Calibration Results for Non-Expected Utility Theories*

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Rabin [22] proved that a low level of risk aversion with respect to small gambles leads to a high, and absurd, level of risk aversion with respect to large gambles. Rabin's arguments strongly depend on expected utility theory, but we show that similar arguments apply to general non-expected utility theories.

Keywords: Risk aversion; Calibration; Non-expected utility theories.

1 Introduction

About ten years ago, Rabin [22] offered a very convincing argument against using expected utility theory by showing how reasonable levels of risk aversion with respect to small lotteries imply absurdly high levels of risk aversion with respect to large lotteries. Our aim is to show that this analysis challenges not only expected utility theory, and similar results can be obtained for all (smooth) preferences.

We assume the following two properties of preferences throughout this paper:

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D1 Actions are evaluated by considering possible *final* wealth levels.

D2 Risk aversion: If lottery Y is a mean preserving spread of lottery X, then X is preferred to Y.

Given these assumptions, we explore the quantitative relationship between the following two additional behaviors.

B3 Rejection of a small actuarially favorable lottery in a certain range. For example, rejection of the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all wealth levels below 300,000.

B4 Acceptance of large, very favorable lotteries. For example, acceptance of the lottery $(-5{,}000, \frac{1}{2}; 10{,}000{,}000, \frac{1}{2})$.

Specifically we explore how the size and form of actuarially favorable lotteries that are rejected and the range of wealth on which they are rejected relate to the size of large lotteries that can be accepted.

Rabin [22] shows the tension between these properties within expected utility theory. If for given small and relatively close $\ell < g$, the decision maker rejects $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels $x \in [a, b]$, then he also rejects $(-L, \frac{1}{2}; G, \frac{1}{2})$ at x^* for some L, G, and $x^* \in [a, b]$ where G is huge while L is not. Rabin shows how stunning these numbers can be. For example, if a risk averse decision maker is rejecting the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all positive wealth levels below 300,000, then at wealth level 290,000 he will also reject the lottery $(-10{,}000, \frac{1}{2}; 5{,}503{,}790, \frac{1}{2})$. [1] Rabin's arguments rely on the properties of expected utility theory and are understood to be a major attack on this theory. [2]

The hypothesis that in risky environments decision makers evaluate actions by considering possible final wealth levels is widely used in expected

[1] For an earlier claim that a low level of risk aversion in the small implies huge risk aversion at the large, although without detailed numerical estimates, see Hansson [14] and Epstein [9].

[2] Palacios-Huerta and Serrano [20] object to this conclusion and claim that expected utility decision makers do not satisfy B3 on such large intervals. But Theorems 1 and 2 show that the intervals needed are much smaller than Rabin's (see Tables 1 and 2 below). In another approach, where decision makers do not maximize expected utility but are rather concerned with not going below a certain wealth level, Foster and Hart [11] show that in Rabin's example, B4 may not be implied by B3 if decision makers consider infinitely many repetitions of the gamble.

utility theory as well as in its applications. Moreover, many of the new alternatives to expected utility, alternatives that were developed during the last twenty five years in order to overcome the limited descriptive power of expected utility, are also based on the hypothesis that only final wealth levels matter.

The final-wealth hypothesis is analytically tractable as it assumes that decision makers behave according to a unique, universal preference relation over final-wealth distributions. Suggested deviations from this hypothesis require much more elaborate and complex analysis. For example, postulating that decision makers ignore final wealth levels and, instead, care about possible gains and losses may require using many preference relations and will necessitate the need for a mechanism that defines the appropriate reference points. However, the poor descriptive power of some of the final-wealth models and, in particular, of final-wealth expected utility, have increased the popularity of gain-losses models such as prospect theory [16], [30] and its offsprings. Using Rabin's results, Cox and Sadiraj [7] and Rubinstein [24] conclude that the final-wealth approach should be dropped.

A less radical conclusion from Rabin's argument is that final-wealth expected utility should be replaced with more general final-wealth theories (see e.g. Rabin [22, p. 1288] and Rabin and Thaler [23]). For example, rank-dependent utility with linear utility (Yaari [31]) is capable of exhibiting both a relatively strong aversion to small gambles and a sensible degree of risk aversion with respect to large gambles (see Section 4 below). The present paper confronts this claim. We show that with small modifications of B3 and B4 one can still show that a rejection of small lotteries with positive expected value leads to the rejection of very attractive large lotteries even if the expected utility hypothesis is dropped.[3] Yaari's model, for example, is inconsistent with our extended requirement for a moderate level of risk aversion in the small (see Proposition 2 in Section 4 below).

