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Fiat Money As A Medium of Exchange

Scott Freeman
Department of Economics
Boston College

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Abstract

This paper presents a completely explicit exchange environment in which agents wishing to exchange one type of good for another choose to trade for fiat money which is then traded for the desired goods. This exchange pattern is chosen over barter because specified properties of fiat money make this pattern less expensive than the search for a double coincidence of wants. Although the transaction services of money directly affect agents' utility, the model's welfare implications are those of the basic overlapping generations models, not those of models with money balances in the utility function.

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Introduction:

In this paper I present a model in which fiat money is valued because it has properties that make it a useful medium of exchange in a specific physical environment. It serves as a medium of exchange in that agents wishing to exchange one type of good for another choose to do so by acquiring fiat money which is then exchanged for the desired good. This endogenously determined transaction pattern is the one described by Clower (1967): "Money buys goods and goods buy money; but goods do not buy goods." The use of fiat money benefits the economy because it is less expensive than barter or the use of other media of exchange. The lower costs of transactions involving fiat money result from two features of the environment: (i) that the costs of search are high if one must search for a "double coincidence of wants" (Jevons, 1875); and (ii) that fiat money has properties--it is non-counterfeitable and costless to hold, count, or transfer--that make its exchange for goods less costly than the exchange of goods for goods. Fiat money may be valued and held even though there exist other possible stores of value with rates of return no less than that of fiat money.

My model is motivated by assertions that money does not serve as a true medium of exchange in overlapping generations models (Tobin, 1980; Hahn, 1981; McCallum, 1983). The criticism is that fiat money is valued in the overlapping generations model solely as a store of value. If there existed any other asset with a greater rate of return, fiat money would not be necessary to achieve a Pareto optimum and would not be valued. The critics contend that in a better model agents would hold money for a transactions motive--that is, because its use as a medium of exchange makes the exchange of goods less costly.

There have been a number of attempts to represent the transaction services of money. It has been assumed a priori that the exchange of money involves lower (unspecified) transaction costs than futures contracts or the exchange of goods (Heller and Starr, 1976); that only money can buy goods (Grandmont and Younes, 1972; Lucas, 1980); or that balances of money are an argument in agents' utility (Sidrausky, 1967; Samuelson, 1968). The implications of each of these models for such issues as the optimal quantity of money and the effects of government bonds differ from those of the overlapping generations model. This has led Tobin, Hahn, and McCallum to contend: i) that the differences are caused by a failure of the overlapping generations model to adequately model the medium-of-exchange role for money; and ii) that the implications of the overlapping generations model should be discounted for this reason. However, in none of these models is the demand for money derived from optimizing behavior in a completely specified physical environment. Therefore, the differences in the implications of these models and models of overlapping generations cannot be traced to differences in environmental assumptions.

In section I of this paper I specify a physical environment in which agents may choose to hold fiat money because of its usefulness as a medium of exchange. Equilibria with valued fiat money display these four phenomena: i) goods are not traded directly for goods; ii) fiat money facilitates multilateral exchange by eliminating the need to search for a "double coincidence of wants"; iii) transaction costs are lower in monetary equilibria than in nonmonetary equilibria; iv) fiat money may have value even if its rate of return is exceeded by those of other assets. In section II I show that the implications of my model do not differ from those of the basic overlapping generations model with respect to the optimum quantity of money. I conclude in section III that it is not justified to ignore the implications of the overlapping generations model simply for its alleged lack of a medium-of-exchange role for money.

I. THE MODEL:

Preferences and endowments:

The model is a version of Samuelson's (1958) overlapping generations model of two-period lived agents. There are J different kinds of consumption goods, each indexed by a number from 1 to J with $J > 2$. The utility of any agent is defined on R^{2J+1}. There are J^2 types of agents $(i,j) \in \Omega^2$, where $\Omega \equiv (1,2,3,\ldots,J)$. An agent h of type (i,j) is born with an endowment of good i only, cares only for good i when he is young, and cares only for good j when he is old. He also cares (in a negative sense) on his total lifetime transaction costs (described below). The utility of a type (i,j) agent can therefore be represented as $U_i(C_{1i}) + V_j(C^i_{2j}) - S$, where C_{1i} represents the units of good i consumed when young, C^i_{2j} represents the units of good j consumed when old, and S represents total lifetime transaction costs. It is assumed that U and V are twice continuously differentiable and that for $i \in \Omega$, $U_i' > 0$, $V_j' > 0$, $U_i'' < 0$, $V_j'' < 0$ $\lim_{C\to 0} U'(C) = \infty$, and both $CV'(C)$ and $\lim_{C\to 0} CV'(C)$ are positive and bounded from zero. (The primes denote derivatives.) When young, an agent h does not exactly know his type. He knows his endowment (Y_i) and the type of good he wishes to consume when young (U_i) but not the type of good he will wish to consume when old [i.e., he knows i but not j]. When young, he has a uniform probability measure on j, independent of i. When old, he knows his type j exactly. Agents are assumed to maximize expected utility. For agents endowed with good i when young expected utility is

$$U_i(C_{1i}) + \frac{1}{J}\sum_{j=1}^{J} V_j(C^i_{2j}) - \bar{S} \qquad : \; \Omega \times R^{j+1}$$

where $\bar{S}$ is expected lifetime search costs. (See appendix B for a description of the economy with a more general utility function.)

single bilateral meeting, the mean number of search attempts is $1/\rho$.

