Formation of Teams in Contests: Tradeoffs Between Inter- and Intra-Team Inequalities*

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Abstract

We examine a team contest in which players exert effort to compete against other teams for a prize, which is then divided among the members of the winning team according to a specified sharing rule. Assuming that players vary in their observable abilities to contribute to team performance, we study the structure of stable teams under different sharing rules. A head-hunting attempt is successful if the targeted player benefits from switching teams and the offering team's probability of winning increases as a result. A team structure is defined as stable when no such successful head-hunting opportunities exist. We show that when all teams adopt an egalitarian prize-sharing rule, complete ability sorting arises, resulting in the greatest inter-team inequality. In contrast, when teams use substantially unequal sharing rules, the stable team structure features lower inter-team inequality but greater intra-team inequality. These results reveal a trade-off between intra-team and inter-team inequality in the formation of stable teams.

Keywords: team contest, CES effort aggregator function, prize sharing rule, head-hunting, stable team structure, intra-team inequality, inter-team inequality
***JEL* Classification Numbers:** C71, C72, C78, D71, D72, D74

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1 Introduction

In team contests, players must exert joint effort to compete against other teams for a prize. Each player's performance is typically influenced by internal team characteristics, including the team's composition (i.e., the ability or productivity of one's teammates), the agreed-upon prize-sharing rule, the degree of complementarity between individual efforts, and the presence of free-riding incentives. These individual contributions are aggregated into total team effort, which, when compared against that of other teams, determines the team's probability of winning according to a Tullock contest success function. Thus, the relative strength and composition of opposing teams can affect not only each player's equilibrium effort choice, but also the formation of teams in the first place.

This paper focuses on a specific form of player mobility across teams: *head-hunting*. We assume that, in an effort to improve their equilibrium winning probability, teams may extend offers to any player—whether currently affiliated with another team or not. Each offer specifies the recruit's position within the team, defined by a predetermined share of the prize. When evaluating such offers, players must weigh multiple factors: (1) the prize share they would receive on the new team; (2) the level of effort they would optimally choose to exert on the new team; (3) the new team's winning probability and the significance of their role in it; (4) how their publicly known ability compares to that of their prospective teammates; and (5) the consequences of vacating their current position, including how it may affect the broader competition. If a player accepts an offer and the offering team's equilibrium winning probability increases as a result, we define the move as a successful head-hunting. A team structure—that is, a matching between teams and players—is said to be *head-hunting-proof* (or *stable*) if no such successful head-hunting opportunity exists.

The aim of this paper is to study the types of stable team structures that may arise under commonly used prize allocation rules. We assume that all teams have the same fixed capacity and adopt a common prize-sharing rule. This rule may be externally imposed—for example, as part of a contest's official guidelines, such as in the Kaggle example discussed below—or it may reflect established industry practice. We differentiate between allocation rules based on the degree of equality in reward distribution. At one extreme is the egalitarian rule, which divides the prize equally among team members. At the other extreme are hierarchical rules that allocate significantly larger shares to higher-ranked members. Before presenting an overview of our findings, we offer several illustrative examples of such allocation rules and the resulting team structures.

Inter-team inequality can coexist with intra-team equality. A clear example comes from Kaggle, a platform that hosts a variety of data science and machine learning competitions. Each competition is self-contained, with a predetermined prize and a global cap of eight players per team (although individual competitions often set lower team size limits). Participation is open to all, and team formation is entirely at the discretion of the contestants. Committing to a team occurs at registration and constitutes a binding agreement. When a team wins, the prize is divided equally among its members, regardless of individual contributions—an egalitarian rule enforced by default on the platform.[1] Notably, all submissions are evaluated and ranked, not just the winning entry. Each player receives a score based on their team's performance, and these publicly available scores help players find teammates in future competitions. Over time, teams have become increasingly sorted by ability, with top-ranked contestants often joining forces and repeatedly achieving strong results. Similar sorting patterns can be observed among mid- and lower-ranked players as well. The ongoing evolution of team formation on Kaggle closely resembles head-hunting and has led to a landscape where inter-team inequality persists alongside enforced intra-team equality.

In contrast, inter-team equality may emerge alongside intra-team inequality. The second example comes from hierarchical reward systems commonly found in white-collar industries. These systems are typically based on the rationale that strong differentiation—whether in pay or recognition—between organizational ranks promotes internal competition for promotion, thereby enhancing individual performance. Although such structures are designed as internal incentive mechanisms, they often lead employers to engage industry specialists, known as head-hunters, to identify candidates from competing organizations. These specialists typically do not conduct the recruitment themselves but focus on locating high-performing individuals for specific senior-level positions.[2] As a result, high-ability individuals are often placed in top roles across competing firms, contributing to a relatively even distribution of talent across organizations.

We incorporate the intuitive structures illustrated in the examples above by explicitly modeling worker heterogeneity. In particular, players differ in their *observable* abilities and, consequently, in their potential contributions to a team. Team output is modeled as the aggregation of individual effort inputs, using a constant elasticity of substitution (CES) aggregator, which allows for varying degrees of complementarity among efforts. Players' efforts are *neither observable nor contractible*; thus, while these efforts influence the team's probability of winning, they do not affect how the prize is shared among team members. Prize shares may differ across players depending on their assigned positions, and we focus specifically on the rate at which lower-ranked positions are discounted relative to higher ones. By combining the approaches of Konishi and Pan (2020, 2021) and Simeonov (2020), we are able to explicitly solve for players' equilibrium payoffs, enabling us to treat head-hunting as a well-defined process of attracting better candidates (see the literature review below).[3]

[1] The well-known Netflix Prize competition was hosted on this platform. Page (2018) describes how the winning team and several rival teams were formed dynamically over the course of the competition.

[2] Some U.S. higher education institutions hire head-hunting firms to assist in the search for new deans. For example, in a dean search for a business school, a head-hunting firm may approach department chairs or senior faculty from other institutions to assess their interest in the role.

[3] In the group contest literature, some authors use convex individual effort cost functions with a simple sum as the effort aggregator, while others adopt a CES aggregator. Both specifications capture effort complementarities among team members. However, the analysis in this paper relies entirely on a constant-returns-to-scale (CRS)

If all teams adopt the egalitarian rule—dividing rewards equally among members—then high-ability players are unlikely to be satisfied on lower-ability teams. Not only might they feel undercompensated relative to their contributions, but their presence may also exacerbate free-riding by lower-ability teammates. As a result, high-ability players may prefer to be headhunted by other high-ability teams. In such an environment, a team matching structure with a high degree of ability sorting emerges as a natural outcome. What, then, would be sufficient to disrupt this sorting—specifically, to induce high-ability players to leave strong teams and join weaker ones? Clearly, a substantial difference in compensation between positions is required to make this shift viable. Only when a high-ability player is offered a significantly better position—and thus a larger share of the prize—on a weaker team would they be willing to leave a lower-ranking role on a stronger team. In this sense, high intra-team inequality becomes a necessary condition for achieving a more balanced distribution of talent across teams.

The main result of this paper is to demonstrate that the trade-off between intra- and inter-team inequality is not coincidental, but rather structurally embedded in the nature of stable team formation. We show that when the egalitarian rule is applied within each team, complete ability sorting across teams is the only stable team structure (Proposition 3). We then focus on a class of hierarchical sharing rules characterized by a common discount factor $\mu \in (0, 1)$ applied to rewards. When $\mu = 1$, the rule corresponds to egalitarian sharing, and we show that for values of μ close to one, an assignment is stable if and only if it exhibits complete ability sorting (Proposition 4). In contrast, when μ is sufficiently close to zero, an assignment is stable if and only if it follows a cyclical pattern of ability assignment across teams (Proposition 5). For intermediate values of of μ, both cyclical and complete sorting assignments can coexist (Example 3), and more generally, stable team structures may involve combinations of cyclical assignment and complete sorting (Example 4). Proposition 6 offers insights into the extent of rent dissipation under these different types of stable assignments. Finally, in the conclusion, we present an additional example (Example 5) that explores how teams' bidding behavior for talented players can be analyzed within our framework, further illustrating its applicability to real-world team markets.

The remainder of the paper is organized as follows. The next subsection provides a brief review of the related literature. Section 2 introduces the model and key assumptions. Section 3 analyzes equilibrium player and team efforts in general team contests. In Section 4, we introduce and formalize the concept of stability in team formation, and present our main results on the trade-offs between intra- and inter-team inequality. Section 5 concludes.

CES aggregator combined with constant (but heterogeneous) marginal costs of effort. The CRS property implies that each player's equilibrium share of total effort is independent of the team's total effort level—an assumption that significantly simplifies our analysis.

1.1 Relations to the Literature

Broadly, this paper contributes to the literature on coalition formation with externalities, in which players' payoffs depend not only on the coalition to which they belong, but also on the composition of other coalitions. Foundational works in this area include Hart and Kurz (1983), Bloch (1996), Yi (1997), Ray and Vohra (1999), and Ray (2008), who provide general frameworks for coalition formation games with externalities. More specific economic applications can be found in Bloch (1995), Yi (1996), and Ray and Vohra (2001), who examine cartel formation, customs unions, and public goods provision, respectively. While our paper fits within this broader literature, it introduces several novel features. In our setting, each coalition (or team) is subject to a membership quota, and prize-sharing rules are fixed in advance but allow for heterogeneous shares. As a result, players not only care about which team they join, but also about the specific position they occupy within the team. This position-dependent preference is a distinguishing feature of our model and represents a new contribution to the coalition formation literature.

More specifically, this paper contributes to the literature on group contests and prize-sharing rules. Assuming that individual efforts are contractible, Nitzan (1991) analyzes how the combination of egalitarian and relative-effort-sharing rules affects players' incentives in both large and small groups. Lee (1995) and Ueda (2002) extend this by endogenizing group sharing rules. Esteban and Ray (2001), as well as Nitzan and Ueda (2011), show that Olson's (1973) group size paradox disappears when the prize can be split into public and private benefits, and when private benefits are allocated via an endogenously determined relative-effort-sharing rule, respectively.

Building on this line of research, Baik and Lee (1997, 2001) endogenize alliance formation in Nitzan's (1991) framework with endogenous group sharing rules, analyzing both two- and multi-alliance settings. They model alliance formation using open-membership games. Bloch et al. (2006) generalize this framework to study the stability of the grand alliance in various alliance formation games. Sanchez-Pages (2007a, 2007b) explores alternative stability concepts in contests with alliance formation, assuming perfect substitutability of efforts among members. These papers generally assume that alliance members can commit to binding sharing contracts upon winning.

In contrast, following Esteban and Sakovics (2003), Konishi and Pan (2020, 2021) analyze equilibrium alliance structures in games with homogeneous players, using a CES aggregator to capture complementarity among efforts.[4] In their setting, alliance sizes are endogenous, and prizes are shared equally due to player homogeneity. By contrast, the current paper examines team formation under fixed membership constraints, allowing for heterogeneous player abilities and unequal sharing rules. Our focus is on how such sharing rules influence the distribution of talent across teams.

[4]In contest games, Kolmar and Rommeswinkel (2013) examine group contests with exogenously formed teams using a CES effort aggregator, allowing for heterogeneous player abilities.

With fixed team sizes, Simeonov (2020) and Kobayashi, Konishi, and Ueda (2024) investigate prize-sharing rules that maximize the team's winning probability under a CES aggregator. Choi, Chowdhury, and Kim (2016) analyze a two-group model in which each group consists of two members who are heterogeneous in their within-group powers. They study the allocation of an indivisible private good, following the framework of Esteban and Sakovics (2003). In contrast, our paper takes the sharing rule as given but allows players to move across teams, focusing on how the sharing rule shapes equilibrium membership structures.[5] Furthermore, unlike in Nitzan (1991) and Nitzan and Ueda (2011), we assume that individual efforts are unobservable or noncontractible, creating scope for free-riding as in Esteban and Ray (2001). For comprehensive surveys of the group contest literature, see Konrad (2009) and Fu and Wu (2019).

Our notion of stability—head-hunting-proofness—is closely related to the concept of pairwise stability in the matching literature, particularly due to the presence of team membership quotas. Gale and Shapley (1962) introduced the seminal two-sided matching problem and the corresponding notion of pairwise stable matching. In their setting, pairwise stability is equivalent to core stability, despite being defined in relatively simple terms. However, when externalities exist across matched pairs, the treatment of a vacancy created by head-hunting becomes crucial. Imamura, Konishi, and Pan (2021) extend the two-sided one-to-one matching model to include such externalities and propose a notion of pairwise stability via swapping, which preserves several desirable properties.[6] Our concept of head-hunting-proofness can be interpreted as a form of pairwise stability, where a team structure is immune to deviations involving a team leader—who seeks to improve the team's winning probability—and a player—who is motivated by individual payoff gains.[7]

This paper also bears a natural connection to the literature on sports economics. Consider, for example, the draft practices used in professional sports leagues such as the NBA. These drafts are typically designed to promote inter-team equality by granting weaker teams priority in selecting new players. As a result, high-ability players often join weaker teams, leading to team compositions with a wide range of player abilities—at least initially. However, such intra-team heterogeneity often coincides with a high degree of salary inequality within teams. For instance, among the top 30 rookie contracts in the NBA in 2019, salaries ranged from \$13 to \$51 million. In this sense, a high degree of intra-team inequality appears to be a prerequisite for achieving a relatively equal distribution of talent across teams. That said, as Rosen and Sanderson

[5] Assuming a continuum of atomless teams, Konishi, Pan, and Simeonov (2024) study the existence of equilibria in which both team structures and sharing rules are determined endogenously. See the conclusion section for discussion.

