MODELING PARKING

(Revised Version)

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Abstract

This paper presents a simple model of parking congestion which focuses on drivers' search for a vacant parking space in a spatially homogeneous metropolis. Individuals' residences are distributed uniformly around an annulus. When at home an individual waits for a trip opportunity which is generated by an exogenous stochastic process. A trip opportunity provides a benefit if she visits a specific location for a specific period of time. The individual decides which trip opportunities to accept and what mode of transport to take. If she drives, she must decide how far from her destination to start cruising for parking, and then takes the first vacant parking spot and walks to her destination. The mean density of vacant parking spaces is endogenous. A parking externality arises because individuals neglect the effect of their parking on the mean density of vacant parking spaces.

The paper examines stochastic stationary-state equilibria and optima in the model. Our main finding is that due to the model's nonlinearity, there may be three equilibria -- stable-congested, unstable, and stable-hypercongested (which is Pareto inferior to the stable-congested equilibrium). The effects of parking fees are, as a result, complex. For one thing, the fee may cause the appearance of a congested equilibrium and/or the disappearance of a hypercongested equilibrium. The important policy insight is that parking pricing is both a delicate and potentially powerful tool for the regulation of traffic congestion.

A variety of extensions are discussed. One is to include flow congestion in car travel. If such congestion is unpriced, determination of the optimal parking fee is an exercise in the second best. Another is to determine the social value of a particular parking information system.

<u>MODELING PARKING</u>

Downtown parking is a significant problem in all major cities. Remarkably, however, there has been very little formal economic analysis of even the most obvious issues. If traffic congestion is efficiently priced, how should parking fees be set? Alternatively, what are the second-best parking fees when, as is realistic, traffic congestion is underpriced? Depending on the pricing of auto congestion and public parking, should private, off-street parking fees be taxed or regulated? For various pricing régimes, how much land should be allocated to parking, both on- and off-street? What is the value of information concerning parking availability? In this paper we develop a simple structural model which provides a conceptual basis to answer such questions.

Various aspects of parking have been considered in the literature. Descriptions of parking patterns, the effects of on-street parking on traffic circulation, and the technology of off-street parking appear (e.g. Highway Research Board (1971), Institute of Transportation Engineers (1982)), as well as discussions of parking policy (e.g. Segelhorst and Kirkus (1973), Miller and Everett (1982), Shoup (1982), U.S. DOT (1982), Adiv and Wang (1987)). Some empirical work has been done identifying the determinants of modal choice and parking location (e.g. Gillen (1977a,b, 1978), Westin and Gillen (1978), Hunt (1988)). Numerous city-specific parking studies have been undertaken (Smith (1967)). And there are high-quality, non-technical economic discussions of parking policy, notably Vickrey (1959) and Roth (1965). But with the exception of a short note by Douglas (1975) and papers by Arnott, de Palma, and Lindsey (1991a), Glazer and Niskanen (1990), and Verhoef, Nijkamp, and Rietveld (1995) no economic model has been developed that considers the potential efficiency gains from parking fees or that incorporates the effects of parking on travel congestion. The effects may be substantial, for in major urban areas the time to find a parking spot and walk from there to work can be an appreciable fraction of total travel time, and parking fees may be comparable to vehicle operating costs (Lansing (1967), Gillen(1977b)). Arnott, de Palma, and Lindsey (1991a) explored the effects of parking fees in a deterministic model of the morning auto commute to the central business district, with bottleneck congestion. They showed that parking fees which vary over location can significantly reduce total travel costs. Glazer and Niskanen (1992) examined simple partial equilibrium models to demonstrate that raising parking fees may <u>increase</u> both local traffic (by encouraging shorter visits) and through traffic. And Verhoef, Nijkamp, and Rietveld (1995) compared parking fees and parking regulations.

This paper presents a quite different model that focuses on the stochasticity of vacant parking spaces. This stochasticity is important to treat for several reasons. First, it results in drivers cruising around looking for a parking space. It has been claimed that, in Boston and major European cities, over one-half the cars driving downtown in rush hour are cruising for parking. Cruising for parking is not only frustrating and time-consuming for the driver but also contributes significantly to traffic congestion, by increasing traffic volume and slowing traffic down. Second, many cities are exploring a variety of information systems that provide information to drivers on parking availability; to evaluate such systems, it is necessary to treat the stochasticity of vacant parking spaces. Third, recent studies (e.g. Small, et al. (1995)) support what intuition suggests -- that unanticipated travel time is disproportionately costly; variability in the time to find a parking spot is a major component.

The aim of this paper is modest. It does not attempt to treat these issues in their full complexity. Rather, it explores perhaps the simplest possible structural model that incorporates the stochasticity of vacant parking spaces. A later section discusses at some length how the model can be extended in the direction of realism. The structure of the basic model is as follows. The city is located on an annulus and is spatially symmetric. At each location there is a fixed amount of land devoted to parking. The demand for parking is derived from the demand for trips. Trip opportunities are generated according to an exogenous, stochastic, time-invariant process. A trip opportunity provides a benefit to a specific individual if she travels to a specific location and visits there for a specified period. An individual sits at home waiting for a trip opportunity. When she receives an opportunity, she decides whether to accept it, and if she does accept it, what mode of transport to take. If she drives, she must decide how far from her destination to start cruising for parking, and then takes the first available parking spot and walks to her destination. The expected walking distance depends on the mean density of vacant parking spaces, which is determined endogenously. A parking externality arises because individuals collectively neglect the effect of their parking on the mean density of vacant parking spaces.

Our main finding is that the model exhibits complex nonlinearity. One consequence is that there may be two stable equilibria which can be Pareto ranked. Which obtains depends presumably on the path of adjustment to equilibrium. Another consequence is that the comparative static properties of the model, including its response to policy variables -- notably parking fees -- are complex. The important policy insight is that even though parking pricing is a potentially powerful tool for the regulation of traffic congestion, it is intrinsically difficult to determine the appropriate level of parking fees.

Section I describes the basic model. Section II examines equilibrium with no parking fee. Section III considers the social optimum. Section IV treats equilibrium with a parking fee, and explores decentralization of the social optimum via parking fees. Section V provides quite a detailed discussion of directions for future research, and provides an illustration of how the model can be employed to determine the social value of parking information systems. Brief concluding comments are given in section VI.

I. The Basic Model

The basic model provides a highly stylized and simplified, but structural and general equilibrium, representation of the downtown parking problem. It is designed to admit numerous extensions. The model has four modules: spatial structure, trip generation technology, technology of parking and travel, and stationary-state conditions.

I.1 Spatial structure

To abstract from complications arising from spatial heterogeneity, it is assumed that the city is spatially symmetric. More specifically, the city occupies a thin annulus of arbitrarily large inner radius r and has the same spatial structure at each location. Population density is Γ per unit length. The number of parking spaces per unit distance, the "density" of parking spaces, is D.

I.2 Trip generation technology

The demand for parking is derived from the demand for trips. An individual takes trips for the benefit she derives at the destination. To avoid complications associated with scheduling, interaction between individuals is ignored and all trips are single-purpose.

When at home an individual receives trip opportunities according to a Poisson process. A trip opportunity states that if she travels immediately to a specific location and visits for a fixed period of time ℓ, she will receive a fixed dollar benefit $\int$. If the individual accepts the trip opportunity, she travels to that location, receives the benefit, returns home, and waits for her next trip opportunity.

The origin of trip opportunities is uniformly distributed around the circle. The Poisson arrival rate of trip opportunities is μ per individual. Thus, the model is temporally, as well as spatially, homogeneous.

Many other specifications of the trip generation technology are possible. The above assumptions were chosen for their simplicity.

I.3 Travel and parking technologies

There are two travel modes - walking and driving, indexed $i = 1, 2$, respectively. Let x denote the distance of a trip opportunity from home. The *expected travel time* to and from x by mode i is $T_i(x)$. Walking speed is a constant w. Thus,

$$T_1(x) = \frac{2x}{w}. \tag{1}$$

Determining auto travel time is more complex. There are two components of car travel time - time spent in the car, and time spent walking from the parking location to the destination and back again. The time spent in the car can in turn be decomposed into time spent cruising for parking and time spent in "regular" car travel. Since the congestion caused by cruising for parking is of central importance to the parking problem, as is walking time when driving, it is important to model car travel with care.

Two simplifying assumptions are made, both of which should be relaxed in more realistic models. The first is that cars travel at a constant speed, independent of the density of cars both in regular traffic and cruising for parking; that is, there is no travel congestion. The second is that car speed is the same whether in regular traffic or cruising for parking, v.

To provide a primitive treatment of the parking technology and to incorporate cruising for parking, it is important to treat the stochastic nature of finding a parking spot. Since traffic is in a stochastic stationary state, it is reasonable to assume that a driver knows the probability of finding a vacant spot between x and $x + dx$, Pdx, where P is the *average density of vacant parking spaces*. To simplify, it is assumed that a driver searching for a parking spot can neither stop and wait for a parking spot to become vacant, nor back-track. For short distances, a driver would start cruising for parking as soon as she leaves home. For longer distances, she would start cruising for parking a distance d from her destination. A simple argument establishes that with a non-negative parking fee, which we assume, walking dominates driving for $x < d$: If the driver starts cruising immediately upon leaving home, then the expected cost of taking a vacant parking space immediately is no greater than the cost of not taking a parking space, which is simply the expected travel cost. The cost of taking a parking space immediately upon leaving home is, in turn, at least as high as the cost of just walking since in both cases she has to walk the same distance. Thus, the expected travel cost of driving and cruising immediately upon leaving home is at least as high as the cost of walking, and so the option of driving and cruising immediately upon leaving home can be ignored. Hence, the individual will walk on shorter trips and drive on longer ones.

Let y be the distance the driver cruises for parking.[1] Pdy is the probability that she finds a vacant parking space in an interval dy. On the assumption that the probabilities of adjacent parking spaces being vacant are independent,[2] the probability that she finds her first vacant parking space between y and $y + dy$ is $Pe^{-Py}dy$. *Expected driving time* on the round-trip journey is therefore

$$R(x,P,d) = \frac{2(x-d)}{\mathrm{v}} + \frac{2}{\mathrm{v}P} \quad \text{for} \quad x \geq d. \tag{2}$$

On the journey from home to the destination, the individual drives a distance $x-d$ before starting to cruise for parking and then cruises for parking for an expected distance $\frac{1}{P}$. Since, by assumption, driving speed is v both in regular traffic and while cruising for parking, expected driving time on the outbound journey is $\frac{x-d}{\mathrm{v}} + \frac{1}{\mathrm{v}P}$. On the homeward journey, the individual travels an expected distance $x-d + \frac{1}{P}$ in regular traffic at speed v. Summing the times on the outbound and homeward journeys gives $R(\cdot)$.

Expected walking time on a car trip is now computed. Let y be the distance the driver cruises for parking. If she finds a parking spot at $y < d$, she must walk a distance $d-y$; and if she finds a spot at $y > d$, she must walk a distance $y-d$. Thus,

$$W(P,d) = 2\int_0^d \frac{d-y}{w} Pe^{-Py}dy + 2\int_d^\infty \frac{y-d}{w} Pe^{-Py}dy$$

[1]It is assumed that the radius of the city is sufficiently large that the probability that a driver will drive more than half way round the circle beyond her destination is negligible.
[2]This assumption is an approximation. The accuracy of the assumption is discussed in section V.5.

$$= \frac{2}{w}\left(\frac{2e^{-Pd}}{P} + d - \frac{1}{P}\right). \qquad (3)$$

Finally, [3]

$$T_2(x,P,d) = R(x,P,d) + W(P,d)$$

$$= \frac{2x}{\mathrm{v}} + \frac{4e^{-Pd}}{wP} + 2\left(d - \frac{1}{P}\right)\left(\frac{1}{w} - \frac{1}{\mathrm{v}}\right). \qquad (4)$$

I.4 Decision variables and stationary state conditions

There are three individual decision variables. The first concerns which offers to accept. Since there is an opportunity cost to the individual's time, she will not accept trip opportunities beyond $\bar{x}$, the *maximum travel distance*. The second relates to her travel mode; she will walk shorter distances, up to the *maximum walking distance* $\tilde{x}$, and drive longer distances. And the third is the *cruising distance* d - the distance from her destination a driver will start cruising for parking. It was argued in the previous subsection that, with a non-negative parking fee, $\tilde{x} \geq d$.

