Money's Role in the Monetary Business Cycle

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1. An Optimizing IS-LM-PC Specification

1.1. Overview

Here, the models of Ireland (1997) and McCallum and Nelson (1999) are modified to focus on the role of money in the monetary business cycle. The economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a monetary authority. During each period $t = 0, 1, 2, ...$, each intermediate goods-producing firm produces a distinct, perishable intermediate good. Hence, intermediate goods may also be indexed by $i \in [0, 1]$, where firm i produces good i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm, identified by the generic index i.

1.2. The Representative Household

The representative household enters period t with money M_{t-1} and bonds B_{t-1}. At the beginning of the period, the household receives a lump-sum nominal transfer T_t from the monetary authority. Next, the household's bonds mature, providing B_{t-1} additional units of money. The household uses some of this money to purchase B_t new bonds at nominal cost B_t/r_t, where r_t denotes the gross nominal interest rate between t and $t + 1$.

The household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0, 1]$, for a total of

$$h_t = \int_0^1 h_t(i)di$$

during period t. The household is paid at the nominal wage rate W_t. The household consumes c_t units of the finished good, purchased at the nominal price P_t from the representative finished goods-producing firm.

At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0,1]$, for a total of

$$D_t = \int_0^1 D_t(i)di.$$

The household then carries M_t units of money into period $t+1$, subject to the budget constraint

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq c_t + \frac{B_t/r_t + M_t}{P_t}. \tag{1}$$

The household's preferences are described by the expected utility function

$$E\sum_{t=0}^{\infty} \beta^t a_t \{u[c_t, (M_t/P_t)/e_t] - \eta h_t\},$$

where $1 > \beta > 0$ and $\eta > 0$. The preference shocks a_t and e_t follow the autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

and

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3}$$

where $1 > \rho_a > -1$, $1 > \rho_e > -1$, $e > 0$, and the zero-mean, serially uncorrelated innovations ε_{at} and ε_{et} are normally distributed with standard deviations σ_a and σ_e.

Thus, the household chooses c_t, h_t, B_t, and M_t for all $t = 0, 1, 2, ...$, to maximize its utility subject to the budget constraint (1) for all $t = 0, 1, 2, ...$ Letting $m_t = M_t/P_t$ denote real balances, $\pi_t = P_t/P_{t-1}$ the inflation rate, $w_t = W_t/P_t$ the real wage rate, and λ_t the nonnegative multiplier on (1), the first-order conditions for this problem are

$$a_t u_1(c_t, m_t/e_t) = \lambda_t, \tag{4}$$

$$\eta a_t = \lambda_t w_t, \tag{5}$$

$$\lambda_t = \beta r_t E_t(\lambda_{t+1}/\pi_{t+1}), \tag{6}$$

$$(a_t/e_t)u_2(c_t, m_t/e_t) = \lambda_t - \beta E_t(\lambda_{t+1}/\pi_{t+1}), \tag{7}$$

and (1) with equality for all $t = 0, 1, 2, ...$

1.3. The Representative Finished Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative finished goods-producing firm uses $y_t(i)$ units of each intermediate good $i \in [0, 1]$, purchased at nominal price $P_t(i)$, to manufacture y_t units of the finished good according to the constant-returns-to-scale technology described by

$$\left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)} \geq y_t,$$

where $\theta > 1$. Thus, the finished goods-producing firm chooses $y_t(i)$ for all $i \in [0, 1]$ to maximize its profits, given by

$$P_t \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)} - \int_0^1 P_t(i) y_t(i) di,$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem are

$$y_t(i) = [P_t(i)/P_t]^{-\theta} y_t$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$

Competition drives the finished goods-producing firm's profits to zero in equilibrium. This zero-profit condition implies that

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)}$$

for all $t = 0, 1, 2, ...$

1.4. The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labor from the representative household to manufacture $y_t(i)$ units of intermediate good i according to the constant-returns-to-scale technology described by

$$z_t h_t(i) \geq y_t(i). \tag{8}$$

The aggregate technology shock z_t follows the autoregressive process

$$\ln(z_t) = (1 - \rho_z) \ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{9}$$

where $1 > \rho_z > -1$ and $z > 0$. The zero-mean, serially uncorrelated innovation ε_{zt} is normally distributed with standard deviation σ_z.