[6] Imamura and Konishi (2023) further support the notion of pairwise stability via swapping by introducing farsighted agents. They show that this concept is equivalent to the largest consistent set of Chwe (1994) in the context of the pairs competition problem developed in Imamura, Konishi, and Pan (2023).

[7] Our stability concept assumes that players are somewhat naïve in their reasoning. In contrast, much of the coalition formation literature with externalities across coalitions assumes highly sophisticated players. See Bloch (1997), Ray (2008), and Ray and Vohra (2014) for comprehensive surveys.

(2001) note, professional sports labor markets are somewhat peculiar, and our model cannot be directly applied to them.[8] Palomino and Sakovics (2004) and Burguet and Sakovics (2019) analyze team bidding behavior for talent in professional sports leagues, explicitly accounting for externalities on other teams and leagues. Although our equilibrium concept is not directly suited for modeling sports labor markets, our framework may nonetheless offer useful insights. In Example 5 (see the conclusion), we demonstrate how our model could be adapted to study such markets.

Lastly, our results on intra- and inter-team inequality are closely related to two contributions in the literature on hedonic games. Hedonic games are characteristic function form games—that is, games without externalities across coalitions—in which a player's payoff depends solely on the coalition they join.[9] Banerjee, Konishi, and Sönmez (2001) show that under the top coalition property, an assortative coalition structure is the unique core allocation. Morelli and Park (2016) study hedonic games where players evaluate both (i) the expected power of the coalition and (ii) their position in the coalition's internal hierarchy. They show that cyclical assignment allocations can be core stable. In our model, both complete sorting and cyclical assignment structures emerge endogenously by varying the prize-sharing rule—within a unified framework that allows for externalities.[10]

2 The Model

We consider a setting with potentially $j = 1, 2, ..., J$ teams, each consisting of M positions. Let (m, j) denote the mth position in j team. Each player $i = 1, ..., N$ is characterized by an *ability level* $a_i > 0$, and we assume without loss of generality that abilities are ordered such that $a_1 \geq a_2 \geq ... \geq a_N$. With slight abuse of notation, we also use M, J, and N to denote the sets of positions, teams, and players, respectively.

An *assignment* is defined as $\varphi = (\varphi_{mj})_{m \in M, j \in J}$, where $\varphi_{mj} \in N \cup \{\emptyset\}$ indicates the player assigned to position m in team j, or $\emptyset$ if the position is vacant. We assume that each player can belong to at most one team, so an assignment is *feasible* if $\varphi_{mj} \neq \varphi_{m'j'}$ for all $(m, j) \neq (m', j')$. Let $\varphi_j = (\varphi_{mj})_{m \in M}$ denote the assignment of players to team j; a player i with $\varphi_{mj} = i$ for some $m \in M$ is a member of team j under assignment φ.

We study team stability within a two-stage team contest framework. In Stage 1, a team structure φ is formed. In Stage 2, a team contest is played given the assignment φ. Players

[8] For example, our model differentiates rewards through prize-sharing rules, while in sports labor markets, compensation is typically structured through salaries. Moreover, while players in our model are motivated to increase their team's winning probability through effort, real-world athletes are often incentivized by repeated interactions and performance bonuses.

[9] See Banerjee, Konishi, and Sönmez (2001) and Bogomolnaia and Jackson (2002) for formal models of hedonic games.

[10] It is worth noting that the solution concept used in this paper differs from core stability. In the presence of extensive externalities, it becomes difficult to define group deviations without imposing substantial additional structure on the game.

anticipate the outcome of Stage 2 when the assignment φ forms in Stage 1. We first describe the contest game in Stage 2; our stability concept, introduced in Section 4, applies to Stage 1 and is best understood after the contest environment is laid out.[11]

Given a feasible assignment φ, players compete in teams for a prize of value V. Each member of team j simultaneously and noncooperatively chooses an effort level e_i. The total effort in team j is aggregated through a CES function $X_j = (\sum_{m\in M} a^{\sigma}_{\varphi_{mj}} e^{\sigma}_{\varphi_{mj}})^{\frac{1}{\sigma}}$, where $0 < \sigma < 1$.[12] This CES function becomes linear (perfect substitutes) when $\sigma = 1$, and converges to a Cobb-Douglas function as $\sigma \to 0$. Let $X = \sum_j X_j$ be the total effort across all teams, and $X_{-j} = \sum_{k\neq j} X_k$ be the total effort of all teams except team j. The probability that team j wins is given by a simple Tullock contest success function:[13]

$$P_j = \frac{X_j}{\sum_{k=1}^{J} X_k} = \frac{X_j}{X_j + X_{-j}}. \tag{1}$$

After a team wins, it distributes the prize V according to a fixed sharing rule $\theta = (\theta_1, ..., \theta_m, ..., \theta_M)$, where $\theta_m \in [0, 1]$ for all m, and $\sum_{m\in M} \theta_m = 1$. Here, θ_m represents the share awarded to the player in position m of any team.[14] We assume this rule θ is identical across all teams, reflecting an industry-wide norm or externally imposed standard. Without loss of generality, we order the positions such that $\theta_1 \geq \theta_2 \geq ... \geq \theta_M$. Each player has a linear cost of effort: $c_i(e_i) = e_i$. Thus, the expected payoff for the player assigned to position m of team j—that is, player i such that $\varphi_{mj} = i$—is

$$U_{\varphi_{mj}} = \theta_m P_j V - e_{\varphi_{mj}}.$$

Each player chooses effort to maximize their expected payoff. Within each team, we assume that players take the aggregate efforts of other teams, X_{-j}, and the efforts of their own teammates as given, and play a Nash equilibrium in team j's effort contribution game. The aggregation of each team's equilibrium efforts is thus a best response to the aggregate efforts of the other teams.

[11] Our solution concept blends cooperative and noncooperative elements: in Stage 2, players choose efforts noncooperatively within teams, while in Stage 1, we look for assignments that are immune to head-hunting deviations.

[12] Kolmar and Rommeswinkel (2013) refer to this CES function as a group impact function.

[13] While the simple Tullock contest success function treats the total efforts of other teams as perfect substitutes, the ability distribution within those teams still matters. Because aggregate effort depends on both ability and effort, differences in ability composition across teams influence the vector $(X_k)_{k\neq j}$, and thus affect team j's winning probability.

[14] Fixed sharing rule θ means that even if some positions are vacant, the player at position m receives exactly θ_m of the prize. For instance, under equal sharing, each occupied position receives $1/M$ of the prize, regardless of vacancies.

3 Team Contest

3.1 Equilibrium Analysis

We begin by analyzing the effort choices in the inter-team contest, assuming for now that all teams and all players exert positive effort in equilibrium.

Then, the first-order condition of any player i with $\varphi_{mj} = i$ is:

$$\frac{\partial U_{\varphi_{mj}}}{\partial e_{\varphi_{mj}}} = \theta_m \frac{(\sum_{m'=1}^{M} a^{\sigma}_{\varphi_{m'j}} e^{\sigma}_{\varphi_{m'j}})^{\frac{1}{\sigma}-1} a^{\sigma}_{\varphi_{mj}} e^{\sigma-1}_{\varphi_{mj}} X_{-j}}{\left(\left(\sum_{m'=1}^{M} a^{\sigma}_{\varphi_{m'j}} e^{\sigma}_{\varphi_{m'j}}\right)^{\frac{1}{\sigma}} + X_{-j}\right)^2} V - 1 = 0. \tag{2}$$

Using the identity $X_j^{1-\sigma} = \left(\sum_{m'=1}^{M} a^{\sigma}_{\varphi_{m'j}} e^{\sigma}_{\varphi_{m'j}}\right)^{\frac{1}{\sigma}-1}$, we can solve for player i's equilibrium effort:

$$e_{\varphi_{mj}} = X_j \left[(1 - P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}} \left(a^{\sigma}_{\varphi_{mj}} \theta_m\right)^{\frac{1}{1-\sigma}}. \tag{3}$$

Let us define the productivity of team j under assignment φ_j as $A_j(\varphi_j) = \left(\sum_{m=1}^{M} a^{\frac{\sigma}{1-\sigma}}_{\varphi_{mj}} \theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$. By aggregating the equilibrium efforts of team members, we obtain the following result. (All proofs are provided in the Appendix.)

Proposition 1. *Suppose all J teams exert positive effort in equilibrium. Then, for each team $j = 1, ..., J$, the equilibrium aggregate effort and winning probability are given by:*

$$X_j = \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}\right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}},$$

and

$$P_j = 1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}. \tag{4}$$

The total effort across all teams, X, is:

$$X = \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}. \tag{5}$$

Remark 1. Note that this model admits the possibility of coordination failures due to the use of a CES production function with $\sigma \in (0, 1)$ and and linear effort costs.[15] a player may choose zero effort if all her teammates also exert zero effort—due to complementarity. On the other hand, if at least some teammates exert positive effort, then equation (3) implies that all other

[15] A CES function with $\sigma \in (0, 1)$ can yield positive output even when some effort inputs are zero (in contrast to the Cobb-Douglas case, i.e., $\sigma = 0$, which requires all inputs to be positive). Since marginal cost is constant, if a player's marginal product is below 1, she will not exert effort.

members will also exert positive effort. Hence, multiple equilibria may arise in each team—one with positive effort and one without. *We focus on the positive-effort equilibrium whenever it exists to avoid this coordination problem.* Under this refinement, the equilibrium is unique.

Using the results above, we can also compute each player's equilibrium payoff in Stage 2.

Proposition 2. *Given a feasible assignment φ, suppose that all J teams exert positive effort in equilibrium. Then, for any prize-sharing rules θ, the equilibrium payoff of player i, where $i = \varphi_{mj}$, is:*

$$U_{\varphi_{mj}} = V \times \underbrace{\theta_m}_{\text{share}} \underbrace{\left[1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}\right]}_{\text{team's winning probability}} \underbrace{\left[1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left(\frac{a_i^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}}{\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'}^{\frac{\sigma}{1-\sigma}}}\right)\right]}_{\text{net benefits (after disutility of effort)}}.$$

Remark 2. The final bracketed term is strictly positive because $\frac{a_i^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}}{\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'}^{\frac{\sigma}{1-\sigma}}} < 1$, and the team's winning probability is given by $1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} = P_j$. Thus, as long as team j exerts positive effort, we have $U_{\varphi_{mj}} \geq V \times \theta_m P_j^2 > 0$.

3.2 Share Function Approach and Active/Inactive Teams

So far, we have assumed that all teams are active in equilibrium—that is, each team exerts a positive aggregate effort. However, because players face constant positive marginal costs of effort, this assumption may not hold, especially if a team consists of players with substantially lower abilities relative to others. To complete the equilibrium analysis, we adopt the *share function approach*, systematically analyzed in Cornes and Hartley (2005), by reformulating the second-stage competition as a Tullock contest among J individual players with heterogeneous marginal costs.[16] Cornes and Hartley (2005) consider a Tullock contest among J players with constant but heterogeneous marginal costs $w_1 \leq w_2 \leq .... \leq w_J$, where each player $j = 1, ..., J$ chooses an effort level X_j at cost $w_j X_j$. Player j's winning probability is given by $P_j = \frac{X_j}{\sum_{k=1}^{J} X_k}$, and her payoff is

$$u_j = \frac{X_j}{\sum_{k=1}^{J} X_k} V - w_j X_j.$$

This payoff function is strictly concave in X_j, and the first-order condition yields:

$$\frac{\left(\sum_{k \neq j} X_k\right)}{\left(\sum_{k=1}^{J} X_k\right)^2} V - w_j = \frac{X_{-j}}{X^2} V - w_j = 0, \tag{6}$$

[16] See also Esteban and Ray (2001) and Ueda (2002), who apply the share function approach in similar contexts.

for $j = 1, ..., J$. A player's best response $X_j > 0$ is uniquely defined by solving:

$$X_j^2 + 2X_{-j}X_j + X_{-j}^2 - \frac{X_{-j}}{w_j}V = 0.$$

It follows that player j's best response to X_{-j} is interior if and only if the solution to the first-order condition is positive. Otherwise, the best response is zero. That is, player j's best response is:

$$\beta_j(X_{-j}) = \max\left\{-X_{-j} + \sqrt{\frac{X_{-j}V}{w_j}}, 0\right\}.$$

Following Cornes and Hartley (2005), we define player j's *replacement function* $r_j(X)$ as the effort level that is the best response to $X - r_j(X)$. Formally, it satisfies $r_j(X) = \beta_j(X - r_j(X))$. Solving this yields:

$$r_j(X) = \max\left\{X - \frac{w_j X^2}{V}, 0\right\}.$$

The *share function* of player j, denoted as $s_j(X)$, is then defined as the fraction of total effort X that player j contributes in equilibrium:

$$s_j(X) = \frac{r_j X}{X} = \max\left\{1 - \frac{w_j X}{V}, 0\right\}.$$

Note that $s_j(X)$ is a decreasing function in X. Let $s(X) = \sum_k s_k(X)$, which is also decreasing in X. The share function $s(X)$ is a piecewise linear function with kinks at $\hat{X}^{n_j} = \frac{V}{w_j}$ for each $j = 1, ..., J$. Figure 1 illustrates the individual share functions $s_j(X)$ and their sum $s(X)$. The equilibrium for this artificial contest is a total effort X^* such that $\sum_k s_k(X^*) = 1$. This equilibrium is unique. Moreover, at X^*, the value $s_j(X^*)$ corresponds to player j's winning probability. As is clear from Figure 1, if $\hat{X}^{n_j} = \frac{V}{w_j} < X^*$, then $s_j(X^*) = 0$, implying that player j is inactive—that is, they exert zero effort in equilibrium. Therefore, only players (or, in our application, teams) with relatively low marginal costs are *active, i.e., exert positive efforts.* The following lemma summarizes the outcome of the Tullock contest with heterogeneous marginal costs $(w_j)_{j=1}^J$.

