Approximate Asymptotic P-Values for Structural Change Tests

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Abstract

Numerical approximations to the asymptotic distributions of recently proposed tests for structural change are presented. This enables easy yet accurate calculation of asymptotic p-values.

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1 Introduction

Recently, Andrews (1993) found the asymptotic distribution of a wide class of tests for structural change in econometric models. In a related paper Andrews and Ploberger (1994) developed an analogous class of tests with stronger optimality properties. The asymptotic distributions of the tests are non-standard, and depend upon two parameters: the number of parameters tested, and the range of the sample which is examined for the break date.

While a selected set of asymptotic critical values have been tabulated, the non-standard nature of these distributions means that p-values cannot be calculated from previously published information. This is a disadvantage in applications, since applied economists are frequently more interested in p-values than in classical Neyman-Pearson significance tests.

This paper presents computationally convenient approximations $p^*(x)$ to the asymptotic p-value functions $p(x)$ for the Andrews and Andrews-Ploberger asymptotic distributions. The approximation methods proposed here may find use in a wide range of non-standard statistical contexts.

Pervious attempts to estimate p-value functions for non-standard test statistics in econometrics were made by Hansen (1992) and MacKinnon (1994). Hansen (1992) set $p^*(x)$ to be a simple polynomial

$$\alpha_v(x \mid \theta) = \theta_0 + \theta_1 x + \cdots \theta_v x^v \tag{1}$$

and fitted the coefficients by a least squares polynomial regression of upper percentiles on quantiles. MacKinnon (1994) improved the approach by setting $p^*(x) = \Phi(\alpha_v(x \mid \theta))$, where $\Phi(\cdot)$ is a leading distribution function of interest (in his case, the standard normal). He fitted the coefficients by a least squares polynomial regression of $\Phi^{-1}(p)$ (where p are upper percentiles) on quantiles.

The methods presented in this paper extend this literature. Similarly to MacKinnon (1994), we set $p^*(x) = \Phi(\alpha_v(x \mid \theta) \mid \eta)$, where $\alpha_v(x \mid \theta)$ is a polynomial and $\Phi(z \mid \eta)$ is a leading distribution of interest. One difference is that we allow the distribution to depend on an unknown parameter η. To fit the approximation, we use a weighted loss function over the p-value space. We find that our approximations are extremely accurate, even though our models are quite parsimonious.

In independent and complementary work, Adda and Gonzalo (1995) use the semi-nonparametric (SNP) approach of Gallant and Nychka (1987) to approximate the asymptotic distribution of the Dickey-Fuller test. While their approximating p-value function is different, their method to fit the coefficients is quite similar to ours.

In Section 2, we review the tests and distribution theory of Andrews (1993) and Andrews and Ploberger (1994). Section 3 presents the methodology used to approximate the p-value function. Section 4 presents the approximations. A Gauss program which computes the test statistics and asymptotic p-values is available on request from the author.

2 Tests for Structural Change

An $m \times 1$ parameter β, describing some aspect of a time series x_t, takes the value β_1 for $t < k$ and the value β_2 for $t \geq k$, where $m \leq k \leq n - m$. Let $F_n(k)$ denote a Wald, LM or LR statistic of the hypothesis of no structural change ($\beta_1 = \beta_2$) for given k. When k (the date of structural change) is known only to lie in the range $[k_1, k_2]$, the Quandt or "Sup" test statistic is

$$\text{SupF}_n = \sup_{k_1 \leq k \leq k_2} F_n(k).$$

The Andrews and Ploberger (1994) "Exp" and "Ave" tests are

$$\text{ExpF}_n = \ln\left(\frac{1}{k_2 - k_1 + 1}\sum_{t=k_1}^{k_2} \exp\left(\frac{1}{2}F_n(k)\right)\right),$$

and

$$\text{AveF}_n = \frac{1}{k_2 - k_1 + 1}\sum_{t=k_1}^{k_2} F_n(k).$$

As shown in Andrews (1993) and Andrews-Ploberger (1994), under a wide set of regularity conditions, these statistics have the asymptotic null distributions

$$\text{SupF}_n \to_d \text{SupF}(\pi_0) = \sup_{\pi_1 \leq \tau \leq \pi_2} F(\tau), \tag{2}$$

$$\text{ExpF}_n \to_d \text{ExpF}(\pi_0) = \ln\left(\frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \exp\left(\frac{1}{2}F(\tau)\right) d\tau\right), \tag{3}$$

$$\text{AveF}_n \to_d \text{AveF}(\pi_0) = \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau, \tag{4}$$

where

$$F(\tau) = \frac{(W(\tau) - \tau W(1))' (W(\tau) - \tau W(1))}{\tau(1-\tau)}, \tag{5}$$