The *expected trip period*, L, will feature prominently in the analysis. This has three components: expected travel time, visit length ℓ, and expected time waiting at home for an accepted trip opportunity. Expected travel time is $\int_0^{\bar{x}} T(x)g(x)dx$, where $g(x)$ is the p.d.f. of x on trips taken and $T(x)$ is travel time to x with the chosen mode. Since the location of trip opportunities is uniform on the circle, and since all trips up to $\bar{x}$ are accepted, $g(x) = \frac{1}{\bar{x}}$. Furthermore, $T(x) = T_1(x)$ for $x \leq \tilde{x}$ and $T(x) = T_2(x,P,d)$ for $x \in (\tilde{x},\bar{x})$. Hence, expected travel time is

$$\frac{1}{\bar{x}}\left[\int_0^{\tilde{x}} T_1(x)dx + \int_{\tilde{x}}^{\bar{x}} T_2(x,P,d)dx\right].$$

Since the arrival rate of trip opportunities is μ and the proportion of trip opportunities accepted is $\frac{2\bar{x}}{2\pi r}$, the arrival rate of accepted trip opportunities is $\mu\left(\frac{\bar{x}}{\pi r}\right)$. The expected time waiting for an accepted trip opportunity between trips is therefore $\frac{\pi r}{\mu\bar{x}}$. Thus,

[3]The partial derivatives of W(P,d) and T_2(x,P,d) are provided in Appendix 1.

$$L(\tilde{x},\bar{x},P,d)=\frac{1}{\bar{x}}\left[\int_0^{\tilde{x}}T_1(x)dx+\int_{\tilde{x}}^{\bar{x}}T_2(x,P,d)dx\right]+\ell+\frac{\pi r}{\mu\bar{x}}. \tag{5}$$

Finally there is a stochastic stationary-state condition, which can be interpreted in various ways. One is that the average rate at which parking spaces become occupied equal the average rate at which they are vacated. In stationary state, the rate at which parking spaces become occupied equals the rate at which car trips are initiated, which equals the rate at which trips are initiated, $\frac{\Gamma}{L}$, times the proportion of trips that are by car, $\frac{\bar{x}-\tilde{x}}{\bar{x}}$. And, also in stationary state, the rate at which parking spaces are vacated equals the density of occupied spaces times the rate at which each occupied space is vacated, $\frac{D-P}{W(P,d)+\ell}$. Thus[4]

$$D-P=\frac{\Gamma\big(W(P,d)+\ell\big)(\bar{x}-\tilde{x})}{L\bar{x}}. \tag{6}$$

II. Equilibrium with No Parking Fee

This is the natural base case.

II.1 Derivation of equilibrium

The individual chooses $\tilde{x}$, $\bar{x}$, and d to maximize benefits per unit time, taking P as fixed. Because P in fact depends on everyone's choice of $\tilde{x}$, $\bar{x}$, and d, there is an uninternalized parking externality. Since the benefit per trip is fixed, maximization of benefits per unit time is equivalent to minimization of the average trip period. Thus, using (5), the individual's optimization problem is

$$\min_{\tilde{x},\bar{x},d}\ \frac{1}{\bar{x}}\left[\int_0^{\tilde{x}}T_1(x)dx+\int_{\tilde{x}}^{\bar{x}}T_2(x,P,d)dx\right]+\ell+\frac{\pi r}{\mu\bar{x}}. \tag{7}$$

The first-order conditions are

$$\tilde{x}:\ \frac{1}{\bar{x}}\left[T_1(\tilde{x})-T_2(\tilde{x},P,d)\right]=0 \tag{8a}$$

$$\bar{x}:\ -\frac{1}{\bar{x}^2}\left[\int_0^{\tilde{x}}T_1(x)dx+\int_{\tilde{x}}^{\bar{x}}T_2(x,P,d)dx+\frac{\pi r}{\mu}\right]+\frac{1}{\bar{x}}T_2(\bar{x},P,d)=0 \tag{8b}$$

or, using (5),

[4] Later we shall have occasion to determine how P responds to changes in $\tilde{x}$, $\bar{x}$, and d. For this purpose we regard (5) and (6) as two equations in the unknowns L and P. The comparative statics of this pair of equations is given in Appendix 2.

$$\frac{1}{\bar{x}}\left[-L(\tilde{x},\bar{x},P,d)+\ell+T_2(\bar{x},P,d)\right]=0 \qquad (8b')$$

$$d\ :\ \frac{1}{\bar{x}}\left[\int_{\tilde{x}}^{\bar{x}}\frac{\partial T_2(x,P,d)}{\partial d}dx\right]=0. \qquad (8c)$$

Eq. (8a) indicates that, in the absence of a parking fee, the individual chooses the mode with the lower travel time. Eq. (8b´) has the following interpretation: The individual will accept a trip opportunity if the benefit from doing so covers the opportunity cost of the expected trip time. The *trip benefit* is β. Since the opportunity cost of time is $\frac{\beta}{L}$, the opportunity cost of expected trip time to x is $\frac{\beta}{L}\left(T(x,P,d)+\ell\right)$. Hence, she is indifferent between accepting and declining a trip opportunity to $\bar{x}$. Eq. (8c) indicates that the individual will choose d to minimize expected travel time by car. An increase in d will decrease expected driving time. Thus, she will choose d so that the decrease in expected driving time from a small increase in d is just offset by an increase in expected walking time. Using (4), (8c) gives

$$d=\frac{\theta}{P}\ \text{ where }\ \theta=-\ln\left[\frac{1}{2}\left(1-\frac{w}{v}\right)\right]. \qquad (8c')$$

The cruising distance is inversely proportional to the density of vacant parking spaces.

The individual's optimization problem is well-behaved. The second-order conditions are satisfied so that the optimum is unique. Furthermore, as long as $\frac{\pi r}{\mu}>\frac{\theta^2}{D^2 w}$, which we assume, then, $\bar{x}>d\geq\tilde{x}$.

The equilibrium with no parking fee is characterized by (8a), (8b´), (8c´), (5), (6), and (3) where the six unknowns are $\tilde{x},\bar{x},d,L,P,$ and W. Unlike the system of equations characterizing the social optimum and positive parking fee equilibria, which shall be examined in subsequent sections, this system of equations can be reduced to two equations in $\tilde{x}$ and $\bar{x}$:

$$H(\tilde{x},\bar{x})\equiv\frac{\bar{x}^2}{v}+\tilde{x}^2\left(\frac{1}{w}-\frac{1}{v}\right)-\frac{\pi r}{\mu}=0 \qquad (9)$$

$$G(\tilde{x},\bar{x})\equiv\left(D-\frac{\theta}{\tilde{x}}\right)\bar{x}\left(2\left(\frac{\bar{x}}{v}+\tilde{x}\left(\frac{1}{w}-\frac{1}{v}\right)\right)+\ell\right)-\Gamma\left(\frac{2\tilde{x}}{\theta}\left(\frac{\theta}{w}-\frac{1}{v}\right)+\ell\right)(\bar{x}-\tilde{x})=0. \qquad (10)$$

The solution to (8a) is[5]

$$\tilde{x} = d = \frac{\theta}{P}. \tag{8a´}$$

Eq. (10) is obtained by substituting (8a´), (8c´), (5), and (3) into (6). Thus, it has the interpretation as the locus of ($\tilde{x}, \bar{x}$) such that parking is in equilibrium. Eq. (9) is obtained from (8a) and (8b). Since (10) incorporates (8a), (9) is appropriately interpreted as the $\bar{x}$ chosen by the individual as a function of the $\tilde{x}$ he chooses. Eq. (9) describes an ellipse with the origin as center. Eq. (10) has a far more complex form. The substitution of (9) into (10) gives a sixth-order polynomial equation (in $\tilde{x}$ or $\bar{x}$).

The rest of this section explores the characteristics of equilibrium.

II.2 Two numerical examples

We have been unable to obtain a complete analytical characterization of the solutions to (9) and (10), though we have proved that there is at least one real solution with $\tilde{x}$ and $\bar{x} > 0$. To gain some insight into the properties of the no-parking-fee equilibrium, we examine two numerical examples.

a) example 1

We employ the following parameter values:[6]

$w = 3.0$ mls./hr.	$D = 200$ spaces/ml.
$v = 12.0$ mls./hr.	$\Gamma = 2533.3$ persons/ml.
$\frac{\pi r}{\mu} = .79052$ ml.-hrs.	$\ell = 0$ hrs.

These parameter values imply that ç — 0.98083.

[5]The intuition for $\tilde{x} \geq d$ with a non-negative parking fee was given earlier. Suppose with a zero parking fee that $\tilde{x} > d$. Then a person driving to $\tilde{x}$ would be better off not taking a parking spot immediately upon leaving home. Since the cost of taking a parking spot immediately upon leaving home is the same as walking, she is better off driving, which is inconsistent with the definition of $\tilde{x}$.

[6]D was chosen such that there is continuous parking on one side of the street, with each parking space 26.4´ long. Ì was chosen on the basis of 6-story apartment buildings on each side of the street, with each apartment having a frontage of 25.01´. Thus, there is one household per 25.01/12 ft., which corresponds to 2533.3 households per mile. $\frac{\pi r}{\mu}$ was chosen so that, with no congestion in parking, the longest trip taken would be approximately 3.08 miles.

Three solutions[7] of economic interest to (9) and (10)were found:

	$\tilde{x}$	$\bar{x}$	P	L	d
①	0.0052382	3.0800	187.25	0.51595	0.0052382
②	0.085619	3.0764	11.456	0.55554	0.085619
③	1.4924	1.6747	0.65722	1.0253	1.4924

Eqs. (9) and (10) are plotted for this example in Figure 1. As already noted, (9) $\left(H(\tilde{x},\bar{x})=0\right)$ is an ellipse. The properties of (10) can be explained by noting that with $\tilde{x}$ fixed, the equation is a quadratic function in $\bar{x}$. The upper curve for (10) corresponds to the positive root of the function and the lower curve to the negative root. Note that $P \leq D$ since the number of vacant parking spaces cannot exceed the number of parking spaces. From (8a´) this implies that $\tilde{x} \geq \frac{\theta}{D}$ so that lower values of $\tilde{x}$ are not of economic interest; relatedly, the singularity in the positive root function of (10) at $\tilde{x} = \frac{\theta}{D}$ is not of economic interest.

Figure 1: The zero-parking-fee equilibrium with ℓ = 0.0

[7]All numerical results are presented with five non-zero digits. The computed accuracy was the maximum allowed by EUREKA, namely to 13 digits.

In Appendix 3, we provide an argument that equilibria ① and③ are stable, while equilibrium ② is unstable. The argument is supported by the fact that comparative static results in the neighborhood of equilibria① and ③ are intuitive, while those in the neighborhood of equilibrium② are perverse. Following the economic terminology applied to road congestion, which we shall explain shortly, we term equilibrium ① the congested equilibrium and ③ the hypercongested equilibrium.

In the congested equilibrium ①, over 90% of parking spaces are vacant. On car trips, individuals start cruising for parking 28´ before reaching their destination, and walk an average distance of 21´ to their destination which takes 4.7 seconds. Since $\ell = 0$, the average time parked on a car trip therefore equals 9.4 seconds. With a zero parking fee, $\tilde{x} = d$; hence individuals drive on trips exceeding 28´. Average trip duration, L, is about $33\frac{1}{2}$ minutes of which about $15\frac{1}{2}$ are spent at home with the rest spent traveling. Evidently, this equilibrium entails very little parking congestion.

The hypercongested equilibrium ③ is very different. Only about one in three hundred parking spaces is vacant. On car trips individuals start cruising for parking almost $1\frac{1}{2}$ miles before reaching their destination, and walk an average distance of 1.1 miles to their destination which takes somewhat over 22 minutes. The average time parked on a car trip therefore equals about 45 minutes. Average trip duration is about $61\frac{1}{2}$ minutes, of which about $28\frac{1}{2}$ minutes are spent at home with the remainder spent in travel. This equilibrium exhibits extreme parking congestion.

Why are there multiple equilibria? To provide an answer, we explore the analogy between flow congestion on a highway and the parking congestion considered in this paper.

Figure 2: Equilibrium with highway flow congestion

A central tenet of traditional highway engineering (Institute of Transportation Engineers (1982)) is that on a given section of road, there is a stable relationship between flow, q, and velocity, v. This relationship has the characteristic that for any flow rate between zero flow and maximum (capacity) flow, there are two velocities; for example, zero flow corresponds to zero velocity (completely jammed traffic) and also to free-flow velocity (when there are no cars on the road). Travel at the lower velocity for a given flow rate is termed hypercongested, and at the higher velocity congested. Now plot travel time against flow -- shown as Ω in Figure 2. The upper portion of the travel time curve corresponds to hypercongested travel, and the lower portion to congested travel. To simplify, ignore the money costs of travel and normalize so that the shadow cost of time equals 1.0. Then the travel time curve can be interpreted as the marginal price cost or user cost curve. Now draw in a set of demand curves which relate the number of trips demanded per unit time to travel time. As drawn, there may be only a congested equilibrium (with d_5), three equilibria (with d_3), or only a hypercongested equilibrium (with d_1). In the case where there are three equilibria, conventional stability arguments imply that the top and the bottom equilibria are stable and the middle one unstable; also the top equilibrium is termed the stable, hypercongested equilibrium and (with a slight abuse of terminology) the bottom one the stable, congested equilibrium.