Lemma 1. [Cornes and Hartley, 2005] *A Tullock contest with heterogeneous marginal costs $(J, (w_j)_{j=1}^J)$ has a unique equilibrium total effort X^* satisfying $\sum_j s_j(X^*) = 1$. Moreover, there exists a threshold index j^* such that for each $j = 1, ..., j^*$, the equilibrium effort is positive and given by $X_j = X^* - \frac{w_j(X^*)^2}{V}$, while for each $j = j^* + 1, ..., J$, we have $\hat{X}^{n_j} \leq X^*$ (i.e., $\sum_k s_k(\hat{X}^{n_j}) \geq 1$), and $X_j = 0$. These players (or teams) are inactive.*

In our original model, recall that from equations (4) and (5), we have:

$$\frac{1}{A_j(\varphi_j)} = \frac{X_{-j}}{X^2} \times V,$$

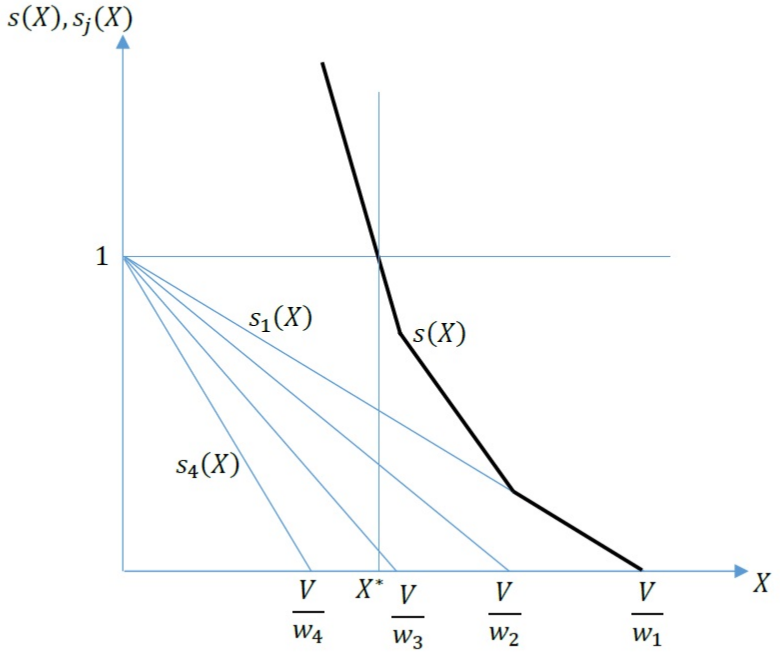

Figure 1: An example with $J = 4$ and $w_1 < w_2 < w_3 < w_4$. Teams 1, 2, and 3 are active. Team 4 is inactive.

which is exactly the first-order condition of the Tullock game, i.e., equation (6), with heterogeneous marginal costs when we define $w_j = \frac{1}{A_j(\varphi_j)}$. That is,

$$\frac{X_{-j}}{X^2}V - \frac{1}{A_j(\varphi_j)} = \frac{\left(\sum_{k\neq j} X_k\right)}{\left(\sum_{k=1}^{J} X_k\right)^2}V - w_j = 0,$$

for $j = 1, ..., J$. Thus, the team with lower marginal cost w_j corresponds to one with higher productivity $A_j(\varphi_j)$. Building on this mapping, we can apply the share function approach to derive closed-form equilibrium solutions for the team contest model, extending results from Kolmar and Rommeswinkel (2013).

Theorem 1. *Given a team profile φ such that team productivities are ordered as $A_1(\varphi_1) \geq A_2(\varphi_2) \geq ... \geq A_J(\varphi_J)$, there exists a unique equilibrium in the inter-team contest, characterized by the share function condition $s(X^*) = 1$. Let $j^* \in \{1, ..., J\}$ be such that $P_j^* = s_j(X^*) > 0$ (active teams) for all $j \leq j^*$ ($\hat{X}_j > X^*$), while $P_j^* = s_j(X^*) = 0$ (inactive teams) for all $j > j^*$ ($\hat{X}_j \leq X^*$). The winning probability of active team j's is given by:*

$$P_j^* = 1 - \frac{(j^* - 1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}},$$

and the equilibrium payoff of player $i = \varphi_{mj}$, occupying position m in team j, is:

$$U_{\varphi_{mj}}(\varphi) = \begin{cases} V \times \theta_m P_j^* \left[1 - \frac{(j^*-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}} \left(\frac{a_{\varphi_{mj}} \theta_m}{A_j(\varphi_j)} \right)^{\frac{\sigma}{1-\sigma}} \right] & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}.$$

The equilibrium total efforts is:

$$X^* = \frac{j^* - 1}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}},$$

and for all active teams $j \leq j^*$, we have:[17]

$$(j^* - 1)\frac{1}{A_j(\varphi_j)} < \sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}.$$

4 Stable Team Structures

In this section, we analyze the stability of a team structure $\varphi = (\varphi_j)_{j \in J}$. We introduce a simple notion of *head-hunting*: given $\varphi_{mj} = i' \in N$, team j offers position (m, j) to another player i, replacing the incumbent player $\varphi_{mj} = i'$ with i. A head-hunting is deemed *successful* if (i) team j's winning probability increases, and (ii) player i, the target of the offer, strictly prefers the new position to her current one. However, because of externalities across teams, we must define this concept with care. We proceed step by step to formalize the definition precisely.

From this point onward, we focus on the case in which all J teams are active.[18] Since we now assume $N = M \times J$, and since leaving positions vacant does not maximize a team's winning probability, it follows that every player will be employed on a team in any feasible assignment. However, when a head-hunting occurs, one position in the poached team becomes vacant, and one player—dismissed from the head-hunting team—is left unemployed. If players are entirely myopic, and if head-hunting decisions are evaluated based solely on the new assignment resulting from the swap, then head-hunting may succeed even in team structures that seem intuitively stable. The following informal example illustrates this issue.

Example 1. *Suppose there are three two-person teams (i.e., three pairs of players) under a common egalitarian sharing rule, where each team member receives 50% of the prize. Let the players be ranked by ability, with player 1 being the highest and player 6 the lowest.*

[17] For inactive teams $j > j^*$, we have $(j^* - 1) \times \frac{1}{A_j(\varphi_j)} \geq \sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}$, implying $X_j = 0$ and $P_j^* = 0$.

[18] This assumption is made for expositional clarity. In a previous version of the paper, we allowed for the possibility that only $j^* < J$ teams are active in equilibrium. However, under our head-hunting definition, it can be shown that players in inactive teams must have lower ability than those in active teams in any stable team structure. Therefore, players in inactive teams are irrelevant for our analysis. Removing these lowest-ability players is equivalent to assuming that all teams are active in all assignments.

Consider an assortative matching: a great team (players 1 and 2), a very good team (players 3 and 4), and a poor team (players 5 and 6). In this configuration, the great team has the highest winning probability, followed by the very good team, while the poor team has a negligible chance of winning. Now suppose players are myopic. Then the great team might consider replacing player 2 with player 3—initiating a head-hunting. In this case, player 2 and player 4 become unassigned, resulting in a new team composition: a semi-great team with players 1 and 3 and a poor team with players 5 and 6. The semi-great team, now significantly stronger and facing no real competition, enjoys a sharp increase in winning probability. Since both conditions for a successful head-hunting are satisfied, this deviation would be considered successful—even though it undermines the apparent stability of the original, ability-sorted assignment.

In the previous example, players 2 and 4 are left alone after the head-hunting, and as single-member teams, they are not competitive enough to remain active. However, note that a team's competitiveness is strictly increasing in the number of occupied positions, and it is always in the interest of a team with a vacancy to recruit the newly unemployed player. Therefore, it is natural to assume that in response to the head-hunting, these players would form a new team.

More generally, if team j head-hunts a player who is currently employed by team k, it is reasonable to assume that team k hires the player fired by team j. In this case, team j anticipates that its fired player will be hired by team k, and evaluates whether the head-hunting is still profitable under this induced assignment.[19] Formally, we define the notion of a successful head-hunting via player swapping as follows:

Definition 1. *Let φ be a feasible assignment. Consider swapping players $i = \varphi_{mj}$ and $h = \varphi_{\ell k}$ between team j and k. Let $\varphi' = (\varphi_{j'})_{j'=1}^{M}$ be the resulting assignment, where (i) $\varphi'_{j'} = \varphi_{j'}$ for all $j' \neq j, k$, (ii) $\varphi'_j = (\varphi_{1j}, ..., \varphi_{m-1j}, h, \varphi_{m+1j}, ..., \varphi_{Mj})$, and (iii) $\varphi'_k = (\varphi_{1k}, ..., \varphi_{\ell-1k}, i, \varphi_{\ell+1k}, ..., \varphi_{Mk})$. We say that this swap is a **successful head-hunting of player** h **by team** j if (a) $P_j(\varphi') > P_j(\varphi)$ and (b) $U_h(\varphi') > U_h(\varphi)$. An assignment φ is **stable** if it admits no successful head-hunting.*

Remark 3. Under this definition, a successful head-hunting by team j implies a Pareto improvement for all members of team j, except for the player being replaced. This follows from Theorem 1: if $A(\varphi'_j) > A(\varphi_j)$, then the payoffs of all remaining members in team j increase. Thus, all team members—except for the dismissed player—*unanimously* accepts player h's entry. Alternatively, we can define a successful head-hunting by prioritizing the preferences of a team leader whose sole objective is to maximize the team's winning probability. If the team leader has full discretion in assigning members to positions, she would allocate positions according to ability in descending order: $a_{\varphi_{1j}} \geq a_{\varphi_{2j}} \geq ... \geq a_{\varphi_{Mj}}$. Suppose player h , currently on team k, is head-hunted by team j for position m. Under the team leader's preference, rather

[19] Knuth (1976) asks whether a sequence of myopic, pairwise deviations via swapping would converge to a stable matching in the marriage problem, where all players are acceptable to the other side. In matching problems with externalities, Imamura, Konishi, and Pan (2023) and Imamura and Konishi (2023) discuss the desirability of using pairwise deviation via swapping to define pairwise stable matchings.

than simply swapping h with φ_{mj}, the leader would instead remove the least able member φ_{Mj} from the team. She would insert player h into position m, and shift the existing players accordingly, i.e., $\varphi'_{mj} = h$ and $\varphi'_{\tilde{m}j} = \varphi_{\tilde{m}-1j}$ for all $\tilde{m} = m+1, ..., M$. In this case, players $\varphi_{mj}, ..., \varphi_{M-1j}$ may not be better off following player h's entry. We can modify our stability concept based on this alternative definition of successful head-hunting. Importantly, our key results—Propositions 4 and 5—remain robust under this alternative formulation.[20]

We now examine the conditions under which a head-hunting—modeled as a swap—is successful. The following lemma identifies the condition of a successful head-hunting when two players in different teams hold the same share and one has strictly higher ability.

Lemma 2. *Let $i = \varphi_{mj}$ and $h = \varphi_{\ell k}$ with $\theta_m = \theta_\ell$. Suppose that $A_j(\varphi_j) \geq A_k(\varphi_k)$, and $a_i < a_h$. Then, under an assignment φ' resulting from swapping i and h, we have $A_j(\varphi'_j) > A_j(\varphi_j)$ and $P_j(\varphi') > P_j(\varphi)$. Hence, this constitutes a successful head-hunting for team j.*

Now, we consider the **egalitarian sharing rule**: $\theta = (\frac{1}{M}, ..., \frac{1}{M})$. That is, all positions in a team receive the same equal share. Repeatedly applying Lemma 2, we obtain the following result.