$W(\tau)$ is an $m \times 1$ vector Brownian motion, $\pi_1 = k_1/n$ and $\pi_2 = k_2/n$. These distributions are non-standard. In addition to m, the distributions depend on π_1 and π_2 through the single index

$$\pi_0 = \frac{1}{1 + \sqrt{\lambda_0}} \tag{6}$$

where

$$\lambda_0 = \frac{\pi_2(1-\pi_1)}{\pi_1(1-\pi_2)}. \tag{7}$$

Note that when the range $[k_1, k_2]$ is symmetric in the sample, $\pi_0 = \pi_1 = 1 - \pi_2$.

3 Methodology

Let T_n denote one of the three tests SupF_n, ExpF_n, or AveF_n for some π_0 and let T denote the associated asymptotic distribution (e.g., $\text{SupF}(\pi_0)$, $\text{ExpF}(\pi_0)$, or $\text{AveF}(\pi_0)$). Let $p(x) = P(T > x)$ denote the "p-value function" of T. Define the inverse function of $p(x)$: $Q(q) = p^{-1}(q)$ which satisfies $q = p(Q(q))$. Note that $Q(1-q)$ is the quantile function of the distribution. For simplicity, we will refer to $Q(q)$ as the quantile function.

While $p(x)$ may be (in principle) calculable, it may be computationally burdensome in applications, so we desire a parametric approximation, valid at least for small p-values. In the following sections we describe how we obtain such an approximation.

3.1 Approximating P-Value Functions

We need a parametric function $p(x \mid \theta)$ which can be made close to the true function $p(x)$ by appropriate selection of the parameter θ. In principle, we would like our functional choice $p(x \mid \theta)$ to have the standard properties of a distribution function (bounded between 0 and 1 and monotonically decreasing in x), although these properties are not essential if the function gives good approximations.

A general approach is to pick a flexible function class with known approximation properties. Let $\alpha_v(x \mid \theta)$ be the v'th-order polynomial in x defined in (1). By the Stone-Weierstrass thereom, any bounded continuous function $f(x)$ can be arbitrarily well approximated on a compact set by $\alpha_v(x \mid \theta)$ for a suitable choice of θ. It thus makes sense to consider setting $p(x \mid \theta) = \alpha_v(x \mid \theta)$, which is the approach of Hansen (1992). An improvement suggested by MacKinnon (1994) is to set $p(x \mid \theta) = 1 - \Phi(\alpha_v(x \mid \theta))$, where $\Phi(\cdot)$ is a distribution function of leading interest. This retains the approximation properties of the polynomial, but may be more parsimonious, at least when $\Phi(\cdot)$ is close to the true distribution function. We extend this idea one step further and allow Φ to depend on an unknown parameter η, viz., $\Phi(\cdot \mid \eta)$, so that our approximating p-value function is $p(x \mid \theta) = 1 - \Phi(\alpha_v(x \mid \theta) \mid \eta)$.

In our specific applications, we set $\Phi(\cdot \mid \eta) = \chi^2(\eta)$, the chi-square distribution with η degrees of freedom, although other distribution functions could be selected in appropriate contexts. In summary, our approximating function is

$$p(x \mid \theta) = 1 - \chi^2\left(\theta_0 + \theta_1 x + \cdots \theta_v x^v \mid \eta\right) \tag{8}$$

where

$$\chi^2\left(z \mid \eta\right) = \int_0^z \frac{y^{\eta/2-1} e^{-y/2}}{\Gamma\left(\eta/2\right) 2^{\eta/2}} dy$$

is the cummulative chi-square distribution, and $\theta = (\theta_0, \theta_1, \ldots, \theta_v, \eta)$.

Why this particular choice? The asymptotic theory of section 2 shows that when $\pi_0 = 1/2$, the SupF(π_0), ExpF (π_0), and AveF (π_0) distributions simplify to the χ^2_m distribution. By continuity, their distributions will be close to the χ^2_m for π_0 close to $1/2$. For other values of π_0, we can get a sense of the distributions through numerical plots. Figures 1, 2, and 3 display estimated plots of the density functions of the SupF(π_0), ExpF (π_0), and AveF (π_0) distributions, respectively, for $m = 1$, 5, 10, and 20, and several values of π_0. The densities appear to resemble those of the chi-square, but with shifts in location and spread. It therefore seems reasonable to use the chi-square distribution as our "leading case" distribution.