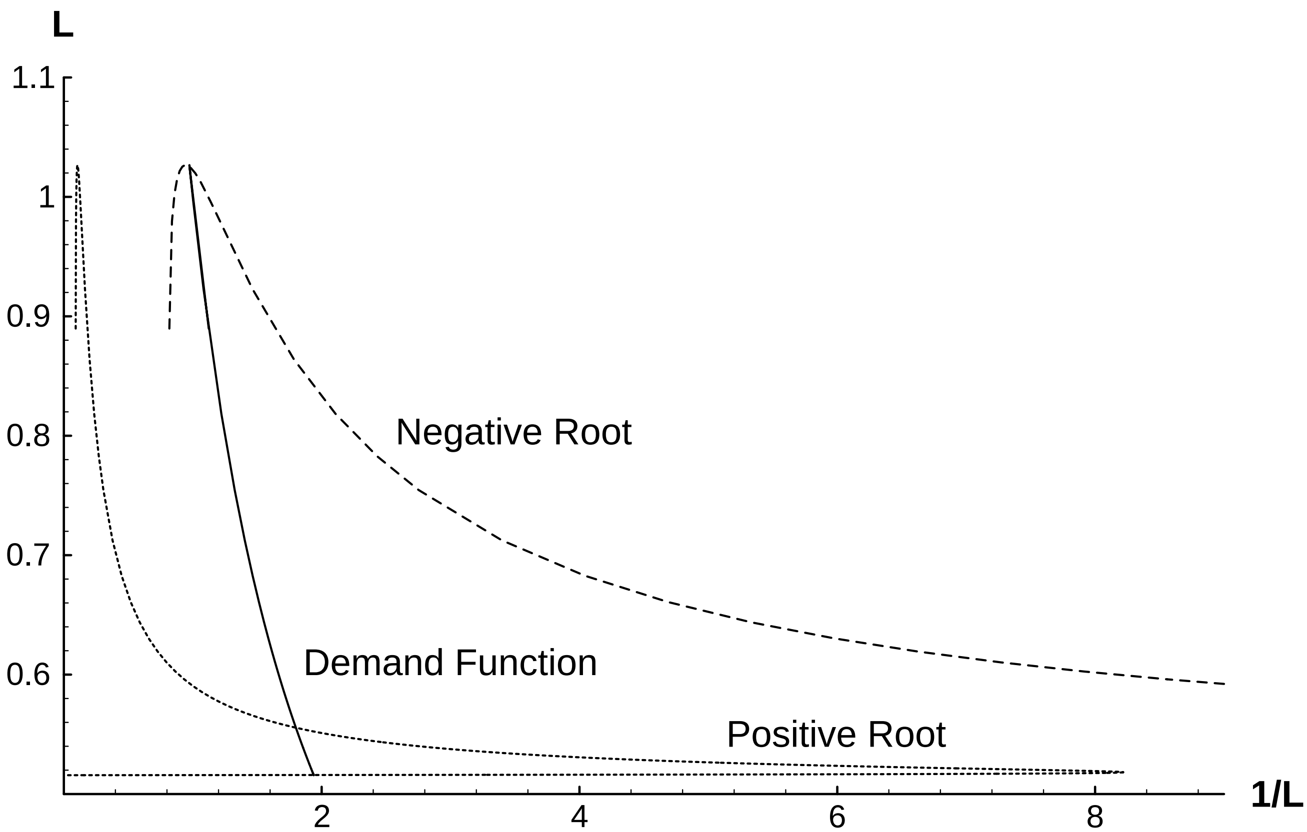

Figure 3: The analog for the parking problem of the 'supply-demand' diagram of flow congestion

We now develop an analogous analysis for the parking problem. We take as the flow rate the expected number of trips by an individual per unit time, and as the travel time the expected trip period. Since under our assumptions the individual is either traveling or waiting to travel, the demand curve is trivial: The flow rate is $\frac{1}{L}$ and travel time L, so that the demand curve is simply the unit rectangular hyperbola. To derive the user cost curve, we first substitute out $\tilde{x}$, $\bar{x}$, and d from (5), using (8a), (8b), and (8c), which gives an equation $L = \hat{L}(P)$. We then substitute out $\tilde{x}$, $\bar{x}$, and d from (6), again using (8a), (8b), and (8c), which gives an equation $S\left(\frac{1}{L}, P\right) = 0$. Then we vary P over the relevant range plotting the L's which satisfy $\hat{L}(\cdot) = 0$ against the $\frac{1}{L}$'s which satisfy

$S(\cdot) = 0$, which yields the analog of a user cost curve. Figure 3 plots the curves for the parameter values of example 1. The congestion technology is evidently more complex than that for highway flow congestion, which is perhaps not surprising considering that it incorporates the three behavioral margins of adjustment, $\tilde{x}$, $\bar{x}$, and d. The upper portion of the user cost curve, shown by the dashed line, corresponds to the negative root of $G(\cdot)$ in Figure 1, and may be interpreted as corresponding to hypercongestion. The lower portion, shown by the dotted line, corresponds to the positive root of $G(\cdot)$ = 0 in Figure 1, and may be interpreted as corresponding to congested travel. Thus, there is an analogy between flow congestion on highways and the parking congestion of the paper, but the analogy is not perfect because of differences in the two technologies.

b) <u>example 2</u>

This example has the same parameters as the previous one, except that $\ell = .25$. Thus, the minimum time parked on a car trip is 15 minutes. If $\tilde{x}$ and $\bar{x}$ were to remain the same as in the congested equilibrium of the previous example, the level of parking congestion would increase very substantially. Thus, one might expect the parameter change to cause the congested equilibrium to more closely resemble the hypercongested equilibrium. In fact, the parameter change eliminates the congested and unstable equilibria. Only the stable, hypercongested equilibrium remains, with $\tilde{x} = 1.4962$, $\bar{x} = 1.6644$, $P = 0.65554$, and $L = 1.2755$. If the analog to Figure 1 were plotted, then $H(\tilde{x}, \bar{x}) = 0$ would remain unchanged, the upper (positive root) portion of $G(\tilde{x}, \bar{x}) = 0$ would lie above $H(\tilde{x}, \bar{x}) = 0$, and the lower (negative root) portion of $G(\tilde{x}, \bar{x}) = 0$ would intersect $H(\tilde{x}, \bar{x}) = 0$ at the hypercongested equilibrium. The equilibrium is very similar to the hypercongested equilibrium ¬ of example 1, except that L is higher by about .25.

II.3 <u>Comments</u>

The comparative static properties of the equilibria can be obtained straightforwardly from (9) and (10), but the analysis is messy. The only simple comparative static exercise is with respect to μ. In terms of Figure 1, an increase in μ, the Poisson arrival rate of trip opportunities, shifts (9) inwards, towards the origin. Consider, for instance, the congested equilibrium ¿ in Figure 1. The inward shift in (9) causes $\bar{x}$ to fall and $\tilde{x}$ to rise. The mechanism is as follows: The immediate effect is that time waiting at home falls, causing the expected trip period L to fall. This in turn has two first-round effects. First, since the opportunity cost of time $\frac{\beta}{L}$ rises, the individual refuses some longer trips that she previously accepted -- $\bar{x}$ falls (eq. (8b´)). Second, trip frequency rises, which increases the parking occupancy rate (eq. (6)). The increased parking congestion in turn causes the individual to walk on some trips on which she would previously have driven -- $\tilde{x}$ rises (eq. (8a)) -- and to increase cruising distance -- d rises (eq. (8c)). The qualitative effects of the full adjustment are the same as for these first-round effects.

The possibility of multiple stable equilibria raises the issue of equilibrium selection. Which equilibrium obtains presumably depends on the path of adjustment to the stationary state. If the economy were previously highly congested, the economy should settle at the hypercongested equilibrium, while if congestion built up towards the stationary state the economy should settle at the congested equilibrium. Unfortunately, this intuition is very difficult to make precise because the transient behavior of the economy is highly complex. For example, in deciding between walking and driving, with perfect foresight an individual would have to take into account that the density of vacant parking spaces would change as she was cruising for parking.

The complexity of the model's solution is discouraging. It is, however, intrinsic to the problem. We chose our assumptions to obtain the simplest structural model that in our opinion captures the essential elements of the problem. Much of the complexity derived from the stochastic nature of finding a parking space. But, without this stochasticity, there would be no cruising for parking, which we judge to be an essential feature of the problem. Fortunately, the first-best welfare economics is relatively straightforward. However, the second-best welfare economics is comparably complex. This suggests that practical parking policy should be investigated employing realistic simulation models.

III. Social Optimum

III.1 First-order conditions and interpretation

The planner's aim is to maximize trip frequency. Unlike individuals, however, the planner takes into account the dependence of the density of vacant parking spaces on $\tilde{x}$, $\bar{x}$ and d. His optimization problem is

$$\min_{\tilde{x},\bar{x},d,P} L \quad \text{s.t} \quad \text{i)} \;\; L = \frac{1}{\bar{x}}\left[\int_0^{\tilde{x}} T_1(x)dx + \int_{\tilde{x}}^{\bar{x}} T_2(x,P,d)dx\right] + \ell + \frac{\pi r}{\mu \bar{x}}$$

$$\text{ii)} \quad \frac{\Gamma\left(W(P,d)+\ell\right)\left(\dfrac{\bar{x}-\tilde{x}}{\bar{x}}\right)}{D-P} = L$$

Constraint i) is the definition of L (eq.(5)), while constraint ii) is the parking equilibrium condition (eq.(6)). Eliminating L, the optimization problem in Langrangean form is

$$\min_{\tilde{x},\bar{x},d,P} \quad Z = \left(\frac{1}{\bar{x}}\left[\int_0^{\tilde{x}} T_1(x)dx + \int_{\tilde{x}}^{\bar{x}} T_2(x,P,d)dx\right] + \ell + \frac{\pi r}{\mu \bar{x}}\right)(1-\lambda)$$

$$+\lambda\left(\frac{\Gamma\left(W(P,d)+\ell\right)\left(\dfrac{\bar{x}-\tilde{x}}{\bar{x}}\right)}{D-P}\right), \qquad (11)$$

where ¬ is the Lagrange multiplier on ii), with L substituted out. The corresponding first-order conditions are:[8]

$$\tilde{x}: \ \frac{1}{\bar{x}}\left[(1-\lambda)\left(T_1(\tilde{x}) - T_2(\tilde{x},P,d)\right) - \lambda\frac{\Gamma\left(W(P,d)+\ell\right)}{D-P}\right] = 0 \qquad (12a)$$

$$\bar{x}: \ \frac{1}{\bar{x}}\left[-(1-\lambda)\left(L(\tilde{x},\bar{x},P,d) - \ell - T_2(\bar{x},P,d)\right) + \lambda\frac{\Gamma\left(W(P,d)+\ell\right)}{D-P}\frac{\tilde{x}}{\bar{x}}\right] = 0 \qquad (12b)$$

$$d: \ \frac{\bar{x}-\tilde{x}}{\bar{x}}\left[(1-\lambda)\frac{\partial T_2(x,P,d)}{\partial d} + \lambda\frac{\Gamma}{D-P}\frac{\partial W(P,d)}{\partial d}\right] = 0 \qquad (12c)$$

$$P: \ \frac{\bar{x}-\tilde{x}}{\bar{x}}\left[(1-\lambda)\frac{\partial T_2(x,P,d)}{\partial P} + \frac{\lambda\Gamma}{D-P}\left(\frac{\partial W(P,d)}{\partial P} + \frac{W(P,d)+\ell}{D-P}\right)\right] = 0. \qquad (12d)$$

Consider first the interpretation of (12a). When an individual walks to $\tilde{x}$, the social time it takes is $T_1(\tilde{x})$. When instead she drives to $\tilde{x}$, the expected social time of the trip equals her expected travel time plus the expected parking congestion externality she imposes through reducing the density of vacant parking spaces and hence increasing the expected travel time of other drivers. Thus, the condition for the optimal choice of $\tilde{x}$ is

$$T_1(\tilde{x}) = T_2(\tilde{x},P,d) + \text{parking congestion externality.} \qquad (12a')$$

[8] We have proved that with P fixed, the minimization problem has a unique interior minimum. We conjecture, but have not proved, that there is a unique interior minimum with P variable in the economically meaningful region of $(\tilde{x},\bar{x},d,P)$. This conjecture is supported by the numerical results reported in fn. 9.