Proposition 3. *Suppose that all teams use the egalitarian sharing rule. Then, an assignment φ is stable if and only if it is a complete ability sorting assignment. That is, ordering teams by their winning probabilities $P_1(\varphi) \geq P_2(\varphi) \geq ... \geq P_J(\varphi)$, we have $a_{\varphi_{mj}} \geq a_{\varphi_{\ell j+1}}$ for all $j = 1, ..., J-1$, and all $m, \ell = 1, ..., M$.*

From this point on, we focus on a family of hierarchical sharing rules, where shares decay geometrically across positions. Specifically, the rule satisfies $\theta_{m+1} = \mu\theta_m$ for all $m = 1, ..., M-1$ with $\mu \in [0, 1]$. We refer to this as a **hierarchical sharing rule with discount factor** μ. Define $\theta(\mu) = (\theta_1(\mu), ..., \theta_M(\mu)) \in \Delta^M$ as follows:

$$\theta_m(\mu) = \frac{\mu^{m-1}}{1 + \mu + ... + \mu^{M-1}} = \frac{(1-\mu)\,\mu^{m-1}}{1-\mu^M}$$

for all $m = 1, ..., M$. If $\mu = 0$, we have $\theta_1 = 1$ and $\theta_m = 0$ for all $m = 2, ..., M$, which is a monopolization rule. If $\mu = 1$, it is the egalitarian rule $\theta_m = \frac{1}{M}$ for all $m = 1, ..., M$. Given sharing rule $\theta(\mu)$ and assignment φ, the winning probability of team j is

$$P_j(\varphi; \mu) = 1 - \frac{\frac{1}{A_j(\varphi_k;\mu)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k;\mu)}},$$

and the equilibrium payoff of player φ_{mj} is

[20] Since Lemma 2 holds under this modified notion of successful head-hunting, Proposition 3 remains unchanged (see footnote 30 in the Appendix). Propositions 4 and 5 are also valid, though the threshold $\bar{\mu}$ would be adjusted accordingly.

$$U_{\varphi_{mj}}(\varphi;\mu) = \theta_m(\mu) P_j(\varphi,\mu)\left(1 - \frac{\frac{1}{A_j(\varphi_j;\mu)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k;\mu)}} \frac{a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}}\theta_m(\mu)^{\frac{\sigma}{1-\sigma}}}{\sum_{\tilde{m}} a_{\varphi_{\tilde{m}j}}^{\frac{\sigma}{1-\sigma}}\theta_{\tilde{m}}(\mu)^{\frac{\sigma}{1-\sigma}}}\right),$$

where $A_j(\varphi_j;\mu) = \left(\sum_{\tilde{m}} a_{\varphi_{\tilde{m}j}}^{\frac{\sigma}{1-\sigma}}\theta_{\tilde{m}}(\mu)^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$.

The next proposition shows that when the sharing rule is sufficiently close to egalitarian (i.e., μ is close to 1), the only stable assignment is the complete ability sorting assignment, denoted by φ^s. Formally, we have

Proposition 4. *Suppose that there are J active teams. Consider a hierarchical sharing rule satisfiying $\frac{\theta_{m+1}}{\theta_m} = \mu$ for all $m = 1, ..., M-1$. Then there exists $\bar{\bar{\mu}} \in (0,1)$ such that for all $\mu \in (\bar{\bar{\mu}}, 1]$, an assignment is stable if and only if it is the complete ability sorting assignment φ^s.*

We now consider an alternative stable structure: the **cyclical assignment**, $\varphi^c = (\varphi_j^c)_{j=1}^J$, where players are assigned by ability level across positions, rather than within teams. Specifically, $a_{\varphi_{11}^c} \geq ... \geq a_{\varphi_{1J}^c} \geq a_{\varphi_{21}^c} \geq ... \geq a_{\varphi_{2J}^c} \geq a_{\varphi_{31}^c} \geq ... \geq a_{\varphi_{3J}^c} \geq ... \geq a_{\varphi_{M1}^c} \geq ... \geq a_{\varphi_{MJ}^c}$. That is, the top J players are assigned to position 1 across the J team, the next J players to position 2, and so on. Team j is composed of players with abilities a_j, a_{j+J}, ..., $a_{j+(M-1)J}$ for all $j = 1, ..., J$.[21] If the sharing rule $\theta(\mu)$ is sufficiently unequal, i.e., when μ is close to 0, this cyclical assignment becomes the only stable team structure.

Proposition 5. *Suppose that there are J active teams. Consider a hierarchical sharing rule satisfiying $\frac{\theta_{m+1}}{\theta_m} = \mu$ for all $m = 1, ..., M-1$. Then there exists $\bar{\mu} \in (0,1)$ such that for all $\mu \in (0, \bar{\mu})$, an assignment is stable if and only if it is the cyclical assignment φ^c.*

In addition to the sharing rule parameter μ, the degree of effort complementarity σ also affects the set of stable assignments. The following example illustrates how the interplay between μ and σ determine determines which assignment—cyclical or sorting—is stable.

Example 2. *Let $V = 1$, and suppose players have the following abilities: $a_1 = a_2 = 1$ and $a_3 = a_4 = \nu$, with $\nu < 1$. For a given $\mu < 1$, the hierarchical sharing rule assigns $\theta_1 = \frac{1}{1+\mu}$ and $\theta_2 = \frac{\mu}{1+\mu}$. Assume there are two teams, each with two positions. A cyclical assignment is $\varphi_1^s = (1,3)$ and $\varphi_2^s = (2,4)$, and a complete sorting assignment is $\varphi_1^c = (1,2)$ and $\varphi_2^c = (3,4)$. Observe that under the complete sorting assignment φ^s, the only possible head-hunting that could destabilize the assignment is player 2 moving to the first position in team 2—resulting in the cyclical assignment. Therefore, unlike Example 3 below, the parameter regions under which φ^s and φ^c are stable must be mutually exclusive.*

Also note that player 2 is the critical player whose payoff comparison determines whether the sorting assignment is stable. In the figure below, we fix $\nu = 0.7$ and plot the contour line

[21] In an interesting coalition formation game, Morelli and Park (2016) show that this type of cyclical assignment is group-stable.

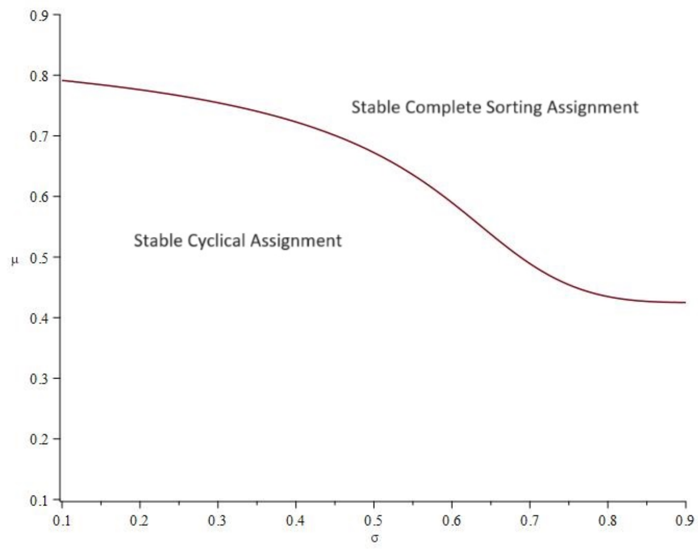

Figure 2: The Parameter Space for Stable Sorting/Cyclical Assignment

where player 2 is indifferent between the two assignments. Specifically, we show the set of (σ, μ) *values such that* $U_2(\varphi^c; \mu) - U_2(\varphi^c; \mu) = 0$.

It is straightforward to see that the $(\mu, \sigma)-$*combinations below the contour curve correspond to parameter values under which the cyclical assignment is stable, whereas combinations above the curve support the stability of the sorting assignment. For any fixed value of* σ*, a more egalitarian sharing rule (i.e., a larger* μ*) tends to support the stability of the sorting assignment. Conversely, for a fixed* μ*, the cyclical assignment is stable when* σ *is relative small, meaning that efforts are more complementary. In this case, differences in team productivity are not large enough to offset the utility loss for player 2 when taking the second position in the sorting assignment. As a result, player 2 prefers the first position in the weaker team, supporting the cyclical assignment. However, when* σ *is large—that is, when efforts are more substitutable—the lower-ability member has stronger incentives to free-ride the high ability member's effort. This reduces player 2's payoff in the cyclical assignment, making the sorting assignment more attractive and hence stable.*

Under the egalitarian sharing rule, intra-team inequality is minimized, while inter-team ability sorting emerges. As a result, team productivities $A_1(\varphi_1), ..., A_J(\varphi_J)$ tend to be highly unequal. In contrast, under a highly hierarchical sharing rule, intra-team inequality is large, but high-ability players are more evenly distributed across teams, reducing the disparity in team productivities.

Propositions 4 and 5 show that when μ is close to one, the complete ability sorting assignment φ^s is the unique stable assignment; whereas when μ is close to zero, the cyclical assignment φ^c is uniquely stable. These results naturally lead to the question: How competitive are stable assignments, depending on the degree of reward inequality? One way to assess this is to examine *rent dissipation*.

To measure rent dissipation, we must carefully define what constitutes waste in the contest setting. If we view team aggregate effort $X_j = (\sum_{m\in M} a^{\sigma}_{\varphi_{mj}} e^{\sigma}_{\varphi_{mj}})^{\frac{1}{\sigma}}$ as the wasted resource, then the rent dissipation measure is given by $\frac{X}{V} = \frac{J-1}{\sum_{j=1}^{J} \frac{1}{A_j(\varphi)}}$. However, the value of $X = \sum_{j=1}^{J} X_j$ is highly sensitive to the complementarity parameter $\sigma \in (0,1)$—especially as $\sigma \to 0$, where even small individual efforts generate high team output due to high complementarity. This makes the measure potentially misleading. Instead, we use the total sum of individual efforts, $E_j \equiv \sum_{m=1}^{M} e_{\varphi_{mj}}$, as a more robust proxy for measuring rent dissipation. To facilitate closed-form analysis, we assume players' abilities follow a geometric decay, i.e., $a_i = \nu^{i-1}$ for $i = 1, ..., MJ$, where $\nu \in (0,1)$ is an ability discounting parameter. With this specification, we can solve for $E(\varphi;\mu,\nu,\sigma) = \sum_{j=1}^{J} E_j(\varphi;\mu,\nu,\sigma)$ in the following manner.

Lemma 3. *Suppose players' abilities are given by $a_i = \nu^{i-1}$ for $i = 1, ..., MJ$, with $\nu \in (0,1)$. Then total effort under the sorting and cyclical assignments can be written as:*

$$E(\varphi^s;\mu,\nu,\sigma) = \frac{J-1}{\frac{1-\mu^M}{1-\mu}} \frac{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}\mu^{\frac{M}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}\mu^{\frac{1}{1-\sigma}}}}{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}}} \left\{ 1 - \frac{(J-1)\frac{1-\nu^{-2JM}}{1-\nu^{-2M}}}{\left(\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right)^2} \right\},$$

and

$$E(\varphi^c;\mu,\nu,\sigma) = \frac{J-1}{\frac{1-\mu^M}{1-\mu}} \frac{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}\mu^{\frac{M}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}\mu^{\frac{1}{1-\sigma}}}}{\frac{1-(\nu^J\mu)^{\frac{\sigma M}{1-\sigma}}}{1-(\nu^J\mu)^{\frac{\sigma}{1-\sigma}}}} \left\{ 1 - \frac{(J-1)\frac{1-\nu^{-2J}}{1-\nu^{-2}}}{\left(\frac{1-\nu^{-J}}{1-\nu^{-1}}\right)^2} \right\}.$$

Moreover, for all $\sigma \in (0,1)$, $\nu \in [0,1)$, and $\mu \in [0,1]$, we have $E(\varphi^s;\mu,\nu,\sigma) < E(\varphi^c;\mu,\nu,\sigma)$.

We have the following result on the rent dissipation of stable assignments.

Proposition 6. *Suppose players' abilities are given by $a_i = \nu^{i-1}$ for $i = 1, ..., MJ$, with $\nu \in (0,1)$. Then for all $\sigma \in (0,1)$ and $\nu \in [0,1)$, (1) if both φ^s and φ^c are stable under a given μ, then $E(\varphi^s;\mu,\nu,\sigma) < E(\varphi^c;\mu,\nu,\sigma)$, and (2) $E(\varphi^s;1,\nu,\sigma) < E(\varphi^c;0,\nu,\sigma)$ holds.*

Propositions 4 and 5 show that in a neighborhood of $\mu = 1$, φ^s is uniquely stable, while for μ close to 0, φ^c is the only stable assignment. Proposition 6 then implies that cyclical assignments—despite being stable under very unequal reward rules—tend to generate more rent dissipation than sorting assignments.[22] The following example shows that both complete ability sorting and cyclical assignments can be stable simultaneously.

Example 3. *Let $\sigma = \frac{1}{2}$, $V = 1$, and $a_i = \nu^{i-1}a$ with $\nu \in (0,1)$. With $\mu < 1$, we have the sharing rule as $\theta_1 = \frac{1}{1+\mu+\mu^2}$, $\theta_2 = \frac{\mu}{1+\mu+\mu^2}$, and $\theta_3 = \frac{\mu^2}{1+\mu+\mu^2}$. Consider two teams with*

[22] Although we cannot compare all stable φ^s and φ^c assignments analytically, our numerical examinations indicate that a stable φ^c tends to involve substantially more rent dissipation than a stable φ^s. See Example 3 for an illustration.