3.2 Loss Function

Given the function $p(x \mid \theta)$ of (8), we need to select θ to make $p(x \mid \theta)$ as close as possible to the true p-value function $p(x)$. Since the object of interest are the p-values themselves, we wish to make the difference $|p(x \mid \theta) - p(x)|$ small. This is equivalent to making $|p(Q(q) \mid \theta) - q|$ small. In principle, we want all errors, not just the "average" error, to be small. The natural metric to measure the statement "all errors are small" is the uniform metric:

$$d_\infty(\theta) = \max_{0 \le q \le 1} |p(Q(q) \mid \theta) - q| . \tag{9}$$

The uniform metric is difficult to implement numerically. A close relative is the L^r norm

$$d_r(\theta) = \left[\int_0^1 |p(Q(q) \mid \theta) - q|^r \, dq \right]^{1/r} \tag{10}$$

for r large. Metric (10) seems inappropriate, however, since it weights all quantiles equally. It seems reasonable to believe that we are more concerned with precision in p-values when the p-values are small. This desire can be incorporated by including a weight function in (10):

$$d_r(\theta) = \left[\int_0^1 |p(Q(q) \mid \theta) - q|^r \, w(q) dq \right]^{1/r} \tag{11}$$

where $w(q) \ge 0$. Beyond the fact that $w(q)$ should be decreasing in q, it is not clear exactly what shape it should take. After some experimentation, I settled on the following choice:

$$w(q) = \begin{cases} 1, & 0 \le q \le 0.1 \\ \left(\dfrac{.8 - q}{.7} \right)^2 & 0.1 \le q \le 0.8 \\ 0, & 0.8 \le q \le 1.0 \end{cases} . \tag{12}$$

The weight function $w(q)$ in (12) has the following features. It gives highest weight to to p-values in the region $[0, 0.1]$, and zero weight to those in the region $[0.8, 1.0]$. It is continuous between these points, with a quadratic decay.

When $Q(q)$ is not analytic, we can replace the continuous region $[0,1]$ by a discrete set $[q_1, ..., q_N]$ to approximate the integral (11). We use the set $\{.001, .002, ..., .999\}$ in the work which follows.

Minimization of $d_r(\theta)$ yields the parameter value which best fits the approximation $p(x \mid \theta)$ to the true p-value function $p(x)$. Let the minimum value be denoted by θ^* :

$$\theta^* = \operatorname*{Argmin}_{\theta \in \Theta} d_r(\theta),$$

the loss-minimizing p-value function by $p^*(x) = p(x \mid \theta^*)$, and the approximate p-values by $p_n^* = p(T_n \mid \theta^*)$.

Alternative choices for loss function (11) and weight function (12) may be made. For example, Hansen (1992) and Gonzalo and Adda (1995) set $r = 2$. This penalizes large errors less severely than our choice. Our choice to set r high implies that we are concerned about large approximation errors, and are not very willing to trade off a few large errors in return for many other small errors. I think this corresponds to our idea that we want a reported p-value in *any* application to be accurate.

One can also view the fitting of $p(Q(q) \mid \theta)$ to q as a regression, a point made in particular by MacKinnon (1994). Since this regression has non-classical statistical properties, MacKinnon (1994) suggests that re-weighting be done to account for heteroskedastic errors. I do not believe this is appropriate. A correctly specified loss function, such as (11)-(12), incorporates all information necessary for loss-minimizing curve fitting. I think it is most constructive to discuss the choice of loss function, rather than the statistical qualities of the regression fit.

3.3 Approximating the Quantile Function

Minimization of the criterion (11) requires the computation of the true quantile function $Q(q)$, which is unknown. It may be numerically approximated using analytic techniques. DeLong (1981) provides expansions for the SupF distribution for $m \leq 4$. Anderson and Darling (1952) provide expansions for the AveF distribution for $m = 1$ and $\pi_0 = 0$. It is possible that these techniques could be generalized to handle our applications. It is not clear, however, that this is desireable. Such numerical approximations involve considerable analytic effort, and in the end still produce approximations (such as truncated infinite sums).

Another approach is to use analytic methods to approximate $p(x)$ for large x (i.e., for small p-values) as in Kim and Siegmund (1989). The downside is that this approach does not necessarily give good estimates for the entire support of the distribution.

