Technological Changes and Equilibrium Wage Structure: A Cooperative Game Approach*[†]

Hideo Konishi[‡] and Ryo Tsukamoto[§]

April 8, 2026

Abstract

In recent years, lumpy technological innovations have occurred with increasing frequency, affecting workers' wages in ways that depend on their skills and abilities. This paper proposes a framework to evaluate which types of workers gain or lose during the interim period following the introduction of a major technological innovation. Using a cooperative game-theoretic approach, we develop a model of large labor markets with finitely many types of atomless workers and a finite set of available technologies, each described by a pair consisting of a labor input vector and an output value. The equilibrium wage structure is characterized by the f-core (the coalition structure core with finite membership coalitions as in Kaneko and Wooders 1986) of an atomless transferable utility game generated by the model. The generically unique equilibrium wage rates can be computed efficiently, as the f-core allocation maximizes total production value. Within this framework, we analyze how the equilibrium wage structure for heterogeneous worker types changes when a new technology becomes available. We show that, under mild conditions, the introduction of a relevant and efficient technology almost always disadvantages at least one type of labor, while benefiting another. However, when there are more than two labor types, identifying which types gain and which lose is generally nontrivial. We also show that if the population of a given worker type increases, holding available technologies fixed, the wage rate for that type weakly decreases.

*Thanks are due to Jim Anderson, Susanto Basu, Enrico Cristforoni, Taiji Furusawa, Sam Kortum, Tomoaki Mikami, Noritsugu Nakanishi, Theo Papageorgiu, Robert Ulbricht, Anne van den Nouweland, Myrna Wooders, and Michael Zierhut for their helpful comments.

[†]**Keywords:** technological innovation, wage structure, f-core, large game, activity analysis; **JEL Codes:** C71, D33, J31

[‡]Boston College, Department of Economics, Chestnut Hill, MA 02467, United States, email: hideo.konishi@bc.edu

[§]Boston College, Department of Economics, Chestnut Hill, MA 02467, United States, email: ryo.tsukamoto@bc.edu

1 Introduction

Technological innovations affect employment and wages differently across workers depending on their skills and educational backgrounds, with important implications for income inequality. In macro labor economics, many researchers have analyzed the relationship between technological change and the relative wages of workers with different skill (or education) levels over the past several decades using aggregate production functions. Early work includes highly aggregated constant elasticity of substitution (CES) models distinguishing between skilled and unskilled workers (Tinbergen 1974; Tinbergen 1975), while more recent research adopts task-based frameworks with multiple worker types differentiated by skill level (Acemoglu and Autor 2011; Acemoglu and Restrepo 2018; Acemoglu and Restrepo 2022).

In this paper, we do not attempt to propose a new mechanism to explain recent changes in market wage structures driven by technological innovation. Instead, we focus on how market wage rates respond to the entry of a major technological innovation in the intermediate run—namely, when labor markets clear but workers have not yet had sufficient time to acquire new skills or education in response to the change. Starting from an initial equilibrium allocation of heterogeneous labor across existing production technologies, we introduce a new technology and ask: How does the arrival of a discrete new technology affect the wage structure in markets with heterogeneous labor? What are the implications for income inequality?

We argue that this exercise is meaningful, as understanding how wage structures respond to discrete and substantial technological innovations is important. However, analyzing such innovations using marginal methods based on a smooth aggregate production function—where wages are equated to marginal products—is no longer appropriate. Instead, we adopt a cooperative game-theoretic approach to overcome this limitation. Specifically, we examine the impact of the introduction of a discrete new technology on the labor market wage structure in an economy with an arbitrary number of heterogeneous labor types, given a set of feasible discrete technology vectors defined over labor input combinations.

Shapley and Shubik (1971) study a transferable utility (TU) version of the two-sided matching problem introduced by Gale and Shapley (1962)—the assignment problem—by allowing for fully heterogeneous players (workers) and characterizing the equilibrium wage structure through the coalition structure core of the TU game. They show that total surplus is maximized in every core allocation and characterize the core wage structure for an arbitrary number of workers. However, their results rely critically on the presence of a two-sided market: workers must be partitioned into two groups, and production requires matching pairs of workers from different groups. In contrast, in a one-sided market, the core may be empty.[1] This is a serious limitation for modeling real-world labor markets.

To overcome this issue, we depart from the assumption of fully heterogeneous workers and instead consider a continuum of atomless workers with a finite (but arbitrary) number of skill types, together with a finite set of technologies. Each technology is described by a vector of labor inputs—specifying the required number of workers of each skill type—and an associated output value. This environment can be naturally represented as a transferable utility cooperative game with atomless players and finite-sized coalitions, allowing us to apply cooperative game-theoretic solution concepts to determine the wage structure. In particular, we adopt the f-core—the coalition-structure core with finite coalition sizes—introduced by Kaneko and Wooders (1986).[2] We show that the f-core exists and is generically unique, and we characterize the unique f-core allocation as the solution to a value maximization problem within a special activity analysis framework. Because this problem can be formulated as a linear programming problem, the f-core allocation can be computed efficiently. Within this framework, a technological innovation can be modeled as the addition of a new activity to the set of feasible activities in the linear programming problem, allowing changes in the equilibrium wage structure induced by the new technology to be computed directly.

We first characterize the f-core in our model and establish its equivalence to a labor market equilibrium (Proposition 1). We also show that the model can be formulated as a multi-input, single-output activity analysis problem (Koopmans 1951). We prove that the f-core is nonempty (Theorem 1) and that the equilibrium wage structure within the f-core is generically unique (Proposition 2).

We then show that, under mild conditions, the adoption of a new efficient technology generates both winners and losers among labor types: at least one type experiences a wage decrease, while another experiences a wage increase (Theorem 2). This result implies that a single discrete technological innovation generally cannot yield a Pareto improvement in wages across all labor types; some workers are inevitably adversely affected.[3] However, identifying the winning and losing labor types is generally difficult, even though changes in the equilibrium wage structure resulting from the introduction of a new technology can be computed easily using linear programming methods.

An important exception arises when there are only two labor types. In this case,

---

[1]For example, if $N = \{1, 2, 3\}$ with $v(1, 2) = v(2, 3) = v(3, 1) = 1$, then there is no core allocation.

[2]Kaneko and Wooders (1986) define the f-core for atomless non-transferable utility (NTU) games. We adapt their solution concept to a TU environment and provide a full characterization.

[3]This does not mean that a sequence of technological innovations cannot achieve Pareto improvement for all types of workers.

the labor type used more intensively by the new technology benefits from the innovation, while the other type loses (Proposition 4). This result holds even when the innovation is large and global. By contrast, when there are more than two labor types, even a small or local innovation may fail to raise the wage of the labor type most intensively used by the new technology. Thus, even marginal technological changes can have non-intuitive distributional effects, making the incidence of technological change difficult to predict (Example 4). We relate these results to the Stolper–Samuelson theorem in classical international trade theory by examining how Hicks-neutral technological improvements affect the equilibrium wage structure.

Finally, we investigate the effect of a change in the population of a given worker type on the wage structure, holding the set of available technologies fixed. Unlike the case of the introduction of a new technology, we show that the wage rate of the worker type whose population increases weakly decreases (Theorem 3).

The remainder of this section provides a brief review of the related literature. Section 2 presents examples of drastic changes in wage structures driven by technological innovations (and other factors). Section 3 introduces the model and the solution concept. Section 4 establishes the foundational results. Section 5 characterizes the equilibrium wage structure, and Section 6 presents local comparative statics to illustrate the non-straightforward nature of equilibrium wage responses to technological change. Section 7 considers the effects of a population change on the wage structure in the labor market. Section 8 concludes.

1.1 Relation to the literature

The relationship between technological change and wage structure has been widely studied in both macroeconomics and labor economics. In Tinbergen (1974) and Tinbergen (1975), a CES production model with skilled and unskilled workers is introduced, in which technological progress takes a factor-augmenting form. This approach has been successful in explaining the wage gap between high school graduates and college graduates from a macro labor economics perspective (see Katz and Murphy (1992), Card and Lemieux (2001), and Goldin and Katz (2008)). Pointing out several limitations of the Tinbergen-type highly aggregated production function, Acemoglu and Autor (2011) introduce a task-based framework with a continuum of tasks, inspired by the Ricardian trade model of Dornbusch, Fischer, and Samuelson (1977), which distinguishes between tasks (occupations) and skills. They endogenize the assignment of tasks to workers with different skill levels and analyze how technological changes affect wage rates across heterogeneous labor types. The task-based model has proven successful in explaining recent trends in income inequality driven by rapid technological change from a macro labor market perspective. Building on this approach, Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2022) study relative wage declines resulting from automation. They show that automation technologies gradually displace certain groups of workers, and they provide robust empirical evidence supporting this mechanism over a long time horizon.

In contrast, we do not assume the existence of differentiable aggregate production function. Our approach can be viewed as a transferable utility (TU) cooperative game with finitely many types of atomless players. The equilibrium concept we employ is the f-core introduced by Kaneko and Wooders (1986). Wang (2020) studies this class of games using an optimal transport framework.

Although our primary focus is on labor markets, our model can also be interpreted as a generalization of assignment games (Shapley and Shubik 1971), in which agents are not necessarily partitioned into two sides. In this interpretation, a modification of a technology's input vector in our labor market model corresponds to a change in buyers' valuations or to the entry of a new buyer or seller, as in Mo (1988).

From a methodological perspective, we provide an alternative proof of equilibrium existence in a linear production economy (Dorfman, Samuelson, and Solow 1958) using a fixed-point theorem, thereby offering a cooperative game–theoretic foundation for the result. This approach builds on recent observations by Konishi and Simeonov (2025), proving the nonemptiness of the f-core in nontransferrable utility games without comprehensiveness.

Our work is closely related to the activity analysis tradition in classical production theory (Koopmans 1951; Simon 1951; Dorfman, Samuelson, and Solow 1958; Gale 1960; Nikaido 1968). Within the activity analysis framework, Simon (1951) classifies technological changes into three categories and studies their effects on output, whereas our focus is on their impact on factor prices. Dorfman, Samuelson, and Solow (1958) develop a method for computing factor valuations using linear programming duality. While we share this methodological foundation, to our knowledge, no existing work examines wage determination across heterogeneous worker types and its relationship to technological change in this setting.

Chambers and Hayashi (2020) ask whether there exists an allocation mechanism that benefits all agents whenever the production possibility frontier expands. Although they reach a similar negative conclusion, their analysis focuses on allocation mechanisms, whereas we examine changes in equilibrium wages.

2 Examples

First consider the following example.

Example 1. There are three types of workers: High-skilled, Medium-skilled, and

Low-skilled workers with population measures $\bar{\nu}_H = 12$, $\bar{\nu}_M = 15$, and $\bar{\nu}_L = 50$, respectively. Initially, there are four technologies: γ_H, γ_M, and γ_L produce 3, 2, and 1 units of output from 1 unit of types H, M, and L workers, respectively, and γ_{HML} produces 20 units of output from 2 type H, 3 type M, and 5 type L workers together. Then, the resources should first go to the most productive technology γ_{HML}, and $100 = 20 \times 5$ units of output are produced by using $2 \times 5 = 10$, $3 \times 5 = 15$, and $5 \times 5 = 25$ units of types H, M, and L workers, respectively. Type M workers are used up, and types H and L have measures 2 and 25 of leftover workers, respectively. Since these leftovers must be employed in production technologies γ_H and γ_L, types H and L workers get wages $w_H = 3$ and $w_L = 1$, respectively. As a result, type M workers employed in technology γ_{HML} get the leftover output $20 - 3 \times 2 - 1 \times 5 = 9$, which is distributed to measure 3 workers, resulting in type M's wage to be $w_M = 3$. Total output is $5 \times 20 + 2 \times 3 + 25 \times 1 = 131$, and they are distributed to three types of workers: $3 \times 12 + 3 \times 15 + 1 \times 50 = 131$.
Now, suppose that another new technology is introduced: γ_{HL} produces 28 units of output from 3 type H and 10 type L workers. Then, the total output is maximized by using technology γ_{HL}: γ_{HL} produces $112 = 4 \times 28$ units of output by using measures $4 \times 3 = 12$ and $4 \times 10 = 40$ of types H and L workers, respectively. Type H workers are used up, and the leftovers are measures 15 and 10 types M and L workers, who are employed in technologies γ_M and γ_L, respectively. Thus, types M and L workers get wages $w_M = 2$ and $w_L = 1$, respectively, and type H's wage is determined as $w_H = (28 - 1 \times 10)/3 = 6$. The total output is $4 \times 28 + 15 \times 2 + 10 \times 1 = 152$, which are distributed to three types of workers: $6 \times 12 + 2 \times 15 + 1 \times 50 = 152$, respectively. That is, by the introduction of technology γ_{HL}, the total output is increased by 21, but only type H workers enjoy the benefits. In contrast, type M workers lose and the loss is also absorbed by type H workers.

