TRADE AND FACTOR RENTALS
WITH MORE THAN TWO FACTORS

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INTRODUCTION

The Heckscher-Ohlin-Samuelson trade model has been thoroughly and intensively explored in the two commodity, two factor case by a number of authors, most notably Samuelson himself. Until recently, however, scant attention has been paid to the many-by-many case outside of the very formal literature establishing sufficient conditions for factor price equalization. It is the intent of this paper to enlarge on recent path-breaking articles by Chipman, [3], and Vanek, [9], by demonstrating a relation between factor endowments and factor prices in the general case that might be termed a weak Stolper-Samuelson theorem.

Part I will review Chipman's demonstration that the Stolper-Samuelson theorem does not hold in the many-by-many case under the sufficient conditions for factor price equalization. Vanek's generalization of the commodity trade (Heckscher-Ohlin) theorem is then discussed and it is seen that it suggests a theorem about the effect of trade on factor prices which is similar to the Stolper-Samuelson theorem. In Part II the theorem is shown to hold under the sufficient conditions for factor price equalization if the utility function is "well behaved". Part III is a brief summary.
Pa. I. The Standard Theorems and Their Extensions

In 1941 Wolfgang Stolper and Paul Samuelson published the first major extension of the Heckscher-Ohlin model, [8], showing that with two goods and two factors the rental of the scarce factor decreased with trade not only relatively, but in real terms as well. Similarly, a tariff increased its real as well as relative income. Subsequent contributions by Samuelson and others clarified the precise assumptions needed for the theorem to hold,¹ but the conjecture in the 1941 paper that the theorem did not hold with more than two factors lay fallow until Chipman picked it up, [2], [3]. Chipman demonstrated that the essence of the Stolper-Samuelson theorem, mathematically, lies in the properties of the inverse of the unit cost mapping of factor rentals into commodity prices. The global univalence of the mapping is necessary for all the standard theorems, but even global univalence is not sufficient to establish the necessary properties in the inverse outside of the 2x2 case. Reviewing briefly,

Let $c(w)$ be the unit cost mapping where $w$ is the vector of factor rentals. At competitive equilibrium we may equate the commodity price vector $p$ with unit costs: $p = c(w)$. $c$ is the matrix of factor-output coefficients at equilibrium factor rentals $w$ multiplied by the vector $w$. Partially differentiating $c$ with respect to $w$, we obtain (by the Wong-Viner envelope theorem) the matrix of factor

¹The well-known assumptions of the H-O-S model sufficient for all standard theorems and repeated here throughout are:

(1) for each good, degree one homogeneous diminishing marginal product production functions identical across countries
(2) fixed endowments of all factors
(3) the number of commodities ≥ number of factors (from the standpoint of the theorems, the number of commodities in excess of the number of factors are superfluous)
(4) identical homothetic utility across countries (an overly strong assumption for the 2x2 case, but true to the spirit of the model)
(5) zero transport costs and no other impediments to free trade
(6) pure competition in all markets
(7) non-specialization by any country in any good
(8) global univalence of the unit cost mapping of factor rentals into commodity prices
(9) all factors are fully employed
output coefficients at \( w \), and denote it \( G \). We can also work in terms of natural logarithms of prices and rentals, writing \( \pi = \ln C(e^u) = \psi(w) \), where \( \pi = \ln P \) and \( \psi = \ln \sigma(d) \). Writing \( \pi = \ln C(e^u) = \psi(w) \), where \( \psi = \ln W \), partially differentiating \( \psi \) with respect to \( w \), we obtain a matrix \( G W = \Gamma \), where \( P \) and \( W \) are diagonal matrices of commodity prices and factor rentals, respectively, the \( ij \)th element of \( \Gamma \) being \( \gamma_{ij} = w_j g_{ij}/p_i \). \( \Gamma \) is the matrix of distributive shares, and thus has all elements non-negative and all row sums equal to unity by Euler's theorem. Now note that the Stolper-Samuelson theorem states that there must be some association of commodities with factors (i.e., some permutation of the rows and columns of \( C \) or \( \psi \)) such that a rise in the \( i \)th commodity price (other prices constant) increases the \( i \)th factor rental proportionally more and decreases all other factor rentals. Factor rental \( i \) thus increases in proportion to every price in the system, erasing the index number problems usually involved in statements about real income. Mathematically the theorem requires \( \partial \ln W_i/\partial P_i > 0 \) and \( \partial \ln W_j/\partial P_i < 0 \). The first condition is equivalent to \( \partial \log w_i/\partial \log P_i > 1 \), while the second is met if \( \partial \log W_j/\partial \log P_i < 0 \). This is precisely the information obtained from \( \Gamma^{-1} \). The sufficient conditions for factor price equalization demand that \( G \) (and hence \( \Gamma \)) possess an inverse for all \( W \) and that its principal minors for any numbering be positive (Gale and Nikaido, [5]). In the 2x2 case, as is well known, this simply means factor intensities must not reverse. Given this condition, \( \Gamma^{-1} \) will have off-diagonal elements negative and diagonal elements greater than unity, since the row sums must still be unity (if \( I \) is the column vector of ones, \( \Gamma^{-1} I = \Gamma^{-1} (\Gamma I) = I \), following Chipman, [3]). However, outside the 2x2 case, not all off-diagonal elements are negative and thus the diagonal elements can be less than one, even under the Gale-Nikaido conditions. Chipman has constructed a counter-example, [2], [3], and along with Kemp and Wegge, [6], [7] has explored additional
conditions which are sufficient in the 3x3 case. No simple rule seems possible for dimensions above this. Thus, the Stolper-Samuelson theorem does not hold in upper dimensions; its requirements demand a link between commodity prices and factor rentals that is much more stringent than the sufficient conditions for factor price equalization.