We took the analytically simpler approach of Monte Carlo simulation. The cost is a relatively heavy use of computer resources. We approximated the distributions (2), (3), and (4) using using a grid on $[0, 1]$ with 1,000 evenly spaced points. This is equivalent to simulating (5) using a sample of size 1,000. We then constructed the empirical quantile function $\hat{Q}(q)$ from $R = 50,000$ independent replications. This should be sufficiently precise for our purposes. Indeed, let $\hat{p}(x) = \hat{Q}^{-1}(x)$ be the empirical p-value function. By the central limit theorem,

$$P\left(|\hat{p}(x) - p(x)| \geq .0044\right) \approx .05.$$

Thus at the 95% confidence level, the maximum simulation error is about .0044. Most simulation errors, of course, are much less than this amount.

To summarize, our p-value approximations involve two separate approximations. First, we estimate the true p-values $p(x)$ by $\hat{p}(x)$ using simulation. Second, we use a parametric function $p(x \mid \theta)$ to approximate the estimated p-values $\hat{p}(x)$. These two errors do not necessarily offset one another. To reduce the total error, we need to make both errors small, which is possible only by (i) increasing the number of simulation replications; and (ii) increasing the order v of the polynomial $\alpha_v(x \mid \theta)$.

3.4 Results

I fit p-value functions of the form (8) to the SupF(π_0), ExpF (π_0), and AveF (π_0) distributions for $m = 1, 2, ..., 40$ and $\pi_0 = .01, .03, .05, ..., .49$. There are thus 3,000 distinct distributions. For each distribution, I selected the polynomial order v to get a good yet parsimonious fit. For the SupF distributions, I found that $p = 1$ was sufficient for all m. For the ExpF distributions, $v = 3$ was necessary for $m = 1$, $v = 2$ was needed for $m = 2$ and $m = 3$, and $v = 1$ was sufficient for $m \geq 4$. For the AveF distributions, $v = 3$ was used for $m = 1$, $v = 2$ for $m = 2$, and $v = 1$ for $m \geq 3$.

For any distribution, the absolute error from our parametric approximation is

$$d(x) = \left|p(x \mid \hat{\theta}) - \hat{p}(x)\right|.$$

Table 1 reports a summary of the errors for 20 p-values of interest. The column "Median Error" reports the median of the absolute errors across the 1000 distributions for each test (SupF, ExpF and AveF. The column "Maximum Error" reports the maximum absolute error across all 1000 distributions. It appears that the errors are quite small. For example, we see that for the 1% p-value, the distributions err at most by 0.0017, with a median error of only

0.0004. Interestingly, the numerical approximations are almost as precise for large p-values. For example, at the 50% p-value, the SupF distribution has a maximal error of 0.0030, and a median of 0.0006. The accuracy of these approximations is much better than necessary for empirical applications.

Despite the parsimony of the fitted models, we still have over 9,000 coefficients to report, which is too many to print in this article. The complete estimates are available in a GAUSS program available from the JBES WEB site[1] or upon request from the author. We report in the paper the coefficients for $\pi_0 = .01, .05,.15,.25,.35$, and for $m = 1, 2, ..., 20, 25, 30, 35$, and 40. Table 2 reports the coefficients for the SupF distributions. Tables 2 and 3 report those for the ExpF distributions, and Tables 4 and 5 for the AveF distributions.

4 Empirical Illustration

To illustrate the usefulness of these p-value approximations, I report two simple applications using autoregressive models. The first application is an AR(6) fit to the growth rate (log-difference) of U.S. monthly total personal income for the period 1946.1-1995.7, extracted from the Citibase file GMPY. The second is an AR(12) fit to the first-difference of the monthly three-month U.S. Treasury Bill rate for the period 1947.1-1995.7, extracted from the Citibase file FYMG3. The results are reported in Table 7. I report the numerical value of the SupLM, ExpLM, and AveLM versions of the tests for constancy of all regression coefficients, setting $\pi_1 = .15$ and $\pi_2 = .85$ (so $\pi_0 = .15$). I also report the asymptotic 10%, 5%, and 1% critical values from Andrews (1993) and Andrews-Ploberger (1994). Finally, I report the approximate asymptotic p-values.