Since the intermediate goods substitute imperfectly for one another in producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market; during each period $t = 0, 1, 2, ...$, the intermediate goods-producing firm sets the nominal price $P_t(i)$ for its output, subject to the requirement that it satisfy the representative finished goods-producing firm's demand. In addition, the intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price, measured in terms of the finished good and given by

$$\frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t,$$

where $\phi > 0$ and where π denotes the steady-state inflation rate.

The cost of price adjustment makes the intermediate goods-producing firm's problem dynamic; it chooses $P_t(i)$ for all $t = 0, 1, 2, ...$ to maximize its total market value, given by

$$E\sum_{t=0}^{\infty}\beta^t \lambda_t [D_t(i)/P_t],$$

where $\beta^t \lambda_t / P_t$ measures the marginal utility value to the representative household of an additional dollar in profits received during period t and where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{P_t(i)}{P_t}\right]^{-\theta}\left(\frac{w_t y_t}{z_t}\right) - \frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t \tag{10}$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem are

$$\begin{aligned} 0 \quad = \quad & (1-\theta)\lambda_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta}\left(\frac{y_t}{P_t}\right) + \theta\lambda_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta-1}\left(\frac{y_t w_t}{z_t P_t}\right) \\ & -\phi\lambda_t \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]\left[\frac{y_t}{\pi P_{t-1}(i)}\right] \\ & +\beta\phi E_t\left\{\lambda_{t+1}\left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right]\left[\frac{y_{t+1}P_{t+1}(i)}{\pi P_t(i)^2}\right]\right\} \end{aligned} \tag{11}$$

for all $t = 0, 1, 2, ...$

1.5. The Monetary Authority

The monetary authority conducts monetary policy by adjusting the nominal interest rate r_t in response to deviations of output y_t, inflation π_t, and money growth

$$\mu_t = M_t / M_{t-1} \tag{12}$$

from their steady-state values y, π, and μ according to the policy rule

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_y \ln(y_{t-1}/y) + \rho_\pi \ln(\pi_{t-1}/\pi) + \rho_\mu \ln(\mu_{t-1}/\mu) + \varepsilon_{rt}, \quad (13)$$

where r is the steady-state value of r_t and where the zero-mean, serially uncorrelated innovation ε_{rt} is normally distributed with standard deviation σ_r.

1.6. Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $y_t(i) = y_t$, $h_t(i) = h_t$, $P_t(i) = P_t$, and $d_t(i) = D_t(i)/P_t = D_t/P_t = d_t$ for all $i \in [0,1]$ and $t = 0,1,2,\ldots.$ In addition, the market-clearing conditions $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ must hold for all $t = 0,1,2,\ldots.$

After imposing these conditions (1)-(13) become

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \quad (1)$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \quad (2)$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \quad (3)$$

$$a_t u_1(c_t, m_t/e_t) = \lambda_t, \quad (4)$$

$$\eta a_t = \lambda_t w_t, \quad (5)$$

$$\lambda_t = \beta r_t E_t(\lambda_{t+1}/\pi_{t+1}), \quad (6)$$

$$(a_t/e_t)u_2(c_t, m_t/e_t) = \lambda_t - \beta E_t(\lambda_{t+1}/\pi_{t+1}), \quad (7)$$

$$y_t = z_t h_t, \quad (8)$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \quad (9)$$

$$d_t = y_t - w_t h_t - \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \quad (10)$$

$$\begin{aligned} 0 \;=\; & (1-\theta)\lambda_t + \theta\lambda_t\left(\frac{w_t}{z_t}\right) - \phi\lambda_t\left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) \quad (11) \\ & + \beta\phi E_t\left[\lambda_{t+1}\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{y_{t+1}}{y_t}\right)\left(\frac{\pi_{t+1}}{\pi}\right)\right], \end{aligned}$$

$$m_{t-1}\mu_t = m_t \pi_t, \quad (12)$$

and

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_y \ln(y_{t-1}/y) + \rho_\pi \ln(\pi_{t-1}/\pi) + \rho_\mu \ln(\mu_{t-1}/\mu) + \varepsilon_{rt}. \quad (13)$$