Let us distinguish two kinds of search strategies, which I will call one-sided and two-sided searches. In a two-sided search, exchange takes place only when each party wants to consume what the other party is offering. This is the double coincidence of wants of Jevons. The probability of a match on any single search attempt is $\rho = 1/J^2$ and the expected cost of searching until a match is found is $\gamma(J^2 - 1)$. Note that the expected number of searches is $J^2 - 1$ because there is a chance that they will not have to search at all (the case of $i = j$).

In a one-sided search, one party is looking to trade for a specific good but the other party will accept any good (or fiat money if it has value) in an exchange. In a one-sided search, $\rho = 1/J$. Expected search costs equal γJ if the agent begins his search holding fiat money and $\gamma(J-1)$ if he begins his search holding a good. (The difference in the expected search costs results from the possibility that an agent may wish to consume the goods he starts with before any search is undertaken.)

When a coincidence of wants is found and exchange takes place, the transfer of the ownership of each type of good between two islands entails a "transfer cost" of λ units of effort per buyer in addition to any search costs. This cost may be thought of as either the cost of physically transferring the good or the cost of ascertaining the quantity or quality of the goods transferred.

Fiat money consists of intrinsically useless, storable pieces of paper costlessly created by the government. It is costless to hold, count, and transfer. Counterfeits can be costlessly identified. As a result of these properties, fiat money has no transfer cost attached to it. While the exchange of goods for goods between two islands costs a total of $2N\lambda$, the trade of fiat money for goods between two islands costs $N\lambda$. At the start of the initial

Agents are located on a countably infinite number of spatially separated islands. On each island there are N agents. All the agents on any single island are identical in age and type. Islands of agents of different types and ages are equally represented in the economy.

All goods are storable over time. For each unit of good i $(i = 1,2,\ldots,J)$ that is stored at t, an agent receives X units of good i at t+1.

Transaction technology:

Agents conducting exchanges with other islands encounter two types of costs: the cost of locating an island willing to make an exchange ("search costs") and the cost of transferring goods from one island to another ("transfer costs"). The sum of these two represents the total lifetime effort expended (or leisure lost) in transaction activities and will be referred to as "transaction costs."

Agents wishing to trade must expend an effort, costly in utility, in order to communicate with another island. An island that makes this effort is then paired at random with another island undertaking a search. Once joined, the two islands describe to each other the goods they have and those they desire. If there is no coincidence of wants, the islands continue their search with another random pairing. Each iteration of this search process uses up γ units of effort (or leisure or "shoe leather") for each agent of both islands involved in the search. An island can costlessly restrict its search to islands of the old or islands of the young. The decision to search again is independent of the number of searches already undertaken because utility is a linear function of total transaction costs. The number of search attempts before a match is found has a geometric distribution. If ρ is the probability of finding a match in any period (period 1) all money is in the hands of the old. Each island of old agents in period 1 begins with M units of fiat money.

Exchanges between two islands are made at competitively determined prices. Each agent is free to purchase as much as he desires at prices he takes as given. Prices are free to adjust to clear markets.

Equilibria:

A search equilibrium is defined to be a set of search strategies, consumption decisions, and prices such that: (i) each agent endowed with good i (i = 1,2,...,J) chooses his contingent consumption pattern $(C_{1i},\{C_{2j}\}_{j=1}^{J})$ to maximize his expected utility taking prices as given; (ii) agents have rational expectations; (iii) markets clear; and (iv) no agent desires to switch his search strategy taking as given the search strategies of others.

Let us first consider search equilibria in the absence of valued fiat money ("nonmonetary equilibria"). Three search strategies are conceivable. Old agents may seek a trade with old agents, old agents may seek a trade with young agents or agents may choose not to search at all. This last possibility occurs when the expected disutility of search is so great that agents decide to forego second period consumption (i.e., when $U_i(Y_i) > U_i(C^*_{1i}) + \frac{1}{J}\sum_{j=1}^{J} V_j(C^*_{2j}) - \overline{S}$, where C^*_{1i} and C^*_{2j} represent the level of C_{1i} and C_{2j} that would be chosen if the agent undertook a search). Because such an autarky is uninteresting I will assume that $\overline{S}$ is not so great that it overcomes the desire to consume something when old.