$i = 1, 2, ..., 6$. *The cyclical assignment is* $\varphi_1^C = (1, 3, 5)$ *and* $\varphi_2^C = (2, 4, 6)$*, and a complete sorting assignment is* $\varphi_1^S = (1, 2, 3)$ *and* $\varphi_2^S = (4, 5, 6)$*. When* $\mu = 0.45$*,* $\nu = 0.8$*, and* $\sigma = 0.5$*, both assignments are stable. Moreover, we have* $E(\varphi^s; 0.45, 0.8, 0.5) = 0.132 < 0.269 = E(\varphi^c; 0.45, 0.8, 0.5)$*. In fact, given* $\nu = 0.8$ *and* $\sigma = 0.5$*, both assignments remain stable when* $\mu \in (0.4, 0.5)$*. Over this interval,* $\max_{\mu \in (0.4,0.5)} E(\varphi^s; \mu, 0.8, 0.5) = 0.1347 < 0.246 = \min_{\mu \in (0.4,0.5)} E(\varphi^s; \mu, 0.8, 0.5)$*, highlighting the effort-saving nature of sorting assignments.*

The following example shows that an assignment combining features of both cyclical and ability sorting structures can also be stable.

Example 4. *Let* $\sigma = \frac{1}{2}$*,* $V = 1$*, and define abilities as* $a_i = (0.9)^{i-1}$*for* $i = 1, ..., 6$*,* $a_i = (0.9)^6$*for* $i = 7, 8, 9$*,* $a_i = (0.9)^7$ *for* $i = 10, 11, 12$*, and* $a_i = (0.9)^8$ *for* $i = 13, 14, 15$*. Let* $M = 5$*,* $\theta_1 = 0.5$*,* $\theta_2 = 0.2$*, and* $\theta_3 = \theta_4 = \theta_5 = 0.1$*. There exists a stable hybrid assignment* $\varphi_1 = (1, 4, 7, 8, 9)$*,* $\varphi_2 = (2, 5, 10, 11, 12)$*, and* $\varphi_3 = (3, 6, 13, 14, 15)$*, which is a combination of cyclical and ability sorting assignments. Their winning probabilities are:* $P_1 = 0.369$*,* $P_2 = 0.332$*, and* $P_3 = 0.299$*. This sharing rule assigns decreasing rewards across positions but treats all lower-ranked positions equally. The resulting assignment reveals a structure in which high-ability players are spread across teams, while low-ability players are sorted by ability within teams. This pattern may resemble the distribution of worker abilities observed in corporate organizations.*

Finally, we illustrate how our results can be extended to settings with multiple categories of positions and distinct skill types of workers. Up to this point, we have assumed that all players belong to a single category and that all team positions are identical, aside from potentially differing prize shares. However, in many real-world environments, teams consist of heterogeneous roles, and players possess skill sets suited to specific positions.[23] For instance, in baseball, team positions include pitchers, catchers, infielders, and outfielders, each requiring different abilities. Similarly, in team formation or matching environments, players often differ in their roles and responsibilities.

To formalize this, we partition both the set of positions M and the player set N into K disjoint categories. With a slight abuse of notation, let the set of categories be $K = \{1, ..., K\}$, and define $M \equiv \cup_{k \in K} M_k$ with $M_k \cap M_\ell = \emptyset$ for any $k \neq \ell$, and $N \equiv \cup_{k \in K} N_k$ with $N_k \cap N_\ell = \emptyset$ for any $k \neq \ell$. Here, M_k and N_k represent, respectively, the set of positions and the set of players associated with category k. Each team j's assignment function $\varphi_j : M \to N$ is category-respecting: for all $k \in K$ and $m \in M_k$, we have $\varphi_{mj} \in N_k$. That is, each position must be filled by a player of the corresponding type. We then define a modified team j's effort

[23] Imamura, Konishi, and Pan (2023) analyze pair formation problems in two-sided matching markets with externalities, such as pairs figure skating or oligopolistic joint ventures. In these contexts, player roles are asymmetric (e.g., male and female in skating, or product development and marketing in business), and the authors introduce role-based categories to model these distinctions.

aggregator function as:

$$X_j = \left(\sum_{k \in K} b_k \left(\sum_{m \in M_k} a^{\sigma}_{\varphi_{mj}} e^{\sigma}_{\varphi_{mj}} \right) \right)^{\frac{1}{\sigma}},$$

where $0 < \sigma < 1$ and $b_k > 0$ denotes the productivity weight of category k's contribution. The prize-sharing vector θ is also partitioned by category: $\theta = (\theta_k)_{k \in K}$, where $\theta_k = (\theta_{1k}, ..., \theta_{M_k k})$ for all $k \in K$.

We can obtain counterparts of Propositions 3 and 4 at no cost by applying Lemma 2 and the same proof argument within each group. That is, when θ_k is close to egalitarian, the assignment is stable if and only if players in category k are sorted by ability. Moreover, if θ_k is highly hierarchical, then an assignment is stable if and only if type k players are spread over teams in a cyclical assignment manner. It is because the above production function preserves additive structure in the first parenthesis, and the effect of swapping players in a category is additively separable from other categories.

5 Concluding Remarks

When players consider moving to another team, they must weigh a wide range of factors. As members of a new group, they work collectively toward a common prize, but their individual positions within the team—and their stakes in the outcome—can vary substantially. Relevant considerations include: their relative position in the team both in terms of their contribution and compensation, the effort they would need to exert and how it would influence their teammates, its effect on the team's probability of success and free-riding incentives, and the resulting impact on the strategies of competing teams. We have constructed a model of collective group contests that incorporates all of these effects while allowing for free mobility of players across teams through head-hunting. Our analysis shows that the resulting stable team structures reveal a fundamental trade-off between inter-team and intra-team inequality.

If all teams use the egalitarian rule to divide rewards equally, high-ability players are attracted to high-productivity teams, forming teams with minimal internal inequality but large differences in team strength. In contrast, if the goal of a contest designer is to ensure a competitive environment with more balanced teams, then allowing or enforcing a greater degree of intra-team inequality becomes necessary.

While this paper does not model how each player's share is determined endogenously, a natural next step would be to explore how team bidding behavior might shape reward allocation—particularly in professional sports labor markets. Below, we illustrate how our model can be adapted to generate insights into how sharing rules might emerge in equilibrium when teams compete to recruit talented players.[24]

[24] In practice, professional athletes receive salaries rather than shares of a winning prize. Nonetheless, analyzing equilibrium shares offers a useful approximation of salary structures, especially given the institutional complexity of real-world sports labor markets.

Example 5. *Suppose that there are four players with* $a_1 \geq a_2 \geq a_3 \geq a_4$ *and two teams (*1 *and* 2*), each with two positions. Let* $\sigma = \frac{1}{3}$*, and assume that each team leader's objective is to maximize their team's winning probability.*

The rule of the game is as follows. In the first round, team leaders simultaneously bid for player 1 by offering θ_1 *and* θ_2*. If player 1 rejects both offers, he becomes ineligible for future bids, so he must accept one of them. If team 1 wins the bid, the sharing rule will be* $(\theta_1, 1-\theta_1)$*. In the second round, knowing the outcome of the first, team 2 offers* $\tilde{\theta}_2$ *to player 2. Player 2 then chooses between the second position in team 1 and the first position in team 2. Players 3 and 4 fill the remaining positions in order, and the team contest takes place. If both teams offer the same* $\theta_1 = \theta_2$*, player 1 accepts either offer with equal probability. We consider two cases to illustrate the point:*[25] *(Case 1) Player 1 is a super star, and the remaining players are mediocre:* $a_1 \gg a_2 \geq a_3 \geq a_4$*; and (Case 2) Players 1 and 2 have the same high ability* a_H*, and players 3 and 4 have the same low ability* a_L*:* $a_1 = a_2 = a_H > a_3 = a_4 = a_L$*.*

(Case 1) Begin with the second round. Suppose team 1 wins player 1 with offer θ_1*. Then, team 2 chooses* $\tilde{\theta}_2$ *to maximize its winning probability with the remaining players—recognizing that player 2 might prefer to take the second position on team 1 instead. Note that if players 2 and 3 join team 2, the winning-probability-maximizing share for player 2 is* $\theta_2^* = \frac{a_2}{a_2+a_3}$ *(see Appendix for the detailed derivation). Assume that, given* θ_1*, the offer* $\tilde{\theta}_2 = \theta_2^*$ *can attract both players 2 and 3 to team 2. In this case,* θ_2^* *is team 2's manager's best response.*

Since player 1 is a super star, we may assume that the resulting contest satisfies $P_1 > P_2$*, implying that both team managers are eager to acquire player 1 in the first round. One natural offer is* $\theta_1^* = \frac{a_1}{a_1+a_4}$ *which maximizes the team's winning probability given player 1's presence. However, if both teams offer* θ_1^**, player 1 accepts either bid with equal probability. Therefore, each team has an incentive to increase* θ_1 *slightly, as long as doing so increases player 1's payoff. Hence, in equilibrium, both teams offer the payoff-maximizing share* θ_1^{**} *to player 1 in the first round, and player 1 accepts one of the offers randomly. In this case, player 1 can receive a quite high share offer in the league.*

As a numerical example, let $a_1 = 2 > a_2 = a_3 = a_4 = 1$*. If team 1 obtains player 1 with offer* θ_1*, then the ability profiles of teams 1 and 2 are* $\{2, 1\}$ *and* $\{1, 1\}$*, respectively. Team 2 optimally chooses* $\tilde{\theta}_2 = \frac{1}{2}$ *in the second round. In the first round, both teams will propose* θ_1^{**} *that maximizes player 1's payoff, expecting that* $\tilde{\theta}_2 = \frac{1}{2}$ *and that team 1 will have a higher winning probability. The equilibrium first-round offer is* $\theta_1^{**} = 0.8867$ *for both teams, resulting in productivities* $A_1 = 2.7832 > A_2 = 2$*. Note that team 1's maximal productivity is* 3*, which exceeds* 2.7832*.*

(Case 2) Start with the second round. If team 1 secures player 1 with offer θ_1*, team 2 will offer* $\theta^* = \frac{a_H}{a_H+a_L}$ *to player 2. If* $\theta_1 \neq \theta^*$*, team 2's winning probability exceeds that of team 1. Hence, in equilibrium, both teams offer* $\theta^* = \frac{a_H}{a_H+a_L}$ *in the first round. Player 1*

[25] In this two-team, two-position case, we can characterize the equilibrium explicitly for any ability distribution. However, extending this analysis to more general cases is significantly more complex.

randomly chooses her team, and team 2 also offers θ^* to player 2. With $a_H = 2$ and $a_L = 1$, $\theta^* = \frac{2}{3} = 0.667$ holds.[26]

Another promising direction for future research involves allowing coexistence of multiple sharing rules. Unfortunately, general existence results of a stable team structure for an arbitrary set of sharing rules are hard to obtain due to integer problems and the presence of externalities across teams. Even our positive results rely on the head-hunting-proofness concept, which explicitly defines what happens after a head-hunting swap (see also Imamura, Konishi, and Pan, 2023, and Imamura and Konishi 2023). One way to sidestep these difficulties is to assume a large market composed of a continuum of atomless teams, so that head-hunting or the entry of new teams with previously nonexisting sharing rules have no impact on the market. This approach follows the framework introduced by Kaneko and Wooders (1986).[27]

In a companion paper, Konishi, Pan, and Simeonov (2024) analyze a competitive market with finitely many player types and a continuum of players forming production teams of finite size. They prove the existence of a free-entry equilibrium—a stable team structure in this idealized large market—where no new team can profitably enter, and multiple sharing rules can coexist.

Building on their result, we can consider the limit of replica problems of our team contest, with a continuum of players and uniform prizes. After teams are formed, the subsequent game is structured as follows: teams are randomly grouped into a large number of K-team leagues, within which they compete for identical prizes. That is, while each contest is ex ante identical, each is ex post played by a different combination of teams. Thus, each player's expected payoff is calculated as a weighted expected payoff of each possible draw of team profile in the contest—effectively, they are playing contests with a distribution of team types in their league.[28] In the team formation stage, each player chooses among available positions based on her expected payoff comparison. Meanwhile, a potential team manager may enter the market by proposing a new sharing rule—one not currently in use—if it offers a profitable opportunity. This leads to a much stronger equilibrium concept, and the resulting allocation is strongly stable. We plan to investigate the properties of this free-entry equilibrium in such large replica contests in future work.

Appendix

We collect most of the formal proofs here.

[26]When $a_1 > a_2 > a_3 = a_4$, the team that fails to acquire player 1 may need to bid up for player 2 to prevent the opposing team from acquiring both top players.

[27]Assuming finite arbitrary types of atomless players, Kaneko and Wooders (1986) proved the nonemptyness of the *f-core*, which is immune to invasions by newly formed team assignments with different sharing rules. See also Konishi and Simeonov (2024).

[28]Thus, each team's winning probability depends on the distribution of other team types, and we need widespread externalities, as formalized in Konishi, Pan, and Simeonov (2024).