This simple numerical example illustrates that wage structure is determined by relative scarcity of different types of labor. An introduction of new technology can change the wage structure completely. The next example shows that even if there are two factors, a one-factor augmented technological progress can reduce the factor price.

Example 2. There are two types of workers with population measures $\bar{\nu}_H = 3$ and $\bar{\nu}_L = 28$. respectively. There are three technologies: γ_H and γ_L produce 3 and 1 units of output from 1 unit of types H and L workers, respectively, and γ_{HL} produces 20 units of output from 1 type H and 10 type L workers together. Thus, equilibrium wages for H and L are $w_H = 3$ and $w_L = 1.7$, respectively. Now, there is a technological improvement in technology γ_{HL}: the new technology $\tilde{\gamma}_{HL}$ still produces 20 units of output from 1 type H but only 9 type L workers, augmenting type L worker's productivity. Then, $w_H = 3.67$ and $w_L = 1$ by increasing output from 56.6 to 61. That is, a factor-augmenting technological progress may not result in benefiting both factor returns.

Finally, the next example shows that a small change in productivity in a technology can impact on the wage structure in a complex manner.

Example 3. There are three types of workers with population measures $\bar{\nu}_H = 3$, $\bar{\nu}_M = 16$, and $\bar{\nu}_L = 30$. respectively. There are five technologies: γ_H, γ_M, and γ_L produce 3, 2, and 1 units of output from 1 unit of types H, M, and L workers, respectively, γ_{HM} produces 10 units of output from 1 type H and 2 type M workers together, and γ_{ML} produces 5 units of output from 1 type M and 2 type L workers together. In this case, equilibrium wages for H, M, and L are $w_H = 4$, $w_M = 3$, and $w_L = 1$, respectively. Now, there is a productivity increase in technology γ_L: $\tilde{\gamma}_L$ now produces 1.2 units of output from 1 type L worker. Then, $w_L = 1.2$ and $w_M = 5 - 1.2 \times 2 = 2.6$, and $w_H = 10 - 2 \times 2.6 = 4.8$. That is, type H workers are not contributing to the improvement of productivity, but they are getting the highest raise in this example.

These three examples show that an introduction of a new technology or a technological improvement can have surprising consequences on equilibrium wage structures.

3 The Model

There is a finite set of worker types T with population mass $\bar{\nu}_t > 0$ of atomless workers for each worker type $t \in T$. We denote the population vector of worker types by $\bar{\nu} \equiv (\bar{\nu}_t)_{t \in T}$. There is a finite set of production technologies Γ, and each production technology $\gamma \in \Gamma$ is characterized by the value of output $V^\gamma > 0$ and a factor (worker) input profile $m^\gamma = (m_t^\gamma)_{t \in T} \in \mathbb{Z}_+^T$: i.e., m_t^γ is a finite nonnegative integer. Since Γ is finite, there is a uniform upper bound $K \in \mathbb{Z}_{++}$ with $m_t^\gamma \leq K$ for all $\gamma \in \Gamma$ and $t \in T$.[4] We assume that each type worker $t \in T$ can engage in a home production $\gamma_t \in \Gamma$, producing $V^{\gamma_t} > 0$ from $m^{\gamma_t} = (m_{t'}^\gamma)_{t' \in T}$ with $m_t^{\gamma_t} = 1$ and $m_{t'}^{\gamma_t} = 0$ for $t' \neq t$ (positive outside option). Let $T^\gamma \equiv \{t \in T : m_t^\gamma > 0\}$ be the set of types of workers who are employed in technology γ. Our model G is summarized by a list $G = \Big(T, \Gamma, (\bar{\nu}_t)_{t \in T}, (V^\gamma, (m_t^\gamma)_{t \in T})_{\gamma \in \Gamma}\Big)$.

[4] An interpretation of this is that production technology γ produces y^γ units of a product using $m_t^\gamma \geq 0$ workers for each type $t \in T$, creating the value $V^\gamma = p^\gamma y^\gamma$, where p^γ is the price of the product.

Remark 1. One can introduce multiple types of products Q and Γ can be partitioned into $(\Gamma_q)_{q\in Q}$ where each $\gamma \in \Gamma_q$ specializes in type q product. Then, the interpretation of this model is that technology $\gamma \in \Gamma_q$ produces y_q^γ units of product q using $m_t^\gamma \geq 0$ workers for each type $t \in T$, generating value $V^\gamma = p_q y_q^\gamma$, where p_q is a market price of product q. Thus, the effects of a market price change of a product or an introduction of a brand new product to the product market can be analyzed in this framework. And although we did not introduce capital input here, we can interpret one of labor type as a capital input as will be done in Example 5 in the conclusion section.

Remark 2. One natural example is when workers are vertically heterogeneous $t_1 < t_2 < ... < t_T$ in the sense that a higher ranked worker type can always work as a lower ranked worker type. This condition can be modeled in our framework as follows: Suppose that $\gamma \in \Gamma$ is characterized by $V^\gamma > 0$ and $m^\gamma = (m_t^\gamma)_{t\in T} \in \mathbb{Z}_+^T$. Then, for any γ' with $V^{\gamma'} = V^\gamma$ and $\left(m_t^{\gamma'}\right)_{t\in T}$ with $\sum_{t=T-\tau}^{T} m_t' \geq \sum_{t=T-\tau}^{T} m_t$ for all $\tau = 0, ..., T-1$, $\gamma' \in \Gamma$ holds.

We call a firm that uses technology γ a type γ firm. A **feasible assignment** for G is a list $(\nu, \mu) \equiv \left((\nu_t^\gamma)_{t\in T,\gamma\in\Gamma}, (\mu^\gamma)_{\gamma\in\Gamma}\right)$, where $\mu^\gamma \geq 0$ and $\nu_t^\gamma \geq 0$ are the measures of type γ firms and of type t workers employed by type γ firms for all $\gamma \in \Gamma$ and all $t \in T$, respectively, such that (1) $m_t^\gamma \mu^\gamma = \nu_t^\gamma$ for all $t \in T$ and all $\gamma \in \Gamma$, and (2) $\sum_{\gamma\in\Gamma} \nu_t^\gamma = \bar{\nu}_t$ for all $t \in T$.[5] A **set of employed technologies** under a feasible assignment (ν, μ) is $\Gamma(\mu) \equiv \{\gamma \in \Gamma : \mu^\gamma > 0\}$. A **feasible allocation** for G is a list $\left(\nu, (w_t^\gamma)_{t\in T,\gamma\in\Gamma(\mu)}, \mu\right)$ such that (i) list (ν, μ) is a feasible assignment, and (ii) $\sum_{t\in T} m_t^\gamma w_t^\gamma = V^\gamma$ for all $\gamma \in \Gamma(\mu)$. We will consider deviations by infinitesimal coalitions from a feasible allocation. A pair $\left(\tilde{\gamma}, (\tilde{w}_t)_{t\in T^{\tilde{\gamma}}}\right)$ with $\tilde{\gamma} \in \Gamma$ is an **improving deviation** from a feasible allocation $\left(\nu, (w_t^\gamma)_{t\in T,\gamma\in\Gamma(\mu)}, \mu\right)$, if (a) $\sum_{t\in T} m_t^{\tilde{\gamma}} \tilde{w}_t = V^{\tilde{\gamma}}$, and (b) for all $t \in T^{\tilde{\gamma}}$, there is $\gamma \in \Gamma(\mu)$ and $\tilde{w}_t > w_t^\gamma$. That is, there are workers who are better off by joining a group that use technology $\tilde{\gamma}$ with wages $(\tilde{w}_t)_{t\in T^{\tilde{\gamma}}}$.[6] An **f-core allocation** for G is a feasible allocation $\left(\nu, (w_t^\gamma)_{t\in T,\gamma\in\Gamma(\mu)}, \mu\right)$ from which there is no improving deviation.[7] That is, a feasible allocation is an f-core allocation if there is no pair $\left(\tilde{\gamma}, (\tilde{w}_t)_{t\in T^{\tilde{\gamma}}}\right)$ that can attract each type $t \in T^{\tilde{\gamma}}$ by offering an improvement in their payoffs.

[5]Condition (2) holds with equality, since we assume that there is a possibility of home production $\gamma_t \in \Gamma$ for each type $t \in T$. Here, we assume that matching sets preserves the measures of the sets (Measure Consistency Condition in Kaneko and Wooders 1986).

[6]That is, there are head-hunting opportunity of workers employed in other firms by using technology $\tilde{\gamma}$.

[7]This solution concept is an f-core allocation in TU games, where the f-core was defined by Kaneko and Wooders (1986) in an NTU game setting. Wang (2020) analyzed the f-core in the TU game setting but with infinitely different types.

Remark 3. This model can be represented as a TU game. Let

$$M \equiv \left\{ m = (m_t)_{t\in T} \in \mathbb{Z}_+^T : \sum_{t\in T} m_t \leq K \right\}$$

be all feasible coalition types, and $V(m) \equiv max_{\gamma\in\left\{\gamma'\in\Gamma:m^{\gamma'}=m\right\}} V^\gamma$ if $\left\{\gamma' \in \Gamma : m^{\gamma'} = m\right\} \neq \emptyset$ and $V(m) = 0$, otherwise. A TU game representation of G is G^V a list $G^V = \left(T, (\bar{\nu}_t)_{t\in T}, V\right)$, where $V : M \to \mathbb{R}_+$. The f-core of problem G coincides with the f-core of G^V as is defined in Kaneko and Wooders (1986).

Remark 4. We can consider a market equilibrium for a labor market problem. In a market equilibrium, for all worker type $t \in T$, all workers receive the same wage w_t irrespective of firm types γ. A **market equilibrium** is a list $(\nu, (w_t)_{t\in T}, \mu)$ such that (i) list (ν, μ) is a feasible assignment, (ii) $\sum_{t\in T} m_t^\gamma w_t = V^\gamma$ for all $\gamma \in \Gamma(\mu)$, and (iii) $\sum_{t\in T} m_t^\gamma w_t \geq V^\gamma$ for all $\gamma \in \Gamma$. Firms in $\Gamma(\mu)$ produce and earn zero profits by paying the market wage rate for each type of workers, and other firms do not operate since it does not pay to produce by hiring workers with the market wage rates. As we will see in the next section, the market equilibrium and the f-core are equivalent in our model.

4 Foundations of the Analysis

The following is a characterization of the f-core of G, implying the equivalence between labor market equilibrium and the f-core.

Proposition 1. *An f-core allocation $\left(\nu, (w_t^\gamma)_{t\in T,\gamma\in\Gamma(\mu)}, \mu\right)$ has common wage rates $(w_t)_{t\in T}$ such that $w_t^\gamma = w_t$ for all $t \in T$and all $\gamma \in \Gamma(\mu)$, and is fully described by a list $(\nu, (w_t)_{t\in T}, \mu)$ such that*
(1) $m_t^\gamma \mu^\gamma = \nu_t^\gamma$ for all $t \in T$ and all $\gamma \in \Gamma$,
(2) $\sum_{\gamma\in\Gamma} \nu_t^\gamma = \bar{\nu}_t$ for all $t \in T$,
(3) $\sum_{t\in T} m_t^\gamma w_t = V^\gamma$ for all $\gamma \in \Gamma(\mu)$, and
(4) $\sum_{t\in T} m_t^\gamma w_t \geq V^\gamma$ for all $\gamma \notin \Gamma(\mu)$.

Thus, in an f-core allocation, identical type workers receive the same payoff. Furthermore, (3) and (4) of the above proposition state that the hyperplane defined by the equilibrium payoff vector separates the equilibrium technologies $\gamma \in \Gamma(\mu)$ and the dormant technologies $\gamma \notin \Gamma(\mu)$ in the sense that the revenue of each $\gamma \notin \Gamma(\mu)$ doesn't cover the payments of the workers employed.

