Markup Shocks, Technology Shocks, and the New Keynesian Phillips Curve

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1. The Model

Here, the framework from Ireland (2002) is modified to focus on the relative importance of markup, or cost-push, shocks and technology shocks in an estimated version of the New Keynesian model. The economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a central bank. During each period $t = 0, 1, 2, ...$, each intermediate goods-producing firm produces a distinct, perishable intermediate good. Hence, intermediate goods may also be indexed by $i \in [0, 1]$, where firm i produces good i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm, identified by the generic index i.

The activities of each agent, and their implications for the evolution of equilibrium prices and quantities, will now be described in turn.

1.1. The Representative Household

The representative household enters each period $t = 0, 1, 2, ...$ with money M_{t-1} and bonds B_{t-1}. At the beginning of the period, the household receives a lump-sum nominal transfer T_t from the central bank. Next, the household's bonds mature, providing B_{t-1} additional units of money. The household uses some of this money to purchase B_t new bonds at nominal cost B_t/r_t, where r_t denotes the gross nominal interest rate between t and $t + 1$.

During period t, the household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0,1]$, for a total of

$$h_t = \int_0^1 h_t(i)di$$

during period t. The household is paid at the nominal wage W_t. The household consumes c_t units of the finished good, purchased at the nominal price P_t from the representative finished goods-producing firm.

At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0,1]$, for a total of

$$D_t = \int_0^1 D_t(i)di.$$

The household then carries M_t units of money into period $t+1$, chosen subject to the budget constraint

$$\frac{M_{t-1} + T_t + B_{t-1} + (1-\tau)(W_t h_t + D_t)}{P_t} \geq c_t + \frac{B_t/r_t + M_t}{P_t}, \tag{1}$$

where τ is the constant income tax rate.

The household's preferences are described by the expected utility function

$$E \sum_{t=0}^{\infty} \beta^t a_t [u(c_t) + e_t v(M_t/P_t) - \eta h_t],$$

where $1 > \beta > 0$ and $\eta > 0$. The preference shocks a_t and e_t follow the autoregressive processes

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

and

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3}$$

where $1 > \rho_a > 0$, $1 > \rho_e > 0$, $e > 0$, and the zero-mean, serially uncorrelated innovations ε_{at} and ε_{et} are normally distributed with standard deviations σ_a and σ_e.

Thus, the household chooses c_t, h_t, B_t, and M_t for all $t = 0, 1, 2, ...$ to maximize its utility subject to the budget constraint (1) for all $t = 0, 1, 2, ...$ Letting $m_t = M_t/P_t$ denote real balances, $\pi_t = P_t/P_{t-1}$ the inflation rate, $w_t = W_t/P_t$

the real wage, and λ_t the nonnegative Lagrange multiplier on (1), the first-order conditions for this problem are

$$a_t u'(c_t) = \lambda_t, \tag{4}$$

$$a_t \eta = (1-\tau)\lambda_t w_t, \tag{5}$$

$$\lambda_t = \beta r_t E_t(\lambda_{t+1}/\pi_{t+1}), \tag{6}$$

$$a_t e_t v'(m_t) = \lambda_t - \beta E_t(\lambda_{t+1}/\pi_{t+1}), \tag{7}$$

and (1) with equality for all $t = 0, 1, 2, ...$

1.2. The Representative Finished Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative finished goods-producing firm uses $y_t(i)$ units of each intermediate good $i \in [0, 1]$, purchased at the nominal price $P_t(i)$, to manufacture y_t units of the finished good according to the constant-returns-to-scale technology described by

$$\left[\int_0^1 y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} \geq y_t,$$

where, as in Smets and Wouters (2002) and Steinsson (2002), θ_t translates into a random shock to the markup of price over marginal cost. Here, this markup shock follows the autoregressive process

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{8}$$

where $1 > \rho_\theta > 0$, $\theta > 1$, and the zero-mean, serially uncorrelated innovation $\varepsilon_{\theta t}$ is normally distributed with standard deviation σ_θ.

