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Solving PDEs arising in Finance with Finite Elements

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Abstract

The Finite Element Method is a well-studied and well-understood method to solve partial differential equation. Its applicability to financial models formulated as pdes is demonstrated. Its advantage concerning the computation of accurate "Greeks" is delineated. This is demonstrated with bond- and option-pricing models.

Contents

1 Introduction

Many pricing models can be cast into continuous time. This naturally leads to partial differential equations. These pdes are usually linear and parabolic. To avoid clutter in notation we restrict our attention to the case of linear models depending on maximal two factors.[1] These models have been solved traditionally with Finite Differences (FD). Many different FD techniques exist ([1], ch. 2); the most important have been introduced to financial problems ([15], ch. 15; [8], ch. 10; [29], ch. 16-22; [9]). The usefulness of Finite Elements (FE) has been recognized by many autors ([14], p. 47; [8], p. 212; [10], p. 1664; [11], p. 582; [13], p. 586; [23]; [6], [30], sec. 2.5.4) but to our knowledge the first to explore this idea in some more detail was [26].

[1]This does *not* include the *nonlinear* model with transaction cost by [19] (see also ([29], ch. 13) and the 3-factor swaption model by Dempster and Hutton [7]. These models can also be solved with FE, but this will not be demostrated here.

2 Derivation

2.1 A pure Finite Element Approach

- results in linear system with positive-definite coefficient matrix
- unusual treatment of time
- hybrid approach is based on this

2.2 A Hybrid Finite Di erences/FiniteElement Approach

- name due to Duffie, typical name in mathematical and engineering literature: time-dependent finite element methods
- results in system of linear ordinary differential equations which is well-known in economics, compare ([25], p. 305)
- exposition here is based on ([3], ch.4) and ([5], ch. 11)

3 Examples

3.1 Barrier Options

3.1.1 Double Barrier

We consider a 1-year up-and-out-down-and-out call option f with continuous monitoring,[2] where the underlying stock S is at 100, the strike X is at 100, volatility σ is 20 %, the risk-free rate r is 10 % (continuous compounding), and the barriers are set at 75 and 130, with a no rebate. This leads to the following well-posed backward parabolic pde problem:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{1}$$

$$f(T,S) = \max(S-X,0) \tag{2}$$

$$f(t,75) = 0 \tag{3}$$

$$f(t,130) = 0 \tag{4}$$

The analytical solution involves a series which goes from $-\infty$ to ∞. For numerical purposes this series has to be cut off after some finite number of terms. It has been shown in [17] that it is sufficient to consider only the terms from -2 to 2 because all other terms are very close to zero. Here, for the analytical solution, we have taken the terms from -5 to 5.[3]

Underlying	Fair Value						
	Analytical	Numerical					
		errlim 0.01		errlim 0.001		errlim 0.0001	
			Error		Error		Error
76	0.27306	0.27376	0.26 %	0.27317	0.04 %	0.27317	0.04 %
80	1.22027	1.22357	0.27 %	1.22092	0.05 %	1.22087	0.05 %
90	2.90287	2.90875	0.20 %	2.90378	0.03 %	2.90378	0.03 %
100	3.52511	3.52456	0.02 %	3.52395	0.03 %	3.52533	0.01 %
110	2.89967	2.89187	0.27 %	2.89670	0.10 %	2.89932	0.01 %
120	1.47489	1.46833	0.44 %	1.47269	0.15 %	1.47458	0.02 %
129	0.13192	0.13137	0.42 %	0.13181	0.08 %	0.13192	0.01 %
Data of FE-Run							
Cycles		25		57		72	
Nodes		223		219		219	
Cells		74		72		130	

Table 1: Double Barrier Option

[2] Solutions to problems with discrete monitoring can be found by applying the adjustment formulae by Glasserman et al. to the continous-monitoring solution.

[3] It is the normal case that analytical solutions to option pricing problems involve infinite series and/or indefinite integrals. This has led ([29], p.) to the recommendation *not* to look for analytical solutions (which are usually not easy to find) but to solve the pde with numerical methods directly.

3.1.2 Single Barrier

The following example is based on the example in ([2], p. 225f). Consider a 6-month up-and-out call option f, where the underlying stock is at 100 S, the strike X is at 100, volatility σ is 20 %, the risk-free rate is r 5 % (continuous compounding), and the barrier is set at 110, with a rebate payment of 10.
This leads to the following well-posed backward parabolic pde problem:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (5)$$
$$f(T,S) = \max(S-X,0) \quad (6)$$
$$f(t,0) = 0 \quad (7)$$
$$f(t,110) = 10 \quad (8)$$

Und.		Fair value	Delta	Gamma	Theta	Vega
80	Analytical	0.43223	0.08507	0.01295	-0.00542	0.08714
	Numerical	0.43221	0.08507	0.01298		
	Error	0.0040 %	0.0000 %	0.1965 %		
90	Analytical	2.10253	0.26128	0.01999	-0.01179	0.16924
	Numerical	2.10252	0.26130	0.01992		
	Error	0.0003 %	0.0068 %	0.3707 %		
100	Analytical	5.60968	0.42205	0.00939	-0.01012	0.12730
	Numerical	5.60975	0.42204	0.00927		
	Error	0.0012 %	0.0014 %	1.3159 %		
105	Analytical	7.79972	0.44635	0.00031	-0.00552	0.06339
	Numerical	7.79971	0.44635	0.00051		
	Error	0.0001 %	0.0000 %	65.707 %		
109	Analytical	9.56930	0.43406	-0.00625	-0.00110	0.011462
	Numerical	9.56929	0.43405	-0.00620		
	Error	0.0001 %	0.0029 %	0.8342 %		