The parking congestion externality is proportional to the length of time parked. Define E to be the *time lost by other drivers per extra minute parked* (note that E is dimensionless). Hence

$$T_1(\tilde{x}) = T_2(\tilde{x}, P, d) + E\big(W(P,d) + \ell\big). \qquad (12a'')$$

Comparing (12a) and (12a´´) yields

$$E = \frac{\lambda}{1-\lambda}\left(\frac{\Gamma}{D-P}\right) \qquad (13a)$$

Eq. (12b) has a similar interpretation. The individual should accept a trip opportunity to x if the expected social time of the trip is not greater than the expected social time until completion of the next accepted trip. Since a trip to $\bar{x}$ is by car, its expected social time is $T_2(\bar{x}, P, d) + \ell + E\big(W(P,d) + \ell\big)$. If the trip opportunity to $\bar{x}$ is declined, the expected social time until completion of the next acceptable trip is L plus the expected parking congestion externality. Since a proportion $\frac{\bar{x}-\tilde{x}}{\bar{x}}$ of trips entail parking, the expected parking congestion externality is $\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)E\big(W(P,d) + \ell\big)$. Thus, $\bar{x}$ is characterized by

$$T_2(\bar{x}, P, d) + \ell + E\big(W(P,d) + \ell\big) = L(\tilde{x}, \bar{x}, P, d) + \left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)E\big(W(P,d) + \ell\big), \qquad (12b')$$

which, using (13a), is consistent with (12b).

Similarly, in deciding on cruising distance the planner takes into account the parking congestion externality. The privately optimal cruising distance minimizes the sum of expected driving time and expected parking time. Since the externality derives from parking, the planner chooses a shorter expected parking time and a longer expected driving time, which entails a shorter cruising distance.

Eq. (12d) gives the value of λ. Combining (13a) and (12d) yields

$$E = \frac{-\dfrac{\partial T_2(x,P,d)}{\partial P}}{\dfrac{W(P,d)+\ell}{D-P} + \dfrac{\partial W(P,d)}{\partial P}}. \qquad (13b)$$

Eq. (9) continues to hold at the social optimum and we shall show in the next section that it holds as well in equilibria with a parking fee. An explanation for why (9) holds in all these situations is as follows: Let $C_i(x)$ be the cost (social cost for the social optimum, and private cost for the equilibria) of travel to x by mode i. The optimality condition with respect to $\tilde{x}$ is

$$C_1(\tilde{x}) = C_2(\tilde{x}), \tag{i}$$

which has an obvious interpretation. That with respect to $\bar{x}$ is

$$C_2(\bar{x}) - \bar{C} = V\frac{\pi r}{\mu \bar{x}}, \tag{ii}$$

where $\bar{C}$ is the average cost of travel on a trip and V is the opportunity cost of time. Eq. (ii) states that $\bar{x}$ is such that the expected travel cost savings from refusing a trip opportunity to $\bar{x}$ and waiting for the next acceptable trip opportunity equals the opportunity cost of the expected time until the next trip opportunity arrives. Now

$$\begin{aligned}\bar{C} &= \frac{1}{\bar{x}}\left[\int_0^{\tilde{x}} C_1(x)dx + \int_{\tilde{x}}^{\bar{x}} C_2(x)dx\right] \\ &= \frac{\tilde{x}}{\bar{x}}\frac{C_1(\tilde{x})}{2} + \frac{\bar{x}-\tilde{x}}{\bar{x}}\left(\frac{C_2(\bar{x}) + C_2(\tilde{x})}{2}\right)\end{aligned} \tag{iii}$$

since $C_1(x)$ and $C_2(x)$ are linear in x and $C_1(0) = 0$.

Combining (i) - (iii) yields

$$\left(\frac{C_2(\bar{x}) - C_2(\tilde{x})}{2}\right)(\bar{x} + \tilde{x}) + \frac{C_1(\tilde{x})\tilde{x}}{2} = \frac{V\pi r}{\mu}. \tag{iv}$$

Now $C_1(\tilde{x}) = \frac{2V\tilde{x}}{w}$ while $C_2(\bar{x}) - C_2(\tilde{x}) = \frac{2V(\bar{x}-\tilde{x})}{v}$. Thus, (iv) reduces to (9).

III.2 <u>Examples</u>

We return to our previous examples and report the corresponding social optima.[9]

a) <u>example 1</u> $(\ell = 0.0)$

Recall that the congested equilibrium ¿ in example 1 of the previous section (shown in Fig. 1) entailed the lowest average trip duration of the three equilibria, and

[9]To solve for the social optimum, we employed two separate packages (EUREKA and GINO) to check for accuracy. We first optimized with respect to $\tilde{x}, \bar{x},$ and d, holding P fixed, and then did a search for the optimal P. In our computational experience, every problem had a unique interior minimum (recall fn. 8).

entailed very little parking congestion. Thus, one would expect the social optimum (s.o.) to closely resemble this equilibrium, and indeed it does.

	$\tilde{x}$	$\bar{x}$	P	L	d
s.o.	0.0056159	3.0800	187.35	0.51595	0.0051148
②	0.0052382	3.0800	187.25	0.51595	0.0052382

Because it takes into account the parking congestion externality, the social optimum entails a larger maximum walking distance and a smaller cruising distance than in equilibrium ② . The level of parking congestion in the no-parking fee equilibrium is, however, so low that the lower average trip duration for the social optimum shows up only in the sixth non-zero digit. Thus, in this example the social loss in the no-parking-fee equilibrium from the uninternalized parking congestion is negligible.

b) <u>example 2</u> $(\ell = 0.25)$

Again, example 2 stands in strong contrast to example 1.

	$\tilde{x}$	$\bar{x}$	P	L	d
s.o.	1.3874	1.9265	20.966	1.0774	0.036637
e	1.4962	1.6644	0.65554	1.2755	1.4962

There are several noteworthy features of the social optimum. In contrast to the no-parking-fee equilibrium (e) where $\tilde{x}$ and d are equal, in the social optimum they are very different. There are two reasons. First, the planner takes into account that driving entails the parking externality, whereas walking does not. Recall from the discussion of (12a) that $T_1(\tilde{x}) - T_2(\tilde{x}, P, d)$ gives the magnitude of the parking externality in time units. At the social optimum, $T_1(\tilde{x}) = \frac{2\tilde{x}}{w} = .92493$ while $T_2(\tilde{x}) = \frac{2\tilde{x}}{\mathrm{v}} + \frac{4e^{-Pd}}{wP} + 2\left(d - \frac{1}{P}\right)\left(\frac{1}{w} - \frac{1}{\mathrm{v}}\right) = .25520$; thus, the parking externality in time units is .66973. Second, the planner favors a shorter cruising distance, since even though this increases travel time, holding P fixed, it reduces the length of time the driver is parked and hence the magnitude of the parking externality. To ascertain the importance of this, we may calculate the cruising distance the driver would choose with P = 20.966 and no parking fee; it is d = .046782, which exceeds the socially optimal value of d, .036637. Another feature of the social optimum is that it entails a considerably lower walking time when driving; in the no-parking-fee equilibrium it is .74323, while in the social optimum it is .022128. At the social optimum the average parking duration on a car trip is .272128, and hence the parking externality per unit time parked, E, is 2.4611, which indicates that for each extra minute a driver parks she causes a time loss to others of 2.4611 minutes. And finally, the social optimum entails a substantial reduction in average trip duration, from 1.2755 to 1.0774 hours, about 15%.

III.3 Comments

The reader may wonder why we have investigated the social optimum at some length when in fact the planner can control none of $\tilde{x}$, $\bar{x}$, d, and P directly. Obviously, doing so provides an insightful benchmark. But more than this, as we shall see, subject to a strong qualification, the social optimum is decentralizable. The only distortion is that the individual fails to take into account the externality associated with her parking, and this can be corrected via a parking fee.

Some analytical comparative static results could be derived. Given the complexity of the analysis, however, it would seem preferable to determine the comparative static properties of the social optimum numerically.

IV. The Equilibrium with a Parking Fee

This section examines both the equilibrium with a positive parking fee[10] and the decentralizability of the social optimum.

IV.1 Derivation of equilibrium

The *parking fee per unit time,* p, is specified in money units. The individual aims to maximize trip benefits net of parking fees per unit time. The average benefit per trip is $\beta - p\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)\left(W(P,d)+\ell\right)$ since $\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)$ is the proportion of trips taken by car and the average parking fee per car trip is the parking fee per unit time multiplied by the average time parked. Thus, the maximand is $\left[\beta - p\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)\left(W(P,d)+\ell\right)\right]\Big/L$. If parking revenues

[10]One can solve for equilibrium with a negative parking fee. But the analysis needs to be altered somewhat to reflect the fact that a person cannot start cruising for parking before she leaves home. We do not investigate negative parking fees since their economic relevance is dubious. For one thing, individuals would then have an incentive to park their cars on the street when at home so as to collect the parking subsidy.

are redistributed, the individual regards the payment as a fixed sum per unit time. The analysis is therefore unaffected by the redistribution of parking revenue.

Thus, the individual's maximization problem is

$$\max_{\tilde{x},\bar{x},d} V(\tilde{x},\bar{x},d;p,P) = \frac{\beta - p\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)\left(W(P,d)+\ell\right)}{\frac{1}{\bar{x}}\left[\int_0^{\tilde{x}} T_1(x)dx + \int_{\tilde{x}}^{\bar{x}} T_2(x,P,d)dx\right] + \frac{\pi r}{\mu \bar{x}} + \ell}. \tag{14}$$

$V(\cdot)$ is the *private value of time*. In undertaking the maximization, the individual regards P as given. Thus, the first-order conditions are:

$$\tilde{x}: \frac{1}{\Delta}\left[p\left(W(P,d)+\ell\right) - V\left(T_1(\tilde{x}) - T_2(\tilde{x},P,d)\right)\right] = 0 \tag{15a}$$

$$\bar{x}: \frac{1}{\Delta}\left[\beta - p\left(W(P,d)+\ell\right) - V\left(T_2(\bar{x},P,d)+\ell\right)\right] = 0 \tag{15b}$$

$$d: \frac{\bar{x}-\tilde{x}}{\Delta}\left[-p\frac{\partial W(P,d)}{\partial d} - V\frac{\partial T_2(\bar{x},P,d)}{\partial d}\right] = 0, \tag{15c}$$

where $\Delta \equiv L\bar{x}$. Each of these conditions has a straightforward interpretation. For example, (15a) indicates that the individual will choose the mode which costs less *in money terms*, where the price of time is its private value. The maximization problem has a unique, interior maximum (see Appendix 3). And the equilibrium is obtained by combining (15a) - (15c) with (5) and the definition of V, giving five equations in five unknowns, $\tilde{x}, \bar{x}, d, P,$ and V. Eq. (9) continues to hold. Comparative static analysis of this system of equations is very messy, so numerical determination of the comparative static properties of the model is justified.

IV.2 Decentralization of the social optimum

We have argued previously that there is only one distortion in the model -- individuals do not pay the full social cost associated with their parking. By setting the parking fee at the appropriate level, it should be possible to correct this distortion.

We proceed by solving for the parking fee which supports the optimum and then turn to a fuller analysis of decentralizability.

Comparison of (12a) - (12c) with (15a) - (15c) indicates that the two sets of first-order conditions can be made consistent by setting

$$\frac{p^*}{V^*} = \frac{\lambda^* \Gamma}{(1-\lambda^*)(D-P^*)} \text{ or } p^* = V^* E^*, \tag{16}$$

where * denotes evaluation at the social optimum (see Appendix 5). Eq. (16) states that the parking fee should be set equal to the parking externality in time units multiplied by the *private* value of time, both evaluated at the social optimum. Observe that the parking fee causes a divergence between the private and social value of time; the latter is $\frac{\beta}{L}$. Since V is a function of p, (16) is an implicit equation. Combining (16) and (14) gives the following explicit equation for p^*:

$$p^* = \frac{E^* \beta}{L^* + E^* \left(\frac{\bar{x}^* - \tilde{x}^*}{\bar{x}^*}\right)(W^* + \ell)}. \tag{16´}$$

Now recall example 1. There were two stable equilibria, one congested and one hypercongested. There was very little parking congestion in the congested equilibrium, and the social optimum was very similar to the congested equilibrium. These observations suggest that application of the optimal parking fee, computed per (16´), should cause the congested equilibrium to coincide with the social optimum but, since the optimal parking fee is so low, should not eliminate the hypercongested equilibrium. As we shall see, such is indeed the case. Thus, application of the optimal parking fee given by (16´) does not necessarily result in attainment of the social optimum.

We provided an intuitive argument earlier that one should expect the hypercongested equilibrium to occur if parking is highly congested in the adjustment to the stationary state, and the congested equilibrium to occur otherwise. This argument suggests that the social optimum can always be attained by an appropriate dynamic parking fee. By setting the parking fee sufficiently high in the adjustment to the stationary state, the planner should be able to ensure that the economy ends up at the social optimum. Investigation of the conjecture will require exploration of the non-stationary dynamics of the model, which we do not explore here.[11]

IV.3 Examples

We return to our previous examples, and explore the effects of parking fees on the equilibrium.