To compute the p-values, I use formula (8) and the coefficients from Tables 2-6 for $\pi_0 = .15$. For example, take the SupLM statistic for personal income. From Table 2, we find that for $m = 7$ (the number of parameters in the regression) and $\pi_0 = .15$ that $\theta_0 = -4.42$, $\theta_1 = 1.10$, and $\eta = 11.0$. Thus we take the test value of 12.5, make the computation $-4.42 + 12.5 * 1.10 = 9.33$, and use the chi-square distribution with $\eta = 11.0$ degrees of freedom evaluated at 9.33 to obtain the p-value of .59. Or consider the AveLM

[1](Assuming this paper is accepted.)

statistic for the T-Bill rate. From Table 6, we find that for $m = 14$ and $\pi_0 = .15$ that $\theta_0 = -6.38$, $\theta_1 = 1.63$ and $\eta = 14.8$. So our approximate p-value for the test value of 18.4 is $1 - \chi^2(-6.38 + 1.63 * 18.4 \mid 14.8) = .07$.

A reading of the table shows that the p-values yield much more information than simply the critical values. Take, for example, the SupLM test applied to the T-Bill series. The test statistic of 25.0 appears to be quite "close" to the 10% critical value of 29.1, so on the basis of the critical values alone a researcher might conclude that the test is "close" to significant. But the asymptotic p-value turns out to be only .27. Similarly, the ExpLM statistic is 10.2, which appears close to the 10% critical value of 11.1, yet has a p-value of only .18. In summary, the p-values are relatively simple to calculate, given the tables provided, yet enable a researcher to come to more informed conclusions than simply on the basis of the asymptotic critical values.

Table 1: Absolute Error in Fitted Distributions

	SupF Distributions		ExpF Distributions		AveF Distributions	
P-Value	Median Error	Maximum Error	Median Error	Maximum Error	Median Error	Maximum Error
0.00	0.0001	0.0006	0.0001	0.0013	0.0001	0.0009
0.01	0.0004	0.0015	0.0004	0.0017	0.0004	0.0014
0.02	0.0005	0.0019	0.0005	0.0021	0.0005	0.0018
0.03	0.0006	0.0023	0.0006	0.0023	0.0006	0.0018
0.04	0.0006	0.0025	0.0006	0.0021	0.0006	0.0018
0.05	0.0006	0.0021	0.0006	0.0019	0.0006	0.0021
0.06	0.0006	0.0021	0.0006	0.0019	0.0005	0.0021
0.07	0.0005	0.0019	0.0005	0.0020	0.0006	0.0020
0.08	0.0005	0.0019	0.0006	0.0024	0.0005	0.0019
0.09	0.0005	0.0018	0.0005	0.0024	0.0005	0.0020
0.10	0.0005	0.0018	0.0005	0.0019	0.0005	0.0018
0.15	0.0005	0.0024	0.0005	0.0020	0.0005	0.0017
0.20	0.0005	0.0020	0.0006	0.0021	0.0005	0.0024
0.25	0.0006	0.0024	0.0006	0.0025	0.0005	0.0019
0.30	0.0005	0.0020	0.0006	0.0027	0.0006	0.0023
0.40	0.0006	0.0024	0.0006	0.0025	0.0006	0.0025
0.50	0.0006	0.0030	0.0006	0.0025	0.0006	0.0023
0.60	0.0006	0.0027	0.0006	0.0027	0.0006	0.0026
0.70	0.0008	0.0024	0.0008	0.0032	0.0008	0.0028
0.80	0.0017	0.0058	0.0019	0.0068	0.0015	0.0066