After such a conclusive demonstration, the road appears to be closed for theoretically deducing the effect of trade on factor prices. Recall, however, that Ohlin's original statement was much less ambitious than Stolper-Samuelson's: Ohlin stated that the relative price (relative to the other factors) of the abundant factor would rise relative to what it had been in isolation (also relative to what it was in the other country in isolation) as a result of trade. We shall see shortly that the many-by-many generalization of this theorem still holds under factor price equalization with the addition of an assumption on demand. The meaning of the theorem will be more evident, however, after first discussing Vanek's generalization of the commodity trade theorem.

It has long been noted that in the many-by-many case the concept of relative factor intensity lost sharpness, thus preventing the linking of goods with factors that permitted the Heckscher-Ohlin prediction that the good relatively intensive in the relatively abundant factor would be exported. It was left for Vanek, [9], to show that the less ambitious statement of Heckscher-Ohlin, that trade in commodities was a complete substitute for trade in goods, was valid in a powerful sense in the many factor case under the sufficient conditions for factor price equalization. In the spirit of the Heckscher-Ohlin-Samuelson model, Vanek assumed, more strictly than the standard assumptions, identical homothetic utility for the two trading countries. He was able to prove the following theorem: let the factor endowment
vector of two countries, A and B be ordered (without loss of generality) such that:

\[
\frac{F_1^A}{F_1^B} \geq \frac{F_2^A}{F_2^B} \geq \cdots \geq \frac{F_r^A}{F_r^B}, \quad F_k^i = \text{physical endowment of factor } K \text{ in country } i
\]

Then the trade in embodied factor services may be ordered \( F_1 \geq F_2 \geq \cdots \geq F_r, F_K^w \text{ trade in factor } K \). At some point in the string of inequalities \( F_K \) will become negative---the listing of imported factors (from the standpoint of A) will begin. The exact point in the string is determined by demand conditions. Vanek does not sufficiently stress the importance of this, which is that the many-by-many Heckscher-Ohlin-Samuelson model does for factors what the Ricardian model does for goods: it provides an ordered listing of factors to be "traded" via commodity trade, the exact point on the listing being determined by reciprocal demands. Relative factor abundance or scarcity is no more a purely physical concept than comparative advantage. With more than two countries, B denotes the rest of the world. The theorem is not valid bilaterally (see Baldwin, [1], for a proof in the 2 factor case).

The proof of Vanek's theorem is straightforward and need not be reproduced. The essence of it is that with commodity and factor price equalization, the consumption of embodied factor services in the two countries must be proportional. World factor endowments are assumed fixed and consumption must equal supply, so when the reciprocal demand determined proportion is found, it determines the direction and ordering of trade in all embodied factor services. This suggests that an ordering of the impact of this trade on relative factor rentals is possible. Again let the factor endowment vectors of A and B be ordered:

\[
\frac{F_1^A}{F_1^B} \geq \frac{F_2^A}{F_2^B} \geq \cdots \geq \frac{F_r^A}{F_r^B}
\]
Then in isolation the relative factor rentals may be ordered:

\[
\frac{w_1^A}{w_1^B} \leq \frac{w_2^A}{w_2^B} \leq \ldots \leq \frac{w_r^A}{w_r^B}
\]