These 13 equations determine equilibrium values for the 13 variables y_t, π_t, m_t, r_t, c_t, h_t, w_t, d_t, λ_t, μ_t, a_t, e_t, and z_t.

Use (4), (5), (8), and (10) to eliminate λ_t, w_t, h_t, and d_t. Then the system can be written more compactly as

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \quad (1)$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \quad (2)$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \quad (3)$$

$$a_t u_1(c_t, m_t/e_t) = \beta r_t E_t[a_{t+1} u_1(c_{t+1}, m_{t+1}/e_{t+1})/\pi_{t+1}], \quad (6)$$

$$r_t u_2(c_t, m_t/e_t) = (r_t - 1) e_t u_1(c_t, m_t/e_t), \quad (7)$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \quad (9)$$

$$\begin{aligned} \theta - 1 \quad &= \quad \theta\left[\frac{\eta}{z_t u_1(c_t, m_t/e_t)}\right] - \phi\left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) \quad (11) \\ &\quad + \beta\phi E_t\left\{\left[\frac{a_{t+1} u_1(c_{t+1}, m_{t+1}/e_{t+1})}{a_t u_1(c_t, m_t/e_t)}\right]\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{y_{t+1}}{y_t}\right)\left(\frac{\pi_{t+1}}{\pi}\right)\right\}, \end{aligned}$$

$$m_{t-1}\mu_t = m_t \pi_t, \quad (12)$$

and

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_y \ln(y_{t-1}/y) + \rho_\pi \ln(\pi_{t-1}/\pi) + \rho_\mu \ln(\mu_{t-1}/\mu) + \varepsilon_{rt}. \quad (13)$$

These 9 equations determine equilibrium values for the 9 variables y_t, π_t, m_t, r_t, c_t, μ_t, a_t, e_t, and z_t.

1.7. The Steady State

In the absence of shocks, the economy converges to a steady state, in which $y_t = y$, $\pi_t = \pi$, $m_t = m$, $r_t = r$, $c_t = c$, $\mu_t = \mu$, $a_t = a$, $e_t = e$, and $z_t = z$. The steady-state values a, e, and z are determined by (2), (3), and (9). The steady-state value π is determined by (13).

The steady-state value r is determined by (6) as

$$r = \pi/\beta.$$

The steady-state value μ is determined by (12) as

$$\mu = \pi.$$

The steady-state value c is determined by (1) as

$$c = y.$$

The steady-state values y and m are determined by (7) and (11):

$$ru_2(y, m/e) = (r-1)eu_1(y, m/e)$$

and

$$u_1(y, m/e) = \left(\frac{\theta}{\theta-1}\right)\left(\frac{\eta}{z}\right).$$

1.8. The Linearized System

The system consisting of (1)-(3), (6), (7), (9), and (11)-(13) can be log-linearized around the steady state in order to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{m}_t = \ln(m_t/m)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\mu}_t = \ln(\mu_t/\mu)$, $\hat{a}_t = \ln(a_t/a)$, $\hat{e}_t = \ln(e_t/e)$, and $\hat{z}_t = \ln(z_t/z)$. The first-order Taylor approximations yield

$$\hat{y}_t = \hat{c}_t, \tag{1}$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{3}$$