Remark 5. This proposition states exactly the equivalence between the f-core and the market equilibrium. Conditions (1) and (2) are the labor market clearing condition for type $t \in T$ workers, and conditions (3) and (4) are the profit maximization conditions. The set $\Gamma(\mu)$ is the set of operating firm types.

The next corollary is the fundamental theorem of linear programming, which is also derived directly from the above Proposition 1.

Corollary 1. *In every equilibrium allocation, the aggregated value (output) $y = \sum_{\gamma\in\Gamma} \mu^\gamma V^\gamma$ is maximized among feasible assignments: i.e., the set of technologies $\Gamma(\mu)$ maximizes the aggregate value (output). Further, in equilibrium, the aggregate value (output) is completely distributed among workers $y = \sum_{\gamma\in\Gamma} \mu^\gamma V^\gamma = \sum_{t\in T}\sum_{\gamma\in\Gamma} \nu_t^\gamma w_t$.*

The following result can be shown as a corollary of a result in Kaneko and Wooders (1986), but here we use a concise and elementary proof following a fixed-point mapping approach by Konishi and Simeonov (2025) and Konishi, Pan, and Simeonov (2025).

Theorem 1. *There exists an f-core allocation $(\nu, (w_t)_{t\in T}, \mu)$.*

5 Equilibrium Wage Structure

We can say more things about equilibrium allocations, using Corollary 1. We can consider the following linear programming problem (Dorfman, Samuelson, and Solow 1958; Nikaido 1968):

$$\max_{\mu} \sum_{\gamma\in\Gamma} V^\gamma \mu^\gamma \text{ subject to } \sum_{\gamma\in\Gamma} m_t^\gamma \mu^\gamma \leq \bar{\nu}_t \text{ for all } t \in T.$$

From Corollary 1, we know $\sum_{t\in T}\sum_{\gamma\in\Gamma} \nu_t^\gamma w_t = \sum_{\gamma\in\Gamma} \mu^\gamma V^\gamma$. This is exactly the duality theorem of the linear programming.

In order to characterize equilibrium wage structure, it is convenient to rewrite the above problem by defining $\tilde{\mu}^\gamma \equiv V^\gamma \mu^\gamma$ and $\tilde{m}^\gamma \equiv \left(\frac{m_t^\gamma}{V^\gamma}\right)_{t\in T}$, and normalizing the output of each technology γ to unity:

$$\max_{\tilde{\mu}} \tilde{y} = \sum_{\gamma\in\Gamma} \tilde{\mu}^\gamma \text{ subject to } \sum_{\gamma\in\Gamma} \tilde{m}_t^\gamma \tilde{\mu}^\gamma = \bar{\nu}_t \text{ for all } t \in T. \tag{1}$$

We make the following genericity assumption:

(G1) For every subset of technologies $S \subset \Gamma$ with $|S| = T - 1$, $\bar{\nu}_t \notin vec(S)$, where $vec(S)$ is the vector space spanned by $\{\tilde{m}^\gamma\}_{\gamma \in S}$.

Condition (G1) requires that at least T technologies must be used with positive amount to exhaust the labor endowment $\bar{\nu}_t$. Since all labor types generate some positive amount of product on its own, this requires every optimal solution to the problem (1) should employ at least T technologies. Furthermore, (G1) ensures the uniqueness of the solution to the dual of the problem (1), which means the uniqueness of the equilibrium factor price vector.

Proposition 2. *Under (G1), the equilibrium factor price vector $w \in \mathbb{R}^T_{++}$ is uniquely determined.*

Proof. Assumption (G1) implies that $\Gamma(\tilde{\mu})$ includes at least T technologies. Choose any set S of T technologies from $\Gamma(\tilde{\mu})$ which already spans $\mathbb{R}^T$. Then, vectors $\{m^\gamma\}_{\gamma \in S}$ are linearly independent. Therefore, the solution to the system of linear equations

$$\sum_{t \in T} \tilde{m}^\gamma_t \widetilde{w}_t = 1 \text{ for all } \gamma \in S$$

is unique and is w. □

We can formulate the problem as a linear programming problem, which means it can be solved computationally. However, since our goal is to investigate the qualitative characteristics of the equilibrium (solution), a more detailed analysis is necessary. To that end, we will introduce a diagram in the next subsection that visualizes the equilibrium along with other auxiliary concepts.

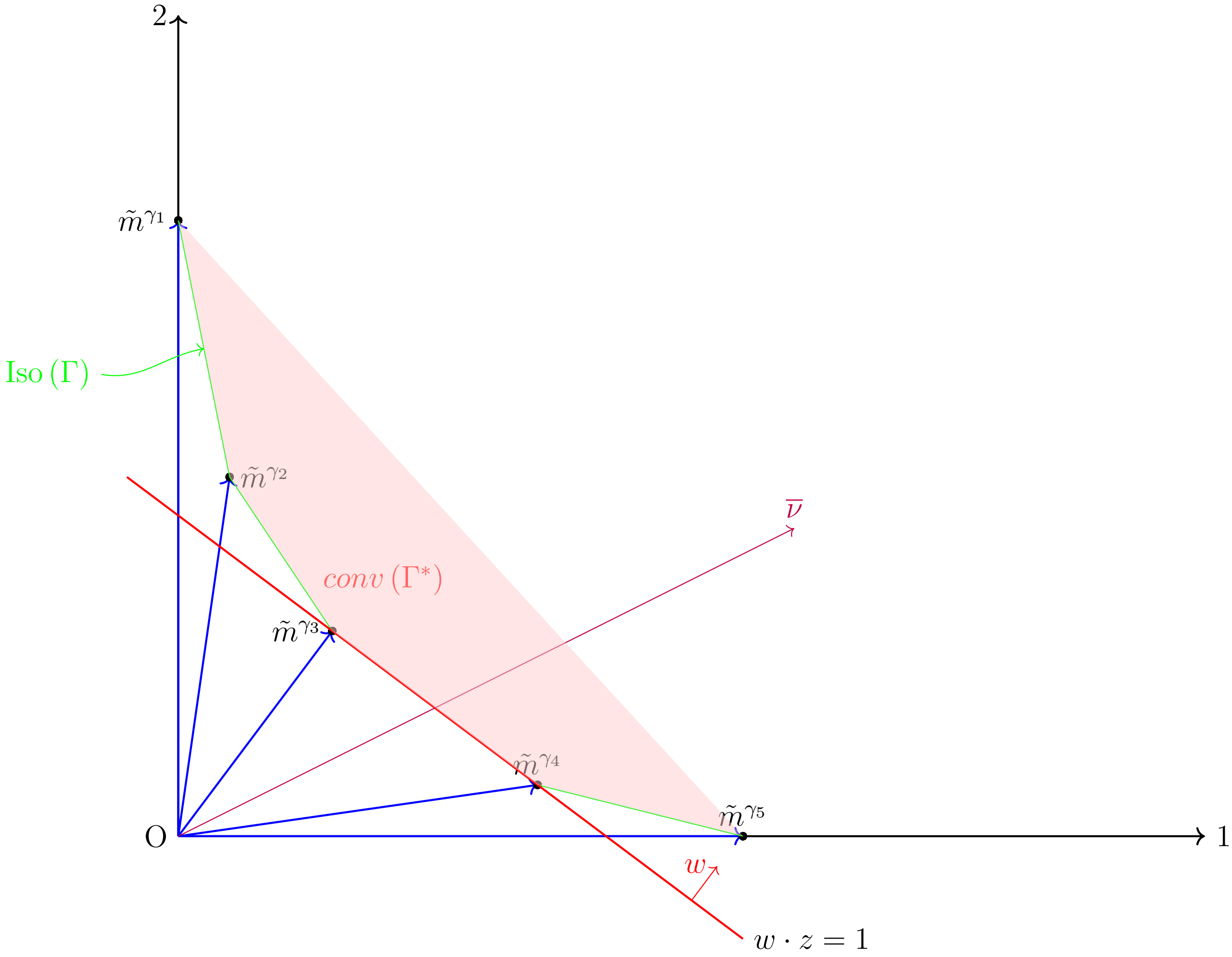
5.1 Diagrammatic Analysis

To ease notational burden, we use $conv(S)$ to denote $conv\left((\tilde{m}^\gamma)_{\gamma \in S}\right)$ for $S \subset \Gamma$. Thus, $conv(\Gamma)$ is the convex hull of unit production vectors, and its efficient frontier, $\overline{conv}(\Gamma) \equiv \left\{\tilde{m} \in conv(\Gamma) : \{\tilde{m}\} + \mathbb{R}^T_{--} \cap conv(\Gamma) = \emptyset\right\}$, is the isoquant technology set (curves) for unit output in this economy. There can be an inefficient technology, say γ, in the sense that γ takes more inputs than some convex combination of technologies: there is $\tilde{m} \in \overline{conv}(\Gamma)$ such that $\tilde{m}^\gamma > \tilde{m}$.[8]. Since the value produced is maximized in equilibrium, we can simply exclude these inefficient technologies from our analysis of equilibrium. Let

$$\Gamma^* \equiv \{\gamma \in \Gamma : \nexists \tilde{m} \in \overline{conv}(\Gamma) \text{ s.t. } \tilde{m}^\gamma > \tilde{m}\}$$

[8]We use conventional definition of inequalities: $\tilde{m}' \leq \tilde{m}$ iff $\tilde{m}'_t \leq \tilde{m}_t$ for all $t \in T$; $\tilde{m}' < \tilde{m}$ iff $\tilde{m}' \leq \tilde{m}$ and $\tilde{m}'_t < \tilde{m}_t$ for some $t \in T$; and $\tilde{m}' \ll \tilde{m}$ iff $\tilde{m}'_t < \tilde{m}_t$ for all $t \in T$.

Figure 1: The diagram when $T = 2$



In this example, Iso (Γ) consists of four segments:

$$conv\left(\{\gamma_1, \gamma_2\}\right), conv\left(\{\gamma_2, \gamma_3\}\right), conv\left(\{\gamma_3, \gamma_4\}\right), conv\left(\{\gamma_4, \gamma_5\}\right).$$

be the set of efficient technologies. So, we have the following relationship on the unit isoquant technology set and efficient technologies:

$$\text{Iso}\left(\Gamma\right) \equiv \overline{conv}\left(\Gamma\right) = \overline{conv}\left(\Gamma^*\right).$$

That is, $\text{Iso}\left(\Gamma^*\right) = \text{Iso}\left(\Gamma\right)$ holds. In a two labor-type case, Figure 1 describes how the isoquant technology set Iso(Γ) is derived, and how each segment of Iso(Γ) is supported by a hyperplane that contains the segment as a part of it.

Since $\tilde{m}^{\gamma_t} < \infty$ for all $t \in T$, Iso (Γ) is compact. Pick $\tilde{m} \in \text{Iso}\left(\Gamma\right)$, and consider set $\left\{r \in \mathbb{R}^T_+ : r < \tilde{m}\right\}$. This is a convex set which has an empty intersection with $conv\left(\Gamma^*\right)$ that is also a convex set. By the separating hyperplane theorem, there is a hyperplane $\{z \in \mathbb{R}^T : z \cdot w = 1\}$ which separates the two sets $conv\,(\Gamma^*)$ and $\{r \in \mathbb{R}^T_+ : r < \tilde{m}\}$. Note that w is a nonnegative (factor price) vector. Let

$$C\,(\Gamma) \equiv \left\{S \subset \Gamma^* : \exists w \in \mathbb{R}^T_+ - \{0\} \text{ s.t. } \tilde{m}^\gamma \cdot w = 1 \text{ for all } \gamma \in S\right\}$$

be the family of sets of technologies which lie in a hyperplane defined by some $w \in \mathbb{R}^T_+ - \{0\}$. Consider those sets in $C\,(\Gamma)$ that are maximal in the set inclusion sense and define the collection of these sets as

$$C^M\,(\Gamma) \equiv \{S \subset \Gamma^* : \nexists S' \in C\,(\Gamma) \text{ s.t. } S \subsetneq S'\}\,.$$

Note that each maximal set $S \in C^M\,(\Gamma)$ must consists of *at least* T technologies. We add another genericity condition in addition to (G1).

(G2) Every $S \in C^M\,(\Gamma)$ has $|S| = T$.