In the first strategy, which I will call a "direct barter equilibrium," agents who wish to consume when old store some of their endowment until they are old and know which good they wish to consume. Then they search among other old agents for a "double coincidence of wants," an island of agents who want what they have and have what they want. When such an island is found, agents trade at the market-clearing competitive prices, $\{d_{ij}\}$. Let d_{ij} represent the price of a unit of good i in units of good j.

The problem of an individual endowed with good i is to choose C_{1i} and C_{2j}^{i} to maximize $U_i(C_{1i}) + \frac{1}{J}\sum_{j=1}^{J} V_j(C_{2j}^{i}) - \bar{S}$ subject to

(1) $Y_i \geq C_{1i} + C_{2j}^{i}/d_{ij}X$ for $i = 1,2,\ldots,J$.

Therefore, the equilibrium conditions can be summarized in the first order conditions (2) and the market clearing conditions (3) for $i,j = 1,2,\ldots,J$:

(2) $-U_i'(C_{1i}) + x\frac{1}{J}\sum_{j=1}^{J} d_{ij}V_j'[(Y_i - C_{1i})d_{ij}] = 0$

(3) $N(Y_i - C_{1i})d_{ij} = N(Y_j - C_{1j})$.

Note that because preferences and the environment are the same in all periods, the direct barter equilibrium features constant consumption choices and prices. The expected lifetime transaction costs equal $(J^2 - 1)\gamma + \lambda$, the costs of a single two-sided search and a single transfer of goods.

The other possible search strategy is for the old to search for a young person endowed with the type of good desired by the old. The young agent stores the good he receives in order to use it to trade when old with the next generation for the good he desires. Let us call this "indirect barter." In the environment presented here, this is not an equilibrium because the young have no incentive to undertake a costly search before they know which good they desire to consume when old. As a result, this search pattern can only be an equilibrium if it is possible for the old to assume all transaction costs. In this case, expected lifetime transaction costs equal $2(J - 1)\gamma + 2(1 - 1/J)\lambda$, the cost of two one-sided searches and two exchanges of goods. The budget constraints, first order conditions, and market clearing conditions are the same as those of the direct barter equilibrium.

Finally, let us consider a search equilibrium with valued fiat money (a "monetary equilibrium"). To consume when old, young agents seek out old agents with whom they can trade some of this endowment for fiat money. The balances of fiat money are held until agents are old, at which time they trade the fiat money for the type of good they desire. As always, present and future prices are consistent with market clearing in every period and are taken as given by each individual agent.

To study this economy under a variety of monetary growth paths, let us assume that the stock of fiat money on each island inhabited by old people, $M(t)$, follows the law of motion $M(t) = zM(t-1)$ for all $t > 1$, where z is a positive constant. Changes in the money stock are accomplished through a lump-sum transfer (tax, if negative) of $T(t) = M(t)(z-1)/N$ units of fiat money to each old agent in each period t. (This is the set of monetary policies considered in Wallace, 1980.)

The problem of an individual agent born at t and endowed with good i is therefore to choose his contingent consumption pattern $(C_{1i}(t), \{C^i_{2j}(t)\}^J_{j=1})$ and his nominal balances of fiat money $(m_i(t)$ to maximize his expected utility subject to his budget constraints:

$$(4) \quad Y_i \geq C_{1i}(t) + \frac{m_i(t)}{P_i(t)}$$

and

$$(5) \quad \frac{m_i(t) + T(t)}{P_j(t+1)} \geq C_{2j}(t)$$

with $C_{1i}(t) \geq 0$, $C_{2j}(t) \geq 0$ for all j, and $m_i(t) \geq 0$. $P_j(t)$ denotes the price of good j at t in terms of units of fiat money.

The first order condition satisfied by a solution with positive $C_{1i}(t)$ and finite $P_j(t)$ (for all j) is

$$(6) \quad -\frac{1}{P_i(t)} U_i'(Y_i - \frac{m_i(t)}{P_i(t)}) + \sum_{j=1}^{J} \frac{1}{J} \frac{1}{P_j(t+1)} V_j'(\frac{m_i(t) + T(t)}{P_j(t+1)}) = 0$$

Note that search costs are independent of the chosen size of real balances and so do not enter into the first order condition.

The market clearing condition at t for good i is

$$(7) \quad N(Y_i - C_{1i}(t))P_i(t) = M(t) \qquad \text{for } i = 1,2,\ldots,J$$

It follows that

$$(8) \quad N(Y_i - C_{1i}(t))d_{ij}(t) = N(Y_j - C_{1j}(t)),$$

which is identical to the market clearing conditions under barter.

The rate of return on fiat money can now be written as

$$(9) \quad \frac{P_i(t)}{P_i(t+1)} = \frac{M(t)/N(Y_i - C_{1i}(t))}{M(t+1)/N(Y_i - C_{1i}(t+1))}$$

Because agents face the same problem in every period I will look only at stationary equilibria, in which agents' decisions on $C_{1i}(t)$ are not time dependent. The rate of return of fiat money in a stationary equilibrium is $1/z$.