Table 2: SupF Distribution, $m \geq 1$

	$\pi_0 = .01$			$\pi_0 = .05$			$\pi_0 = .15$			$\pi_0 = .25$			$\pi_0 = .35$		
m	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-1.79	1.17	4.5	-1.39	1.07	3.6	-0.99	1.02	3.0	-0.73	0.98	2.5	-0.50	0.96	2.1
2	-3.06	1.18	6.1	-2.38	1.11	5.4	-1.65	1.06	4.7	-1.16	1.02	4.1	-0.78	0.97	3.5
3	-4.09	1.21	7.8	-3.31	1.10	6.5	-2.05	1.13	6.8	-1.61	1.03	5.5	-1.06	1.01	4.9
4	-5.33	1.21	8.9	-4.08	1.14	8.2	-2.52	1.11	8.0	-1.91	1.04	7.0	-1.45	0.97	5.7
5	-6.39	1.18	9.4	-4.84	1.15	9.3	-3.46	1.07	8.3	-2.63	1.02	7.5	-1.82	1.00	7.0
6	-7.08	1.26	11.8	-5.37	1.19	11.2	-4.05	1.08	9.5	-2.94	1.05	9.0	-1.79	1.03	8.6
7	-8.49	1.17	11.1	-6.21	1.21	12.6	-4.42	1.10	11.0	-3.23	1.05	10.1	-2.21	1.01	9.3
8	-9.20	1.17	12.2	-7.24	1.13	11.9	-5.36	1.08	11.3	-3.65	1.06	11.4	-1.69	1.10	12.2
9	-10.22	1.14	12.3	-8.07	1.11	12.4	-5.43	1.10	13.1	-4.38	1.01	11.3	-2.83	1.00	11.1
10	-11.01	1.14	13.3	-8.84	1.11	13.2	-6.47	1.06	12.8	-4.97	1.01	12.0	-2.92	1.05	13.0
11	-11.90	1.11	13.4	-9.56	1.06	13.1	-6.79	1.04	13.5	-4.62	1.05	14.4	-3.26	1.01	13.4
12	-12.88	1.06	12.8	-10.35	1.09	14.5	-7.80	1.02	13.6	-5.32	1.05	15.1	-3.91	1.00	13.8
13	-13.88	1.09	14.1	-11.07	1.07	14.8	-7.93	1.07	15.9	-5.80	1.04	15.8	-4.14	1.00	14.9
14	-14.61	1.15	16.6	-11.52	1.11	16.8	-8.54	1.05	16.1	-5.90	1.05	17.2	-4.06	1.02	16.5
15	-15.49	1.04	14.1	-12.44	1.08	16.6	-9.05	1.05	17.2	-6.59	1.04	17.6	-3.10	1.08	20.0
16	-16.34	1.15	17.8	-12.27	1.20	21.4	-9.13	1.09	19.3	-7.00	1.04	18.5	-4.79	0.99	17.6
17	-17.20	1.15	18.8	-13.73	1.15	20.1	-10.45	1.05	18.3	-7.23	1.05	19.8	-5.01	1.02	19.1
18	-18.10	1.17	20.0	-14.15	1.14	21.2	-10.63	1.05	19.5	-7.76	1.04	20.2	-5.11	1.02	20.2
19	-18.19	1.04	17.2	-14.94	0.97	16.3	-12.14	0.90	14.9	-9.84	0.89	15.3	-7.09	0.91	16.8
20	-18.99	1.02	17.0	-16.09	0.99	17.0	-12.14	0.97	18.3	-8.87	1.00	20.5	-5.94	1.00	21.3
25	-23.42	1.06	21.0	-19.06	1.06	23.4	-14.16	1.05	25.0	-10.65	1.03	25.7	-6.57	1.02	27.1
30	-27.30	1.03	22.5	-22.91	1.04	25.1	-17.06	1.03	27.8	-11.51	1.07	32.8	-6.79	1.05	34.1
35	-30.01	0.92	20.0	-25.88	0.97	25.2	-20.09	0.98	28.4	-15.78	0.97	30.2	-10.44	0.98	33.1
40	-34.24	0.97	24.8	-29.24	0.98	28.0	-21.65	1.05	36.7	-14.18	1.07	42.7	-11.95	0.94	35.0

Table 3: ExpF, $m = 1, 2, 3$

	$m=1$					$m=2$				$m=3$			
π_0	θ_0	θ_1	θ_2	θ_3	η	θ_0	θ_1	θ_2	η	θ_0	θ_1	θ_2	η
0.01	-0.74	5.23	-1.16	0.17	2.3	-1.34	3.32	-0.14	3.1	-2.10	3.54	-0.13	4.9
0.05	-0.61	4.66	-1.01	0.16	2.1	-1.07	3.05	-0.11	2.9	-1.66	3.29	-0.11	4.6
0.15	-0.42	3.75	-0.65	0.10	1.7	-0.77	2.66	-0.07	2.6	-1.16	2.91	-0.08	4.2
0.25	-0.30	3.33	-0.54	0.09	1.5	-0.55	2.32	-0.03	2.3	-0.67	2.74	-0.07	4.1
0.35	-0.19	2.81	-0.41	0.08	1.2	-0.38	2.13	-0.02	2.1	-0.56	2.08	-0.01	3.0