Condition (G2) ensures that each face of Iso(Γ) has exactly T vertices. This implies that $(\tilde{\mu}^\gamma)_{\gamma\in\Gamma}$ is uniquely determined. Together with Propositions 1 and 2, and Theorem 1, conditions (G1) and (G2) imply the uniqueness of the f-core allocation.

Proposition 3. *Under assumptions (G1) and (G2), the f-core allocation is uniquely determined as* $(\nu, (w_t)_{t\in T}, \mu)$.

Hence, we assume (G1) and (G2) throughout the remainder of the paper. We will project the unit isoquant curve Iso$\,(\Gamma)$ onto the standard simplex

$$\Delta^T \equiv \left\{x \in \mathbb{R}^T_+ : \sum_{t\in T} x_t = 1\right\}$$

while preserving the grouping pattern $C^M\,(\Gamma)$. Note that Iso$\,(\Gamma)$ is isomorphic to Δ^T, since we can construct a one-to-one-onto mapping $x : \text{Iso}\,(\Gamma) \to \Delta^T$as $x\,(\tilde{m}) = \frac{\tilde{m}}{\|\tilde{m}\|} \in \Delta^T$ for all $\tilde{m} \in \text{Iso}\,(\Gamma)$.[9][10] Thus, for each $\gamma \in \Gamma$, let $x^\gamma \equiv \frac{\tilde{m}^\gamma}{\|\tilde{m}^\gamma\|} \in \Delta^T$. Since $\{x^\gamma\}_{\gamma\in\Gamma^*}$satisfies $conv\left(\{x^\gamma\}_{\gamma\in\Gamma^*}\right) = \Delta^T$, $\{x^\gamma\}_{\gamma\in\Gamma}$ generates a partition $\mathscr{S}\,(\Gamma)$ of Δ^T such that

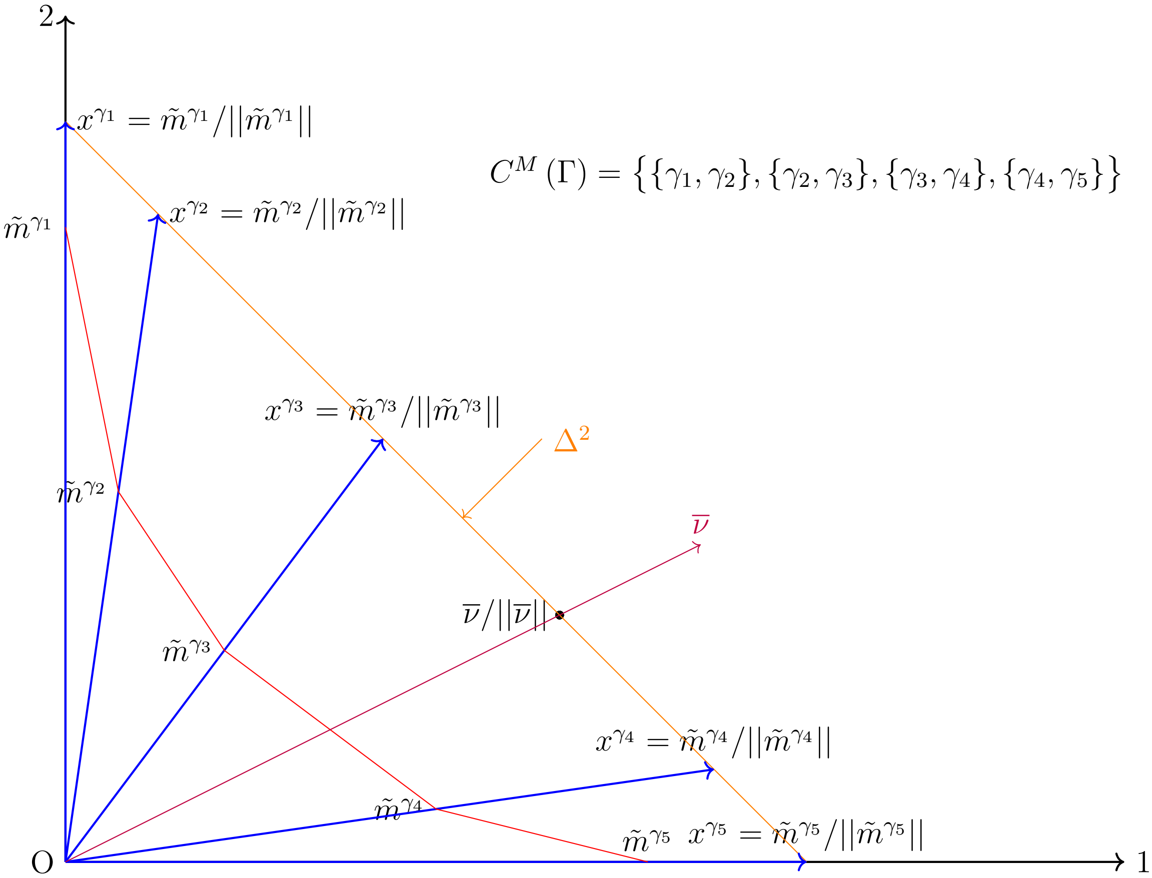
$$\mathscr{S}\,(\Gamma) \equiv \left\{conv\left(\{x(\tilde{m}^\gamma)\}_{\gamma\in S}\right) : S \in C^M\,(\Gamma)\right\},$$

of which example is described in Figure 2.

Since $\Gamma^* = \{\gamma \in \Gamma : \tilde{m}^\gamma \in \text{Iso}(\Gamma)\}$ is a collection of vertices of Iso$\,(\Gamma)$, the labor market clearing $\overline{\nu} = \sum_{\gamma\in\Gamma^*} \tilde{m}^\gamma \tilde{\mu}^\gamma$ must occur at the product value maximizing allocation. At the value maximizing allocation, the maximum value (output level) $y > 0$

[9] Since each worker type $t \in T$ has an outside option production technology $\gamma_t \in \Gamma$.
[10] We denote the Euclidean norm of the vector x by $\|x\|$.

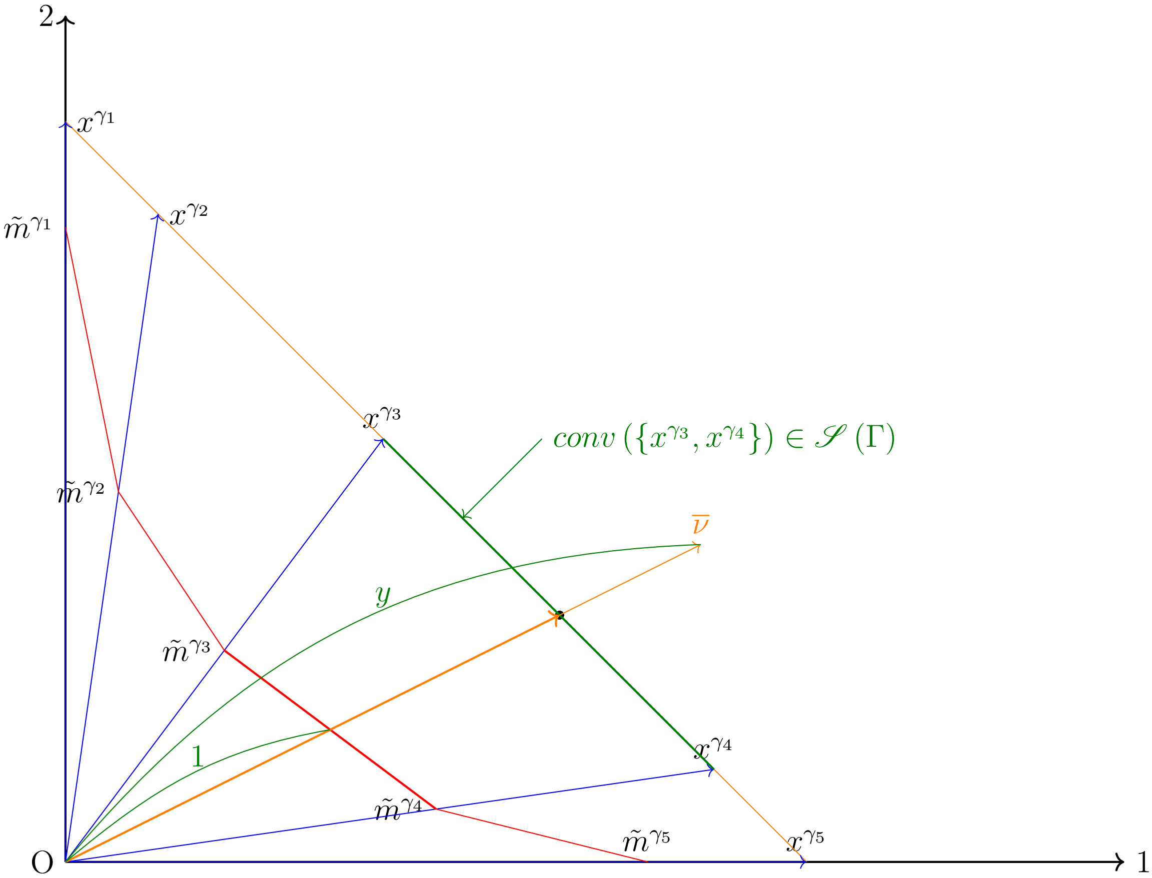
Figure 2: Partition $\mathscr{S}(\Gamma)$ on the unit simplex Δ^T



The partition $\mathscr{S}(\Gamma)$ consists of four segments on Δ^2:

$$\mathscr{S}(\Gamma) = \left\{ conv\left(\{x^{\gamma_1}, x^{\gamma_2}\}\right), conv\left(\{x^{\gamma_2}, x^{\gamma_3}\}\right), conv\left(\{x^{\gamma_3}, x^{\gamma_4}\}\right), conv\left(\{x^{\gamma_4}, x^{\gamma_5}\}\right) \right\}.$$

Figure 3: The characterization of the maximum output value y



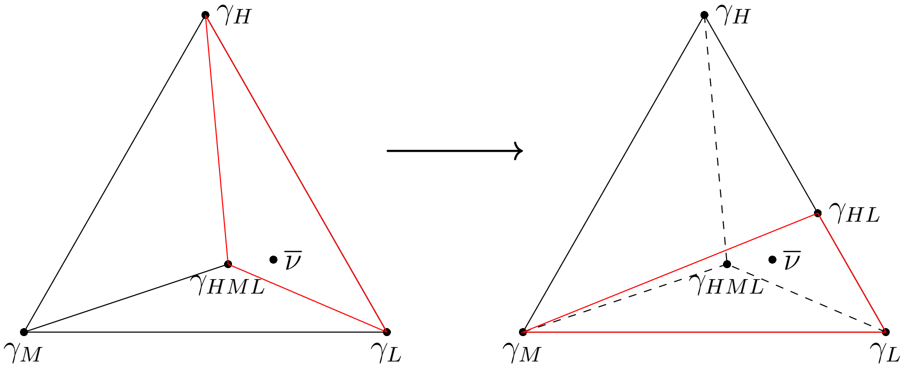
must satisfy $\bar{\nu} \in y\mathrm{Iso}\,(\Gamma)$: an isoquant curve for output y contains the population vector (of worker types) $\bar{\nu}$ (Figure 3).

Proposition 4. *The maximum output y is characterized by $\bar{\nu} \in y\mathrm{Iso}\,(\Gamma)$. A normalized endowment vector $\frac{\bar{\nu}}{\|\bar{\nu}\|}$ and a partition $\mathscr{S}(\Gamma)$ of Δ^T by $\{x^\gamma\}_{\gamma\in\Gamma^*}$ characterize the technologies used for the equilibrium production: a subset of technologies $S \in C^M(\Gamma)$ are used in production, if $\frac{\bar{\nu}}{\|\bar{\nu}\|}$ belongs to $conv\,(S) \in \mathscr{S}(\Gamma)$. Moreover, $\mathscr{S}(\Gamma)$ is a simplicial partition, and $\frac{\bar{\nu}}{\|\bar{\nu}\|}$ belongs to unique partition component $conv\,(S) \in \mathscr{S}(\Gamma)$.*

For each $\gamma \in S$ that is used in equilibrium, the unit value produced is distributed to employed worker types, $T^\gamma = \{t \in T : \tilde{m}^\gamma_t > 0\}$, through wages $(w_t)_{t\in T}$, which defines the separating hyperplane that includes $conv\,(S)$. In three worker-type case, Example 1 illustrates how $\mathscr{S}(\Gamma)$ is generated diagrammatically.