Note that the relative price at t of good i in units of good j is $d_{ij}(t) = P_i(t)/P_j(t)$. In the equilibrium described, relative prices are constant ($d_{ij}(t) = d_{ij}$ for $t \geq 1$). We are now able to rewrite the individual's first order condition (6) as:

$$(10)\quad -U_i' + \frac{1}{zJ} \sum_{j=1}^{J} d_{ij} V_j' = 0$$

Note that for a fixed stock of fiat money ($z = 1$) and storage with no physical appreciation or depreciation ($x = 1$), the first order conditions (as well as the market clearing conditions) are exactly those of the barter equilibrium (3). In appendix A I establish sufficient conditions for the existence of such equilibria.

When $x = 1$ and $z = 1$, the sole difference between the monetary and non-monetary equilibria is the expected value of lifetime transaction costs. The expected transaction costs of the monetary equilibrium are those of two one-sided searches and one receipt of goods: $2\gamma J + \lambda$. For $J > 2$, the monetary equilibrium features lower transaction costs than the barter equilibrium $[(J^2 - 1)\gamma + \lambda]$ for exactly the reason discussed by Jevons: when the number of goods is large, the search for a double coincidence of wants (a two-sided search) is more expensive than the combination of a search for a buyer who will pay money for your goods and a search for a seller of the goods you desire (two one-sided searches).

The monetary equilibrium may also have lower expected transaction costs than indirect barter (which were $2(J - 1)\gamma + 2(1 - 1/J)\lambda$). The use of fiat money instead of indirect barter reduces the expected number of times goods, which are more costly than fiat money to transfer, must be exchanged. Indirect barter has lower expected search costs because of the chance that agents turn out to desire the good they have stored. The larger the number of types of goods (J), the less significant is this possibility. The exact necessary condition for monetary exchange to have lower transaction costs than indirect barter is $\lambda(1 - 2/J) > 2\gamma$.

In describing the monetary and nonmonetary equilibria, I have so far ignored the fourth equilibrium condition: that given the search strategy of others, no agent desires to switch his own search strategy. Consider first the case of direct barter. (Indirect barter has already been shown not to be an equilibrium.) If all others are following the search pattern of direct barter (old search for exchanges among the old and the young do not search), and if it is costly to undertake an unsuccessful search, no single old agent has the incentive to search for a young agent for a monetary trade because no young agents are searching. For similar reasons no single young agent has the incentive to search for an exchange with an old agent. In the case of monetary exchange it is also true that no agent desires to switch his search strategy. If all others in his generation trade for fiat money, a young person has no incentive to store goods until old and then look for a (nonexistent) barter trade. Also, if all the young seek a trade for fiat money, an old agent with fiat money has every incentive to sell his fiat money.

Note that in each case, no single agent will change his search strategy simply because if he is the only one to do so, he will not find a trading partner. For this reason the incentive not to deviate from the trading pattern followed by all others is unrelated to the relative rates of return and transaction costs of the two strategies. Even if one strategy dominates another, no single isolated agent can arrange for the economy to follow any particular trading pattern. Only if communication among islands is allowed would it be possible to agree to follow the exchange pattern that maximizes utility.

It is nevertheless possible to Pareto rank the alternative equilibria. When the stock of fiat money is fixed ($z = 1$) and goods have no net physical rate of return ($x = 1$), the monetary equilibrium is Pareto superior to the direct barter equilibrium. The old in the initial period are better off because their initial fiat money balances have value and all other generations are better off because their expected lifetime transaction costs are lower than in the barter equilibrium but there is no difference in rates of return. Of course, if the rate of return on storage should exceed the rate of return on fiat money ($x > 1/2$), the benefits of lower transaction costs may be offset for fiat money's lower rate of return. The old in the initial period will always prefer the equilibrium with valued fiat money.

It should be noted that fiat money can be a useful medium of exchange in this economy because the government has the ability to print pieces of paper that are noncounterfeitable and costless to hold, count, or transfer. There is no reason to believe that these pieces of paper will lose these properties if they are called "bonds" instead of "money." Therefore, as in the simpler overlapping generations models [Wallace (1980)], unless the government has some reason to lessen the liquidity of the paper called bonds, bonds and money are redundant. This redundancy is of course overlooked by models that start by assuming that money gives utility but bonds do not or that only money may be used to purchase goods.

II. THE OPTIMUM QUANTITY OF MONEY

In the basic ovdrlapping generations model, the stationary equilibrium with a fixed stock of fiat money maximizes steady-state lifetime utility. This result is also an implication of the model of this paper, as shown in the following proposition. It is established for more general utility and transaction cost functions in appendix B.