Table 4: ExpF Distribution, $m \geq 4$

	$\pi_0 = .01$			$\pi_0 = .05$			$\pi_0 = .15$			$\pi_0 = .25$			$\pi_0 = .35$		
m	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
4	-2.48	2.41	4.5	-2.05	2.34	4.5	-1.47	2.26	4.4	-1.09	2.12	4.1	-0.72	2.03	4.0
5	-3.19	2.34	5.2	-2.66	2.29	5.2	-1.95	2.20	5.2	-1.49	2.08	4.9	-0.89	2.05	5.0
6	-3.94	2.20	5.5	-3.34	2.17	5.6	-2.45	2.13	5.8	-1.77	2.09	5.9	-1.18	2.00	5.8
7	-4.67	2.23	6.3	-3.88	2.20	6.6	-2.73	2.19	7.1	-1.90	2.12	7.1	-1.26	2.05	7.0
8	-5.26	2.20	7.1	-4.37	2.19	7.5	-2.91	2.22	8.5	-1.86	2.23	9.0	-0.80	2.20	9.3
9	-6.08	2.17	7.6	-5.07	2.20	8.3	-3.58	2.20	9.1	-2.58	2.12	9.0	-1.46	2.10	9.4
10	-6.74	2.17	8.4	-5.56	2.16	9.0	-4.03	2.11	9.4	-3.08	2.00	9.0	-1.78	2.03	9.8
11	-7.49	2.25	9.6	-6.21	2.23	10.1	-4.66	2.15	10.3	-3.13	2.15	11.1	-1.44	2.18	12.2
12	-8.19	2.10	9.2	-6.86	2.11	10.0	-5.20	2.07	10.5	-3.79	2.01	10.7	-2.24	2.00	11.4
13	-8.89	2.07	9.6	-7.39	2.12	10.9	-5.49	2.04	11.3	-4.16	2.00	11.5	-2.55	2.01	12.3
14	-9.65	2.16	11.0	-7.79	2.25	13.0	-5.56	2.17	13.5	-3.68	2.17	14.5	-2.15	2.11	14.6
15	-10.51	2.07	10.6	-8.97	2.13	12.0	-6.66	2.11	13.1	-5.27	2.00	12.7	-3.57	1.98	13.3
16	-11.26	2.00	10.5	-9.63	2.01	11.5	-6.92	2.11	14.2	-4.82	2.13	15.6	-2.48	2.11	16.7
17	-11.89	2.13	12.5	-10.07	2.11	13.4	-7.58	2.08	14.4	-5.34	2.09	15.8	-2.53	2.15	18.2
18	-12.57	1.98	11.5	-10.62	2.00	12.9	-8.15	1.94	13.5	-5.96	2.00	15.5	-4.05	1.96	15.9
19	-13.17	2.01	12.6	-11.26	1.98	13.2	-8.78	1.93	14.0	-6.51	1.99	16.0	-4.47	1.96	16.5
20	-13.84	2.03	13.4	-11.91	2.00	14.1	-8.96	2.02	16.1	-6.43	2.06	18.0	-4.17	2.01	18.6
25	-17.50	1.97	15.0	-15.05	1.92	15.8	-12.18	1.88	16.7	-9.27	1.91	18.9	-6.04	1.97	21.7
30	-20.92	1.90	16.4	-18.16	1.91	18.3	-14.19	1.93	21.0	-11.25	1.89	21.9	-8.47	1.85	22.6
35	-23.97	1.80	16.9	-20.86	1.87	20.2	-16.50	1.95	24.7	-12.00	2.01	28.8	-7.10	2.07	33.3
40	-27.29	1.84	19.9	-23.96	1.97	25.1	-18.64	1.93	27.8	-14.44	1.92	30.2	-5.07	2.20	43.8

Table 5: AveF, $m = 1, 2)$

	$m=1$					$m=2$			
π_0	θ_0	θ_1	θ_2	θ_2	η	θ_0	θ_1	θ_2	η
0.01	-1.02	5.39	-0.95	0.11	3.2	-1.78	3.12	-0.10	4.0
0.05	-0.74	4.95	-0.84	0.09	3.2	-1.42	2.80	-0.08	3.8
0.15	-0.47	3.63	-0.51	0.05	2.5	-0.94	2.21	-0.05	3.2
0.25	-0.35	2.56	-0.30	0.03	1.8	-0.62	1.70	-0.02	2.7
0.35	-0.22	1.79	-0.16	0.02	1.3	-0.41	1.21	0.00	2.0