Proof of Proposition 1. From (2), we have:

$$e_{\varphi_{mj}}^{1-\sigma} = X_j^{1-\sigma}\left[(1-P_j)\frac{V}{X}\right] a_{\varphi_{mj}}^{\sigma}\theta_m$$

which implies:

$$e_{\varphi_{mj}} = X_j\left[(1-P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}}\left(a_{\varphi_{mj}}^{\sigma}\theta_m\right)^{\frac{1}{1-\sigma}}.$$

Note that if $X_j = 0$, then $e_i = 0$ for all $i \in N_j$. Raise both sides to the power σ and multiply by $a_{\varphi_{mj}}^{\sigma}$ to get:

$$a_{\varphi_{mj}}^{\sigma}e_{\varphi_{mj}}^{\sigma} = X_j^{\sigma}\left[(1-P_j)\frac{V}{X}\right]^{\frac{\sigma}{1-\sigma}} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}}\theta_m^{\frac{\sigma}{1-\sigma}}$$

(since the exponent on a_i is $\frac{\sigma^2}{1-\sigma}+\sigma = \frac{\sigma}{1-\sigma}$). Summing up all positions m in team j and raising it to the power $\frac{1}{\sigma}$, we get:

$$X_j = X_j\left[(1-P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}}\left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}}\theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1}{\sigma}}.$$

Thus:

$$1 = \left[\frac{X_{-j}V}{X^2}\right]^{\frac{1}{1-\sigma}}\left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}}\theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1}{\sigma}}.$$

Rearranging:

$$\frac{1}{\left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}}\theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}} = \frac{X_{-j}}{X^2}\times V. \tag{7}$$

Letting $A_j = \left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}}\theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$ and summing over all active teams, we have:

$$\frac{1}{A_j(\varphi_j)} = \frac{X_{-j}}{X^2}\times V, \tag{8}$$

or

$$X = \frac{(J-1)\,V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}.$$

Substitute this into (7) and recall $\frac{X_{-j}}{X} = 1-P_j$. We have:

$$1-P_j = \frac{\frac{1}{A_j(\varphi_j)}}{V}\frac{(J-1)\,V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}} = \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}$$

and

$$P_j = 1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}. \tag{9}$$

Equation (9) shows that the equilibrium winning probability of team j is increasing function in $A_j(\varphi_j)$, which depends only on φ_j and the share vector θ.

Combining (8) and (9), we get:

$$X_j = P_j X = \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}.$$

That completes the proof of Proposition 1.□

Proof of Proposition 2. We begin by computing the equilibrium effort level. Recalling equation (3), we have:

$$\begin{aligned} e_{\varphi_{mj}} &= X_j \left[(1-P_j)\frac{V}{X} \right]^{\frac{1}{1-\sigma}} \left(a_{\varphi_{mj}}^{\sigma} \theta_m \right)^{\frac{1}{1-\sigma}} \\ &= \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left[\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \frac{V}{\frac{(J-1)V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}} \right]^{\frac{1}{1-\sigma}} \left(a_{\varphi_{mj}}^{\sigma} \theta_m \right)^{\frac{1}{1-\sigma}} \\ &= \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left[\frac{1}{A_j(\varphi_j)} \right]^{\frac{1}{1-\sigma}} \left(a_{\varphi_{mj}}^{\sigma} \theta_m \right)^{\frac{1}{1-\sigma}}. \end{aligned} \tag{10}$$

This implies that the payoff of player $i = \varphi_{mj}$ is:

$$\begin{aligned} U_{\varphi_{mj}} &= P_j \theta_m V - e_{\varphi_{mj}} \\ &= \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \theta_m V - \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left[\frac{1}{A_j(\varphi_j)} \right]^{\frac{1}{1-\sigma}} \left(a_{\varphi_{mj}}^{\sigma} \theta_m \right)^{\frac{1}{1-\sigma}} \\ &= \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \left[\theta_m V - \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left[\frac{1}{A_j(\varphi_j)} \right]^{\frac{1}{1-\sigma}} \left(a_{\varphi_{mj}}^{\sigma} \theta_m \right)^{\frac{1}{1-\sigma}} \right] \\ &= \left\{ 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right\} \left[\theta_m V - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)} V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left[\frac{1}{A_j(\varphi_j)} \right]^{\frac{\sigma}{1-\sigma}} \theta_m \left(a_{\varphi_{mj}}^{\sigma} \theta_m^{\sigma} \right)^{\frac{1}{1-\sigma}} \right] \\ &= V \times \theta_m \left[1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \right] \left[1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left(\frac{a_i^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}}{\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'}^{\frac{\sigma}{1-\sigma}}} \right) \right]. \end{aligned}$$

This completes the proof.□

Proof of Lemma 2. We assume $A_j(\varphi_j) \geq A_k(\varphi_k)$, which implies $\tilde{A}_j(\varphi_j) = \sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'j}^{\frac{\sigma}{1-\sigma}} \geq \tilde{A}_k(\varphi_k) = \sum_{m'=1}^{M} a_{\varphi_{m'k}}^{\frac{\sigma}{1-\sigma}} \theta_{m'k}^{\frac{\sigma}{1-\sigma}}$. Let $\theta := \theta_m = \theta_\ell$. Define $\tilde{a}_i \equiv a_i^{\frac{\sigma}{1-\sigma}} \theta_\ell^{\frac{\sigma}{1-\sigma}} = a_i^{\frac{\sigma}{1-\sigma}} \theta^{\frac{\sigma}{1-\sigma}}$ and $\tilde{a}_h = a_h^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}} = a_h^{\frac{\sigma}{1-\sigma}} \theta^{\frac{\sigma}{1-\sigma}}$. Since $a_h > a_i$, it follows that $\tilde{a}_h > \tilde{a}_i$. As a result, we obtain the following inequalities: $\tilde{A}_j(\varphi'_j) > \tilde{A}_j(\varphi_j) \geq \tilde{A}_k(\varphi_k) > \tilde{A}_k(\varphi'_k)$, which implies $A_j(\varphi'_j) > A_j(\varphi_j) \geq$

$A_k(\varphi_k) > A_k(\varphi'_k)$. Note that

$$\frac{\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}} > \frac{\frac{1}{A_j(\varphi'_j)}}{\sum_{s\neq j,k}\frac{1}{A_s(\varphi_s)}+\frac{1}{A_j(\varphi'_j)}+\frac{1}{A_k(\varphi_k)}} > \frac{\frac{1}{A_j(\varphi'_j)}}{\sum_{s\neq j,k}\frac{1}{A_s(\varphi_s)}+\frac{1}{A_j(\varphi'_j)}+\frac{1}{A_k(\varphi'_k)}}.$$

The first inequality follows from $\frac{1}{A_j(\varphi'_j)} < \frac{1}{A_j(\varphi_j)}$, and the second from $\frac{1}{A_k(\varphi'_k)} > \frac{1}{A_k(\varphi_k)}$. Since the winning probability is given by $P_j(\varphi) = 1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}}$, we conclude that $P_j(\varphi') > P_j(\varphi)$. Therefore, condition (a) for a successful head-hunting is satisfied.

Now consider condition (b). From Theorem 1, we know:

$$U_h(\varphi) = V \times \theta P_k(\varphi)\left[1 - \frac{(J-1)\frac{1}{A_k(\varphi_k)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}}\left(\frac{a_h\theta}{A_k(\varphi_k)}\right)^{\frac{\sigma}{1-\sigma}}\right]$$

and

$$U_h(\varphi') = V \times \theta P_j(\varphi')\left[1 - \frac{(J-1)\frac{1}{A_j(\varphi'_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi'_{j'})}}\left(\frac{a_h\theta}{A_j(\varphi'_j)}\right)^{\frac{\sigma}{1-\sigma}}\right].$$

Since $P_j(\varphi') > P_j(\varphi) \geq P_k(\varphi)$, $\frac{\frac{1}{A_k(\varphi_k)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}} > \frac{\frac{1}{A_j(\varphi'_j)}}{\sum_{s\neq j,k}\frac{1}{A_s(\varphi_s)}+\frac{1}{A_j(\varphi'_j)}+\frac{1}{A_k(\varphi'_k)}} = \frac{\frac{1}{A_j(\varphi'_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi'_{j'})}}$, and $A_j(\varphi'_j) > A_k(\varphi_k)$, we conclude $U_h(\varphi') > U_h(\varphi)$. Thus, condition (b) is also satisfied.[29] $\square$

Proof of Proposition 3. Lemma 2 establishes that if φ does not exhibit complete ability sorting, then there exists a head-hunting opportunity under the egalitarian sharing rule. Therefore, it suffices to show that no such opportunity arises under the egalitarian rule when the assignment is a complete ability sorting assignment φ^s.

Consider a player i in team j contemplating a move to an inferior team $k > j$. Let φ' denote the assignment after swapping player h in team k with player i in team $j < k$ with $a_i > a_h$. Since $a_i > a_h$ and $j < k$, we have $A_j(\varphi^s) > A_k(\varphi') > A_k(\varphi^s)$ and $A_k(\varphi^s) < A_j(\varphi') < A_j(\varphi^s)$. This implies:

$$\begin{aligned} P_j(\varphi^s) &= 1 - \frac{\frac{1}{A_j(\varphi^s)}}{\frac{1}{A_j(\varphi^s)}+\frac{1}{A_k(\varphi^s)}+\sum_{j'\neq j,k}\frac{1}{A_{j'}(\varphi^s)}} \\ &> 1 - \frac{\frac{1}{A_k(\varphi')}}{\frac{1}{A_j(\varphi')}+\frac{1}{A_k(\varphi')}+\sum_{j'\neq j,k}\frac{1}{A_{j'}(\varphi^s)}} = P_k(\varphi'). \end{aligned}$$

[29]Given the above proof, it suffices to verify that $\tilde{A}_j(\varphi'_j) > \tilde{A}_j(\varphi_j) \geq \tilde{A}_k(\varphi_k) > \tilde{A}_k(\varphi'_k)$ holds after the swap. This inequality also holds under the alternative definition of successful head-hunting discussed in Remark 3. Therefore, Propositions 3 and 4 remain valid under the alternative definition as well.

Furthermore, since $\sum_{\tilde{m}}(\frac{1}{M})^{\frac{\sigma}{1-\sigma}} a_{\varphi^s_{\tilde{m}j}}^{\frac{\sigma}{1-\sigma}} > \sum_{\tilde{m}}(\frac{1}{M})^{\frac{\sigma}{1-\sigma}} a_{\varphi'_{\tilde{m}k}}^{\frac{\sigma}{1-\sigma}}$ and $P_k(\varphi^s) < P_j(\varphi')$, we have:

$$\begin{aligned} U_i(\varphi^s) &= \frac{1}{M} P_j(\varphi^s) \left(1 - \frac{\frac{1}{A_j(\varphi^s)}}{\frac{1}{A_j(\varphi^s)} + \frac{1}{A_k(\varphi^s)} + \sum_{j' \neq j,k} \frac{1}{A_{j'}(\varphi^s)}} \frac{(\frac{1}{M})^{\frac{\sigma}{1-\sigma}} a_i^{\frac{\sigma}{1-\sigma}}}{\sum_{\tilde{m}}(\frac{1}{M})^{\frac{\sigma}{1-\sigma}} a_{\varphi^s_{\tilde{m}j}}^{\frac{\sigma}{1-\sigma}}} \right) \\ &> \frac{1}{M} P_k(\varphi') \left(1 - \frac{\frac{1}{A_k(\varphi')}}{\frac{1}{A_j(\varphi')} + \frac{1}{A_k(\varphi')} + \sum_{j' \neq j,k} \frac{1}{A_{j'}(\varphi^s)}} \frac{(\frac{1}{M})^{\frac{\sigma}{1-\sigma}} a_i^{\frac{\sigma}{1-\sigma}}}{\sum_{\tilde{m}}(\frac{1}{M})^{\frac{\sigma}{1-\sigma}} a_{\varphi'_{\tilde{m}k}}^{\frac{\sigma}{1-\sigma}}} \right) = U_i(\varphi'). \end{aligned}$$

Thus, no player in φ^s can be head-hunted.□

Proof of Proposition 4. We first show that there exists $\hat{\mu} \in [0, 1)$ such that, for all $\mu \in (\hat{\mu}, 1]$, the complete ability sorting assignment φ^s is stable. Consider the assignment φ' obtained by swapping player φ^s_{mj} in team j with player $\varphi^s_{m'k}$ in team $j < k$. Clearly, we have $a_{\varphi^s_{mj}} > a_{\varphi^s_{m'k}}$. Let $i = \varphi^s_{mj}$. From the proof of Proposition 3, we know that under the egalitarian rule ($\mu = 1$), player i's payoff decreases: $U_i(\varphi^s; 1) > U_i(\varphi'; 1)$. Define $\xi(\mu; \varphi^s, mj, m'k) \equiv U_i(\varphi^s; \mu) - U_i(\varphi'; \mu)$, and we have $\xi(1; \varphi^s, mj, m'k) > 0$. If $\xi(\mu; \varphi^s, mj, m'k) > 0$ for all $\mu \in [0, 1]$, let $\hat{\mu}_{mj,m'k} = 0$. If $\xi(0; \varphi^s, mj, m'k) < 0$, then continuity implies there exists a solution $\mu \in [0, 1)$ such that $\xi(\mu) = 0$. Let $\hat{\mu}_{mj,m'k}$ be the largest such solution. Let $\hat{\mu} \equiv \max_{m',m=1,\dots,M; j,k=1,\dots,J, j<k} \hat{\mu}_{mj,m'k}$. Then, for all $\mu \in [\hat{\mu}, 1]$, it holds that $U_i(\varphi^s; \mu) \geq U_i(\varphi'; \mu)$ for all players i. Hence, the complete ability sorting assignment over J teams is stable for all $\mu \in [\hat{\mu}, 1]$.