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[11]Support for this conjecture is provided by Figure 4a, which plots equilibria as a function of the parking fee for $\ell = 0$. If the parking fee is set sufficiently high, the hypercongested equilibrium disappears.

a) example 1 $(\ell = 0.0)$

We start by computing the optimal parking fee and then solving for the corresponding equilibria. The optimal parking fee, with $\beta = \$10,$ is $p^* = \$1.4232$ / hr. The results are presented in the table below. A heavy dot in a cell indicates that the values are the socially optimal ones. The table contains few surprises. Since the congested no-parking-fee equilibrium was very little congested, the optimal parking fee is low. Application of the fee does indeed result in the congested equilibrium coinciding with the social optimum. But since the optimal fee is low, its application does not substantially alter the other equilibria. It is, however, worthy of note that application of the parking fee *increases* trip duration for the unstable equilibrium.[12]

		$\tilde{x}$	$\bar{x}$	P	L	d
①	p=p*	0.0056162	3.0800	187.35	0.51595	0.0051149
	p=0	0.0052382	3.0800	187.25	0.51595	0.0052382
②	p=p*	0.093515	3.0757	11.315	0.55608	0.084541
	p=0	0.085619	3.0764	11.456	0.55554	0.085619
③	p=p*	1.4878	1.6967	0.75598	1.0132	1.2425
	p=0	1.4924	1.6747	0.65722	1.0253	1.4924

We next investigate the effects of imposing a non-optimal parking fee on the equilibrium.[13] These effects can be explained using Figure 4a, which plots the equilibrium $\tilde{x}$ as a function of p. A feature of special interest is that for p > \$56.45/hr., the unstable and hypercongested equilibria disappear. Thus, a sufficiently high parking fee "unlocks" the hypercongested equilibrium. As the parking fee is increased, for each stable equilibrium type the density of vacant parking spaces increases. Whether $\tilde{x}$ is positively or negatively related to p depends on two competing effects. On one hand, a higher parking fee makes driving less attractive; on the other hand, it makes parking less congested which makes driving more attractive. In the examples the former effect dominates for the congested equilibrium (since there is little congestion) and the latter

[12]Since perverse comparative static results are characteristic of unstable equilibria, this result supports the argument that the intermediate equilibrium is unstable.
[13]We determined these equilibria by employing a combination of MATHEMATICA and numerical comparative statics. Using MATHEMATICA alone, we encountered serious difficulties in the numerical solution. We developed a solution procedure supplementing MATHEMATICA which circumvented these numerical problems. The procedure is described in Arnott and Rowse(1995).

effect dominates for the hypercongested equilibrium (since an increase in the parking fee reduces congestion substantially).

Figure 4a: Equilibrium as a function of the parking fee, ℓ = 0.0

b) <u>example 2</u> $(\ell = 0.25)$

The optimal parking fee is $p^* =$ \$19.459/hr. Application of the optimal parking fee decentralizes the social optimum. Prior to the application of the parking fee, benefit per hour is $V = \frac{\beta}{L}$ = \$7.843. With the optimal parking fee, benefit per hour without redistribution of parking fee revenues is $V = \frac{\beta - p\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right)(W+\ell)}{L}$ = \$7.906. Thus, even without redistribution of parking fee revenues, individuals are made better off by the parking fee. In contrast, in example 1 individuals are worse off in the decentralized social optimum if toll revenues are not redistributed than in the no-parking-fee equilibrium ①. These results are consistent with the economics of congestion pricing, where unredistributed optimal tolls help drivers if the pre-toll equilibrium is hypercongested but hurt them if it is congested.

The effects of imposing a non-optimal parking fee can be explained using Figure 4b. For p between 0 and \$18.90/hr., $\tilde{x}$ decreases with p -- driving becomes more attractive since the reduction in congestion more than compensates for the higher parking fee. For p between \$18.90/hr. and \$23.80/hr., $\tilde{x}$ increases with p -- the reduction in congestion does not compensate for the higher parking fee. At a parking fee slightly above \$23.80/hr., no one drives. The parking fee is so high that even though there is no congestion, the cost of parking on a car trip ($p\ell \approx \$5.95$) makes walking cheaper, even on the longest trips.

Figure 4b: Equilibrium as a function of the parking fee, $\ell = 0.25$

The effects of the parking fee were so different for $\ell = 0.0$ and $\ell = 0.25$ that we explored an intermediate case, $\ell = 0.03$.

c) <u>example 3</u> $(\ell = 0.03)$

Figure 4c: Equilibrium as a function of the parking fee, $\ell = 0.03$

Figure 4c plots p against $\tilde{x}$ for this example. With a zero parking fee, there is only a single equilibrium which is hypercongested. As the parking fee is raised, a critical parking fee, $\hat{p}$, is reached at which the congested and unstable equilibria appear. As p is raised further, another critical parking fee, $\hat{\hat{p}}$, is reached at which the unstable and hypercongested equilibria disappear.[14] For parking fees above $\hat{\hat{p}}$, there is only a single equilibrium which is congested. This example is interesting since it demonstrates that

[14]Thus, there are three equilibria for $p \in \left(\hat{p}, \hat{\hat{p}}\right)$. For example, with p = 61.5, $\tilde{x}$ = .804 for the hypercongested equilibrium, .489 for the unstable equilibrium, and .414 for the congested equilibrium.

raising the parking fee may not only cause a hypercongested equilibrium to disappear, as in example 1, but may also cause a congested equilibrium to appear where one did not exist in the absence of a parking fee.

The results for this example are broadly analogous to those obtained in the economic analysis of flow congestion on highways. Refer to Figure 2, and suppose that the demand curve in the absence of a toll is d_1. Then initially there is a single equilibrium, which is stable and hypercongested. Raising the toll causes the demand curve to shift down. When the demand curve falls to d_2, two new equilibria appear, one an unstable, hypercongested equilibrium, the other a stable, congested equilibrium. When the toll is further increased to the level associated with d_4, both the hypercongested equilibria disappear.

V. Extensions

We have deliberately kept the model as simple as possible, both to elucidate basic points and to keep the algebra manageable. Numerous extensions in the direction of realism are possible. Since the algebra for even the basic model is quite difficult, with many of the extensions numerical solution will probably be necessary. In this section, we simply discuss most of the extensions. We do, however, present one extension in some detail since the model is so well-suited to treating it -- the provision of information on parking availability and its effect on equilibrium.

V.1 Demand

It is useful to think of an individual as deriving utility from activities -- e.g. commuting to work, attending a baseball game, playing with the kids, eating out with friends. Each of these activities can be undertaken at only a subset of locations and over certain intervals of time. Furthermore, the utility from an activity depends on the length of time spent at it as well as the goods which are purchased in conjunction with it. According to this conceptualization, the individual's utility maximization entails solving a scheduling cum budget allocation problem, the demand for parking is derived from the solution to this problem, and trip chaining -- whereby the individual undertakes several different activities on a trip from home and back again -- occurs. The problem is further complicated if account is taken of the uncertainties associated with travel time and the length of time needed to undertake various activities, for then the individual continuously updates the solution to a stochastic scheduling problem.

While there is broad agreement among experts that this is the "right" way to think about transport demand, because of the complexity of its implementation this conceptualization has not been made operational. A less conceptually satisfying but more operational method is to enrich the approach taken in this paper, whereby an individual receives trip opportunities according to a stochastic process. Trip length and trip benefit could be made random variables. Alternatively, a trip opportunity could be characterized by a random function, the realization of which would specify trip benefit as a function of length of time spent at the destination, and the individual would choose how long to spend at the destination. A trip opportunity could also specify the period of time over which the opportunity was in effect, which would permit the treatment of trip chaining.

In this paper, the individual allocates her time so as to maximize per unit time the expected benefit received at trip destinations, which implies that the individual's value of time is the same whether she is driving on a freeway, cruising for parking, walking, or at home. It is well-documented, however, that individuals value time in different activities significantly differently. The model can be augmented to take this into account. A simple way of doing this is to assume that the individual maximizes benefits minus costs per unit time, where the costs includes the time cost in various activities (relative to, say, being at home). A slightly more sophisticated approach is to assume that the individual maximizes utility, which is a function of the time spent in various activities.

It should be possible to calibrate the specification of demand in such a way that the pattern of trips, by origin/destination/parking duration is close to that observed, which will permit calculation of how the pattern of parking demand is altered in response to policy changes.

V.2 Parking supply

In the model, all parking was operated by the planner and the number of parking spaces was fixed. It would be relatively straightforward to provide a more sophisticated treatment of parking supply. At each location the supply of land for roads and for other uses would be specified. The land for roads would be used for either traffic or on-street parking. Allocating more land to on-street parking would increase the availability of parking, but would exacerbate traffic congestion. The land for other uses would be allocated between housing and off-street parking. Increasing the amount of off-street parking would cause construction of housing at higher density, which would increase housing costs. The allocation of land between these uses could be optimized.

While on-street parking is provided by the government, much off-street parking, at least in North American cities, is operated by the private sector. Even if traffic congestion were efficiently priced, the private pricing of off-street parking would likely be inefficient since the friction of space provides private parking operators with market power. And if traffic congestion were inefficiently priced, parking operators would base their decisions on distorted prices. Since land values would be distorted, the market allocation of land between off-street parking and housing would be inefficient. Also, in making their pricing decisions, private parking operators would collectively neglect the effect of their pricing on traffic congestion. Thus, the taxation/regulation of private, off-street parking is an important issue in parking policy.

V.3 <u>Traffic congestion</u>

The model ignored traffic congestion -- car travel speed was assumed to be independent of the density or flow of cars on the road. Incorporating traffic congestion would not be difficult.[15] Flow congestion can be treated by assuming that travel speed in regular traffic depends on the density of cars in regular traffic as well as on effective capacity which is influenced by the amount of on-street parking, the rate at which cars enter and leave on-street parking spaces, the volume of pedestrian traffic, and the density of cars cruising for parking. Cruising-for-parking congestion is also important and can be modeled similarly to flow congestion. Other forms of congestion which may be desirable to incorporate include parking entry-and-exit congestion, both for on- and off-street parking, pedestrian congestion and, in network models, intersection congestion and gridlock.

The incorporation of traffic congestion is not only quantitatively very important, but also, when- -as is typically the case- -traffic congestion is underpriced, qualitatively changes the economics since parking policy is then an exercise in the second best. Suppose, for example, that the only forms of congestion are traffic flow congestion and parking congestion, and that the distribution of parking duration is independent of the distance driven on the trip. Then the second-best parking fee would include a fixed,

[15]Flow congestion can be incorporated straightforwardly into the model by making car speed depend on traffic density which can be calculated straightforwardly.

second-best component approximately equal to the average flow congestion externality associated with car trip[16] plus a first-best component linear in time parked to cover the parking congestion externality. More generally, the second-best parking fee would be set to minimize the deadweight loss associated with the myriad forms of unpriced congestion.

V.4 Alternative parking strategies

In the paper it was assumed that an individual starts cruising for parking a certain distance from her destination and then continues cruising on the single road until she finds a parking spot. But with the single road the individual can also backtrack, sit and wait for a spot to become available, or simply double park. If the model were extended to two dimensions -- a Manhattan grid network of streets, for example -- the range of parking strategies would be considerably greater. It would be interesting to explore which parking strategies are privately and socially optimal under different traffic conditions, and also to solve for the optimal fine for double-parking.[17]

V.5 Stochasticity

Parking may be modeled at varying degrees of sophistication. At the simplest, one can suppress the stochasticity associated with finding a parking space and posit a function $s = s(\phi)$ which gives time spent parked on a car trip as a function of the occupancy rate of parking spaces, ϕ. With such a specification, one would obtain most of the qualitative results we have found -- the optimal parking fee equals the parking congestion externality, there may be multiple equilibria, etc. The danger of such an implicit approach is that important insights may be lost. In the current context, for example, if we had started with the function $s(\phi)$ and assumed it to be technological, we would have overlooked the efficiency gains from congestion pricing that derive from its impact on the modal choice and the cruising-for-parking decisions.

At the other extreme, one can provide an exact mathematical treatment of the stochasticity associated with finding a parking spot by employing stochastic queuing theory (Cooper (1981), Syski (1986)). Each parking spot can be regarded as a separate

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[16]Short trips then would be priced above marginal cost and long trips below marginal cost. Whether the second-best component would be greater than or less than the average flow congestion externality would depend on the relative demand elasticities of short and long trips.

[17]This was discussed by Polinsky and Shavell (1979). Their aim was to provide an explanation for why the optimal fine for all crimes is not infinite. There are situations in which the social benefit from double parking exceeds the social cost.

server, with the service time equal to the length of time parked and the distribution of service times endogenously determined. Servers are spatially ordered and the customer travels from server to server at a specified speed until she finds one that is idle. This is not a conventional stochastic-queuing-theoretic problem. While it can presumably be solved from first principles, doing so would be difficult and the solution would be messy.