$$\begin{aligned} \hat{y}_t \quad = \quad & E_t\hat{y}_{t+1} - \omega_1(\hat{r}_t - E_t\hat{\pi}_{t+1}) + \omega_2(\hat{m}_t - E_t\hat{m}_{t+1}) \\ & -\omega_2(\hat{e}_t - E_t\hat{e}_{t+1}) + \omega_1(\hat{a}_t - E_t\hat{a}_{t+1}), \end{aligned} \tag{6}$$

$$\hat{m}_t = \gamma_1 \hat{y}_t - \gamma_2 \hat{r}_t + \gamma_3 \hat{e}_t, \tag{7}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{9}$$

$$\hat{\pi}_t = \left(\frac{\pi}{r}\right) E_t \hat{\pi}_{t+1} + \psi \left[\left(\frac{1}{\omega_1}\right) \hat{y}_t - \left(\frac{\omega_2}{\omega_1}\right) \hat{m}_t + \left(\frac{\omega_2}{\omega_1}\right) \hat{e}_t - \hat{z}_t \right], \tag{11}$$

$$\hat{m}_{t-1} + \hat{\mu}_t = \hat{m}_t + \hat{\pi}_t, \tag{12}$$

and

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \rho_y \hat{y}_{t-1} + \rho_\pi \hat{\pi}_{t-1} + \rho_\mu \hat{\mu}_{t-1} + \varepsilon_{rt}, \tag{13}$$

where

$$\omega_1 = -\frac{u_1(y, m/e)}{y u_{11}(y, m/e)},$$

$$\omega_2 = -\frac{(m/e) u_{12}(y, m/e)}{y u_{11}(y, m/e)},$$

$$\gamma_1 = \left(\frac{y r \omega_2}{m \omega_1} + \frac{r-1}{\omega_1} \right) \gamma_2,$$

$$\gamma_2 = \frac{r}{(r-1)(m/e)} \left[\frac{u_2(y, m/e)}{(r-1) e u_{12}(y, m/e) - r u_{22}(y, m/e)} \right],$$

$$\gamma_3 = 1 - (r-1)\gamma_2,$$

and

$$\psi = \frac{\theta - 1}{\phi}.$$

Equation (6) is the IS curve, equation (7) is the LM curve, equation (11) is the Phillips curve, and equation (13) is the policy rule. Use (1) to eliminate c_t, and rewrite the system as

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{3}$$

$$\begin{aligned} \hat{y}_t \quad = \quad & E_t \hat{y}_{t+1} - \omega_1 (\hat{r}_t - E_t \hat{\pi}_{t+1}) + \omega_2 (\hat{m}_t - E_t \hat{m}_{t+1}) \\ & - \omega_2 (1 - \rho_e) \hat{e}_t + \omega_1 (1 - \rho_a) \hat{a}_t, \end{aligned} \tag{6}$$

$$\hat{m}_t = \gamma_1 \hat{y}_t - \gamma_2 \hat{r}_t + \gamma_3 \hat{e}_t, \tag{7}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{9}$$

$$\hat{\pi}_t = \left(\frac{\pi}{r}\right) E_t \hat{\pi}_{t+1} + \psi \left[\left(\frac{1}{\omega_1}\right) \hat{y}_t - \left(\frac{\omega_2}{\omega_1}\right) \hat{m}_t + \left(\frac{\omega_2}{\omega_1}\right) \hat{e}_t - \hat{z}_t \right], \tag{11}$$

$$\hat{m}_{t-1} + \hat{\mu}_t = \hat{m}_t + \hat{\pi}_t, \tag{12}$$

and

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \rho_y \hat{y}_{t-1} + \rho_\pi \hat{\pi}_{t-1} + \rho_\mu \hat{\mu}_{t-1} + \varepsilon_{rt}. \tag{13}$$

2. Solving the Model

Let

$$f_t^0 = \left[\begin{array}{ccc} \hat{m}_t & \hat{r}_t & \hat{\mu}_t \end{array} \right]',$$

$$s_t^0 = \left[\begin{array}{ccccccc} \hat{y}_{t-1} & \hat{m}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{\mu}_{t-1} & \hat{y}_t & \hat{\pi}_t \end{array} \right]',$$

and

$$v_t = \left[\begin{array}{cccc} \hat{a}_t & \hat{e}_t & \hat{z}_t & \varepsilon_{rt} \end{array} \right]'.$$

Then (7), (12), and (13) can be written as

$$Af_t^0 = Bs_t^0 + Cv_t, \tag{14}$$

where A is 3×3, B is 3×7, and C is 3×4.