Next, we will show that there exists $\check{\mu} \in [0, 1)$ such that, for all $\mu \in (\hat{\mu}, 1]$, any stable assignment must be a complete ability sorting assignment. Consider a non-sorting assignment φ and, for any given μ, reorder team indices such by their winning probabilities. From Lemma 2, if $P_1(\varphi; \mu) > P_2(\varphi; \mu) > \dots > P_J(\varphi; \mu)$, then we may assume $a_{\varphi_{m1}} \geq a_{\varphi_{m2}} \geq \dots \geq a_{\varphi_{mJ}}$ for each position $m = 1, \dots M$, since all $\theta_m(\mu)$ are identical across teams.

Since φ is a non-sorting assignment, there must exist mj and $m'k$ with $k < j$ and $a_{\varphi_{mj}} > a_{\varphi_{m'k}}$. Let $i = \varphi_{mj}$ and consider the assignment φ' obtained by swapping players i and $\varphi_{m'k}$. It follows that $A_j(\varphi'; \mu) < A_j(\varphi; \mu) < A_k(\varphi; \mu) < A_k(\varphi'; \mu)$, and we have

$$\frac{\frac{1}{A_j(\varphi;\mu)}}{\frac{1}{A_j(\varphi;\mu)} + \frac{1}{A_k(\varphi;\mu)} + \sum_{j' \neq j,k} \frac{1}{A_{j'}(\varphi;\mu)}} > \frac{\frac{1}{A_k(\varphi';\mu)}}{\frac{1}{A_j(\varphi';\mu)} + \frac{1}{A_k(\varphi';\mu)} + \sum_{j' \neq j,k} \frac{1}{A_{j'}(\varphi;\mu)}}$$

for any μ. Thus, $P_k(\varphi'; \mu) > P_j(\varphi; \mu)$. Now consider the player's payoffs:

$$U_i(\varphi; \mu) = \theta_m(\mu) P_j(\varphi; \mu) \left(1 - \frac{\frac{1}{A_j(\varphi;\mu)}}{\frac{1}{A_j(\varphi;\mu)} + \frac{1}{A_k(\varphi;\mu)} + \sum_{j' \neq j,k} \frac{1}{A_{j'}(\varphi;\mu)}} \frac{\theta_m^{\frac{\sigma}{1-\sigma}}(\mu) a_i^{\frac{\sigma}{1-\sigma}}}{\sum_{\tilde{m}} \theta_{\tilde{m}}^{\frac{\sigma}{1-\sigma}}(\mu) a_{\varphi^s_{\tilde{m}j}}^{\frac{\sigma}{1-\sigma}}} \right)$$

and

$$U_i(\varphi';\mu)=\theta_{m'}(\mu)P_k(\varphi';\mu)\left(1-\frac{\frac{1}{A_k(\varphi',\mu)}}{\frac{1}{A_j(\varphi';\mu)}+\frac{1}{A_k(\varphi';\mu)}+\sum_{j'\neq j,k}\frac{1}{A_{j'}(\varphi';\mu)}}\frac{\theta_{m'}^{\frac{\sigma}{1-\sigma}}(\mu)a_i^{\frac{\sigma}{1-\sigma}}}{\sum_{\tilde{m}}\theta_{m}^{\frac{\sigma}{1-\sigma}}(\mu)a_{\varphi'_{\tilde{m}k}}^{\frac{\sigma}{1-\sigma}}}\right).$$

Define $\chi(\mu;\varphi,mj,m'k)=U_i(\varphi';\mu)-U_i(\varphi;\mu)$. We have $\chi(1;\varphi,mj,m'k)>0$. If $\chi(\mu;\varphi,mj,m'k)>0$ for all $\mu\in[0,1]$, let $\check{\mu}_\varphi=0$. If $\chi(0;\varphi,mj,m'k)<0$, continuity guarantees a solution $\chi(\mu;\varphi,mj,m'k)=0$ in interval $(0,1)$. Let $\check{\mu}_{\varphi,mj,m'k}$ be the largest such solution. Define $\Gamma_\varphi=\{(mj,m'k)|P_k(\varphi;1)>P_j(\varphi;1)$ and $a_{\varphi_{mj}}>a_{\varphi_{m'k}}\}$ and $\check{\mu}_\varphi\equiv max_{(mj,m'k)\in\Gamma_\varphi}\check{\mu}_{\varphi,mj,m'k}$. Since the Γ_φ is finite, we have $\check{\mu}_\varphi<1$. Then, let $\check{\mu}\equiv\max_{\varphi\in\Phi^{ns};mj,m'k}\check{\mu}_\varphi$, where Φ^{ns} is a set of non-sorting assignments. Since the number of such assignments is finite, we have $\check{\mu}<1$. Therefore, for all $\mu\in(\check{\mu},1]$, $U_i(\varphi';\mu)>U_i(\varphi;\mu)$ holds for player i, and thus no non-sorting assignment is stable.

Let $\bar{\bar{\mu}}=\max\{\check{\mu},\hat{\mu}\}$, we have completed the proof.□

Proof of Proposition 5. First, we will prove that there exists $\tilde{\mu}>0$ such that a cyclical assignment is stable under any $\mu\in(0,\tilde{\mu})$. Given $\theta(\mu)=(\theta_1(\mu),...,\theta_M(\mu))$, we evaluate the stability of a cyclical assignment of players over J teams.

Construct the cyclical assignment φ^c by assigning players to teams in order of decreasing ability, cycling through the teams. Let $T_m=\{(m-1)J+j'|j'=1,...,J\}$ denote the set of players assigned to position m across teams. Players in T_m will not be head-hunted for a higher position, so we focus on a player $\varphi^c_{mj}=i\in T_m$ in team, who considers moving to position $m+1$ in team $k\neq j$. Since all teams are active, $P_j(\varphi^c,\mu)>0$ and $P_k(\varphi^c;\mu)>0$. By Theorem 1 and Remark 2, we have:

$$U_i(\varphi^c;\mu)>\theta_m(\mu)\left(P_j(\varphi^c;\mu)\right)^2.$$

Suppose player $i=\varphi^c_{mj}$ swaps places with $\varphi^c_{(m+1)k}$ and takes position $m+1$ of team k, resulting in a new assignment φ'. Her payoff becomes:

$$\begin{aligned}U_i(\varphi';\mu)&=\theta_{m+1}(\mu)P_k(\varphi';\mu)\left[1-(J-1)\frac{\frac{1}{A_k(\varphi'_k;\mu)}}{\sum_{j'=1}^J\frac{1}{A_{j'}(\varphi'_{j'};\mu)}}\left(\frac{a_i\theta_{m+1}(\mu)}{A_k(\varphi'_k;\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right]\\&<\theta_{m+1}P_k(\varphi';\mu).\end{aligned}$$

Thus, $U_i(\varphi;\mu)\geq U_i(\varphi';\mu)$ holds if:

$$\theta_{m+1}(\mu)P_k(\varphi';\mu)\leq\theta_m(\mu)\left(P_j(\varphi^c;\mu)\right)^2,$$

or equivalently,

$$\mu=\frac{\theta_{m+1}(\mu)}{\theta_m(\mu)}\leq\frac{\left(P_j(\varphi^c;\mu)\right)^2}{P_k(\varphi';\mu)}.$$

Note that the right-hand side is also a function of μ. Define $\psi(\mu;\varphi^c,mj,(m+1)k)\equiv\frac{(P_j(\varphi^c;\mu))^2}{P_k(\varphi';\mu)}-$

μ. We have $\psi(0;\varphi^c,mj,(m+1)k)>0$. If $\psi(\mu;\varphi^c,mj,(m+1)k)>0$ for all $\mu\in[0,1]$, let $\tilde{\mu}_{mj,(m+1)k}=1$. Otherwise, if $\psi(1;\varphi^c,mj,(m+1)k)<0$, then by continuity, there exists a solution $\psi(\mu)=0$ in $[0,1]$. Let $\tilde{\mu}_{mj,(m+1)k}$ be the smallest such solution, and define $\tilde{\mu}\equiv\min_{m=1,...,M-1;k,j=1,...,J}\tilde{\mu}_{mj,(m+1)k}$. Then, for all $\mu\in[0,\tilde{\mu}]$, we have $U_i(\varphi^c;\mu)\geq U_i(\varphi';\mu)$ for all players i. Hence, the cyclical assignment φ^c is stable for $\mu<\tilde{\mu}$.

Next, we show that there exists $\hat{\mu}>0$ such that any stable assignment must be cyclical for all $\mu\in(0,\hat{\mu})$. Let φ be a non-cyclical assignment. We aim to find conditions under which φ is not stable. Reorder team indices such that $P_1(\varphi;\mu)\geq P_2(\varphi;\mu)\geq...\geq P_J(\varphi;\mu)$. From Lemma 2, we may assume $a_{\varphi_{m1}}\geq a_{\varphi_{m2}}\geq...\geq a_{\varphi_{mJ}}$ for each m; otherwise, the assignment is already unstable. Since assignment φ is non-cyclical, there exist positions (m,k) and $(\tilde{m},j)$ such that $\tilde{m}>m$ and $a_{\varphi_{mk}}<a_{\varphi_{\tilde{m}j}}$. We will show that player $\varphi_{\tilde{m}j}$ would be head-hunted for position (m,k) under some μ. Let the assignment φ' obtained by swapping φ_{mk} and $\varphi_{\tilde{m}j}$. For player $i=\varphi_{\tilde{m}j}$, we have:

$$\begin{aligned}
U_i(\varphi';\mu) &= \theta_m(\mu)P_k(\varphi';\mu)\left[1-(J-1)\frac{\frac{1}{A_k(\varphi'_k;\mu)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi'_{j'};\mu)}}\left(\frac{a_i\theta_m(\mu)}{A_k(\varphi'_k;\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right]\\
&> \theta_m(\mu)P_k(\varphi';\mu)\left[1-(J-1)\frac{\frac{1}{A_k(\varphi'_k;\mu)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi'_{j'};\mu)}}\right]\\
&= \theta_m(\mu)\left(P_k(\varphi';\mu)\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
U_i(\varphi;\mu) &= \theta_{\tilde{m}}(\mu)P_j(\varphi;\mu)\left[1-(J-1)\frac{\frac{1}{A_j(\varphi_j;\mu)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'};\mu)}}\left(\frac{a_i\theta_m(\mu)}{A_j(\varphi_j;\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right]\\
&< \theta_{\tilde{m}}(\mu)P_j(\varphi;\mu).
\end{aligned}$$

Note that since $\tilde{m}>m$, $U_i(\varphi';\mu)>U_i(\varphi;\mu)$ for all $\mu\in(0,\hat{\mu})$ for some $\hat{\mu}>0$ if $\theta_m(\mu)\left(P_k(\varphi';\mu)\right)^2>\theta_{\tilde{m}}(\mu)P_j(\varphi;\mu)$ or equivalently, $\frac{\theta_{\tilde{m}}(\mu)}{\theta_m(\mu)}\leq\mu<\frac{(P_k(\varphi';\mu))^2}{P_j(\varphi;\mu)}$. Define $\phi(\mu;\varphi,mk,\tilde{m}j)\equiv\frac{(P_k(\varphi';\mu))^2}{P_j(\varphi;\mu)}-\mu$. We have $\phi(0)>0$. If $\phi(\mu;\varphi,mk,\tilde{m}j)>0$ for all $\mu\in[0,1]$, let $\hat{\mu}_{\varphi,mk,\tilde{m}j}=1$. If $\phi(1;\varphi,mk,\tilde{m}j)<0$, then continuity implies the existence of a solution $\phi(\mu)=0$ in interval $[0,1]$. Let $\hat{\mu}_{\varphi,mk,\tilde{m}j}$ be the smallest such solution. Note that $\hat{\mu}_{\varphi,mk,\tilde{m}j}>0$ depends on the non-cyclical assignment φ, the deviator $\varphi_{\tilde{m}j}$, and her destination mk. Since the number of players and assignments is finite, define $\hat{\mu}\equiv\min_{\varphi\in\Phi^{NC};m,\tilde{m}=1,...,M;k,j=1,...,J}\hat{\mu}_{\varphi,mk,\tilde{m}j}>0$, where Φ^{NC} is the set of non-cyclical assignments. By construction, for all $\mu\in(0,\hat{\mu})$, there is no non-cyclical stable assignment.