We adopted an intermediate strategy, explicitly treating individuals' behavioral decisions and the stochasticity of parking availability, but making the approximation that the probabilities of adjacent parking spaces being vacant are statistically independent, which permitted application of the simple mathematics of Poisson processes. An analogous approximation is made in intersection congestion analysis (Institute of Transportation Engineers (1982)), where it is assumed that the arrival rate of cars at an intersection is generated by a Poisson process, even though it evidently is not (if cars were delayed by a traffic light at an upstream intersection, for example, they will have traveled as a platoon to the intersection under consideration).

In the paper, parking fees were independent of the particular realization of vacant parking spaces. Back in the fifties, Vickrey (1959) proposed <u>responsive</u> pricing for parking, whereby the parking fee for a parking space would be based on the realized availability of nearby vacant parking spaces. The efficiency gains from responsive parking pricing are larger the better-informed is the individual concerning the pattern of vacant parking spaces when making her usage decisions. At one extreme, when the individual is perfectly informed, responsive pricing provides the first-best allocation of parking spaces; at the other extreme, where the individual knows only the mean density of parking spaces when making her trip and parking decisions, responsive pricing simply adds insult to injury -- if, by bad luck, all the parking spaces close to her destination are occupied, she has to pay more for parking.

Because it explicitly treats the stochasticity of vacant parking spaces, our model is particularly well-suited for examining the value of parking information. Many cities in Europe and Japan put signs on major arterial roads indicating the availability of parking at the major parking lots. And there is talk of providing parking information to drivers via computer either before they start a trip or when they are in transit (Asakura and Kashiwadani (1995)).

To illustrate how the model can be adapted to deal with "informatics," we now consider a particularly simple parking information system (PIS) in which each driver is informed, at the time she receives a trip opportunity, of the available parking spot that is closest to her destination in terms of travel time. If she decides to drive, the parking spot is then reserved for her and she must take that spot. There is no parking fee.

Since the individual is tentatively assigned a parking spot when she receives a trip opportunity, she knows exactly the duration of the trip opportunity -- the lesser of the travel times walking and driving. Thus, she will adopt a reservation travel time rule in accepting trip opportunities. It is shown in Appendix 6 that the probability that travel time for a trip opportunity is less than t is[18]

$$H(t) = \frac{1}{\pi r}\left\{\frac{vt}{2} - \frac{1}{A}\left(1 - e^{-\frac{A(v-w)t}{2}}\right)\right\}, \quad A = \frac{2Pvw}{v^2 - w^2}, \tag{17}$$

(with $h(t)$ denoting the corresponding pdf) where P is the density of available parking spaces. A parking space is available if it is vacant and unreserved.

The individual chooses t', reservation travel time, to minimize expected trip period

$$L(t') = T(t') + \ell + \frac{1}{\mu H(t')}, \tag{18}$$

where $T(t')$ is average travel time as a function of t':

$$\begin{aligned} T(t') &= \frac{1}{H(t')}\int_0^{t'} th(t)dt \\ &= \frac{1}{H(t')}\left(\left(tH(t)\right)_0^{t'} - \mathfrak{H}(t')\right), \left(\mathfrak{H}(t') = \int_0^{t'} H(t)dt\right) \\ &= t' - \frac{\mathfrak{H}(t')}{H(t')}. \end{aligned} \tag{19}$$

The first-order condition is

$$T'(t') - \frac{h(t')}{\mu H(t')^2} = 0,$$

which using (19) reduces to $\mathfrak{H}(t') = \frac{1}{\mu}$, or

[18]Again under the working assumption that the probabilities of adjacent parking spaces being available are statistically independent.

$$\frac{v(t')^2}{4} - \frac{t'}{A} + \frac{2}{A^2(v-w)}\left(1 - e^{-\frac{A(v-w)}{2}t'}\right) = \frac{\pi r}{\mu}. \tag{20}$$

From (18), (19) and $\mathfrak{H}(t') = \frac{1}{\mu}$, $L(t') = t' + \ell$. If the individual is offered a trip with travel time t', she is indifferent between accepting and refusing it. If she accepts it, the duration of the trip is $t' + \ell$; if she rejects it, the expected time waiting for the next acceptable trip opportunity plus the expected duration of the trip is $L(t')$.

Then there is the stationary-state condition. This can be calculated as "the ratio of unavailable parking spaces to the population equals the ratio of average "parking time" on a trip, $K(t',P)$, to average trip period," where the parking time on a trip equals the time the parking spot is occupied plus the time the parking spot is unoccupied but reserved, while the individual is driving from home to the parking spot:

$$\frac{D-P}{\Gamma} = \frac{K(t',P)}{L}. \tag{21}$$

The average parking time on a trip is calculated as

$$K(t',P) = \int_0^\infty \int_0^{t'} k(t,x) g(t|x) f(x) dt\, dx \div H(t'), \tag{22}$$

where $k(t,x)$ is the average parking time on a journey to x with travel time t, and $g(t|x) f(x)$ is the joint p.d.f. of x and t, which are given in Appendix 6. With $K(t',P)$ substituted out, (20) - (22) provide two equations in t' and P.

We now investigate a numerical example. Recall that, in the absence of PIS, there were three equilibria: a stable, congested equilibrium with (P,L) = (187.25, 0.51595), an unstable, congested equilibrium with (P,L) = (11.456, 0.55554), and a stable, hypercongested equilibrium with (P,L) = (0.65722, 1.0253). With the PIS

P = .042052 $\qquad\qquad$ $L(=t')$ = 0.98042.

Introduction of the PIS causes the two congested equilibria to disappear, but results in a reduction in expected trip time of about 4.4% in the hypercongested equilibrium.

The PIS is very inefficient when without it the parking occupancy rate would be low. To see why, suppose that the economy is in the stable, congested equilibrium without the PIS, and that <u>one</u> individual is provided with the PIS. Her trip length would be reduced by about three seconds but this would come at the social cost of holding an unoccupied parking spot reserved on each of her outbound car trips, for an average of about nine minutes, which would generate a parking externality of about 40 seconds (calculated as E times nine minutes). In fact, the PIS does even worse than this calculation suggests -- it eliminates the stable, congested equilibrium.

But when the parking occupancy rate would be high without the PIS, the PIS is welfare improving. The gains from directing drivers to the closest parking spaces and from having them reject trips they would have taken without parking information more than offsets the costs of unoccupied but reserved parking spaces.

Intuition suggests that this parking information system would be improved by reserving a parking spot for an individual only if parking close to her destination is scarce, and would be further improved by updating the reserved parking spot as parking spots become available while the individual is en route. This intuition is not completely correct, however, since when prices are distorted (here parking is underpriced) better information is not always welfare-improving.[19] A possible further refinement would entail the planner assigning the individual the parking spot with the lowest expected social cost rather than the lowest expected private cost.

Responsive pricing and the provision of parking information are synergistic. The policies are technologically complementary since information on pricing and the availability of parking would presumably be provided by the same computer system. Also, as noted earlier, responsive pricing is more effective the better informed are users, while with responsive pricing (under which the individual would face expected social cost on all parking-related decisions) better information would always be welfare-improving.

V.6 Practical complications

Traffic engineers have recently been devoting considerable effort to the development of parking simulation models for actual road/parking networks (Muromachi et al. (1995) and the references therein). The level of sophistication of these models is impressive. Car trips by destination are generated by time-dependent Poisson processes, which can be made cost-sensitive; parking duration is stochastic; each user decides which parking lot to try first on the basis of imperfect information on parking availability at the various lots and, if unpleasantly surprised when she arrives, may adopt one of several

[19] This point is made in the context of bottleneck congestion on a road by Arnott, de Palma, and Lindsey (1991b).

parking search strategies, etc. Economists can make two sorts of contributions to the development of such models. First, economic theory can be employed to strengthen their behavioral underpinnings. And second, analytical modeling along the lines indicated in this section can provide insight into the simulation results.

VI. Conclusion

This paper has developed a model of parking. The model was constructed with four principal considerations in mind. First, the model was primitive or structural, rather than reduced-form, *viz.* the demand for parking and the congestion cost function were derived rather than assumed. Second, the model was designed so that it can be extended to incorporate a host of realistic complications, and so that, appropriately elaborated, it can be employed in practical policy analysis. Third, the model was general equilibrium so as to ensure a rigorous conceptual basis for welfare analysis. And fourth, the model was designed to focus on stochastic aspects of parking, especially cruising for parking. The model was the simplest we could think of which satisfied these four criteria. This simplicity was achieved by assuming spatial and temporal homogeneity and by ignoring the flow congestion of cars.

The most remarkable feature of the model was that, despite extreme simplification, its behavior was complex. In particular, multiple stable equilibria are possible; which equilibrium obtains depends presumably on the history of the economy prior to reaching the stationary state. This is discouraging since it suggests that non-stationary-state analysis is needed, but this will be difficult. The comparative static properties of the model are complex as well; recall the sensitivity of even the qualitative properties of the solutions to changes in the parking fee and in parking duration. The welfare economics of the model was, however, relatively straightforward. The parking fee should be set at the value of the parking congestion externality. However, setting the parking fee at that level which decentralizes the social optimum does not guarantee attainment of the optimum; the economy may remain stuck at a hypercongested equilibrium. Unfortunately, the complexity of the model appears to be intrinsic to the parking problem rather than an artifact. Thus, sound *analytical* work on parking would appear to be discouragingly difficult. Despite computational problems deriving from the "highly" nonlinear nature of the model, numerical solution would appear to be a more fruitful avenue for future research, and the model we have constructed should provide a useful starting point for the construction of practical parking simulation models.

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Appendix 1

Partial Derivatives of $W(P,d)$ and $T_2(x,P,d)$

$$\frac{\partial W}{\partial d} = \frac{2}{w}\left(-2e^{-Pd} + 1\right) \tag{A1.1}$$

$$\frac{\partial W}{\partial P} = \frac{2}{w}\left(\frac{-2de^{-Pd}}{P} - \frac{2e^{-Pd}}{P^2} + \frac{1}{P^2}\right) \tag{A1.2}$$

$$\frac{\partial^2 W}{\partial d^2} = \frac{\partial^2 T_2}{\partial d^2} = \frac{2}{w}\left(2Pe^{-Pd}\right) > 0 \tag{A1.3}$$

$$\frac{\partial^2 W}{\partial d \partial P} = \frac{\partial^2 T_2}{\partial d \partial P} = \frac{2}{w}\left(2de^{-Pd}\right) > 0 \tag{A1.4}$$

$$\frac{\partial T_2}{\partial d} = \frac{\partial W}{\partial d} - \frac{2}{v} = -\frac{4e^{-Pd}}{w} + 2\left(\frac{1}{w} - \frac{1}{v}\right) \tag{A1.5}$$

$$\frac{\partial T_2}{\partial P} = -\frac{4e^{-Pd}}{w}\left(\frac{d}{P} + \frac{1}{P^2}\right) + \frac{2}{P^2}\left(\frac{1}{w} - \frac{1}{v}\right) \tag{A1.6}$$

$$\frac{\partial^2 T_2}{\partial P^2} = \frac{4e^{-Pd}}{wP}\left(d^2 + \frac{2d}{P} + \frac{2}{P^2}\right) - \frac{4}{P^3}\left(\frac{1}{w} - \frac{1}{v}\right) \tag{A1.7}$$

Appendix 2

Comparative Statics of (5) and (6)

$$L\bar{x} - \int_0^{\tilde{x}} T_1(x)dx - \int_{\tilde{x}}^{\bar{x}} T_2(x,P,d)dx - \ell\bar{x} - \frac{\pi r}{\mu} = 0 \qquad (5)$$

$$(D-P)L\bar{x} - \Gamma(W+\ell)(\bar{x}-\tilde{x}) = 0 \qquad (6)$$

Total differentiation of this pair of equations gives

$$\begin{bmatrix} -(\bar{x}-\tilde{x})\dfrac{\partial T_2}{\partial P} & \bar{x} \\ -L\bar{x} - \Gamma\dfrac{\partial W}{\partial P}(\bar{x}-\tilde{x}) & (D-P)\bar{x} \end{bmatrix} \begin{bmatrix} dP \\ dL \end{bmatrix} = \begin{bmatrix} T_1(\tilde{x}) - T_2(\tilde{x},P,d) \\ -\Gamma(W+\ell) \end{bmatrix} d\tilde{x}$$

$$+\begin{bmatrix} -L+\ell+T_2(\bar{x},P,d) \\ -(D-P)L+\Gamma(W+\ell) \end{bmatrix} d\bar{x} + \begin{bmatrix} \dfrac{\partial T_2}{\partial d}(\bar{x}-\tilde{x}) \\ \Gamma\dfrac{\partial W}{\partial d}(\bar{x}-\tilde{x}) \end{bmatrix} dd + \begin{bmatrix} 0 \\ -L\bar{x} \end{bmatrix} dD$$

Define Ô – $\begin{bmatrix} -(\bar{x}-\tilde{x})\dfrac{\partial T_2}{\partial P} & \bar{x} \\ -L\bar{x} - \Gamma\dfrac{\partial W}{\partial P}(\bar{x}-\tilde{x}) & (D-P)\bar{x} \end{bmatrix}$ for use in Appendix 5.