Thus, during period t, the finished goods-producing firm chooses $y_t(i)$ for all $i \in [0, 1]$ to maximize its profits, which are given by

$$P_t\left[\int_0^1 y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} - \int_0^1 P_t(i) y_t(i) di.$$

The first-order conditions for this problem are

$$y_t(i) = [P_t(i)/P_t]^{-\theta_t} y_t$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$

Competition drives the finished goods-producing firm's profits to zero in equilibrium. This zero-profit condition implies that

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t} di\right]^{1/(1-\theta_t)}$$

for all $t = 0, 1, 2, \ldots$.

1.3. The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, \ldots$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labor from the representative household to manufacture $y_t(i)$ units of intermediate good i according to the constant returns to scale technology described by

$$z_t h_t(i) \geq y_t(i). \tag{9}$$

The aggregate technology shock z_t follows the autoregressive process

$$\ln(z_t) = (1 - \rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{10}$$

where $1 > \rho_z > 0$, $z > 0$, and the zero-mean, serially uncorrelated innovation ε_{zt} is normally distributed with standard deviation σ_z.

Since the intermediate goods substitute imperfectly for one another in producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market; during period t, the intermediate goods-producing firm sets the nominal price $P_t(i)$ for its output, subject to the requirement that it satisfy the representative finished goods-producing firm's demand. In addition, the intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price, measured in terms of the finished good and given by

$$\frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t,$$

where $\phi > 0$ and where $\pi > 1$ measures the gross steady-state inflation rate.

The cost of price adjustment makes the intermediate goods-producing firm's problem dynamic; it chooses $P_t(i)$ for all $t = 0, 1, 2, \ldots$ to maximize its total value, given by

$$[(1-\tau)/\lambda_0] E \sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t],$$

where $\beta^t \lambda_t / P_t$ measures the marginal utility to the representative household of an additional unit of profits received during period t and where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} y_t - \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} \left(\frac{w_t y_t}{z_t}\right) - \frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t \tag{11}$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem are

$$\begin{aligned} 0 \quad = \quad & (1-\theta_t)\lambda_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} \left(\frac{y_t}{P_t}\right) + \theta_t \lambda_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t - 1} \left(\frac{w_t y_t}{z_t P_t}\right) \\ & -\phi\lambda_t \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right] \left[\frac{y_t}{\pi P_{t-1}(i)}\right] \\ & +\beta\phi E_t \left\{\lambda_{t+1} \left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right] \left[\frac{y_{t+1} P_{t+1}(i)}{\pi P_t(i)^2}\right]\right\} \end{aligned} \tag{12}$$

for all $t = 0, 1, 2, ...$

1.4. Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $y_t(i) = y_t$, $h_t(i) = h_t$, $P_t(i) = P_t$, and $d_t(i) = D_t(i)/P_t = D_t/P_t = d_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ In addition, the market-clearing conditions $M_t = M_{t-1} + T_t - \tau(W_t h_t + D_t)$ and $B_t = B_{t-1} = 0$ must hold for all $t = 0, 1, 2, ...$

After imposing these equilibrium conditions, (1)-(12) become

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3}$$

$$a_t u'(c_t) = \lambda_t, \tag{4}$$

$$a_t \eta = (1-\tau)\lambda_t w_t, \tag{5}$$

$$\lambda_t = \beta r_t E_t(\lambda_{t+1}/\pi_{t+1}), \tag{6}$$

$$a_t e_t v'(m_t) = \lambda_t - \beta E_t(\lambda_{t+1}/\pi_{t+1}), \tag{7}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{8}$$

$$z_t h_t = y_t, \tag{9}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{10}$$

$$d_t = y_t - w_t h_t - \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{11}$$

and

$$\begin{aligned} 0 \quad = \quad & (1-\theta_t)\lambda_t + \theta_t\lambda_t\left(\frac{w_t}{z_t}\right) - \phi\lambda_t\left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) \\ & +\beta\phi E_t\left[\lambda_{t+1}\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{\pi_{t+1}}{\pi}\right)\left(\frac{y_{t+1}}{y_t}\right)\right] \end{aligned} \tag{12}$$

for all $t = 0, 1, 2, ...$

Use (3)-(5), (7), (9), and (11) to eliminate e_t, λ_t, w_t, m_t, h_t, and d_t. Then the system consisting of (1)-(12) can be written more compactly as