Example 1 (Diagram). The old set of unit-output technologies $\Gamma^{old} \equiv \{\gamma_H, \gamma_M, \gamma_L, \gamma_{HML}\} = \left(\Gamma^{old}\right)^*$ are: $\tilde{m}^{\gamma_H} = (\frac{1}{3}, 0, 0)$, $\tilde{m}^{\gamma_M} = (0, \frac{1}{2}, 0)$, $\tilde{m}^{\gamma_L} = (0, 0, 1)$, and $\tilde{m}^{\gamma_{HML}} = (\frac{1}{10}, \frac{3}{20}, \frac{1}{4})$, which form the old isoquant curve $\mathrm{Iso}\left(\Gamma^{old}\right)$, and are projected to $x^{\gamma_H} =$

Figure 4: Illustration of the equilibrium change in Example 1



$(1,0,0)$, $x^{\gamma_M} = (0,1,0)$, $x^{\gamma_L} = (0,0,1)$, and $x^{\gamma_{HML}} = (\frac{1}{5}, \frac{3}{10}, \frac{1}{2})$ on Δ^3, which are described in Figure 4. Thus, simplicial partition $\mathscr{S}\left(\Gamma^{old}\right)$ of Δ^3 is formed by three triangles $x^{\gamma_H}x^{\gamma_M}x^{\gamma_{HML}}$, $x^{\gamma_M}x^{\gamma_L}x^{\gamma_{HML}}$, and $x^{\gamma_L}x^{\gamma_H}x^{\gamma_{HML}}$, and the population vector $\frac{\bar{\nu}}{\|\bar{\nu}\|}$ belongs to the last triangle. This means that type H workers are employed by technologies γ_H and γ_{HML}, and type Lworkers by γ_L and γ_{HML}, but type M workers are employed by γ_{HML} only, implying that all the surpluses are absorbed by type M workers. Now, introduce a new technology γ_{HL}. Let $\Gamma^{new} \equiv \{\gamma_H, \gamma_M, \gamma_L, \gamma_{HML}, \gamma_{HL}\}$. Since new technology $\tilde{m}^{\gamma_{HL}} = (\frac{3}{28}, 0, \frac{3}{10})$ is efficient, and *conv* $(\{\tilde{m}^{\gamma_H}, \tilde{m}^{\gamma_M}, \tilde{m}^{\gamma_L}, \tilde{m}^{\gamma_{HL}}\})$ contains $\tilde{m}^{\gamma_{HML}}$ in its interior: $\{\gamma_H, \gamma_M, \gamma_L, \gamma_{HL}\} = (\Gamma^{new})^*$ holds, and γ_{HML} is no longer an efficient technology. Thus, $\mathscr{S}(\Gamma^{new})$ is described as a collection of two triangles $x^{\gamma_M}x^{\gamma_L}x^{\gamma_{HL}}$, and $x^{\gamma_M}x^{\gamma_H}x^{\gamma_{HL}}$, and the unit population vector $\frac{\bar{\nu}}{\|\bar{\nu}\|}$ belongs to the former triangle, which are described in Figure 4. This means that type H workers are employed by technologies γ_{HL} only, implying that all the surpluses are absorbed by type L workers.$\square$

We will use these characterization results to investigate how a technological innovation affects the equilibrium structure in the next section.

6 Comparative Statics

We return to the original question: the welfare effect of introducing a new technology. This question can be addressed through comparative statics in our model. Specifically, we add new technology $\gamma^{new} \notin \Gamma$ and compare the equilibria before and after its introduction. *We assume that conditions (G1) and (G2) continue to hold for the expanded set of technologies* $\Gamma \cup \{\gamma^{new}\}$ *in the rest of this section.*

Intuitively, the introduction of a new technology changes the equilibrium wage structure and may render some previously used technologies unaffordable. Such technologies are then replaced either by γ^{new} or by existing technologies that were not used in the old equilibrium.

In the graphical representation, the introduction of a new technology alters the set of efficient technologies and, consequently, the isoquant curve. This, in turn, induces a different partition of Δ^T. The resulting equilibrium wage vector is then determined by the set of technologies corresponding to the vertices of the simplex that contains the normalized endowment vector.

6.1 Winners and Losers from an Innovation: A General Analysis

Formally, let $S^{old} \subset \Gamma$ and $S^{new} \subset \Gamma \cup \{\gamma^{new}\}$ denote the sets of technologies employed in old and new equilibria, respectively. By Proposition 4, they satisfy

$$\frac{\bar{\nu}}{\|\bar{\nu}\|} \in conv\left(S^{old}\right) \text{ and } \frac{\bar{\nu}}{\|\bar{\nu}\|} \in conv\left(S^{new}\right).$$

If the new technology is not used in the new equilibrium i.e.,$\gamma^{new} \notin S^{new}$, then it does not affect the wage structure. We therefore focus on the case of $\gamma^{new} \in S^{new}$, which implies

$$\sum_{t \in T} \tilde{m}_t^{\gamma^{new}} w_t^{old} < 1 \text{ and } \sum_{t \in T} \tilde{m}_t^{\gamma^{new}} w_t^{new} = 1.$$

Hence, there exists $\hat{t} \in T^{\gamma^{new}}$ such that $w_{\hat{t}}^{new} > w_{\hat{t}}^{old}$.

If there exists $\gamma' \in S^{new} \setminus S^{old}$ such that $\gamma' \neq \gamma^{new}$, then the old equilibrium condition

$$\sum_{t \in T} \tilde{m}_t^{\gamma'} w_t^{old} > 1$$

together with the new equilibrium condition

$$\sum_{t \in T} \tilde{m}_t^{\gamma'} w_t^{new} = 1$$

implies that there exists another type $\bar{t} \neq \hat{t}$ such that $w_{\bar{t}}^{new} < w_{\bar{t}}^{old}$.

Intuitively, when γ^{new} is sufficiently efficient, it reshapes the equilibrium by replacing existing technologies and shifting labor demand. This can create a surplus of workers previously employed by displaced technologies, forcing them into less desirable technologies that were not used in the old equilibrium, necessarily reducing at least one type suffers from a wage reduction. We refer to this case as a ***global*** change.

We now turn to ***local*** changes. Suppose $S^{new} \backslash \{\gamma^{new}\} \subset S^{old}$. Then, all remaining technologies were already active in the old equilibrium. For each such $\gamma' \in S^{new} \setminus \{\gamma^{new}\}$, both the old and new equilibrium conditions hold with equality. Since

$w_{\hat{t}}^{new} > w_{\hat{t}}^{old}$, if any such γ' employs both $\hat{t}$ and another $t \neq \hat{t}$, then the wage of some other type must strictly decrease.

Thus, a local change does not reduce any wages only if all technologies employing type $\hat{t}$ are replaced by γ^{new}. This means that type $\hat{t}$ is employed by only one technology in both old and new equilibria. In that case, the remaining technologies do not use type $\hat{t}$, and the wages of all other types are determined independently of $w_{\hat{t}}$. Hence, all gains from the innovation are absorbed by type $\hat{t}$, while the wages of all other types remain unchanged. Thus, the only possible case for a Pareto improvement from a technological innovation takes this form.

We summarize the above analysis in the following result.

Theorem 2. *Suppose that $S^{new} \neq S^{old}$. Then, there is a type $\hat{t} \in T$ with $w_{\hat{t}}^{new} > w_{\hat{t}}^{old}$. Also, there is at least one type $\bar{t} \in T$ with $w_{\bar{t}}^{new} < w_{\bar{t}}^{old}$ unless the following two conditions are simultaneously met:*

(a) the change is local i.e., $\{\gamma^{new}\} = S^{new} \setminus S^{old}$, and

(b) $\hat{t}$ is employed only by the replaced technology and the new technology i.e., $m_{\hat{t}}^{\gamma} = 0$ for all $\gamma \in S^{old} \cap S^{new}$.

Note that conditions (a) and (b) are very demanding, especially together. Even if one of them is violated, an innovation inevitably hurts the welfare of some labor type.

6.2 Two Types of Workers

We have shown that the introduction of a new technology creates both winners and losers. However, Theorem 2 is silent on which types of labor experience wage increases or decreases. We next investigate which types are better or worse off.

In the case of $T = 2$, we can the identify the winner by using the relative intensity of the factor demand of the new technology. To evaluate the effect of introducing new technology γ^{new}, consider the hyperplane

$$\left\{z \in \mathbb{R}^T : w \cdot z = 1\right\},$$

spanned by $(\tilde{m}^{\gamma})_{\gamma \in S}$, whose normal vector is the wage vector $w = (w_t)_{t \in T}$. Let the intersection between this hyperplane $\{z : w \cdot z = 1\}$ and the ray $\left\{\alpha \tilde{m}^{\gamma^{new}} : \alpha \geq 0\right\}$ be given by $\tilde{m}^{\gamma'} \equiv \bar{\alpha} \tilde{m}^{\gamma^{new}}$, where $\bar{\alpha} = \frac{1}{w \tilde{m}^{\gamma^{new}}}$. We introduce a family of auxiliary technologies $\{\gamma(\alpha) : \alpha \in [1, \bar{\alpha}]\}$ with $\gamma(\alpha)$ defined by its unit output vector $\tilde{m}^{\gamma(\alpha)} \equiv \alpha \tilde{m}^{\gamma^{new}}$. That is, $\gamma(\alpha)$ takes the same inputs as γ^{new} and produces value $\frac{1}{\alpha} \leq 1$. As α goes down from $\bar{\alpha}$ to 1, the isoquant curve $\mathrm{Iso}(\Gamma \cup \{\gamma(\alpha)\})$ and the corresponding equilibrium technology set $conv(S(\alpha)) \in \mathscr{S}(\Gamma \cup \{\gamma(\alpha)\})$, satisfying

$\frac{\bar{\nu}}{\|\bar{\nu}\|} \in conv\,(S(\alpha))$, changes. Generically, $\frac{\bar{\nu}}{\|\bar{\nu}\|} \in int\,(conv\,(S(\alpha)))$ holds for almost all $\alpha \in [1, \bar{\alpha}]$. Intuitively, this construction allows us to scale the productivity of the new technology continuously from its break-even level under the old wage to its full efficiency.

When there are only two labor types, $T = \{1, 2\}$, one can obtain an intuitive result with a simple graphical analysis: if type 2 workers are more intensively demanded by a new efficient technology γ^{new}, their new equilibrium wage increases. This increase, in turn, implies a (weak) reduction in the wage of the type 1 through the other equilibrium technologies that are used in the new equilibrium.