Proposition: A fixed stock of fiat money yields the greatest steady-state lifetime utility in the class of monetary policies with a constant rate of growth of the stock of fiat money achieved by equal lump-sum transfers to each agent.

Proof: Define $W_i(z)$ as the lifetime utility of an agent endowed with good i in a stationary monetary equilibrium for a given z. Define also $q_i(z)$ as the value of the money balances in terms of good i chosen by an agent endowed with good i for a given z. Note that in a stationary equilibrium,

$$C_{2j}(t) = \frac{m_i(t) + T(t)}{P_j(t+1)}$$

$$= \frac{m_i(t) + (z-1)\frac{M(t)}{N}}{P_j(t+1)}$$

$$= \frac{zm_i(t)}{P_j(t+1)}$$

$$(11) \quad = \frac{m_i(t)}{P_j(t)} = \frac{m_i(t)}{P_i(t)} d_{ij} = q_i(z)d_{ij}$$

Therefore, $W_i(z)$ can be formulated as

$$(12) \quad W_i(z) = U_i(Y_i - q_i(z)) + \frac{1}{J}\sum_{j=1}^{J} V_j(q_i(z)d_{ij}) - \overline{S}.$$

Therefore

$$(13) \quad W_i'(z) = -U_i'q_i' + \frac{1}{J}\sum_{j=1}^{J} V_i'q_i'd_{ij}$$

From (10),

$$(14) \quad W_i'(z) = -U_i'q_i' + zU_i'q_i' = U_i'q_i'(-1 + z)$$

To evaluate this we must find q_i'. The function $q_i(z)$ satisfies the agent's first order condition (10), rewritten here as a function of q_i and z:

$$(15)\quad F(q_i,z) = -U_i'(Y_i - q_i) + \frac{1}{zJ}\sum_{j=1}^{J} d_{ij}\, V_j'(q_i d_{ij})$$

Then by the implicit function theorem,

$$(16)\quad q_i'(z) = \frac{-\dfrac{\partial F}{\partial z}}{\dfrac{\partial F}{\partial q_i}} = \frac{\dfrac{1}{z^2 J}\displaystyle\sum_{j=1}^{J} d_{ij}\, V_j'}{U_i'' + \dfrac{1}{zJ}\displaystyle\sum_{j=1}^{J} (d_{ij})^2 V_j''}$$

which is negative by the assumptions on U_i and V_j.

Therefore, $W_i'(z)$ is positive for $z < 1$, zero at $z = 1$ and negative for $z > 1$, indicating that $W_i(z)$ has a unique global maximum at $z = 1$. ∇

The above proposition establishes that this model does not duplicate the central welfare result of Friedman (1969), McCallum (1983) and the others who have included money balances in the utility function--that only deflation of the money stock is optimal. The principal justification for the inclusion of money balances in the utility function is that the holding of fiat money reduces transaction costs and thereby increases utility (see Samuelson, 1968). Although holding money leads to increased utility in the environment I present, it is a constant stock of fiat money, not deflation, that maximizes steady-state utility.

As in other models of overlapping generations, a fixed stock of fiat money is optimal because only then does the rate of return on fiat money accurately reflect the rate of growth of the economy [Samuelson's (1958) "biological interest rate"]. Only then does the individual's budget set reflect the set of steady-state feasible allocations so that the optimal point in the budget set is also the best of all feasible points. [See Wallace (1980)].

An examination of the proof of this proposition reveals that the optimal rate of return on fiat money is independent of the rate of return of storage/capital (x). If the rate of return on storage is sufficiently high, generations born after period 0 may prefer a barter equilibrium, but among equilibria with valued fiat money, a fixed stock of fiat money is optimal regardless of the rate of return on storage.

The source of the disagreement about the optimum quantity of money seems to lie in the marginal effect of holding greater real money balances. In Friedman's model of infinitely lived agents and McCallum's overlapping generations model, an increment in real balances reduces transaction costs up to the point of satiation. In my model, it is the economy-wide adoption of money that reduces transaction costs; incremental increases in real balances only increase the purchases made. That is, the use of fiat money at the grocery store will be less cumbersome than barter, but using an additional dollar will only increase the amount purchased, not the ease of shopping.

I have described a world in which fiat money serves as a medium of exchange but increments in real money balances do not reduce transaction costs. The obvious question is now whether there might be economies in which increments might actually reduce the costs of conducting transactions. Cases that come to mind would be when greater balances imply that more individuals now accept fiat money in transactions or that individuals now accept fiat money in more types of transactions. In either of these cases, additions to real money balances reduce transaction costs by increasing the number of successful matches. Further investigation may reveal that such an economy may share the welfare properties of the Friedman model. However, neither case applies to economies in which all exchanges involve fiat money. In such economies increments in money balances will affect utility through the increases in the consumption of goods purchased with fiat money but will not affect utility through declines in transaction costs. This suggests that challenges to the optimality of a fixed stock of fiat money should be pursued in models where agents face a choice at the margin of the portion of exchange that will involve fiat money.