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$a_t u'(c_t) = \beta r_t E_t[a_{t+1}u'(c_{t+1})/\pi_{t+1}], \tag{6}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{8}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{10}$$

and

$$\begin{aligned} \theta_t - 1 \quad = \quad & \left(\frac{\eta}{1-\tau}\right)\left[\frac{\theta_t}{z_t u'(c_t)}\right] - \phi\left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) \\ & +\beta\phi E_t\left\{\left[\frac{a_{t+1}u'(c_{t+1})}{a_t u'(c_t)}\right]\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{\pi_{t+1}}{\pi}\right)\left(\frac{y_{t+1}}{y_t}\right)\right\} \end{aligned} \tag{12}$$

for all $t = 0, 1, 2, ...$

1.5. The Steady State

In the absence of shocks, the economy converges to a steady state, in which $y_t = y$, $c_t = c$, $\pi_t = \pi$, $r_t = r$, $a_t = a$, $z_t = z$, and $\theta_t = \theta$. The steady-state values $a = 1$, z, and θ are determined by (2), (8), and (10), while the steady-state value π will be determined by the central bank.

The steady-state values c and r are determined by (1) and (6) as

$$c = y$$

and

$$r = \pi/\beta.$$

Finally, the steady-state value y is determined by (12) as the solution to

$$u'(y) = \left(\frac{\eta}{z}\right)\left(\frac{1}{1-\tau}\right)\left(\frac{\theta}{\theta-1}\right).$$

1.6. The Linearized System

The system consisting of (1), (2), (6), (8), (10), and (12) can be log-linearized around the steady state to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{a}_t = \ln(a_t/a)$, $\hat{z}_t = \ln(z_t/z)$, and $\hat{\theta}_t = \ln(\theta_t/\theta)$. A first-order Taylor approximation to (1) reveals that $\hat{c}_t = \hat{y}_t$, allowing $\hat{c}_t$ to be eliminated from the system. First-order approximations to (2), (6), (8), (10), and (12) then yield

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$\hat{y}_t = E_t\hat{y}_{t+1} - (1/\sigma)(\hat{r}_t - E_t\hat{\pi}_{t+1}) + (1/\sigma)(1-\rho_a)\hat{a}_t, \tag{6}$$

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t}, \tag{8}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{10}$$

and

$$\phi\hat{\pi}_t = \beta\phi E_t\hat{\pi}_{t+1} + (\theta-1)(\sigma\hat{y}_t - \hat{z}_t) - \hat{\theta}_t, \tag{12}$$

for all $t = 0, 1, 2, ...$, where

$$\sigma = -yu''(y)/u'(y) > 0.$$

1.7. Efficient and Inefficient Shocks

A social planner for this economy chooses y_t, m_t, and $h_t(i)$ for all $i \in [0,1]$ and $t = 0, 1, 2, ...$ to maximize

$$E \sum_{t=0}^{\infty} \beta^t a_t \left[u(y_t) + e_t v(m_t) - \eta \int_0^1 h_t(i) di \right],$$

subject to

$$z_t \left[\int_0^1 h_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\theta_t/(\theta_t - 1)} \geq y_t$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem are

$$v'(m_t^*) = 0 \tag{13}$$

and

$$\eta = z_t u'(y_t^*) \tag{14}$$

for all $t = 0, 1, 2, ...$

Equation (13), of course, indicates that the representative household should be satiated with real balances, which can be produced at zero nominal cost. Since (4), (6), and (7) imply that

$$e_t v'(m_t) = u'(c_t) \left(1 - \frac{1}{r_t} \right)$$

in equilibrium, implementing this part of the optimal allocation requires that the central bank follow the Friedman (1969) rule by setting $r_t = 1$ for all $t = 0, 1, 2, ...$

Now consider (14), which defines the efficient level of output y_t^*. In a steady state, $y_t^* = y^*$, where

$$u'(y^*) = \eta / z.$$

And since, as shown above,

$$u'(y) = \left(\frac{\eta}{z} \right) \left(\frac{1}{1 - \tau} \right) \left(\frac{\theta}{\theta - 1} \right),$$

the efficient level of steady-state output can be achieved by setting

$$\tau = -\frac{1}{\theta - 1} < 0$$

that is, by subsidizing production.