Trade brings relative (and absolute) factor rental equalization so that relative to country B, factor 1 gains relative to every other factor, factor 2 relative to every factor but 1, and so forth. Also, we are able by the same reasoning to order A's factor rentals in free costless trade relative to A's factor rentals in isolation. This is because in free trade with identical homothetic utility, factor rentals are determined as if there were only one country, using the world endowment for its production. We can order A's endowment relative to the A plus B situation:

\[
\frac{F_1^A}{F_1^{A+B}} = 1 + \frac{F_1^A}{F_1^B} > 1 + \frac{F_2^A}{F_2^B} > \ldots > 1 + \frac{F_r^A}{F_r^B}
\]

Then the factor rentals may be ordered:

\[
\frac{w_1^A}{w_1^{A+B}} \leq \frac{w_2^A}{w_2^{A+B}} \leq \ldots \leq \frac{w_r^A}{w_r^{A+B}}, \quad w_k^{A+B} = \text{free trade rental of factor } K
\]

Part II. Proof of the Theorem

For the theorem to hold, it is necessary to impose regularity on the shared utility surface beyond the usual assumption of homotheticity: we must require neutrality of other goods such that the ratios of quantities demanded at two points on the surface depend only on the ratios of own prices. That is, \(\frac{p_i^*}{p_j^*} \geq \frac{p_i}{p_j}\)

must imply \(\frac{X_i^*}{X_j^*}\) where \(X_i^*, X_j^*, X_i, X_j\) are quantity pairs and \(p_i^*, p_j^*, p_i, p_j\) and \(X_i, X_j\) are their associated prices. This regularity is obviously not met in general even under homotheticity. A CES utility surface where the elasticity of
substitution is the same between all pairs of goods satisfies the conditions (since
by definition substitution with all other goods is the same for each member of the
pair), however, so a non-trivial class of surfaces is permitted. And in any event,
a perhaps not so honorable tradition holds that it is useful to see how restrictive
one's assumptions need to be in order to achieve definite results.

Proceeding with the proof of the theorem, we write the unit cost mapping into
prices for two countries, A and B, in isolation:

(1) \( P^A = G^A W^A \), \( G^A, B \) = matrix of factor output coefficients in A and B respectively
(2) \( P^B = G^B W^B \)

Without loss of generality, we arrange the rows and columns of (1) and (2) so that:

\[
\begin{align*}
\frac{p_1^A}{p_1^B} & > \frac{p_2^A}{p_2^B} \; > \; \cdots \; > \; \frac{p_n^A}{p_n^B} \quad \text{and} \quad \frac{f_1^A}{f_1^B} & > \frac{f_2^A}{f_2^B} \; > \; \cdots \; > \; \frac{f_r^A}{f_r^B}, \quad n = r \quad \text{(if} \; n > r \; \text{we neglect the superfluou excess)}
\end{align*}
\]

We may express the two chains of inequalities in matrix notation:

(3) \( P^A = AP^B \), A is diagonal with \( a_{11} > a_{22} \geq \cdots \geq a_{rr} > 0 \)
(4) \( P^A = BP^B \), B is diagonal with \( b_{11} > b_{22} \geq \cdots \geq b_{nn} > 0 \)

By our assumption on the regularity of the utility surface we can order the
quantities demanded and produced at the two points:

(5) \( x^A = CX^B, \quad 0 < c_{11} \leq c_{22} \leq \cdots \leq c_{nn} \)

No. denote the unknown diagonal matrix which relates factor prices at the two
points as D.:

(6) \( W^A = DW^B \)

We complete the relations necessary for the proof by noting that the full employ-
ment assumption of the standard model implies:

(7) \( Cx^A = \bar{x}^A \)
(8) \( Bx^B = x^B \)
Note that \( P^A = BP^B = BG^B B = (BG^D)^{-1}W^A = GW^A \)

Also, \( P^A = AP^B = AG^B X = [AG^B C^{-1}]X^A = G^{A^+} X^A \)

Now note that:

\[
[AG^B C^{-1}] = [C^{-1} \ G^B \ A] = [BG^B D^{-1}]
\]