III. CONCLUSION

The dependence of the overlapping generations model's implications on the absence of a stronger medium-of-exchange role for money, although asserted by Hahn, McCallum, and Tobin, has not been demonstrated in any entirely explicit physical environment. To provide a first test of this dependence, the model of this paper displays the phenomena that they argue are essential features of monetary economies but missing in overlapping generations models: (i) goods are not traded directly for goods; (ii) money facilitates multilateral exchange by eliminating the need to search for a "double coincidence of wants"; (iii) transaction costs are lower in monetary equilibria; (iv) fiat money may have value even if its rate of return is exceeded by those of other assets. Nevertheless, the implications of the model concerning the optimal quantity of money and the redundancy of government bonds and money are those of the overlapping generations model, not those of models like the ones with money in the utility function that claim, but do not explicitly model, a medium-of-exchange role for money.

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APPENDIX A

Existence of a stationary equilibrium:

Let $x = 1$ and $z = 1$. (This proof may easily be extended to other values of x and z.) Define $k_j \equiv Y_j - C_{1j}$ for each $j \in \Omega \equiv (1,2,3,...,J)$ and define the $J \times 1$ vector $K = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_J \end{bmatrix}$

The equilibrium conditions, (6) and (7), of a stationary monetary equilibrium [as well as the equilibrium conditions, (2) and (3), of the stationary barter equilibrium] can now be combined and written as

$$(A.1) \quad -U_i'(Y_i - k_i)k_i + \frac{1}{J} \sum_{j=1}^{J} k_j V_j' (k_j) = 0 \qquad \forall_i \in \Omega$$

or

$$(A.2) \quad k_i = \phi^i(K) \equiv \frac{\sum_{j=1}^{J} k_j V_j' (k_j)}{JU_i'(Y_i - k_i)} \qquad \forall_i \in \Omega$$

Since $V_j'(\cdot)$ and $U_i'(\cdot)$ are continuous and $U_i'(\cdot) > 0$ for $k \in Y \equiv \bigotimes_{j=1}^{J} (0,Y_j)$, $\phi^i(K)$ is well-defined and continuous on Y.

For $\delta > 0$ with $\delta < \min\{Y_j\}_{j=1}^{J}$ and $0 < \varepsilon < \min\{Y_j - \delta\}_{j=1}^{J}$, define

$$B_{\varepsilon,\delta} \equiv \bigotimes_{i=1}^{J} [\varepsilon, Y_i - \delta]$$

Define $\hat{\phi}^i_{\varepsilon,\delta}(\cdot) : B_{\varepsilon,\delta} \to B_{\varepsilon,\delta}$ as follows:

$$\hat{\phi}^i_{\varepsilon,\delta}(K) = \begin{cases} \varepsilon & \text{if } \phi^i(K) \leq \varepsilon \\ \phi^i(K) & \text{if } \varepsilon < \phi^i(K) < Y_i - \delta \\ Y_i - \delta & \text{if } \phi^i(K) \geq Y_i - \delta \end{cases}$$

Notice that $\hat{\phi}^i_{\varepsilon,\delta}$ is continuous. Define $\Phi(K) : Y \to R^J_+$ and $\hat{\Phi}_{\varepsilon,\delta}(K) : B_{\varepsilon,\delta} \to B_{\varepsilon,\delta}$ as follows:

$$\Phi(K) \equiv \begin{bmatrix} \phi^1(K) \\ \phi^2(K) \\ \vdots \\ \phi^J(K) \end{bmatrix} \qquad \hat{\Phi}_{\varepsilon,\delta}(K) \equiv \begin{bmatrix} \hat{\phi}^1_{\varepsilon,\delta}(K) \\ \hat{\phi}^2_{\varepsilon,\delta}(K) \\ \vdots \\ \hat{\phi}^J_{\varepsilon,\delta}(K) \end{bmatrix}$$

A stationary equilibrium exists on the open set $Y \equiv \bigotimes_{i=1}^{J} (0, Y_i)$ if there exists some $K^* \in Y$ such that $K^* = \Phi(K^*)$. Because Φ is defined on an open set, and does not map the domain back on itself, we cannot directly apply the Brouwer fixed point theorem. For this reason the vector of functions $\hat{\Phi}_{\varepsilon,\delta}(\cdot)$ was constructed. The proof proceeds by establishing that there exists a fixed point of the function $\hat{\Phi}_{\varepsilon,\delta}$, where ε and δ can be chosen to be any positive numbers satisfying $\delta < \min\{Y_j\}_{j=1}^{J}$ and $\varepsilon < \min\{Y_j - \delta\}_{j=1}^{J}$. Then it is shown that when sufficiently small (but positive) values of ε and δ are chosen, the fixed point is such that $\hat{\Phi}^i_{\varepsilon,\delta}(\bar{K}) = \Phi^i(\bar{K})$ $\forall$ $i \in \Omega$, hence $\bar{K}$ is a fixed point of Φ, which demonstrates the existence of a stationary equilibrium. These steps are detailed in Lemmas 1, 2, and 3.