Next, let $\hat{y}_t^* = \ln(y_t^*/y^*)$. A first-order Taylor approximation to (14) then yields

$$y_t^* = (1/\sigma^*)\hat{z}_t \tag{14}$$

for all $t = 0, 1, 2, ...$, where

$$\sigma^* = -y^* u''(y^*)/u'(y^*).$$

Assume from now on that $\sigma^* = \sigma$, either because production is subsidized so as to make $y = y^*$ or because the representative household's utility function over consumption takes the constant relative risk aversion form. Under either of these two conditions, (14) simplifies to

$$y_t^* = (1/\sigma)\hat{z}_t. \tag{14}$$

Now define the output gap g_t as the ratio of the equilibrium and efficient levels of output,

$$g_t = y_t/y_t^*, \tag{15}$$

and define the natural real rate of interest q_t based on the household's intertemporal marginal rate of substitution, evaluated at the efficient levels of output,

$$a_t u'(y_t^*) = \beta q_t E_t[a_{t+1} u'(y_{t+1}^*)]. \tag{16}$$

Once again, let $g_t = g$ and $q_t = q$ in steady state, and let $\hat{g}_t = \ln(g_t/g)$ and $\hat{q}_t = \ln(q_t/q)$. Continuing under the assumption that $\sigma^* = \sigma$, first-order Taylor approximations to (15) and (16) yield

$$\hat{g}_t = \hat{y}_t - \hat{y}_t^* = \hat{y}_t - (1/\sigma)\hat{z}_t \tag{15}$$

and

$$\hat{q}_t = (1 - \rho_a)\hat{a}_t - (1 - \rho_z)\hat{z}_t \tag{16}$$

for all $t = 0, 1, 2, ...$

In terms of these new variables, the IS curve (6) and the Phillips curve (12) can be rewritten as

$$\hat{g}_t = E_t \hat{g}_{t+1} - (1/\sigma)(\hat{r}_t - E_t \hat{\pi}_{t+1} - \hat{q}_t) \tag{6}$$

and

$$\phi\hat{\pi}_t = \beta\phi E_t \hat{\pi}_{t+1} + (\theta - 1)\sigma\hat{g}_t - \hat{\theta}_t \tag{12}$$

for all $t = 0, 1, 2, ...$

Equations (6) and (12) reveal that in the absence of markup shocks, when $\hat{\theta}_t = 0$ for all $t = 0, 1, 2, ...$, the central bank can stabilize both inflation and the output gap, with $\hat{\pi}_t = 0$ and $\hat{g}_t = 0$ for all $t = 0, 1, 2, ...$, by letting the nominal interest rate track movements in the natural rate, with

$$\hat{r}_t = \hat{q}_t$$

for all $t = 0, 1, 2, ...$ More specially, in light of (16), the nominal rate should rise in response to the preference shock $\hat{a}_t$ and fall in response to a technology shock $\hat{z}_t$. However, unlike $\hat{a}_t$ and $\hat{z}_t$, which impact on the efficient level of output $\hat{y}_t^*$ and the natural rate of interest $\hat{q}_t$, the markup, or cost-push, shock $\hat{\theta}_t$ generates a trade-off between inflation and output-gap stabilization. For details, see Clarida, Gali, and Gertler (1999), Gali (2002), and Woodford (2001*a*). Gali (2002) and Woodford (2001*b*) show how a stochastic term like $\hat{\theta}_t$ appears in New Keynesian Phillips curves like (12) when nominal wages, as well as nominal prices, adjust sluggishly to the shocks that hit the economy, as in Kim (2000) and Erceg, Henderson, and Levin (2000); in this case, however, the additional term is no longer exogenous. Rotemberg and Woodford (1995) is an earlier paper in which exogenous markup shocks are considered, albeit in the context of a purely real business cycle model; exogenous markup shocks also play a key role in Mankiw and Reis' (2002) sticky-price model. Finally, inefficient shocks of another kind appear in Dupor's (2002) sticky-price model.