Table 2: Double Barrier Option

3.2 Plain Vanilla European Call

In contrast to the barrier problems which posses boundary conditions by their very nature this is not the case with simple European calls and puts. The boundaries have to be approximated. Here we use the approximation by ([29], p.).[4] This also leads to a well-posed backward parabolic pde problem:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (9)$$
$$f(T,S) = \max(S-X,0) \quad (10)$$

[4] This point is discussed in some more detail in ([18], ch. 3) and -with numerous numerical studies- in [27].

$$f(t,0) = 0 \tag{11}$$
$$f(t,100) = S - Xe^{-rt} \tag{12}$$

U.	Solution	Fair V.	Delta	Gamma	Speed	Vega	Theta	Rho
30	Analytical	0.09141	0.05370	0.02573		2.31569	-0.61511	0.75985
	Numerical	0.09139	0.05372	0.02539		2.31605	-0.61510	0.76045
	Error [%]	0.02002	0.03465	1.30772		0.01561	0.00129	0.07882
33	Analytical	0.41154	0.17463	0.05516		6.00707	-1.73655	2.67569
	Numerical	0.41153	0.17465	0.05469		6.01050	-1.73650	2.68050
	Error [%]	0.00190	0.00811	0.85089		0.05706	0.00300	0.17983
36	Analytical	1.22015	0.37420	0.07443		9.64623	-3.15435	6.12554
	Numerical	1.22013	0.37422	0.07438		9.64500	-3.15500	6.13000
	Error [%]	0.00203	0.00400	0.07336		0.01278	0.02046	0.07279
39	Analytical	2.68005	0.59686	0.07019		10.67579	-4.194921	0.29884
	Numerical	2.68006	0.59687	0.07066		10.67500	-4.195001	0.30000
	Error [%]	0.00024	0.00022	0.66413		0.00736	0.00180	0.01129
42	Analytical	4.75942	0.77913	0.04996		8.81342	-4.55909	13.98205
	Numerical	4.75942	0.77912	0.04975		8.81000	-4.56000	13.98500
	Error [%]	0.00006	0.00120	0.42786		0.03875	0.01991	0.02112
45	Analytical	7.28782	0.89564	0.02845		5.76025	-4.45367	16.50808
	Numerical	7.28782	0.89565	0.02823		5.76000	-4.46000	16.51000
	Error [%]	0.00001	0.00111	0.75287		0.00427	0.14223	0.01162
48	Analytical	10.07750	0.95669	0.01354		3.11957	-4.20826	17.92174
	Numerical	10.07750	0.95671	0.01334		3.10000	-4.20000	17.90000
	Error	0.00005	0.00259	1.47273		0.62736	0.19635	0.12133
51	Analytical	12.99433	0.98391	0.00558		1.45049	-4.00860	18.59253
	Numerical	12.99430	0.98390	0.00556		1.45000	-4.00000	18.60000
	Error	0.00019	0.00096	0.34530		0.03402	0.21465	0.04018

Table 3: Plain Vanilla European Call

3.3 Capped Power Option

There is a closed-form solution to power option. But within the market place only *capped* power options are traded to which an analitical solution is not known.

3.4 Term Structure Models

3.4.1 Single Factor: Vasicek

Face value	T	Analytical solution	Numerical solution with $PDEase2D^{TM}$	Error
5	3.5	4.7343	4.7340	0.0003
5	4.0	4.4838	4.4836	0.0002
5	4.5	4.2475	4.2474	0.0001
105	5.0	84.5408	84.5455	0.0047

Table 4: Differences in computing the Vasicek model with FE/FD and analytically with interest rates between 0 and 21 %

3.4.2 Two Factors: Duffie/Kan

		Grid Size N			Exact
Short rate x_0	Long rate x_1	111	221	331	∞
0.1070	0.1584	0.8464	0.8531	0.8532	0.8535
0.0336	0.0791	0.9179	0.9251	0.9246	0.9246
0.0710	0.0593	0.9411	0.9420	0.9421	0.9424

Table 5: The Duffie-Kan model - an FD approach by Duffie-Kan

		Number of Elements			Exact
Short rate x_0	Long rate x_1	38	97	2476	∞
0.1070	0.1584	0.8256	0.8421	0.8533	0.8535
0.0336	0.0791	0.8992	0.9163	0.9247	0.9246
0.0710	0.0593	0.9223	0.9411	0.9419	0.9424

Table 6: The Duffie-Kan model - a hybrid FE/FD approach by the author

3.5 Rainbow Options

3.5.1 Call on the Maximum of two Risky Assets

3.5.2 Basket Option

4 Conclusions

A Further Computations

This appendix includes further computations and remarks thereupon.

B Codes

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