Formally, given an existing technology set Γ and endowment $\overline{\nu}$, type 2 is **relatively more intensively used** by γ^{new} if

$$\frac{\overline{\nu}_1}{m_1^{\gamma^{new}}} > \frac{\overline{\nu}_2}{m_2^{\gamma^{new}}}.$$

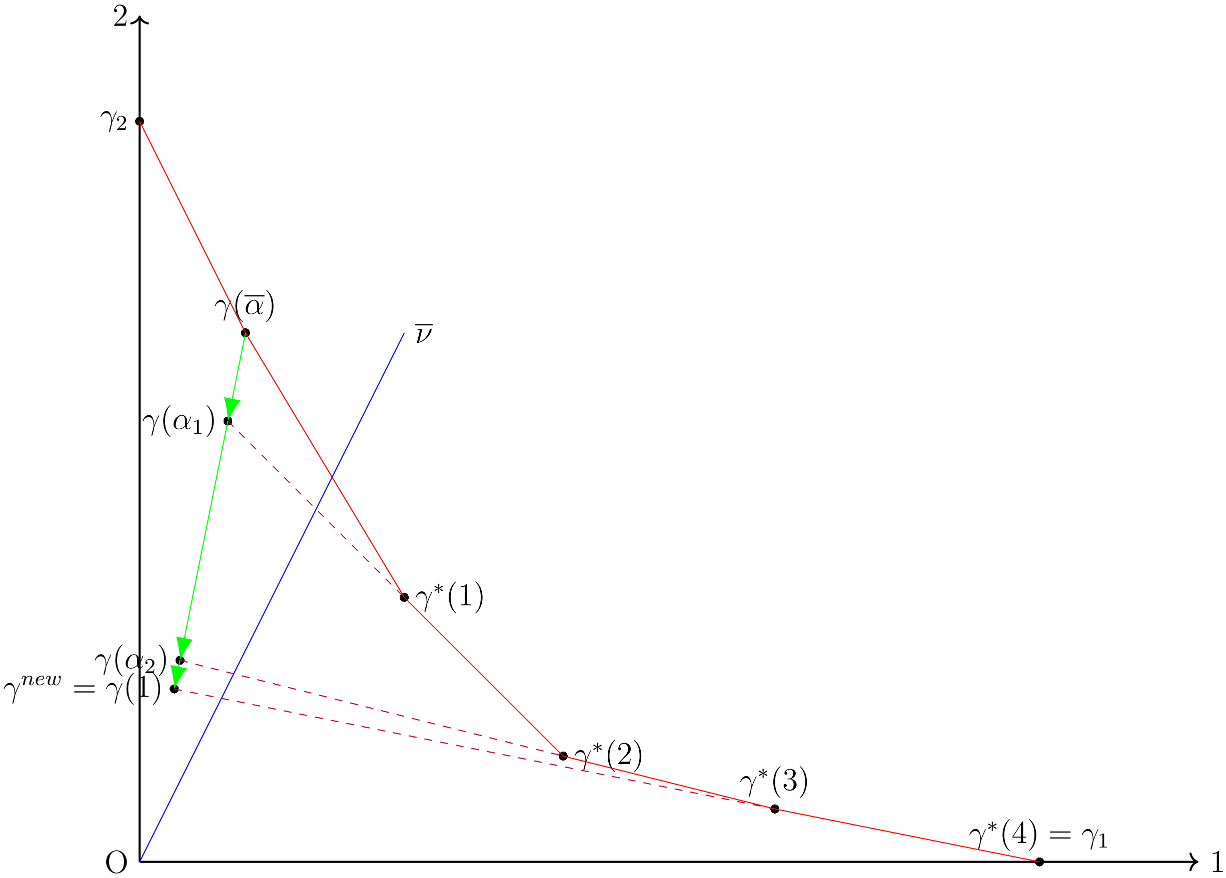
Proposition 5. *Assume that there are only two types of workers, and that type 2 is relatively more intensively used by a new technology γ^{new}. Let w^{old} and w^{new} be the old and new equilibrium wage vectors, respectively. Then, $w_1^{old} \geq w_1^{new}$ and $w_2^{old} < w_2^{new}$.*

Figure 5 provides a graphical illustration of Proposition 5. We trace the equilibrium technology set $S(\alpha)$ as α continuously decreases. Initially, $S(\overline{\alpha}) = \{\gamma(\overline{\alpha}), \gamma^*(1)\} \in C(\Gamma \cup \{\gamma(\overline{\alpha})\})$ is the set of equilibrium technologies. For $\alpha \in [\alpha_1, \overline{\alpha}]$, the equilibrium set $S(\alpha) = \{\gamma(\alpha), \gamma^*(1)\}$ changes locally. When α falls below α_1, $\gamma^*(1)$ is replaced with $\gamma^*(2)$, which was a non-cost-minimizing technology. For $\alpha \in [\alpha_2, \alpha_1]$, $S(\alpha) = \{\gamma(\alpha), \gamma^*(2)\}$. At α_2, another replacement occurs, and for $\alpha \in [1, \alpha_2]$, $S(\alpha) = \{\gamma(\alpha), \gamma^*(3)\}$. It should be clear from Figure 5 that the relative wage $-\frac{w_1}{w_2}$ which is the slope of the dashed red line decreases as α goes down, implying the wage for w_2 increases. Furthermore, Theorem 2 implies that the wage for w_1 decreases.

Of course, the symmetric argument holds for type 1 worker by switching the role of the two types. Graphically, relatively more intensively used worker type is determined by the relative position of γ^{new} and $\overline{\nu}$. In Figure 5, type 2 is more intensively used factor since γ^{new} lies above the ray $\overline{\nu}$.

This result resembles the Stolper-Samulelson theorem in the classical theory of international trade (see Jones 1965). The Stolper-Samuelson theorem says that in a two-commodity-two factor economy, if the price of a commodity increases then the factor used more (less) intensively for the commodity production will have an increase (decrease) in the factor price. In our model, there is a unique output, but if we interpret the amount of output as the value created by using a technology, then

Figure 5: An Innovation with 2 Types



The existing technology set is $\Gamma = \{\gamma_2, \gamma(\overline{\alpha}), \gamma^*(1), \gamma^*(2), \gamma^*(3), \gamma^*(4)\}$. At two values, $\alpha_2 < \alpha_1 \in (1, \overline{\alpha})$, one equilibrium technology is replaced by another.

a Hicks-neutral technological progress is the same as the price increase of the output produced by the technology. Thus, we can interpret Proposition 5 as the (global) Stolper-Samuleson theorem.

Unfortunately, the same intuition does not generalize with more than two types of workers. In the case of $T \geq 3$, we obtain only a subtle relationship between relative intensity and the identity of winner(s). Consider the following example in which equilibrium changes only locally.

Example 4. (See Figures 6 and 7)There are three types $T = \{1, 2, 3\}$ with population ray $\overline{\nu} = (1, 1, 1)$. There are six normalized technologies whose input vectors are give as follows:

$$\begin{aligned}
m^{\gamma_1} &= (20, 0, 0) \\
m^{\gamma_2} &= (0, 20, 0) \\
m^{\gamma_3} &= (0, 0, 20) \\
m^{\gamma^1} &= (3, 1, 0.99) \\
m^{\gamma^2} &= (2.1, 1.9, 2.11) \\
m^{\gamma^3} &= (1.2, 3.1, 3.1001)\,.
\end{aligned}$$

We will introduce new normalized technology γ^{new} with input vector

$$m^{\gamma^{new}} = \frac{1}{1.01}\,(1.2, 3.1, 3.1001) = \frac{1}{1.01} m^{\gamma^3}.$$

Note that γ^{new} uses type 3 worker most intensively relative to endowment $\overline{\nu} = (1, 1, 1)$. The old equilibrium technologies are $S^{old} = \{\gamma^1, \gamma^2, \gamma^3\}$ and the old equilibrium wage vector is

$$w^{old} = (0.259.., 0.067.., 0.154..)\,.$$

The new equilibrium is $S^{new} = \{\gamma^{new}, \gamma^1, \gamma^2\}$ and the new wage is $w^{new} = (0.258.., 0.091.., 0.134..)$. Type 3 and type 1 get worse off from this innovation while type 2 receives higher wage.

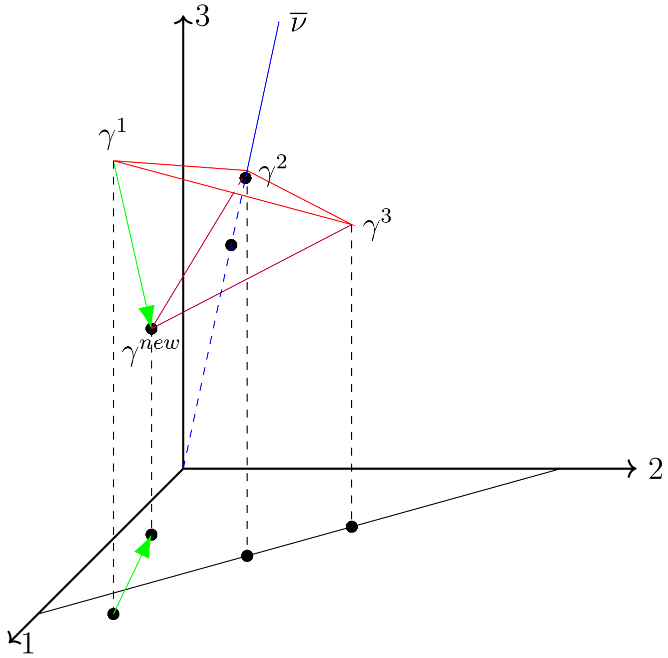
In the example above, it was found that when there are three or more types, the price of the relatively most used factor does not necessarily increase and can decrease. Below, we will attempt to provide graphical explanations for this.

Consider arbitrary hyperplane with normal vector $w > 0$:

$$\left\{z \in \mathbb{R}^3 : w_1 z_1 + w_2 z_2 + w_3 z_3 = 1\right\}.$$

The intercept of this hyperplane with type 3 axis is $x_3 = (w_3)^{-1}$. So, instead of comparing two wage vectors w^{old} and w^{new}, one can consider the direction in which the intercepts shift when the equilibrium change occurs. This can be easily

Figure 6: Example 4



illustrated in the diagram we create in subsection 5.1. Figure 7 is the projection of Example 4. The replacement of γ^3 with γ^{new} corresponds to the contraction of γ^3. Graphically, shortening γ^3 results in the hyperplane rotating around line $\gamma^2 - \gamma^1$ as its axis. The key point in Example 4 is that γ^3 is located on the opposite side of type 3 axis with respect to line $\gamma^2 - \gamma^1$. Therefore, when the hyperplane rotates, the intercept with type 3 axis shifts upward, implying $\left(w_3^{old}\right)^{-1} < \left(w_3^{new}\right)^{-1}$.[11]

7 Population Changes

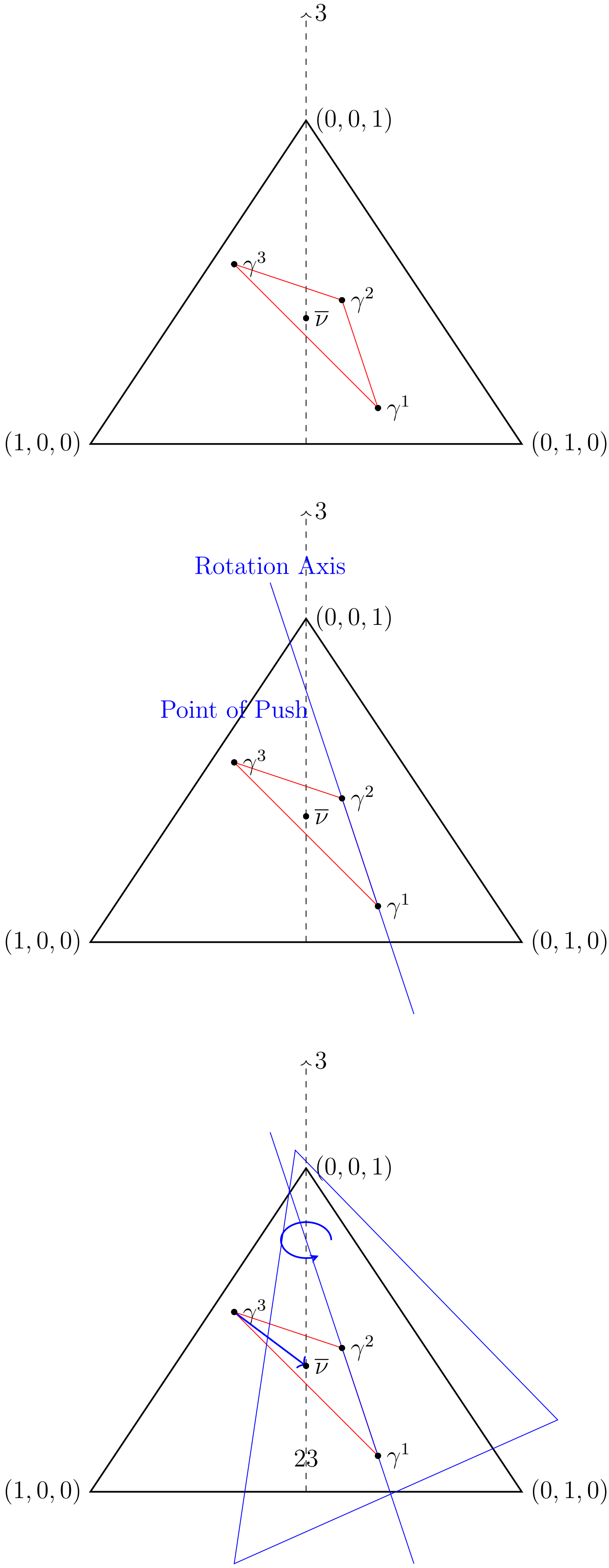
Although the main focus of this paper is on how technological innovation affects the equilibrium wage structure, the same framework also allows us to study comparative statics with respect to the population vector of worker types. In this section, we consider an increase in the population of type-T workers, holding all else fixed.