<u>Lemma 1</u>: For any $\delta > 0$ with $\delta < \min\{Y_j\}_{j=1}^{J}$ and $0 < \varepsilon < \min\{Y_j - \delta\}_{j=1}^{J}$ there exists $\bar{K} \in B_{\varepsilon,\delta}$ such that $\bar{K} = \hat{\Phi}_{\varepsilon,\delta}(\bar{K})$.

Define $\bar{k}_i$ as the i^{th} element of the fixed point $\bar{K}$.

<u>Lemma 2</u>: There exists some $\delta^* > 0$ with $\delta^* < \min\{Y_j\}_{j=1}^{J}$ such that for all ε satisfying $0 < \varepsilon < \min\{Y_j - \delta^*\}_{j=1}^{J}$, $\bar{K} = \hat{\Phi}_{\varepsilon,\delta^*}(\bar{K})$ implies that for each $i \in \Omega$, $\bar{k}_i \neq Y_i - \delta^*$.

Lemma 3: There exists ε^* satisfying $0 < \varepsilon^* < \min\{Y_j - \delta^*\}_{j=1}^{J}$ such that $\bar{K} = \hat{\Phi}_{\varepsilon^*,\delta^*}(\bar{K})$ implies that for each $i \in \Omega$ $\bar{k}_i \neq \varepsilon^*$.

It follows from Lemmas 2 and 3 that there exist ε^* and δ^* such that any fixed point of $\hat{\Phi}_{\varepsilon^*,\delta^*}$ is also a fixed point of Φ. Since Lemma 1 establishes the existence of a fixed point for $\hat{\Phi}_{\varepsilon^*,\delta^*}$, the existence of a stationary equilibrium (a fixed point of Φ) is established.

Proof of Lemma 1: The set $B_{\varepsilon,\delta}$ is a non-empty, compact, and convex subset of R^J. Since $\hat{\Phi}_{\varepsilon,\delta}(\cdot)$ is a continous function mapping $B_{\varepsilon,\delta}$ into $B_{\varepsilon,\delta}$, it has a fixed point $\bar{K} = \hat{\Phi}_{\varepsilon,\delta}(\bar{K})$ (by Brouwer's theorem).

Proof of Lemma 2: For all $i \in \Omega$, $\lim_{k_i \to Y_i} \phi^i(K) = 0$ for all $k_j \varepsilon (0, Y_j)$ (for all $j \neq i$) since $\lim_{k_i \to Y_i} U_i'(Y_i - k_i) = \infty$ and $\frac{1}{J} \sum_{j=1}^{J} k_j V_j'(k_j)$ is bounded for all $K \varepsilon Y$.

Therefore there exists $\delta^* > 0$ such that for any fixed $i \in \Omega$, if $K \varepsilon Y$ and $k_i = Y_i - \delta^*$, then $\phi^i(K) < Y_i - \delta^*$. Thus for any positive $\varepsilon < \min\{Y_j - \delta^*\}_{j=1}^{J}$ and for any fixed $i \in \Omega$, if $K \varepsilon B_{\varepsilon,\delta^*}$, and $k_i = Y_i - \delta^*$, then $\hat{\phi}^i_{\varepsilon,\delta^*} \neq Y_i - \delta^*$. Thus if $\bar{K}$ is a fixed point of $\hat{\Phi}_{\varepsilon,\delta^*}$, then for each $i \in \Omega$, $\bar{k}_i \neq Y_i - \delta^*$.

Proof of Lemma 3: By assumption, $k_j V_j'(k_j)$ is positive for $k_j > 0$ and $\lim_{k_j \to 0} k_j V_j'(k_j)$ is bounded above zero for all $j \in \Omega$. Therefore, for each $i \in \Omega$, $\lim_{k_i \to 0} \phi^i(K) = \lim_{k_i \to 0} \frac{\sum_{j=1}^{J} k_j V_j'(k_j)}{JU'(Y_i - k_i)}$ is bounded above 0 for all $k_j \in (0, Y_j)$ $\forall j \neq i$. Then there exists ε^* satisfying $0 < \varepsilon^* < \min\{Y_j - \delta^*\}_{j=1}^{J}$ such that for any fixed $i \in \Omega$, if $K \in Y$ and $k_i = \varepsilon^*$, then $\phi^i(K) > \varepsilon^*$. Thus for any fixed $i \in \Omega$, if $K \in B_{\varepsilon^*,\delta^*}$ and $k_i = \varepsilon^*$, then $\hat{\phi}^i_{\varepsilon^*,\delta^*}(K) \neq \varepsilon^*$. Thus if $\bar{K}$ is a fixed point of $\hat{\phi}_{\varepsilon^*,\delta^*}$ then for each $i \in \Omega$, $\bar{k}_i \neq \varepsilon^*$.