Equation (7) implies

$a_{11} = 1$

$a_{12} = \gamma_2$

$b_{16} = \gamma_1$

$c_{12} = \gamma_3$

Equation (12) implies

$a_{21} = 1$

$a_{23} = -1$

$b_{22} = 1$

$b_{27} = -1$

Equation (13) implies

$a_{32} = 1$

$b_{31} = \rho_y$

$b_{33} = \rho_\pi$

$b_{34} = \rho_r$

$b_{35} = \rho_\mu$

$c_{34} = 1$

Equations (6) and (11) can be written as

$$DE_t s^0_{t+1} + FE_t f^0_{t+1} = Gs^0_t + Hf^0_t + Jv_t, \tag{15}$$

where D and G are 7×7, F and H are 7×3, and J is 7×4.

Equation (6) implies

$d_{16} = 1$

$d_{17} = \omega_1$

$f_{11} = -\omega_2$

$g_{16} = 1$

$h_{11} = -\omega_2$

$h_{12} = \omega_1$

$j_{11} = -\omega_1(1 - \rho_a)$

$j_{12} = \omega_2(1 - \rho_e)$

Equation (11) implies

$d_{27} = \pi/r$

$g_{26} = -\psi/\omega_1$

$g_{27} = 1$

$h_{21} = \psi(\omega_2/\omega_1)$

$j_{22} = -\psi(\omega_2/\omega_1)$

$j_{23} = \psi$

The presence of lagged values m_{t-1} in s_t^0 implies

$d_{31} = 1$

$g_{36} = 1$

$d_{42} = 1$

$h_{41} = 1$

$d_{53} = 1$

$g_{57} = 1$

$d_{64} = 1$

$h_{62} = 1$

$d_{75} = 1$

$h_{73} = 1$

Equations (2), (3), and (9) can be written as

$$v_t = Pv_{t-1} + \varepsilon_t, \tag{16}$$

where

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 \\ 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}'.$$

Rewrite (14) as

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}Cv_t.$$

When substituted into (15), this last result yields

$$(D + FA^{-1}B)E_t s^0_{t+1} + FA^{-1}CPv_t = (G + HA^{-1}B)s^0_t + (J + HA^{-1}C)v_t$$

or, more simply,

$$E_t s^0_{t+1} = Ks^0_t + Lv_t, \tag{17}$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(J + HA^{-1}C - FA^{-1}CP).$$

If the 7×7 matrix K has five eigenvalues inside the unit circle and two eigenvalues outside the unit circle, then the system has a unique solution. If K has more than two eigenvalues outside the unit circle, then the system has no solution. If K has less than two eigenvalues outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980).

Assuming from now on that there are exactly two eigenvalues outside the unit circle, write K as

$$K = M^{-1}NM,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of N are the eigenvalues of K, with those in the 5×5 matrix N_1 inside the unit circle and those in the 2×2 matrix N_2 outside the unit circle. The columns of M^{-1} are the eigenvectors of K; M_{11} is 5×5, M_{12} is 5×2, M_{21} is 2×5, and M_{22} is 2×2. In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where L_1 is 5×4 and L_2 is 2×4.