Since we are interested in the wage structures when the set of technologies $S \in \mathscr{S}(\Gamma)$ are used for the population vector of worker types $\bar{\nu}$, we again consider a hyperplane spanned by $(\tilde{m}^{\gamma})_{\gamma \in S}$ as well as its normal (wage vector) w. This hyperplane is given by

$$\left\{z \in \mathbb{R}^T : w \cdot z = 1\right\}.$$

[11]In the context of the Stolper-Samuelson theorem, Jones, Majit, and Mitra (1993) show that under their "strong factor intensity" condition, the Stolper-Samuelson result still holds, but the condition is so much stronger than the factor intensity condition.

Figure 7: Example 4: Graphical Explanations



Let the intersection between the hyperplane $\{z : w \cdot z = 1\}$ and the population vector $\bar{\nu}$ be $\tilde{\nu} \equiv \frac{\bar{\nu}}{y}$, where y is the total output level. This $\tilde{\nu}$ denotes the portion of the population vector to produce one unit of output using equilibrium technologies in S.

Theorem 3. *Consider a change in population vector from $\bar{\nu}^{old}$to $\bar{\nu}^{new}$, keeping Γ unchanged. Suppose that the population vectors satisfy $\bar{\nu}_T^{new} > \bar{\nu}_T^{old}$, and $\bar{\nu}_t^{new} = \bar{\nu}_t^{old}$ for all $t \neq T$. Then, the corresponding equilibrium wage vectors w^{new} and w^{old} satisfy $w_T^{new} \leq w_T^{old}$.*

Proof. Since the set of available technologies Γ remains unchanged, $\mathrm{Iso}(\Gamma)$ and $\mathscr{S}(\Gamma)$ are unchanged. Under conditions (G1) and (G2), there are $S^{old}, S^{new} \in \mathscr{S}(\Gamma)$ such that

$$\frac{\bar{\nu}^{old}}{\|\bar{\nu}^{old}\|} \in conv\left(S^{old}\right) \text{ and } \frac{\bar{\nu}^{new}}{\|\bar{\nu}^{new}\|} \in conv\left(S^{new}\right).$$

Notice that if $S^{old} = S^{new}$, then $w^{old} = w^{new}$, thus $w_T^{old} = w_T^{new}$ must hold. So, we will assume $S^{old} \neq S^{new}$ in the rest of the proof.

Denoting equilibrium output levels by y^{old} and y^{new}, let $\tilde{\nu}^{old} \equiv \frac{\bar{\nu}^{old}}{y^{old}}$ and $\tilde{\nu}^{new} \equiv \frac{\bar{\nu}^{new}}{y^{new}}$. We have $\tilde{\nu}_{-T}^{new} = \frac{y^{old}}{y^{new}}\tilde{\nu}_{-T}^{old}$. Since $y^{new} > y^{old}$, $\tilde{\nu}_{-T}^{old} > \tilde{\nu}_{-T}^{new}$ must hold. Since conditions (G1) and (G2) hold before and after the population change, $w^{old} \neq w^{new}$ are uniquely determined. Since $\tilde{\nu}^{old}, \tilde{\nu}^{new} \in Iso(\Gamma)$, cost-minimization conditions imply

$$w^{old} \cdot \tilde{\nu}^{new} > w^{old} \cdot \tilde{\nu}^{old} = 1 \text{ and } w^{new} \cdot \tilde{\nu}^{old} > w^{new} \cdot \tilde{\nu}^{new} = 1.$$

Rewriting them, we obtain

$$(w_{-T}^{old}, w_T^{old}) \cdot (\tilde{\nu}_{-T}^{new}, \tilde{\nu}_T^{new}) > (w_{-T}^{old}, w_T^{old}) \cdot (\tilde{\nu}_{-T}^{old}, \tilde{\nu}_T^{old}) = 1 \tag{2}$$

and

$$(w_{-T}^{new}, w_T^{new}) \cdot (\tilde{\nu}_{-T}^{old}, \tilde{\nu}_T^{old}) > (w_{-T}^{new}, w_T^{new}) \cdot (\tilde{\nu}_{-T}^{new}, \tilde{\nu}_T^{new}) = 1. \tag{3}$$

Rearranging (2), we obtain

$$w_T^{old}\left(\tilde{\nu}_T^{new} - \tilde{\nu}_T^{old}\right) > w_{-T}^{old} \cdot \left(\tilde{\nu}_{-T}^{old} - \tilde{\nu}_{-T}^{new}\right).$$

Using $\tilde{\nu}_{-T}^{new} = \frac{y^{old}}{y^{new}}\tilde{\nu}_{-T}^{old}$, the above inequality can be written as

$$\tilde{\nu}_T^{new} - \tilde{\nu}_T^{old} > \frac{1}{w_T^{old}}\left(1 - \frac{y^{old}}{y^{new}}\right) w_{-T}^{old} \cdot \tilde{\nu}_{-T}^{old}. \tag{4}$$

Similarly, rearranging (3), we obtain

$$\tilde{\nu}_T^{old} - \tilde{\nu}_T^{new} > \frac{1}{w_T^{new}} \left(\frac{y^{old}}{y^{new}} - 1 \right) w_{-T}^{new} \cdot \tilde{\nu}_{-T}^{old}. \tag{5}$$

Summing (4) and (5) up and noting $\frac{y^{old}}{y^{new}} < 1$, we obtain

$$\frac{1}{w_T^{new}} w_{-T}^{new} \cdot \tilde{\nu}_{-T}^{old} > \frac{1}{w_T^{old}} w_{-T}^{old} \cdot \tilde{\nu}_{-T}^{old}. \tag{6}$$

For future purpose, it is convenient to rewrite (4) as

$$\left(\frac{y^{old}}{y^{new}} \right) \frac{1}{w_T^{old}} w_{-T}^{old} \cdot \tilde{\nu}_{-T}^{old} > \tilde{\nu}_T^{old} - \tilde{\nu}_T^{new} + \frac{1}{w_T^{old}} w_{-T}^{old} \cdot \tilde{\nu}_{-T}^{old}. \tag{7}$$

Now, since $w^{new} \cdot \tilde{\nu}^{new} = w_T^{new} \tilde{\nu}_T^{new} + w_{-T}^{new} \cdot \tilde{\nu}_{-T}^{new} = 1$, we have

$$\begin{array}{rcc} \frac{1}{w_T^{new}} & = & \tilde{\nu}_T^{new} + \frac{1}{w_T^{new}} w_{-T}^{new} \cdot \tilde{\nu}_{-T}^{new} \\ & = & \tilde{\nu}_T^{new} + \left(\frac{y^{old}}{y^{new}} \right) \frac{1}{w_T^{new}} w_{-T}^{new} \cdot \tilde{\nu}_{-T}^{old} \\ & > & \tilde{\nu}_T^{new} + \left(\frac{y^{old}}{y^{new}} \right) \frac{1}{w_T^{old}} w_{-T}^{old} \cdot \tilde{\nu}_{-T}^{old} \\ & > & \tilde{\nu}_T^{new} + \tilde{\nu}_T^{old} - \tilde{\nu}_T^{new} + \frac{1}{w_T^{old}} w_{-T}^{old} \cdot \tilde{\nu}_{-T}^{old} \\ & = & \frac{1}{w_T^{old}} w^{old} \cdot \tilde{\nu}^{old} \\ & = & \frac{1}{w_T^{old}}. \end{array} \tag{8}$$

Here, we used $\tilde{\nu}_{-T}^{new} = \frac{y^{old}}{y^{new}} \tilde{\nu}_{-T}^{old}$ in the second line, (6) in the third line, (7) in the fourth line, and $w^{old} \cdot \tilde{\nu}^{old} = 1$ in the sixth line. Hence, we conclude $w^{new} < w^{old}$. $\square$

This result can be understood intuitively by slicing Iso(Γ) by a plane spanned by two vectors $\bar{\nu}$ and($\bar{\nu}_{-T}$,0). Notice that both $\tilde{\nu}^{new}$ and $\tilde{\nu}^{old}$ belong to this subspace, and the restriction of Iso(Γ) to this subspace is a two-dimensional downward-sloped kinked isoquant curve. Thus, the worker-type population vector changes from $\tilde{\nu}^{old}$ to steeper $\tilde{\nu}^{new}$ by increasing type T's population, the segment of Iso(Γ) to which $\tilde{\nu}^{new}$ belongs has a steeper slope than the one $\tilde{\nu}^{old}$ belongs to, implying that w_T decreases.

8 Concluding Remarks

We introduce a simple framework to assess the impact of technological changes on the equilibrium wage structure, allowing for the analysis of innovations of arbitrary forms. While the framework yields several counter intuitive examples, we have two general results. First, whenever a new technological innovation is introduced, there is at least one type of workers suffer from a wage reduction (Theorem 2). This implies that even if the introduced technological innovation is quite substantial, it is almost impossible to hope that all types of workers benefit from it. In order to achieve Pareto improvement, a sequence of technological innovations is needed. Thus, temporary welfare reduction for some types of workers are inevitable. Second, with an increase in the population of one type of workers, they suffer from a wage reduction (Theorem 3). This is a clear-cut result supporting the law of supply. However, it is not straightforward to see how an increase in the population of one type of workers affect the welfare of other types of workers. We also developed analytical and graphical methods that are potentially useful for studying these problems.

Our analysis can be extended to environments with additional factors of production, such as capital with an elastic supply. The next example provides a slight modification of Example 1 by incorporating capital as an additional input and examining how a decline in the market interest rate, which reduces the return to capital, affects equilibrium wages.

Example 5. There are three types of workers and capital with population measures $\bar{\nu}_H = 12$, $\bar{\nu}_M = 15$, and $\bar{\nu}_L = 50$. respectively, and capital is abundant and its return rate is determined outside of the model. There are five technologies: γ_H, γ_M, and γ_L produce 3, 2, and 1 units of output from 1 unit of types H, M, and L workers alone, respectively, γ_{HMLK} produces 24 units of output from 2 type H, 3 type M, and 5 type L workers together with 2 units of capital K, and γ_{HLK} produces 38 units of output from 3 type H and 10 type L workers together with 10 units of capital K. Initially, the market capital return rate is 2. Since capital is perfectly elastically supplied at the market capital return rate, the relevant factor markets are only for types H, M, and L workers. Then, the output net of market capital return for γ_{HMLK} and γ_{HLK} are 20 and 18, respectively, generating equilibrium wages $w_H = 3$, $w_M = 3$, and $w_L = 1$, respectively. With this wage structure, technology γ_{HLK} is not in use, since it does not pay to hire workers ($3 \times 3 + 1 \times 10 = 19 > 18$).
Now, suppose that capital return goes down to 1. Then, the output net of market capital return for γ_{HMLK} and γ_{HLK} become 22 and 28, respectively, generating equilibrium wages $w_H = 6$, $w_M = 2$, and $w_L = 1$. With this wage structure, technology γ_{HMLK} is not in use, since it does not pay to hire workers ($6 \times 2 + 2 \times 3 + 1 \times 5 = 23 > 22$).

This example shows that a fall in market interest rate can result in a drastic change in wage structures through changes in employed technologies in production. Our model appears to provide a useful tool to investigate the effects of changes in market interest rate and other factor prices (energy price etc.) on the market wage structure as well.

References

Acemoglu, Daron and David Autor (2011). "Skills, Tasks and Technologies: Implications for Employment and Earnings". In: *Handbook of Labor Economics*. Vol. 4. Elsevier, pp. 1043–1171. DOI: 10.1016/S0169-7218(11)02410-5.

Acemoglu, Daron and Pascual Restrepo (2018). "The Race between Man and Machine: Implications of Technology for Growth, Factor Shares, and Employment". In: *American Economic Review* 108.6, pp. 1488–1542. DOI: 10.1257/aer.20160696.

— (2022). "Tasks, Automation, and the Rise in U.S. Wage Inequality". In: *Econometrica* 90.5, pp. 1973–2016. DOI: 10.3982/ECTA19815.

Card, David and Thomas Lemieux (2001). "Can Falling Supply Explain the Rising Return to College for Younger Men? A Cohort-Based Analysis". In: *The Quarterly Journal of Economics* 116.2, pp. 705–746. DOI: 10.1162/00335530151144140.

Chambers, Christopher P. and Takashi Hayashi (2020). "Can Everyone Benefit from Innovation?" In: *Journal of Mathematical Economics* 88, pp. 187–191. DOI: 10.1016/j.jmateco.2020.02.001.