1.8. The Central Bank

The central bank conducts monetary policy by adjusting the short-term nominal interest rate according to the modified Taylor (1993) rule

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \rho_y \hat{y}_{t-1} + \rho_\pi \hat{\pi}_{t-1} + \varepsilon_{rt}. \tag{17}$$

The lagged interest rate $\hat{r}_{t-1}$ is included among the determinants of the current-period interest rate $\hat{r}_t$, to allow for a gradual response of policy to changes in output and inflation. A sufficiently vigorous long-run response of the interest rate to inflation, as measured by $\rho_\pi/(1-\rho_r)$, is required to insure that this policy rule is consistent with the existence of a unique rational expectations equilibrium; for details, see Parkin (1978), McCallum (1981), Kerr and King (1996), and Clarida, Gali, and Gertler (2000). Finally, in (17), the zero-mean, serially uncorrelated innovation ε_{rt} is normally distributed with standard deviation σ_r.

1.9. Equilibrium Conditions

The workings of the model can now be summarized by the system consisting of the IS curve

$$\hat{y}_t = E_t\hat{y}_{t+1} - (1/\sigma)(\hat{r}_t - E_t\hat{\pi}_{t+1}) + (1/\sigma)(1-\rho_a)\hat{a}_t, \tag{6}$$

the Phillips curve

$$\phi\hat{\pi}_t = \beta\phi E_t\hat{\pi}_{t+1} + (\theta - 1)(\sigma\hat{y}_t - \hat{z}_t) - \hat{\theta}_t, \tag{12}$$

and the policy rule

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + \rho_y\hat{y}_{t-1} + \rho_\pi\hat{\pi}_{t-1} + \varepsilon_{rt}. \tag{17}$$

together with the laws of motion for the three exogenous shocks

$$\hat{a}_t = \rho_a\hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$\hat{\theta}_t = \rho_\theta\hat{\theta}_{t-1} + \varepsilon_{\theta t}, \tag{8}$$

and

$$\hat{z}_t = \rho_z\hat{z}_{t-1} + \varepsilon_{zt}. \tag{10}$$

More specifically, these 6 equations determine the behavior of the 6 variables $\hat{y}_t$, $\hat{r}_t$, $\hat{\pi}_t$, $\hat{a}_t$, $\hat{\theta}_t$, and $\hat{z}_t$.

2. Solving the Model

Let

$$f_t^0 = \begin{bmatrix} \hat{r}_t \end{bmatrix}',$$

$$s_t^0 = \begin{bmatrix} \hat{r}_{t-1} & \hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{y}_t & \hat{\pi}_t \end{bmatrix}',$$

and

$$v_t = \begin{bmatrix} \hat{a}_t & \hat{\theta}_t & \hat{z}_t & \varepsilon_{rt} \end{bmatrix}'.$$

Then (17) can be written as

$$Af_t^0 = Bs_t^0 + Cv_t, \tag{18}$$

where A is 1×1, B is 1×5, and C is 1×4. More specifically, (17) implies

$a_{11} = 1$

$b_{11} = \rho_r$

$b_{12} = \rho_y$

$b_{13} = \rho_\pi$

$c_{14} = 1$

Equations (6) and (12) can be written as

$$DE_t s^0_{t+1} + FE_t f^0_{t+1} = Gs^0_t + Hf^0_t + Jv_t, \tag{19}$$

where D and G are 5×5, F and H are 5×1, and J is 5×4.

Equation (6) implies

$d_{14} = 1$

$d_{15} = 1/\sigma$

$g_{14} = 1$

$h_{11} = 1/\sigma$

$j_{11} = -(1/\sigma)(1 - \rho_a)$

Equation (12) implies

$d_{25} = \beta\phi$

$g_{24} = -(\theta - 1)\sigma$

$g_{25} = \phi$

$j_{22} = 1$

$j_{23} = \theta - 1$

The presence of lagged values in s^0_t implies

$d_{31} = 1$

$h_{31} = 1$

$d_{42} = 1$

$g_{44} = 1$

$d_{53} = 1$

$g_{55} = 1$

Equations (2), (8), and (10) can be written as

$$v_t = Pv_{t-1} + \varepsilon_t, \tag{20}$$

where

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & \rho_\theta & 0 & 0 \\ 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{\theta t} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}'.$$