Now (17) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s^0_{t+1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^0_t + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} v_t$$

or

$$E_t s^1_{1t+1} = N_1 s^1_{1t} + Q_1 v_t \tag{18}$$

and

$$E_t s^1_{2t+1} = N_2 s^1_{2t} + Q_2 v_t, \tag{19}$$

where

$$s^1_{1t} = M_{11} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\mu}_{t-1} \end{bmatrix} + M_{12} \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}, \tag{20}$$

$$s^1_{2t} = M_{21} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\mu}_{t-1} \end{bmatrix} + M_{22} \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}, \tag{21}$$

$$Q_1 = M_{11} L_1 + M_{12} L_2,$$

and

$$Q_2 = M_{21} L_1 + M_{22} L_2.$$

Since the eigenvalues in N_2 lie outside the unit circle, (19) can be solved forward to obtain

$$s^1_{2t} = -N_2^{-1} R v_t,$$

where the 2×4 matrix R is given by

$$\begin{aligned} vec(R) &= vec \sum_{j=0}^{\infty} N_2^{-j} Q_2 P^j = \sum_{j=0}^{\infty} vec(N_2^{-j} Q_2 P^j) \\ &= \sum_{j=0}^{\infty} [P^j \otimes (N_2^{-1})^j] vec(Q_2) = \sum_{j=0}^{\infty} [P \otimes N_2^{-1}]^j vec(Q_2) \\ &= \sum_{j=0}^{\infty} [I_{(8\times 8)} - P \otimes N_2^{-1}]^{-1} vec(Q_2) \end{aligned}$$

Use this result, along with (21), to solve for

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\mu}_{t-1} \end{bmatrix} + S_2 v_t, \tag{22}$$

where

$$S_1 = -M_{22}^{-1} M_{21}$$

and

$$S_2 = -M_{22}^{-1} N_2^{-1} R.$$

Equation (20) now provides a solution for s_{1t}^1:

$$s_{1t}^1 = (M_{11} + M_{12}S_1) \begin{bmatrix} \hat{y}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\mu}_{t-1} \end{bmatrix} + M_{12}S_2 v_t.$$

Substitute this result into (18) to obtain

$$\begin{bmatrix} \hat{y}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \\ \hat{\mu}_t \end{bmatrix} = S_3 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\mu}_{t-1} \end{bmatrix} + S_4 v_t, \tag{23}$$

where

$$S_3 = (M_{11} + M_{12}S_1)^{-1} N_1 (M_{11} + M_{12}S_1)$$

and

$$S_4 = (M_{11} + M_{12}S_1)^{-1} (Q_1 + N_1 M_{12} S_2 - M_{12} S_2 P).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1} B s_t^0 + A^{-1} C v_t \\ &= A^{-1} B \begin{bmatrix} I_{(5\times5)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\mu}_{t-1} \end{bmatrix} + A^{-1} B \begin{bmatrix} 0_{(5\times4)} \\ S_2 \end{bmatrix} v_t + A^{-1} C v_t, \end{aligned}$$

which can be written more simply as

$$f_t^0 = S_5 \hat{m}_{t-1} + S_6 v_t, \tag{24}$$

where

$$S_5 = A^{-1}B \begin{bmatrix} I_{(5\times5)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}B \begin{bmatrix} 0_{(5\times4)} \\ S_2 \end{bmatrix} + A^{-1}C.$$

Equations (16) and (22)-(24) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{25}$$

and

$$f_t = Us_t, \tag{26}$$

where

$$s_t = \begin{bmatrix} \hat{y}_{t-1} & \hat{m}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{\mu}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{z}_t & \varepsilon_{Rt} \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{m}_t & \hat{r}_t & \hat{\mu}_t & \hat{y}_t & \hat{\pi}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} S_3 & S_4 \\ 0_{(4\times5)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(5\times4)} \\ I_{(4\times4)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

3. Estimating the Model

Suppose that data are available on output y_t, real balances m_t, inflation π_t, and the interest rate r_t. These data can be used to construct a series $\{d_t\}_{t=1}^T$, where

$$d_t = \begin{bmatrix} \hat{y}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(y_t) - \ln(y) \\ \ln(m_t) - \ln(m) \\ \ln(\pi_t) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}.$$

