguish innovated assets from their synthetic counterparts. See, for example, Detemple and Murthy (1997).

2. Portfolio choice and option hedging in the presence of proportional transactions costs have been studied, respectively, by Constantinides (1986), Davis and Norman (1990), and Dumas and Luciano (1991) with a somewhat different optimality criteria, and by Leland (1985), Merton (1989), Shen (1990), and Boyle and Vorst (1991) without an explicit optimality criteria.

3. Leverage and nonnegative wealth constraints are analyzed by Grossman and Vila (1992) and Cox and Huang (1989), respectively, with a somewhat different optimality criterion.

4. In continuous time mathematical finance literature, an abstract stochastic control representation for the minimum cost hedging problem is derived and some bounds and complicate approximation schemes for computing them are provided.

5. Some results from this perspective can be inferred from Santos and Woodford (1997) with a constraint that portfolio net worth be nonnegative, Huang and Werner (1998) with an assumption of no uncertainty, and Huang (1998) with general constraints on portfolio values.

6. A subset of a finite dimensional Euclidean space is a polyhedral if it is the intersection of a finite number of supporting half-spaces. See, for example, Sposito and David (1971, 1972).

7. This no-arbitrage characterization for polyhedral cones remains valid for general closed cones, provided that an adapted Slater condition is satisfied. See, for example, Sposito (1989) and Sposito and David (1971, 1972).

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