Rewrite (18) as

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}Cv_t.$$

When substituted into (19), this last result yields

$$(D + FA^{-1}B)E_t s_{t+1}^0 = (G + HA^{-1}B)s_t^0 + (J + HA^{-1}C - FA^{-1}CP)v_t$$

or, more simply,

$$E_t s_{t+1}^0 = Ks_t^0 + Lv_t, \tag{21}$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(J + HA^{-1}C - FA^{-1}CP).$$

If the 5×5 matrix K has three eigenvalues inside the unit circle and two eigenvalues outside the unit circle, then the system has a unique solution. If K has more than two eigenvalues outside the unit circle, then the system has no solution. If K has less than two eigenvalues outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980).

Assuming from now on that there are exactly two eigenvalues outside the unit circle, write K as

$$K = M^{-1}NM,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of N are the eigenvalues of K, with those in the 3×3 matrix N_1 inside the unit circle and those in the 2×2 matrix N_2 outside the unit circle. The columns of M^{-1} are the eigenvectors of K; M_{11} is 3×3, M_{12} is 3×2, M_{21} is 2×3, and M_{22} is 2×2. In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where L_1 is 3×4 and L_2 is 2×4.

Now (21) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s^0_{t+1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^0_t + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} v_t$$

or

$$E_t s^1_{1t+1} = N_1 s^1_{1t} + Q_1 v_t \tag{22}$$

and

$$E_t s^1_{2t+1} = N_2 s^1_{2t} + Q_2 v_t, \tag{23}$$

where

$$s^1_{1t} = M_{11} \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + M_{12} \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}, \tag{24}$$

$$s^1_{2t} = M_{21} \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + M_{22} \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}, \tag{25}$$

$$Q_1 = M_{11}L_1 + M_{12}L_2,$$

and

$$Q_2 = M_{21}L_1 + M_{22}L_2.$$

Since the eigenvalues in N_2 lie outside the unit circle, (23) can be solved forward to obtain

$$s_{2t}^1 = -N_2^{-1}Rv_t,$$

where the 2×4 matrix R is given by

$$\begin{aligned} vec(R) &= vec\sum_{j=0}^{\infty} N_2^{-j}Q_2P^j = \sum_{j=0}^{\infty} vec(N_2^{-j}Q_2P^j) \\ &= \sum_{j=0}^{\infty}[P^j \otimes (N_2^{-1})^j]vec(Q_2) = \sum_{j=0}^{\infty}[P \otimes N_2^{-1}]^j vec(Q_2) \\ &= [I_{(8\times 8)} - P \otimes N_2^{-1}]^{-1}vec(Q_2) \end{aligned}$$

Use this result, along with (25), to solve for

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + S_2v_t, \tag{26}$$

where

$$S_1 = -M_{22}^{-1}M_{21}$$

and

$$S_2 = -M_{22}^{-1}N_2^{-1}R.$$

Equation (24) now provides a solution for s_{1t}^1:

$$s_{1t}^1 = (M_{11} + M_{12}S_1) \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + M_{12}S_2v_t.$$

Substitute this result into (22) to obtain

$$\begin{bmatrix} \hat{r}_t \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = S_3 \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + S_4v_t, \tag{27}$$

where

$$S_3 = (M_{11} + M_{12}S_1)^{-1}N_1(M_{11} + M_{12}S_1)$$

and

$$S_4 = (M_{11} + M_{12}S_1)^{-1}(Q_1 + N_1 M_{12} S_2 - M_{12} S_2 P).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1}Bs_t^0 + A^{-1}Cv_t \\ &= A^{-1}B \begin{bmatrix} I_{(3\times3)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + A^{-1}B \begin{bmatrix} 0_{(3\times4)} \\ S_2 \end{bmatrix} v_t + A^{-1}Cv_t, \end{aligned}$$

which can be written more simply as

$$f_t^0 = S_5 \begin{bmatrix} \hat{r}_{t-1} \\ \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \end{bmatrix} + S_6 v_t, \tag{28}$$

where

$$S_5 = A^{-1}B \begin{bmatrix} I_{(3\times3)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}B \begin{bmatrix} 0_{(3\times4)} \\ S_2 \end{bmatrix} + A^{-1}C.$$