Appendix B

For expositional simplicity I proved in the main body of this paper the optimality of a fixed money stock only for the case of additively separable utility functions and a fixed transfer cost. Consider now a more general case where for $i,j, \epsilon\ \Omega$, the utility of an agent of type (i,j) is given by

$$U_{ij}(C_{1i}, C_{2j}, S)$$

and the transfer cost functions $\lambda_j(g_j)$ for $j = 1, 2, \ldots, J$, where g_j is the quantity transferred of good j. These functions are twice-continuously differentiable. The utility functions are increasing in the first two arguments and decreasing in the third. They need not be additively separable. As before, an agent of type (i,j) wishes to consume when young only the type of good with which he is endowed. He does not know j, the type of good he wishes to consume when old, until he is old.

In a stationary monetary equilibrium, the expected utility of an agent endowed with the good can be expressed as the following function of real money balances in terms of good i $(q_i = m_i(t)/P_i(t))$:

$$V(q_i) = \frac{1}{J}\sum_{j=1}^{J} U_{ij}[y_i - q_i,\ q_i\frac{d_{ij}}{z} + T_j,\ 2\gamma J + \lambda_j(q_i\frac{d_{ij}}{z} + T_j)]$$

where $T_j = \frac{T(t)}{P_j(t+1)}$, the value of the money transfer in terms of good j.

Assume that the functions U_{ij} and λ_j are such that $V''(q_i) < 0$ for $q_i > 0$ (i.e., that there exists a monetary equilibrium that meets the standard second order sufficiency conditions for a maximum). In addition to the assumptions made above, a set of strongly sufficient conditions for $V'' < 0$ comprises:

$$\frac{\partial^2 U_{ij}}{\partial C_{1i}\, \partial C_{2j}} \geqslant 0$$

$$\frac{\partial^2 U_{ij}}{\partial C_{1i}^2} < 0$$

$$\frac{\partial^2 U_{ij}}{\partial C_{2j}^2} < 0$$

$$\frac{\partial^2 U_{ij}}{\partial C_{1i}\,\partial S} \geq 0$$

$$\frac{\partial^2 U_{ij}}{\partial C_{2j}\,\partial S} \geq 0$$

and that λ'' and $\frac{\partial^2 U_{ij}}{\partial S^2}$ agree in sign.

The first order condition for a stationary monetary equilibrium in this case can be written:

(B.1) $-Eu_{i1} + \frac{1}{z}[Eu_{i2} + Eu_{i3}] = 0$

where $Eu_{i1} \equiv \frac{1}{J} \sum_{j=1}^{J} \frac{\partial U_{ij}(C_{1i}, C_{2j}, S)}{\partial C_{1i}}$

$$Eu_{i2} \equiv \frac{1}{J} \sum_{j=1}^{J} d_{ij} \frac{\partial U_{ij}(C_{1i}, C_{2j}, S)}{\partial C_{2j}}$$

$$Eu_{i3} \equiv \frac{1}{J} \sum_{j=1}^{J} \lambda_j' d_{ij} \frac{\partial U_{ij}(C_{1i}, C_{2j}, S)}{\partial S}$$

The above assumptions are sufficient to establish more generally the optimality of a fixed stock of fiat money. The proof proceeds in the manner of the proof presented in Section II. Following the steps of that proof leads us to an expression [equivalent to equation (11)] for the marginal effect on steady-state lifetime utility of a change in z:

(B.2) $W_i'(z) = Eu_{i1} q_i'(z)(-1 + z)$

Again, $q_i'(z)$ can be found by applying the implicit function theorem to the agent's first order condition (A.1). Note that $C_{2j} = q_i d_{ij}$ from equation (8) so that Eu_{i1}, Eu_{i2}, and Eu_{i3} are not direct functions of z. Denote the first order condition as $F(q_i, z) = 0$.

$$\text{Then } q_i'(z) = \frac{-\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial q_i}}$$

The denominator equals $V''(q_i)$, which is negative by the agent's second order condition for a maximum. Therefore, $q_i'(z)$ has the same sign as $\frac{\partial F}{\partial Z}$.

$$\frac{\partial F}{\partial z} = \frac{-1}{z^2}[Eu_{i2} + Eu_{i3}] = \frac{-1}{z^2} Eu_{i1} \qquad \text{from (B.1)}$$

Eu_{i1} is a sum of positive first derivatives. Therefore, $q_i'(z) < 0$ and $W_i'(z)$ is positive for $z < 1$, zero at $z = 1$, and negative for $z > 1$, indicating that $W_i(z)$ has a unique global maximum at $z = 1$.