Equations (25) and (26) then given rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{27}$$

and

$$d_t = Cs_t, \tag{28}$$

where $A = \Pi$, $B = W$, C is formed from the rows of U as

$$C = \begin{bmatrix} U_4 \\ U_1 \\ U_5 \\ U_2 \end{bmatrix},$$

and the vector of serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{at+1} & \varepsilon_{et+1} & \varepsilon_{zt+1} & \varepsilon_{rt+1} \end{bmatrix}'$$

is assumed to be normally distributed with zero mean and diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}.$$

The model defined by (27) and (28) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t | d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (27) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(9\times1)} \tag{29}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1 s_1') = [I_{(81\times81)} - A \otimes A]^{-1} vec(BVB'). \tag{30}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (28) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned} \hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t. \end{aligned}$$

Hence, from (27),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (27) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_tu_t' = C\Sigma_tC' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_tC'(C\Sigma_tC')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_tA' - A\Sigma_tC'(C\Sigma_tC')^{-1}C\Sigma_tA',$$

and initial conditions $\hat{s}_1$ and Σ_1 are provided by (29) and (30).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -2T\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T}u_t'\Omega_t^{-1}u_t.$$

4. Variance Decompositions

Begin by considering (27), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty}A^jB\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty}A^jB\varepsilon_{t+k-j},$$

$$E_ts_{t+k} = \sum_{j=k}^{\infty}A^jB\varepsilon_{t+k-j},$$

$$s_{t+k} - E_ts_{t+k} = \sum_{j=0}^{k-1}A^jB\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (27) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(81\times81)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (26) and (28), which imply that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U',$$

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U',$$

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = C\Sigma_k^s C',$$

and

$$\Sigma^d = \lim_{k\to\infty} \Sigma_k^d = C\Sigma^s C'.$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that the elements of Σ_k^s, Σ^s, Σ_k^f, Σ^f, Σ_k^d, and Σ^d can all be expressed as nonlinear functions of Θ:

$$\Sigma = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\nabla g H \nabla g'.$$

In practice, the gradient ∇g can be evaluated numerically, as suggested by Runkle (1987).

5. Vector Autocorrelations

5.1. Model

Note first that (27) implies

$$s_t = A^k s_{t-k} + \sum_{j=0}^{k-1} A^j B \varepsilon_{t-j}.$$

Hence,

$$E s_t s'_{t-k} = A^k E s_{t-k} s'_{t-k} = A^k \Sigma^s,$$

where

$$vec(\Sigma^s) = [I_{(81\times 81)} - A \otimes A]^{-1} vec(BVB').$$

Equation (28) then implies

$$\Gamma_k = E d_t d'_{t-k} = C A^k \Sigma^s C'.$$

Thus, the autocorrelations can be computed as

$$\frac{\Gamma_k(i,j)}{[\Gamma_0(i,i)]^{1/2} [\Gamma_0(j,j)]^{1/2}}.$$

5.2. Data

Consider the using the data

$$d_t = \begin{bmatrix} \hat{y}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(y_t) - \ln(y) \\ \ln(m_t) - \ln(m) \\ \ln(\pi_t) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}$$

to estimate the autoregression

$$d_t = A d_{t-1} + B \varepsilon_t,$$

where more than one lag of d_t can be accommodated by writing the system in companion form.

Then, as above,

$$d_t = A^k d_{t-k} + \sum_{j=0}^{k-1} A^j B \varepsilon_{t-j}.$$

Hence,

$$\Gamma_k = E d_t d'_{t-k} = A^k E d_{t-k} d'_{t-k} = A^k \Sigma^d,$$

where

$$vec(\Sigma^d) = [I_{(4p \times 4p)} - A \otimes A]^{-1} vec(BVB')$$

and $p + 1$ is the total number of lags in the autoregression.

Thus, the autocorrelations can be computed as

$$\frac{\Gamma_k(i,j)}{[\Gamma_0(i,i)]^{1/2} [\Gamma_0(j,j)]^{1/2}}.$$