Table 6: AveF Distribution, $m \geq 3$

	$\pi_0 = .01$			$\pi_0 = .05$			$\pi_0 = .15$			$\pi_0 = .25$			$\pi_0 = .35$		
m	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
3	-2.41	2.24	4.3	-2.03	2.06	4.2	-1.41	1.70	3.7	-1.01	1.40	3.2	-0.59	1.22	3.1
4	-3.28	2.20	5.5	-2.76	2.02	5.3	-1.94	1.68	4.8	-1.29	1.45	4.5	-0.72	1.26	4.3
5	-3.91	2.24	7.3	-3.27	2.07	7.1	-2.23	1.72	6.4	-1.59	1.44	5.6	-0.89	1.24	5.3
6	-4.85	2.26	8.7	-4.09	2.07	8.4	-2.87	1.71	7.4	-1.86	1.48	7.0	-1.05	1.28	6.6
7	-5.30	2.21	10.2	-4.45	2.02	9.7	-3.24	1.63	8.2	-2.41	1.34	7.0	-1.50	1.17	6.7
8	-6.24	2.17	11.1	-5.27	1.99	10.7	-3.84	1.60	9.0	-2.95	1.32	7.6	-1.84	1.16	7.5
9	-7.39	2.07	11.3	-6.17	1.94	11.3	-4.34	1.62	10.2	-2.80	1.41	9.9	-1.77	1.21	9.1
10	-7.88	2.11	13.3	-6.65	1.94	12.8	-4.65	1.61	11.5	-3.20	1.38	10.6	-1.77	1.22	10.4
11	-8.45	2.17	15.5	-7.25	1.98	14.6	-4.74	1.70	14.0	-3.72	1.39	11.6	-2.37	1.21	10.9
12	-9.41	2.15	16.4	-7.94	1.98	15.9	-5.48	1.66	14.4	-3.45	1.45	14.0	-1.44	1.31	14.2
13	-10.43	2.16	17.7	-8.93	1.97	16.7	-6.38	1.63	14.8	-4.47	1.40	13.7	-2.69	1.22	13.2
14	-11.32	2.02	17.0	-9.89	1.82	15.6	-6.90	1.58	15.2	-4.35	1.43	15.6	-3.18	1.17	13.3
15	-12.01	2.17	20.4	-10.36	1.97	19.1	-7.19	1.67	17.9	-5.21	1.42	16.1	-2.98	1.26	15.9
16	-10.48	2.47	28.9	-8.66	2.25	27.3	-5.34	1.87	24.5	-3.24	1.57	21.8	-1.88	1.31	19.1
17	-12.92	2.22	24.9	-10.95	2.03	23.5	-7.44	1.71	21.5	-4.39	1.52	21.3	-1.75	1.34	21.0
18	-13.31	2.25	27.1	-10.94	2.09	26.6	-8.82	1.58	19.6	-5.92	1.39	19.1	-3.52	1.21	18.3
19	-13.02	2.36	31.9	-10.53	2.17	30.8	-8.30	1.68	23.6	-5.69	1.43	21.5	-4.37	1.17	17.9
20	-14.41	2.23	30.1	-12.65	1.97	26.8	-9.69	1.54	21.2	-6.95	1.33	19.6	-5.05	1.13	17.6
25	-19.33	2.10	33.2	-16.14	1.95	32.6	-11.79	1.60	28.1	-9.03	1.33	24.1	-5.88	1.17	23.3
30	-24.32	2.11	39.1	-19.98	2.01	40.4	-14.53	1.64	34.7	-10.24	1.39	31.5	-2.41	1.36	38.4
35	-26.94	2.19	49.8	-22.85	2.01	47.3	-16.38	1.66	41.6	-8.60	1.55	45.4	-4.38	1.33	42.0
40	-31.58	2.04	50.0	-27.55	1.83	45.8	-19.54	1.57	43.0	-12.15	1.42	44.5	-3.76	1.33	49.5

Table 7: Empirical Applications

	Personal Income ($m = 7$)					T-Bill Rate ($m = 13$)				
	Test Statistic	Critical Value 10%	5%	1%	Approximate P-Value	Test Statistic	Critical Value 10%	5%	1%	Approximate P-Value
SupLM	12.5	19.7	21.8	26.3	.59	25.0	29.1	31.8	37.0	.27
ExpLM	4.6	6.7	7.7	9.5	.41	10.2	11.1	12.3	14.6	.18
AveLM	7.4	10.3	11.5	14.3	.37	18.4	17.2	18.8	22.5	.07

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