Equations (20) and (26)-(28) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{29}$$

and

$$f_t = Us_t, \tag{30}$$

where

$$s_t = \begin{bmatrix} \hat{r}_{t-1} & \hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{a}_t & \hat{\theta}_t & \hat{z}_t & \varepsilon_{rt} \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{r}_t & \hat{y}_t & \hat{\pi}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{\theta t} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} S_3 & S_4 \\ 0_{(4\times3)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(3\times4)} \\ I_{(4\times4)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

Notice that, in this solution, the rows of U should simply reproduce the rows of Π: $U_1 = \Pi_1$, $U_2 = \Pi_2$, and $U_3 = \Pi_3$.

3. Estimating the Model

The model has implications for three observable variables: output, inflation, and the short-term nominal interest rate. The model's parameters are β, σ, η, τ, ϕ, π, ρ_r, ρ_y, ρ_π, θ, z, ρ_a, ρ_θ, ρ_z, σ_a, σ_θ, σ_z, and σ_r.

Since π is the steady-state rate of inflation, its value can be chosen to match the average inflation rate in the data. And since, according to the model, the steady-state nominal interest rate is given by $r = \pi/\beta$, a value for β can be chosen so that the implied value of r equals the average nominal interest rate in the data. The three parameters η, τ, z serve only to pin down the steady-state level of output; hence, values for these parameters can be chosen to match the average level of detrended output in the data. Set $\sigma = 1$, implying the same level of relative risk aversion implied by a utility function that is logarithmic in consumption. Set $\theta = 6$, implying a steady-state markup of 20 percent, and set $\phi = 50$, the same value chosen by Ireland (2000, 2002). Finally, since z_t corresponds to the real business cycle model's technology shock, set $\rho_z = 0.95$ and $\sigma_z = 0.007$, as suggested by Cooley and Prescott (1995).

The model's remaining 8 parameters, ρ_r, ρ_y, ρ_π, ρ_a, ρ_θ, σ_a, σ_θ, and σ_r, can be estimated by maximum likelihood. To begin, let $\{d_t\}_{t=1}^{T}$ denote the series for the logarithmic deviations of detrended output, inflation, and the short-term nominal interest rate from their average, or steady-state, values:

$$d_t = \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(y_t) - \ln(y) \\ \ln(\pi_t) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}.$$

Equations (29) and (30) then give rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{31}$$

and

$$d_t = Cs_t, \tag{32}$$

where $A = \Pi$, $B = W$, C is formed from the rows of U as

$$C = \begin{bmatrix} U_2 \\ U_3 \\ U_1 \end{bmatrix},$$

and the vector of serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{at+1} & \varepsilon_{\theta t+1} & \varepsilon_{zt+1} & \varepsilon_{rt+1} \end{bmatrix}'$$

is assumed to be normally distributed with zero mean and diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon'_{t+1} = \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 & 0 \\ 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}.$$

The model defined by (31) and (32) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t | d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (31) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(7\times1)} \tag{33}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1s_1') = [I_{(49\times49)} - A \otimes A]^{-1} vec(BVB'). \tag{34}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (32) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned}
\hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\
&= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.
\end{aligned}$$

Hence, from (31),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (31) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1} C\Sigma_t A',$$

and initial conditions $\hat{s}_1$ and Σ_1 are provided by (33) and (34).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T} u_t'\Omega_t^{-1}u_t.$$

4. Evaluating the Model

4.1. Variance Decompositions

Begin by considering (31), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (31) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(49\times 49)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (30) and (32), which imply that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U',$$

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U',$$

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = C\Sigma_k^s C',$$

and

$$\Sigma^d = \lim_{k\to\infty} \Sigma_k^d = C\Sigma^s C'.$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that the elements of Σ_k^s, Σ^s, Σ_k^f, Σ^f, Σ_k^d, and Σ^d can all be expressed as nonlinear functions of Θ:

$$\Sigma = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\nabla g H \nabla g'.$$

In practice, the gradient ∇g can be evaluated numerically, as suggested by Runkle (1987).

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