Dorfman, Robert, Paul A. Samuelson, and Robert M. Solow (1958). *Linear Programming and Economic Analysis*. McGraw-Hill.

Dornbusch, Rüdiger, Stanley Fischer, and Paul A. Samuelson (1977). "Comparative Advantage, Trade, and Payments in a Ricardian Model with a Continuum of Goods". In: *American Economic Review* 67.5, pp. 823–839.

Gale, David (1960). *The Theory of Linear Economic Models*. New York: McGraw-Hill.

Gale, David and Lloyd S. Shapley (1962). "College Admissions and the Stability of Marriage". In: *The American Mathematical Monthly* 69.1, pp. 9–15. DOI: 10.2307/2312726.

Goldin, Claudia and Lawrence F. Katz (2008). *The Race between Education and Technology*. Cambridge, MA: Harvard University Press.

Jones, Ronald W. (1965). "The Structure of Simple General Equilibrium Models". In: *Journal of Political Economy* 73.6, pp. 557–572. DOI: 10.1086/259084.

Kaneko, Mamoru and Myrna Holtz Wooders (1986). "The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results". In: *Mathematical Social Sciences* 12.2, pp. 105–137. DOI: 10.1016/0165-4896(86)90032-6.

Katz, Lawrence F. and Kevin M. Murphy (1992). "Changes in Relative Wages, 1963–1987: Supply and Demand Factors". In: *The Quarterly Journal of Economics* 107.1, pp. 35–78. DOI: 10.2307/2118323.

Konishi, Hideo, Chien-Yu Pan, and Dimitar Simeonov (2025). *Competition Teams in Large Markets: Free Entry Equilibrium with (Sub-)Optimal Contracts*. SSRN working paper.

Konishi, Hideo and Dimitar Simeonov (2025). "Nonemptiness of the f-Core Without Comprehensiveness". In: *Journal of Public Economic Theory* 27.5. in honor of Myrna Wooders, e70072. DOI: 10.1111/jpet.70072.

Koopmans, Tjalling C. (1951). "An Analysis of Production as an Efficient Combination of Activities". In: *Activity Analysis of Production and Allocation*. Ed. by Tjalling C. Koopmans. Cowles Commission for Research in Economics, Monograph No. 13. Wiley, pp. 33–97.

Mo, Jie-Ping (1988). "Entry and Structures of Interest Groups in Assignment Games". In: *Journal of Economic Theory* 46.1, pp. 66–96. DOI: 10.1016/0022-0531(88)90150-0.

Nikaido, Hukukane (1968). *Convex Structures and Economic Theory*. Vol. 51. Elsevier.

Shapley, Lloyd S. and Martin Shubik (1971). "The Assignment Game I: The Core". In: *International Journal of Game Theory* 1.1, pp. 111–130. DOI: 10.1007/BF01753437.

Simon, Herbert A. (1951). "Effects of Technological Change in a Linear Model". In: *Activity Analysis of Production and Allocation*. Ed. by Tjalling C. Koopmans. Cowles Commission for Research in Economics, Monograph No. 13. Wiley, pp. 33–97.

Tinbergen, Jan (1974). "Substitution of Graduate by Other Labour". In: *Kyklos* 27.2, pp. 217–226. DOI: 10.1111/j.1467-6435.1974.tb01903.x.

— (1975). *Income Differences: Recent Research*. Amsterdam: North-Holland Publishing Company.

Wang, Xinyang (2020). *Cooperation in Small Groups: An Optimal Transport Approach*. arXiv: 2005.11244 [econ.TH].

9 Appendix: Proofs

9.1 Proofs for Section 4

9.1.1 Proof for Proposition 1

Proof. We start with the existence of $(w_t)_{t\in T}$ such that $w_t^\gamma = w_t$ holds for all $t \in T$ and for all $\gamma \in \Gamma(\mu)$. Suppose not. By the supposition, there is at least one type $t' \in T$ and $\tilde{\gamma}, \gamma \in \Gamma$ with $\mu^{\tilde{\gamma}} > 0$ and $\mu^\gamma > 0$ such that $w_{t'}^{\tilde{\gamma}} > w_{t'}^\gamma$. Let $\epsilon \equiv \frac{m_t^\gamma}{\sum_{t\in T^{\tilde{\gamma}}} m_t^{\tilde{\gamma}}} \left(w_{t'}^{\tilde{\gamma}} - w_{t'}^\gamma\right)$, and let $\hat{w}^{\tilde{\gamma}} = \left(\hat{w}_t^{\tilde{\gamma}}\right)_{t\in T^\gamma}$ be such that (i) $\hat{w}_{t'}^{\tilde{\gamma}} = w_{t'}^\gamma + \epsilon$, and

(ii) $\hat{w}_t^{\tilde{\gamma}} = w_t^{\tilde{\gamma}} + \epsilon$ for $t \in T^{\tilde{\gamma}} \backslash \{t'\}$. This implies that $\hat{w}_t^{\tilde{\gamma}} > w_t^{\gamma_t}$ for all $t \in T^{\tilde{\gamma}}$, and $\sum_{t \in T^{\tilde{\gamma}}} m_t^{\tilde{\gamma}} \hat{w}_t^{\tilde{\gamma}} = V^{\tilde{\gamma}}$ hold. Thus, there is an improving coalitional deviation from the equilibrium allocation $(\nu, (w_t^\gamma)_{t \in T, \gamma \in \Gamma(\mu)}, \mu)$. This is a contradiction. This proves the existence of a market wage vector $(w_t)_{t \in T}$. Properties (1) and (2) follow directly from the definition of feasible assignment. By the definition of feasible allocation, (3) follows. Property (4) holds since firm type γ can deviate from an equilibrium unless it is satisfied. Thus, again, by contradiction, (4) holds. □

9.1.2 Proof for Corollary 1

Proof. First, assume there is a feasible assignment $(\tilde{\nu}, \tilde{\mu})$ with $\sum_{\gamma \in \Gamma} \mu^\gamma V^\gamma < \sum_{\gamma \in \Gamma} \tilde{\mu}^\gamma V^\gamma$. From (3) and (4) in Proposition 1, $\sum_{t \in T^\gamma} m_t^\gamma w_t \geq V^\gamma$ for all $\gamma \in \Gamma$. Thus,

$$\sum_{\gamma \in \Gamma} \mu^\gamma V^\gamma = \sum_{\gamma \in \Gamma} \mu^\gamma m_t^\gamma w_t = \sum_{t \in T} \sum_{\gamma \in \Gamma} \nu_t^\gamma w_t = \sum_{t \in T} \bar{\nu}_t w_t$$

and

$$\sum_{\gamma \in \Gamma} \tilde{\mu}^\gamma V^\gamma \leq \sum_{\gamma \in \Gamma} \tilde{\mu}^\gamma m_t^\gamma w_t = \sum_{t \in T} \sum_{\gamma \in \Gamma} \tilde{\nu}_t^\gamma w_t = \sum_{t \in T} \bar{\nu}_t w_t.$$

In summary, we have

$$\sum_{t \in T} \bar{\nu}_t w_t = \sum_{\gamma \in \Gamma} \mu^\gamma V^\gamma < \sum_{\gamma \in \Gamma} \tilde{\mu}^\gamma V^\gamma \leq \sum_{t \in T} \bar{\nu}_t w_t.$$

This is a contradiction as desired. Thus, the set of technologies $\Gamma(\mu)$ maximizes the aggregate value. The equilibrium aggregate value is obtained in the above discussion. □

9.1.3 Proof for Theorem 1

Proof. For each $\gamma \in \Gamma$, let $T^\gamma = \{t \in T : m_t^\gamma > 0\}$ and let $\varphi^\gamma : \Pi_{t \in T} \left(\bar{\nu}^t \times \Delta^\Gamma\right) \twoheadrightarrow V^\gamma \times \Delta^T$ be such that $\varphi^\gamma((\nu_t^\gamma)_{t \in T}) = \{(w_t^\gamma)_{t \in T} \in V^\gamma \times \Delta^T$: (i) $w_t^\gamma = 0$ for all $t \notin T^\gamma$, (ii) $(w_t^\gamma)_{t \in T^\gamma} = \arg\min_{\left(w_t^\gamma\right)_{t \in T^\gamma}} \sum_{t \in T^\gamma} w_t^\gamma \times \frac{\nu_t^\gamma}{m_t^\gamma}\}$. Condition (ii) says that $w_t^\gamma > 0$ holds for all $t \in T^\gamma$ only when $\frac{\nu_{t'}^\gamma}{m_{t'}^\gamma} = \frac{\nu_t^\gamma}{m_t^\gamma}$ holds for all $t, t' \in T^\gamma$ For each $t \in T$, let $\psi^t : \Pi_{\gamma \in \Gamma} \left(V^\gamma \times \Delta^T\right) \twoheadrightarrow \bar{\nu}^t \times \Delta^\Gamma$ be such that $\psi^t((w_t^\gamma)_{t \in T}) = \arg\max_{(\nu_t^\gamma)_{\gamma \in \Gamma}} \sum_{\gamma \in \Gamma} w_t^\gamma \nu_t^\gamma$. Let $\varphi : \Pi_{t \in T} \left(\bar{\nu}^t \times \Delta^\Gamma\right) \twoheadrightarrow \Pi_{\gamma \in \Gamma} \left(V^\gamma \times \Delta^T\right)$ be a Cartesian product of φ^γs, and let $\psi : \Pi_{\gamma \in \Gamma} \left(V^\gamma \times \Delta^T\right) \twoheadrightarrow \Pi_{t \in T} \left(\bar{\nu}^t \times \Delta^\Gamma\right)$ be a Cartesian product of ψ^ts. Letting ξ be a Cartesian product of φ and ψ, we have a fixed point mapping $\xi : \Pi_{\gamma \in \Gamma} \left(V^\gamma \times \Delta^T\right) \times \Pi_{t \in T} \left(\bar{\nu}^t \times \Delta^\Gamma\right) \twoheadrightarrow \Pi_{\gamma \in \Gamma} \left(V^\gamma \times \Delta^T\right) \times \Pi_{t \in T} \left(\bar{\nu}^t \times \Delta^\Gamma\right)$. Since sets $\Pi_{\gamma \in \Gamma} \left(V^\gamma \times \Delta^T\right)$ and $\Pi_{t \in T} \left(\bar{\nu}^t \times \Delta^\Gamma\right)$ are nonempty, compact, and convex, and functions $\sum_{t \in T^\gamma} w_t^\gamma \times \frac{\nu_t^\gamma}{m_t^\gamma}$ and $\sum_{(\nu_t^\gamma)_{\gamma \in \Gamma}} w_t^\gamma \nu_t^\gamma$ are continuous, and convex in w_t^γ and concave in ν_t^γ, respectively. Thus, ξ is nonempty-valued, upper hemicontinuous, and convex-valued, and all the conditions of Kakutani's fixed point theorem are satisfied, and there is a fixed point $((\nu_t^\gamma)_{t\in T,\gamma\in\Gamma}, (w_t^\gamma)_{t\in T,\gamma\in\Gamma})$ of ξ.

Let $w_t \equiv \max_{\gamma\in\Gamma} w_t^\gamma$. Then, by construction of ψ^t, $\nu_t^\gamma > 0$ only if $w_t = w_t^\gamma$ for all $t \in T$. Thus, type t players get w_t almost everywhere. For all $\gamma \in \Gamma$, we have $\frac{\nu_{t'}^\gamma}{m_{t'}^\gamma} = \frac{\nu_t^\gamma}{m_t^\gamma}$ for all $t, t' \in T^\gamma$, since by the construction of φ^γ, if $\frac{\nu_{t'}^\gamma}{m_{t'}^\gamma} > \min_{t\in T^\gamma} \frac{\nu_t^\gamma}{m_t^\gamma}$ holds then $w_{t'}^\gamma = 0$ and $\nu_{t'}^\gamma = 0$ must occur, which cannot be a fixed point by the assumption that there is a positive outside option for all $t \in T$ ($V^{\gamma_t} > 0$). Let $\mu^\gamma \equiv \frac{\nu_t^\gamma}{m_t^\gamma}$ for all $t \in T^\gamma$. Hence, the fixed point of ξ is an f-core allocation, since there is no improving deviation from the fixed point allocation that satisfies (a) and (b). $\square$