Note that for a given $\tilde{x}, \bar{x},$ and d, there may be more than one economically sensible solution to (5) and (6).

Appendix 3

Stability Analysis

This appendix provides an argument that equilibria ① and ③ in Fig. 1 are stable, while equilibrium ② is unstable. We start at an equilibrium, introduce a perturbation, and ascertain whether the economy returns to the equilibrium. A natural perturbation is a below-average realization of trip demand for an extended period which has resulted in P remaining steady above its equilibrium level for some time. We say that an equilibrium is stable if $\frac{d\dot{P}}{dP} < 0$ evaluated at the equilibrium, and unstable if $\frac{d\dot{P}}{dP} > 0$.

Now, $\dot{P}$ equals the number of drivers leaving parking spaces per unit distance minus the number entering. The temporal evolutions of P and $\dot{P}$ are determined by a complex dynamic stochastic process. Consequently, it is very difficult to provide a precise characterization of the disequilibrium adjustment process. We can, however, obtain an approximation to it, on the assumption that the current values of $\tilde{x}, \bar{x}$, W, and L are based on the current value of P. Then,

$$\dot{P} = \frac{D-P}{W+\ell} - \frac{\Gamma}{L}\left(\frac{\bar{x}-\tilde{x}}{\bar{x}}\right) \tag{A3.1}$$

with

$$\tilde{x} = \frac{\theta}{P} \qquad \text{(from (8a´))} \tag{A3.2}$$

and

$$W = \frac{2}{P}\left(\frac{\theta}{w} - \frac{1}{v}\right). \qquad \text{(from (3) and (8c´)} \tag{A3.3}$$

We wish to conduct our analysis in $\tilde{x}-\bar{x}$ space. To do this, we substitute out W, P, and $\dot{P}$ from (A3.1), using (A3.2) and (A3.3), to obtain

$$\dot{\tilde{x}} = \left(\frac{\Gamma}{L\bar{x}}(\bar{x}-\tilde{x}) - \frac{D-\frac{\theta}{\tilde{x}}}{\frac{2\tilde{x}}{\theta}\left(\frac{\theta}{w}-\frac{1}{v}\right)+\ell}\right)\frac{\tilde{x}^2}{\theta}. \tag{A3.4}$$

Using the definitions of $H(\tilde{x},\bar{x})$ and $G(\tilde{x},\bar{x})$ from (9) and (10), (A3.4) reduces to

$$\dot{\tilde{x}} = \left(-G(\tilde{x},\bar{x}) + H(\tilde{x},\bar{x})\left(D - \frac{\theta}{\tilde{x}}\right)\right)\frac{\tilde{x}^2}{\theta} \Big/ \left[L\bar{x}\left(\frac{2\tilde{x}}{\theta}\left(\frac{\theta}{w} - \frac{1}{v}\right) + \ell\right)\right]. \tag{A3.5}$$

According to our assumptions that $\tilde{x}$, $\bar{x}$, and d are based on the current value of P, from (8a) - (8c), $H(\tilde{x},\bar{x}) = 0$ even out of equilibrium. Then differentiating $\dot{\tilde{x}}$ with respect to $\tilde{x}$, varying $\bar{x}$ such that $H(\tilde{x},\bar{x}) = 0$, and evaluating at an equilibrium:

$$\frac{d\dot{\tilde{x}}}{d\tilde{x}} = \left(-\left.\frac{dG}{d\tilde{x}}\right|_{(9)}\right)\frac{\tilde{x}^2}{\theta} \Big/ \left[L\bar{x}\left(\frac{2\tilde{x}}{\theta}\left(\frac{\theta}{w} - \frac{1}{v}\right) + \ell\right)\right]. \tag{A3.6}$$

From (10), $G(\tilde{x},\bar{x}) > 0$ on (9) along the 45° line which is south-east of the hypercongested equilibrium ③ in Figure 1. Hence $\left.\frac{dG}{d\tilde{x}}\right|_{(9)} > 0$ at ③. That $\left.\frac{dG}{d\tilde{x}}\right|_{(9)} < 0$ at ② and $\left.\frac{dG}{d\tilde{x}}\right|_{(9)} > 0$ at ① follow immediately.

Thus, $\frac{d\dot{\tilde{x}}}{d\tilde{x}} < 0 \Leftrightarrow \frac{d\dot{P}}{dP} < 0$ at ① and ③, and $\frac{d\dot{\tilde{x}}}{d\tilde{x}} > 0 \Leftrightarrow \frac{d\dot{P}}{dP} > 0$ at ②. Hence, given the assumed disequilibrium adjustment process, equilibria ① and ③ are stable, while equilibrium ② is unstable.

Appendix 4

The solution to the individual's maximization problem is unique

We ignore corner solutions since they are not of practical interest. Hence, we need only show that the second-order conditions for the individual's maximization problem are satisfied. The first-order conditions are given in (15). The corresponding second-order conditions are:

$$V_{\tilde{x}\tilde{x}}: \quad -\frac{V}{\Delta}\left(\frac{\partial T_1(\tilde{x})}{\partial \tilde{x}} - \frac{\partial T_2(\tilde{x},P,d)}{\partial \tilde{x}}\right) = -\frac{V}{\Delta}\left(\frac{2}{w} - \frac{2}{\mathrm{v}}\right) < 0 \quad \text{(using (1) and (4), and } \Delta > 0\text{)}$$

$$V_{\tilde{x}\overline{x}}: \quad 0$$

$$V_{\tilde{x}d}: \quad \frac{1}{\Delta}\left[p\frac{\partial W}{\partial d} + V\frac{\partial T_2}{\partial d}\right] = 0 \qquad \text{(from (15c))}$$

$$V_{\overline{x}\overline{x}}: \quad -\frac{V}{\Delta}\ \frac{\partial T_2(\overline{x},P,d)}{\partial \overline{x}} = -\frac{V}{\Delta}\ \frac{2}{v} < 0$$

$$V_{\overline{x}d}: \quad \frac{1}{\Delta}\left[-p\frac{\partial W}{\partial d} - V\frac{\partial T_2}{\partial d}\right] = 0 \qquad \text{(from (15c))}$$

$$V_{dd}: \quad \frac{\overline{x} - \tilde{x}}{\Delta}\left[-p\frac{\partial^2 W}{\partial d^2} - V\frac{\partial^2 T_2}{\partial d^2}\right] < 0 \quad \text{since}$$

$$\frac{\partial^2 W}{\partial d^2} = \frac{\partial^2 T_2}{\partial d^2} = \frac{4P}{w}e^{-Pd} > 0 \quad \text{(from Appendix 1).}$$

Thus, the Hessian matrix for V is negative definite when the first-order conditions are satisfied.

Appendix 5

Proof that there exists a parking fee such that the first-order conditions of the individual's maximization problem are consistent with the first-order conditions of the planning problem

Let * denote the value of a variable at the social optimum and $P(\tilde{x}, \bar{x}, d)$ denote the optimal value of P consistent with (5) and (6) for a given $\tilde{x}$, $\bar{x}$, and d. Then the social optimization problem can be written as

$$\min_{\tilde{x},\bar{x},d} \quad L = \frac{1}{\bar{x}}\left[\int_0^{\tilde{x}} T_1(x)dx + \int_{\tilde{x}}^{\bar{x}} T_2\big(x, P(\tilde{x}, \bar{x}, d), d\big)dx\right] + \ell + \frac{\pi r}{\mu \bar{x}}. \tag{A5.1}$$

With this formulation of the social optimization problem, the first-order condition with respect to $\tilde{x}$ is

$$T_1(\tilde{x}^*) - T_2\big(\tilde{x}^*, P^*(\tilde{x}^*, \bar{x}^*, d^*), d^*\big) + (\bar{x}^* - \tilde{x}^*)\frac{\partial T_2^*}{\partial P}\frac{\partial P^*}{\partial \tilde{x}} = 0. \tag{A5.2}$$

From Appendix 2,

$$\frac{\partial P(\tilde{x}^*, \bar{x}^*, d^*)}{\partial \tilde{x}} = \frac{\begin{vmatrix} T_1(\tilde{x}^*) - T_2(\tilde{x}^*, P^*, d^*) & \bar{x}^* \\ -\Gamma(W^* + \ell) & (D - P)\bar{x}^* \end{vmatrix}}{\Xi^*}. \tag{A5.3}$$

Substituting (A5.2) into (A5.3) yields

$$\frac{\partial P(\tilde{x}^*, \bar{x}^*, d^*)}{\partial \tilde{x}} = \frac{\begin{vmatrix} -(\bar{x}^* - \tilde{x}^*)\dfrac{\partial T_2^*}{\partial P}\dfrac{\partial P^*}{\partial \tilde{x}} & \bar{x}^* \\ -\Gamma(W^* + \ell) & (D - P^*)\bar{x}^* \end{vmatrix}}{\Xi^*}. \tag{A5.4}$$

Factoring out $\frac{\partial P^*}{\partial \tilde{x}}$ gives

$$\frac{\partial P^*}{\partial \tilde{x}} = \frac{\bar{x}^* \Gamma(W^* + \ell)}{\Omega^*}, \tag{A5.5}$$

where

$$\begin{aligned}\Omega^* &\equiv \Xi^* + (D - P^*)\bar{x}^*(\bar{x}^* - \tilde{x}^*)\frac{\partial T_2^*}{\partial P} \\ &= \bar{x}^*\left(L^*\bar{x}^* + \Gamma\frac{\partial W^*}{\partial P}(\bar{x}^* - \tilde{x}^*)\right).\end{aligned} \tag{A5.6}$$

Comparing (15a) and (A5.2), we have that (15a) is consistent with the social optimum if

$$-\frac{p}{V^*}\left(W(P^*, d^*) + \ell\right) = (\bar{x}^* - \tilde{x}^*)\frac{\partial T_2^*}{\partial P}\frac{\partial P^*}{\partial \tilde{x}}. \tag{A5.7}$$

Let p' be the value of p solving (A5.7). Substituting (A5.5), (A5.6), (13b) and (6) into (A5.6) yields

$$p' = V^* E^*. \tag{A5.8}$$

We proceed analogously for $\bar{x}$ and d. Let p'' be the parking fee such that the first-order condition for $\bar{x}$ in the individual's maximization problem is consistent with the corresponding first-order condition in the planning problem, both evaluated at the social optimum. And let p''' be the corresponding parking fee for d. Then it is straightforward to show that $p' = p'' = p'''$. Denote this p by p^*.

Now, from (14)

$$V^* = \frac{\beta - p^*\left(\frac{\bar{x}^* - \tilde{x}^*}{\bar{x}^*}\right)(W^* + \ell)}{L^*}. \tag{A5.9}$$

Substituting (A5.9) into (A5.8) gives (16´), which is the parking fee which decentralizes the social optimum.

<u>Appendix 6</u>

This appendix is organized as follows.

– calculation of the cdf of travel time, t, conditional on x.

– calculation of the unconditional cdf of travel time.

– calculation of the expected parking time on trips with $t < t'$.

A6.1 <u>Cdf of travel time conditional on x</u>

Let z denote parking location relative to the destination (so that $z < 0$ corresponds to parking before reaching the destination) and $t(z,x)$ (return journey) travel time as a function of z and x. Now

$$t(z,x) = \begin{cases} \frac{2x}{v} - 2\left(\frac{1}{w} - \frac{1}{v}\right)z & \text{for} \quad z \in [-x, 0) \\ \frac{2x}{v} + 2\left(\frac{1}{w} + \frac{1}{v}\right)z & \text{for} \quad z \in [0, \bar{z}) \\ \frac{2x}{w} & \text{for} \quad z \geq \bar{z} \end{cases} \tag{A6.1}$$

where $$\bar{z} = \frac{x(v-w)}{v+w} \tag{A6.2}$$

is the parking distance beyond the destination for which walking to the destination takes the same time as driving.

For $z \in [-x, 0)$, the individual drives a distance $x + z$ and then walks a distance $-z$ to his destination, and also on the return journey; thus, $t(z,x) = \frac{2(x+z)}{v} - \frac{2z}{w}$. For $z \in [0, \bar{z})$, the individual drives a distance $x + z$, then walks a distance z to his destination, and also on the return journey; thus, $t(z,x) = \frac{2(x+z)}{v} + \frac{2z}{w}$. By definition, $\bar{z}$ satisfies $\frac{2x}{w} = \frac{2(x+\bar{z})}{v} + \frac{2\bar{z}}{w}$, implying that $\bar{z} = \frac{x(v-w)}{v+w}$. Beyond $\bar{z}$, it takes less time to walk to the destination than to drive to $x + z$ and then walk from $x + z$ to x; thus, for $z > \bar{z}$, $t(z,x) = \frac{2x}{w}$.