Considering just the diagonal elements of \( G^A \), we know that \( C \) and \( B \) have reverse order on their diagonals; hence, \( C^{-1} \) and \( B \) have the same order. This implies that \( A \) and \( D^{-1} \) have the same order, further implying that \( D \) is ordered opposite to \( A \), thus proving our theorem. Specifically, writing out both matrices we get:

\[
C^{-1} G^B A =
\begin{pmatrix}
    c_{11}^{-1} g_{11} a_{11} & c_{11}^{-1} g_{12} a_{22} & \cdots & c_{11}^{-1} g_{1r} a_{rr} \\
    c_{22}^{-1} g_{21} a_{11} & c_{22}^{-1} g_{22} a_{22} & \cdots & c_{22}^{-1} g_{2r} a_{rr} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{nn}^{-1} g_{nn} a_{11} & \cdots & c_{nn}^{-1} g_{nr} a_{rr}
\end{pmatrix}
\]

\[
AG^B D^{-1} =
\begin{pmatrix}
    b_{11}^{-1} g_{11} d_{11} & b_{11}^{-1} g_{12} d_{22} & \cdots & b_{11}^{-1} g_{1r} d_{rr} \\
    b_{22}^{-1} g_{21} d_{11} & b_{22}^{-1} g_{22} d_{22} & \cdots & b_{22}^{-1} g_{2r} d_{rr} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{nn}^{-1} g_{nn} d_{11} & \cdots & b_{nn}^{-1} g_{nr} d_{rr}
\end{pmatrix}
\]

Taking the first two diagonal elements we have:

\[
c_{11}^{-1} g_{11} a_{11} = b_{11}^{-1} g_{11} d_{11} \\
c_{22}^{-1} g_{22} a_{22} = b_{22}^{-1} g_{22} d_{22}
\]
This implies that:
\[
\begin{pmatrix}
\frac{c_{11}}{c_{22}} & \frac{a_{11}}{a_{22}} \\
\frac{b_{11}}{b_{22}} & \frac{d_{11}}{d_{22}}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{b_{11}}{b_{22}} \\
\frac{d_{11}}{d_{22}}
\end{pmatrix}
\]

From our imposed ordering, \( \frac{b_{11}}{b_{22}} > 1 \). Also, \( \frac{c_{11}}{c_{22}} > 1 \), and \( \frac{a_{11}}{a_{22}} > 1 \). Therefore, \( \frac{d_{11}}{d_{22}} < 1 \), implying that \( \frac{d_{11}}{d_{22}} < 1 \). The same manipulation can be performed for any pair of d's, so we have shown that \( d_{11} \leq d_{22} \leq \cdots \leq d_{xx} \), which is opposite to the order of A.

Re-capitulating, we have shown that the standard trade model produces, with the addition of a restriction on demand, an ordering on relative factor rents compared between two trading countries in isolation that is inverse to the ordering on their relative factor endowments. Trade results in relative and absolute factor rental equalization with non-specialization and global univalence of the unit cost mapping, so that relative factor rental gainers and losers compared between the two countries follow the ordering of relative factor endowments. The same proof (as previously noted) also establishes that comparing the free trade and isolation factor rental vectors for one country results in an ordering on relative factor gainers and losers that follows the country’s factor endowment ordering relative to the world factor endowment. Obviously, all “exported” factors gain relative to all “imported” factors. What is not established is any statement on real gains or losses to factor owners; Chipman has conclusively shown no such statement is possible. Also not implied is any statement comparing one trade position with another, the theorem strictly compares no trade with free trade. A statement on
relative factor gains and losses due to tariffs (of a well ordered nature) would be possible only if the transformation surface could be assumed to have the same neutrality property imposed on the utility surface. This is because the C matrix needed for the proof must relate goods produced. Given this condition, a tariff vector ordered so that the comparatively most disadvantaged good is protected most (and has its price rise most—we rule out Metzler effects) and so forth, will increase relative factor rentals in an order inverse to the relative (to the world) factor endowment ordering. The conditions under which the transformation surface would possess this property are not readily apparent (as the recent effective protection literature makes clear, see Ethier, [3]), so the problem is merely posed. Note that it is an important problem; a number of simple statements about the effects of taxes become possible if the conditions, once discovered, may be assumed to hold for all commodities or for groups of commodities.

Part III. Summary

Although Chipman conclusively demonstrated that the Stolper-Samuelson theorem does not hold in upper dimensions, we have seen that a weak version of it more in line with Ohlin's original statement is still possible under an added assumption about demand. Though less cutting than the very powerful Stolper-Samuelson statement, the weaker theorem on the impact of trade on relative factor rewards is still of considerable interest.
References