Let $\bar{\mu}\equiv\min\{\bar{\mu},\hat{\mu}\}>0$. Then, for any $\mu\in(0,\bar{\mu})$, under $\theta(\mu)$, an assignment φ is stable if and only if it is cyclical. We have completed the proof.□

Proof of Lemma 3. We demonstrate the calculation for the complete ability sorting assignment φ^s. Recall that in φ^s, members in team j has the ability profile $\{\nu^{(j-1)M+(m-1)}\}_{m=1}^{M}$. The productivity of team j is given by

$$A_j^s = \left(\frac{1-\mu}{1-\mu^M}\right)\nu^{(j-1)M}\left[\frac{1-(\nu\mu)^{\frac{\sigma M}{1-\sigma}}}{1-(\nu\mu)^{\frac{\sigma}{1-\sigma}}}\right]^{\frac{1-\sigma}{\sigma}}.$$

Moreover, we have:

$$\sum_{k=1}^{J}\frac{1}{A_k^s} = \frac{1-\mu^M}{1-\mu}\left[\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right]\left[\frac{1-(\nu\mu)^{\frac{\sigma}{1-\sigma}}}{1-(\nu\mu)^{\frac{\sigma M}{1-\sigma}}}\right]^{\frac{1-\sigma}{\sigma}},$$

and

$$P_j = 1-\frac{(J-1)\frac{1}{A_j^s}}{\sum_{k=1}^{J}\frac{1}{A_k^s}} = 1-\frac{\frac{J-1}{\nu^{(j-1)M}}}{\frac{1-\nu^{-JM}}{1-\nu^{-M}}}.$$

Using (10), we can calculate individual effort level:

$$\begin{aligned}
e_{\varphi_{mj}^s} &= P_j\frac{J-1}{\sum_{k=1}^{J}\frac{1}{A_k^s}}\left[\frac{\frac{1-\mu^M}{1-\mu}}{\nu^{(j-1)M}}\left[\frac{1-(\nu\mu)^{\frac{\sigma}{1-\sigma}}}{1-(\nu\mu)^{\frac{\sigma M}{1-\sigma}}}\right]^{\frac{1-\sigma}{\sigma}}\right]^{\frac{1}{1-\sigma}}\left(\nu^{\sigma[(j-1)M+(m-1)]}\frac{\mu^{m-1}}{\frac{1-\mu^M}{1-\mu}}\right)^{\frac{1}{1-\sigma}}\\
&= P_j\frac{J-1}{\sum_{k=1}^{J}\frac{1}{A_k^s}}\left[\frac{1-(\nu\mu)^{\frac{\sigma}{1-\sigma}}}{1-(\nu\mu)^{\frac{\sigma M}{1-\sigma}}}\right]^{\frac{1}{\sigma}}\nu^{\frac{(1-j)M}{1-\sigma}+\frac{\sigma[(j-1)M+(m-1)]}{1-\sigma}}\mu^{\frac{m-1}{1-\sigma}}\\
&= \left\{1-\frac{\frac{J-1}{\nu^{(j-1)M}}}{\frac{1-\nu^{-JM}}{1-\nu^{-M}}}\right\}\frac{(J-1)\left[\frac{1-(\nu\mu)^{\frac{\sigma}{1-\sigma}}}{1-(\nu\mu)^{\frac{\sigma M}{1-\sigma}}}\right]}{\frac{1-\mu^M}{1-\mu}\left[\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right]}\nu^{(1-j)M+\frac{\sigma(m-1)}{1-\sigma}}\mu^{\frac{m-1}{1-\sigma}}.
\end{aligned}$$

Summing over m, we get:

$$\begin{aligned}
\sum_{m=1}^{M}e_{\varphi_{mj}^s} &= \left\{1-\frac{\frac{J-1}{\nu^{(j-1)M}}}{\frac{1-\nu^{-JM}}{1-\nu^{-M}}}\right\}\frac{(J-1)\left[\frac{1-\nu^{\frac{\sigma}{1-\sigma}}}{1-\nu^{\frac{M\sigma}{1-\sigma}}}\right]}{\frac{1-\mu}{1-\mu^M}\left[\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right]}\nu^{(1-j)M}\sum_{m=1}^{M}\nu^{\frac{\sigma(m-1)}{1-\sigma}}\mu^{\frac{m-1}{1-\sigma}}\\
&= \left\{1-\frac{\frac{J-1}{\nu^{(j-1)M}}}{\frac{1-\nu^{-JM}}{1-\nu^{-M}}}\right\}\frac{(J-1)\left[\frac{1-\nu^{\frac{\sigma}{1-\sigma}}}{1-\nu^{\frac{M\sigma}{1-\sigma}}}\right]}{\frac{1-\mu^M}{1-\mu}\left[\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right]}\nu^{(1-j)M}\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}\mu^{\frac{M}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}\mu^{\frac{1}{1-\sigma}}}.
\end{aligned}$$

Therefore,

$$E(\varphi^s;\mu,\nu,\sigma) = \sum_{j=1}^{J}\sum_{m=1}^{M} e_{\varphi^s_{mj}} = \frac{J-1}{\frac{1-\mu^M}{1-\mu}\frac{1-\nu^{-JM}}{1-\nu^{-M}}} \frac{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}\mu^{\frac{M}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}\mu^{\frac{1}{1-\sigma}}}}{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}}} \sum_{j=1}^{J}\left\{1-\frac{\frac{J-1}{\nu^{(j-1)M}}}{\frac{1-\nu^{-JM}}{1-\nu^{-M}}}\right\}\nu^{(1-j)M}$$

$$= \frac{J-1}{\frac{1-\mu^M}{1-\mu}} \frac{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}\mu^{\frac{M}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}\mu^{\frac{1}{1-\sigma}}}}{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}}} \left\{1-\frac{(J-1)\frac{1-\nu^{-2JM}}{1-\nu^{-2M}}}{\left(\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right)^2}\right\}.$$

For the cyclical assignment, team j has the ability profile $\{\nu^{(j-1)M+(m-1)}\}_{m=1}^M$. Following a similar derivation, we obtain $A_j^s = \left(\frac{1-\mu}{1-\mu^M}\right)\left[\frac{\nu^{\frac{\sigma}{1-\sigma}j}\left(1-\left(\nu^J\mu\right)^{\frac{\sigma M}{1-\sigma}}\right)}{1-(\nu^J\mu)^{\frac{\sigma}{1-\sigma}}}\right]^{\frac{1-\sigma}{\sigma}}$ and

$$E(\varphi^c;\mu,\nu,\sigma) = \frac{J-1}{\frac{1-\mu^M}{1-\mu}} \frac{\frac{1-\nu^{\frac{M\sigma}{1-\sigma}}\mu^{\frac{M}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}\mu^{\frac{1}{1-\sigma}}}}{\frac{1-(\nu^J\mu)^{\frac{\sigma M}{1-\sigma}}}{1-(\nu^J\mu)^{\frac{\sigma}{1-\sigma}}}} \left\{1-\frac{(J-1)\frac{1-\nu^{-2J}}{1-\nu^{-2}}}{\left(\frac{1-\nu^{-J}}{1-\nu^{-1}}\right)^2}\right\}.$$

Observe that $\frac{1-x^M}{1-x}$ is the sum of a geometric progression with common ratio x. Therefore, $\frac{1-x^M}{1-x}$ is strictly decreasing in $x<1$. Hence, we have $\frac{1-\left(\nu^J\mu\right)^{\frac{\sigma M}{1-\sigma}}}{1-(\nu^J\mu)^{\frac{\sigma}{1-\sigma}}} < \frac{1-\nu^{\frac{M\sigma}{1-\sigma}}}{1-\nu^{\frac{\sigma}{1-\sigma}}}$. Also, $\left(\frac{1-y}{1-y^J}\right)^2\frac{1-y^{2J}}{1-y^2} = \frac{1+y^J}{1-y^J}\frac{1-y}{1+y}$ is increasing $y>1$. Since $\nu^{-M}>\nu^{-1}$, we have $\frac{\frac{1-\nu^{-2JM}}{1-\nu^{-2M}}}{\left(\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right)^2} > \frac{\frac{1-\nu^{-2J}}{1-\nu^{-2}}}{\left(\frac{1-\nu^{-J}}{1-\nu^{-1}}\right)^2}$. Combining these two inequalities, we conclude that $E(\varphi^s;\mu,\nu,\sigma) < E(\varphi^c;\mu,\nu,\sigma)$.□

Proof of Proposition 6. We first show that $E(\varphi^s;1,\nu,\sigma) < E(\varphi^s;0,\nu,\sigma)$. By L'Hôspital's Rule, $\lim_{\mu\to1}\frac{1-\mu^M}{1-\mu} = \lim_{\mu\to1}\frac{M\mu^{M-1}}{1} = M$. Therefore

$$\begin{aligned}E(\varphi^s;1,\nu,\sigma) &= \frac{J-1}{M}\left\{1-\frac{(J-1)\frac{1-\nu^{-2JM}}{1-\nu^{-2M}}}{\left(\frac{1-\nu^{-JM}}{1-\nu^{-M}}\right)^2}\right\}\\ &< (J-1)\left\{1-\frac{(J-1)\frac{1-\nu^{-2J}}{1-\nu^{-2}}}{\left(\frac{1-\nu^{-J}}{1-\nu^{-1}}\right)^2}\right\}\\ &= E(\varphi^s;0,\nu,\sigma) < E(\varphi^c;0,\nu,\sigma),\end{aligned}$$

where the last inequality follows from Lemma 3. We have completed the proof.□

Calculations on Example 5. Let $a_1\geq a_2\geq a_3\geq a_4$, and suppose there are two teams (team 1 and team 2), each with two positions.

In general, if team j consists of players i and h, the team's productivity is defined as:

$$A_j = \left(a_i^\eta \theta_j^\eta + a_h^\eta (1-\theta_j)^\eta\right)^{\frac{1}{\eta}},$$

where $\eta \equiv \frac{\sigma}{1-\sigma} > 0$, and $\theta_j = \theta_{ij}$ is the share allocated to player i, so that $1-\theta_j = \theta_{hj}$. In the case of two teams, team j's winning probability is:

$$P_j = 1 - \frac{\frac{1}{A_j}}{\frac{1}{A_j} + \frac{1}{A_k}} = \frac{\frac{1}{A_k}}{\frac{A_j + A_k}{A_j A_k}} = \frac{A_j}{A_j + A_k},$$

where $k \neq j$. Then, player i's payoff in team j is:

$$u_i = \theta_j \times \frac{A_j}{A_j + A_k} \times \left(1 - \frac{A_k}{A_j + A_k} \times \frac{a_i^\eta \theta_j^\eta}{a_i^\eta \theta_j^\eta + a_h^\eta (1-\theta_j)^\eta}\right).$$

Now suppose team j can choose θ_j in order to maximize its own winning probability. Since the opposing team's productivity A_k is fixed, this is equivalent to maximizing A_j with respect to θ_j. The first-order condition is:

$$\frac{dA_j}{d\theta_j} = \frac{1}{\eta}\left(a_i^\eta \theta_j^\eta + a_h^\eta (1-\theta_j)^\eta\right)^{\frac{1}{\eta}-1}\left(\eta a_i^\eta \theta_j^{\eta-1} - \eta a_h^\eta (1-\theta_j)^{\eta-1}\right) = 0,$$

or equivalently,

$$\left(\frac{a_i}{a_h}\right)^\eta = \left(\frac{\theta_j}{1-\theta_j}\right)^{1-\eta}.$$

Thus, we obtain the optimal share θ_j^* as:

$$\theta_j^* = \begin{cases} \frac{a_i^{\frac{\eta}{1-\eta}}}{a_i^{\frac{\eta}{1-\eta}} + a_h^{\frac{\eta}{1-\eta}}}, & \text{if } \eta > 0 \text{ (or if } \sigma \in (0, \tfrac{1}{2})) \\ 1, & \text{otherwise.} \end{cases}$$

For example, when $\sigma = \frac{1}{3}$, we have $\eta = \frac{1}{2}$ and $\theta_j^* = \frac{a_i}{a_i + a_h}$. Moreover, we have:

$$A_j^* = \left(\frac{a_i}{(a_i + a_h)^{\frac{1}{2}}} + \frac{a_h}{(a_i + a_h)^{\frac{1}{2}}}\right)^2 = \frac{(a_i + a_h)^2}{a_i + a_h} = a_i + a_h.$$

Now consider team k, , which consists of players ℓ and m, with player ℓ holding share θ_k. If team j adopts the optimal θ_j^*, player ℓ obtains:

$$u_\ell = \theta_k \times \frac{A_k}{a_i + a_h + A_k} \times \left(1 - \frac{a_i + a_h}{a_i + a_h + A_k} \times \frac{\sqrt{a_\ell \theta_k}}{\sqrt{a_\ell \theta_k} + \sqrt{a_m (1-\theta_k)}}\right),$$

where $A_k = \left(\sqrt{a_\ell \theta_k} + \sqrt{a_m (1-\theta_k)}\right)^2 \leq a_\ell + a_m$ with equality when $\theta_k = \theta_k^* = \frac{a_\ell}{a_\ell + a_k}$, which maximizes team k's productivity.

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