Let $G(t|x)$ be the probability that travel time to x is less than t, and $g(t|x)$ be the corresponding pdf. Note that

– $G(t|x) = 0$ for $t < \frac{2x}{v}$ since $\frac{2x}{v}$ is the minimum time to travel to x and back.

– $G(t|x)$ has a mass point at $t = \frac{2x}{w}$, since $t(z,x) = \frac{2x}{w}$ for $z > \bar{z}$.

Let P denote the density of available parking spaces. A parking space is available if it is neither occupied nor reserved. (Recall that if an individual drives, she reserves a parking space on the outbound journey when she leaves home.) Then, from (A6.1), for $t \in \left(\frac{2x}{v}, \frac{2x}{w}\right)$, 1 - $G(t|x)$ is the probability that there is no available parking space for $z \in \left(-\frac{t-\frac{2x}{v}}{2\left(\frac{1}{w}-\frac{1}{v}\right)}, \frac{t-\frac{2x}{v}}{2\left(\frac{1}{w}+\frac{1}{v}\right)}\right)$. Since, by assumption, available parking spaces are generated by a Poisson process at rate P per unit distance:

$$G(t|x) = \begin{cases} 0 & \text{for} \quad t < \frac{2x}{v} \\ 1 - \exp\left(-\frac{P\left(t-\frac{2x}{v}\right)v^2 w}{v^2 - w^2}\right) & \text{for} \quad t \in \left(\frac{2x}{v}, \frac{2x}{w}\right) \\ 1 & \text{for} \quad t > \frac{2x}{w} \end{cases} \tag{A6.3}$$

and

$$g(t|x) = \begin{cases} 0 & \text{for} \quad t < \frac{2x}{v} \\ \frac{Pv^2 w}{v^2 - w^2} \exp\left(-\frac{P\left(t-\frac{2x}{v}\right)v^2 w}{v^2 - w^2}\right) & \text{for} \quad t \in \left(\frac{2x}{v}, \frac{2x}{w}\right) \\ 0 & \text{for} \quad t > \frac{2x}{w} \end{cases} \tag{A6.4}$$

with probability mass $\exp\left(-\frac{2Pxv}{v+w}\right)$ at $t = \frac{2x}{w}$.

A6.2 <u>Unconditional cdf of travel time</u>

Let $H(t')$ denote the unconditional probability that travel time on a trip opportunity is less than t'. Then

$$1 - H(t') = \int_x \Pr\left(t > t'|x\right) f(x) dx,$$

where $f(x) = \frac{1}{\pi r}$ is the pdf of x since trip opportunities are uniformly distributed over $[0, \pi r]$ by assumption. Using (A6.3),

$$1 - H(t') = \int_0^{\frac{wt'}{2}} 0 \frac{1}{\pi r} dx + \int_{\frac{wt'}{2}}^{\frac{vt'}{2}} e^{-\frac{P\left(t'-\frac{2x}{v}\right)v^2 w}{v^2 - w^2}} \frac{1}{\pi r} dx + \int_{\frac{vt'}{2}}^{\pi r} \frac{1}{\pi r} dx \tag{A6.5}$$

At distances $x < \frac{wt'}{2}$, travel time is always less than t' since the individual can walk to the destination and back in less than t'; at distances $x > \frac{vt'}{2}$ travel time is always greater than t' since even if the closest parking spot is right at the destination, travel time exceeds t'; for intermediate distances, $\Pr\left(t > t'|x\right)$ is given by $1 - G(t|x)$.

Now simplify the expression for $1-H(t')$:

$$1-H(t')=\frac{1}{\pi r}\left[\int_{\frac{wt'}{2}}^{\frac{vt'}{2}} e^{-\frac{P\left(t'-\frac{2x}{v}\right)v^2 w}{v^2-w^2}}dx+\left(\pi r-\frac{vt'}{2}\right)\right].$$

Let $A=\frac{2Pvw}{v^2-w^2}$ and make the transformation $x'=\frac{vt'}{2}-x$. Then

$$1-H(t')=\frac{1}{\pi r}\left[\int_0^{\frac{(v-w)t'}{2}} e^{-Ax'}dx'+\pi r-\frac{vt'}{2}\right]$$

$$=\frac{1}{\pi r}\left[\frac{1}{A}\left(1-e^{-\frac{A(v-w)t'}{2}}\right)+\left(\pi r-\frac{vt'}{2}\right)\right]$$

Then replacing t' by t

$$H(t)=\frac{1}{\pi r}\left[\frac{vt}{2}-\frac{1}{A}\left(1-e^{-\frac{A(v-w)t}{2}}\right)\right],\quad A=\frac{2Pvw}{v^2-w^2} \tag{A6.6}$$

and

$$h(t)=\frac{1}{2\pi r}\left[v-(v-w)e^{-\frac{A(v-w)t}{2}}\right]. \tag{A6.7}$$

A6.3 Expected parking time on trips with $t<t'$

This can be calculated as

$$k(t',P)=\int_0^{\frac{vt'}{2}}\int_{\frac{2x}{v}}^{t'} k(t,x)g(t|x)f(x)dt\,dx \div H(t') \tag{A6.8}$$

where $k(t,x)$ is the average parking time on a journey to x with travel time t. The upper limit on the first integral reflects that all trips with $x>\frac{vt'}{2}$ have travel time greater than t', while the lower limit on the second integral reflects that the minimum travel time to x is $\frac{2x}{v}$.

The computation of $k(t,x)$ is complicated by the fact that, corresponding to a given (t,x), there may be two parking locations, one before the destination, the other beyond the destination. Let $k^{-}(t,x)$ denote the parking time on a journey to x with travel time t with parking before the destination, and $k^{+}(t,x)$ the corresponding parking time with parking beyond the destination.

For parking before the destination $(z<0)$, parking time as a function of z and x is $\tilde{k}^{-}(z,x)=\frac{x+z}{v}-\frac{2}{w}z+\ell$; the first term on the RHS is the time the individual takes to drive <u>to</u> the parking location when the parking spot is reserved for her; the second term is her walking time; and the third, the time spent at the destination. Also, from (A6.1), $t=\frac{2(x+z)}{v}-\frac{2z}{w}$. Solving for z as a function of x and t from this equation, and substituting into $\tilde{k}^{-}(z,x)$ yields

$$k^{-}(t,x)=t\left(\frac{2v-w}{2(v-w)}\right)-\frac{x}{v-w}+\ell$$

Account must be taken of two complications. First, the individual has the option to walk; but she will never exercise the option if a parking space is available before the destination. Second, since the maximum travel time with parking before the destination occurs when parking is right at the origin (z = - x), travel time cannot exceed $\frac{2x}{w}$. Thus,

$$k^{-}(t,x)=\begin{cases} t\left(\frac{2v-w}{2(v-w)}\right)-\frac{x}{v-w}+\ell & \text{for } t<\frac{2x}{w} \\ 0 & \text{for } t\geq\frac{2x}{w} \end{cases} \qquad \text{(A6.9a)}$$

For parking beyond the destination $(z>0)$, parking time as a function of z and x is $\tilde{k}^{+}(z,x)=\frac{x+z}{v}+\frac{2}{w}z+\ell$. Also, from (A6.1), $t=\frac{2(x+z)}{v}+\frac{2z}{w}$. Solving for z as a function of x and t from this equation, and substituting into $\tilde{k}^{+}(z,x)$ yields

$$k^{+}(t,x)=t\left(\frac{2v+w}{2(v+w)}\right)-\frac{x}{v+w}+\ell$$

Account must be taken of the individual's option to walk. Thus,

$$k^{+}(t,x)=\begin{cases} t\left(\frac{2v+w}{2(v+w)}\right)-\frac{x}{(v+w)}+\ell & \text{for } t<\frac{2x}{w} \\ 0 & \text{for } t\geq\frac{2x}{w} \end{cases} \qquad \text{(A6.9b)}$$

Now, $k(t,x)$ is a weighted average of $k^{-}(t,x)$ and $k^{+}(t,x)$, where the weights are the probabilities that a car trip to x with travel time t entails parking before and beyond the destination, respectively.

The ratio of the probability of parking before the destination to that of parking beyond the destination equals the ratio of $\left|\frac{dz}{dt}\right|$ for $z\,t[-x,0]$ to $\left|\frac{dz}{dt}\right|$ for $z\,t[0,z]$, which from (A6.1) equals $\frac{1}{\frac{1}{w}-\frac{1}{v}}$ to $\frac{1}{\frac{1}{w}+\frac{1}{v}}$ or $\frac{1}{v-w}$ to $\frac{1}{v+w}$. Thus, the probabilities are $\frac{v+w}{2v}$ and $\frac{v-w}{2v}$, and so

$$k(t,x) = \begin{cases} t\left[\left(\frac{2v-w}{2(v-w)}\right)\left(\frac{v+w}{2v}\right) + \frac{(2v+w)}{2(v+w)}\left(\frac{v-w}{2v}\right)\right] - x\left[\left(\frac{1}{v-w}\right)\left(\frac{v+w}{2v}\right) + \left(\frac{1}{v+w}\right)\left(\frac{v-w}{2v}\right)\right] + \ell & \text{for } t < \frac{2x}{w} \\ 0 & \text{for } t \geq \frac{2x}{w} \end{cases}$$

$$= \begin{cases} t\left[\frac{v^2}{v^2-w^2}\right] - \frac{x}{v}\left[\frac{v^2+w^2}{v^2-w^2}\right] + \ell & \text{for } t < \frac{2x}{w} \\ 0 & \text{for } t \geq \frac{2x}{w} \end{cases} \tag{A6.10}$$

Thus,

$$K(t',P) = \frac{1}{H(t')}\left[\int_0^{\frac{wt'}{2}} \int_{\frac{2x}{v}}^{\frac{2x}{w}} k(t,x) g(t|x) f(x) dt dx \right.$$

$$\left. + \int_{\frac{wt'}{2}}^{\frac{vt'}{2}} \int_{\frac{2x}{v}}^{t'} k(t,x) g(t|x) f(x) dt dx \right]. \tag{A6.11}$$

The first double integral corresponds to those locations for which walking time is less than t', the second to those locations for which walking time exceeds t'.

These integrals can be simplified somewhat by making the transformation of variables $u = t - \frac{2x}{v}$. Then

$$K(t',P) = \frac{1}{H(t')}\left[\int_0^{\frac{wt'}{2}} \int_0^{2x\left(\frac{1}{w}-\frac{1}{v}\right)} \hat{k}(u,x) \hat{g}(u|x) f(x) du dx \right.$$

$$\left. + \int_{\frac{wt'}{2}}^{\frac{vt'}{2}} \int_0^{t'-\frac{2x}{v}} \hat{k}(u,x) \hat{g}(u|x) f(x) du dx \right],$$

where $$\hat{k}(u,x) = u\frac{v^2}{v^2-w^2} + \frac{x}{v} + \ell \tag{A6.12}$$

$$\hat{g}(u|x) = \tilde{P}e^{-\tilde{P}u}, \quad \tilde{P} = \frac{Pv^2w}{v^2-w^2}$$

$$f(x) = \frac{1}{\pi r}$$

and $H(t')$ is given by (A6.6).

Notational Glossary

(in order of first use in paper)

r	radius of annulus
Γ	population density (per unit distance)
D	density of parking spaces (per unit distance)
ℓ	length of a visit
β	dollar benefit from a visit
μ	Poisson arrival rate of telephone calls
x	distance from home
$T_1(x)$	time to walk to x and back
w	walking speed
P	density of vacant parking spaces (per unit distance)
v	car speed
d	distance from destination that cruising for parking starts ("cruising distance")
y	distance cruising for parking
$R(x,P,d)$	expected driving time on a round-trip car journey to x
$W(P,d)$	expected walking time on a round-trip car journey
$T_2(x,P,d)$	expected travel time on a round-trip car journey to x
$\bar{x}$	maximum travel distance
$\tilde{x}$	maximum walking distance
L	expected period between trips
θ	intermediate variable
λ	Lagrange multiplier on (6)
E	shadow parking fee in time units
p	parking fee per unit time
V	private value or opportunity cost of time
Δ	$\equiv L\bar{x}$
$*$	(superscript) value at social optimum
$p*$	parking fee consistent with social optimum

z	parking location relative to destination
$t(z,x)$	travel time as a function of z and x
$G(t\|x)$	probability that travel time to x is less than t ($g(t\|x)$ the corresponding pdf)
$H(t)$	probability that travel time on a trip is less than t ($h(t)$ the corresponding pdf)
$f(x)$	pdf of location of trip opportunities (= $\frac{1}{\pi r}$)
A	intermediate variable
$K(t,P)$	average parking time on trips with travel time less than t given P
$k(t,x)$	average parking time on a journey to x with travel time t
u	intermediate variable
$\tilde{P}$	intermediate variable