The technical tool we use is local utilities (Machina [17]). Theorem 1 shows how to use local utilities to obtain calibration results. But the conditions of this theorem are too strong in the sense that they are not satisfied by some non expected utility preferences and by some empirical tests. The main results of the paper show how to obtain inconsistency of the four desired properties with weaker assumptions than those used in Theorem 1. In

[3]Our claims hold provided some minimal degree of smoothness of preferences is assumed. Formally, we assume that preferences are Gâteaux differentiable.

section 3 we show the inconsistency of the four properties under some assumptions concerning the way local utilities change from one distribution to another. In section 4 we modify properties B3 and B4 to get general results. The paper's analysis applies to all (Gâteaux) differentiable models, including Chew's [3] weighted utility, (the differentiable versions of) betweenness (Dekel [8], Chew [3]), quadratic utility (Machina [17], Chew, Epstein, and Segal [4]), and rank-dependent utility (Quiggin [21]). In Section 4 we also show that non expected utility models satisfying constant absolute and relative risk aversion cannot satisfy our modified version of property B3.

The analysis of the paper leaves us with the choice between several controversial conclusions. People do not reject small risks, people reject excellent large risk, or, explanations that seem to be more likely, people are not globally risk averse or people do not utilize just one preference relation.

2 Calibration and Local Utilities

We assume throughout that preferences over distributions are representable by a functional V which is risk averse with respect to mean-preserving spreads, monotonically increasing with respect to first order stochastic dominance, continuous with respect to the topology of weak convergence, and Gâteaux differentiable (see below).[4] Denote the set of all such functionals by $\mathcal{V}$.

According to the context, utility functionals are defined over lotteries (of the form $X = (x_1, p_1; \ldots ; x_n, p_n)$) or over cumulative distribution functions (denoted F, H). Degenerate cumulative distribution functions are denoted δ_x. For x,ℓ, and g, $H_{x,\ell,g}$ denotes the distribution of the lottery $(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$. When ℓ and g are fixed, we write H_x instead.

The functional V is Gâteaux differentiable if for every F and H, the derivative

$$\frac{\partial}{\partial \varepsilon} V((1-\varepsilon)F + \varepsilon H)\bigg|_{\varepsilon=0}$$

exists and is linear in H (see Zeidler [32]). If V is Gâteaux differentiable then

[4] It is of course possible to create non-differentiable examples (see e.g. Dekel [8]), but all the standard models in the literature are Gâteaux (if not Fréchet) differentiable.

there are local utilities $u(\cdot; F)$[5] such that for all F and H

$$V((1-\varepsilon)F+\varepsilon H)-V(F) = \varepsilon\left[\int u(x;F)dH - \int u(x;F)dF\right] + o(\|\varepsilon\|) \qquad (1)$$

Throughout the paper, the local utilities enable us to carry over accumulated levels of risk aversion from one point to another. Proposition 1 links risk attitudes of the functional V with properties of the local utilities $u(\cdot; F)$.

The following lemma is needed since the calibration results for expected utility rely on the concavity of the vNM function. It extends Machina's [17] result for Fréchet differentiable functionals to the class of all Gâteaux differentiable functionals.

Lemma 1 *All local utilities of a Gâteaux differentiable functional $V \in \mathcal{V}$ are concave.*

Suppose that V is expected utility with the vNM utility u. For every F, the function u is also the local utility function of V at F. If the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at a given x, that is, if

$$u(x) > \tfrac{1}{2}u(x-\ell) + \tfrac{1}{2}u(x+g) \qquad (2)$$

then by the nature of expected utility theory, for every distribution F and $\varepsilon > 0$,

$$V\left((1-\varepsilon)F + \varepsilon\delta_x\right) \geqslant V\left((1-\varepsilon)F + \varepsilon H_x\right) \qquad (3)$$

Proposition 1 shows that the equivalence of eqs. (2) and (3) holds for the local utilities of general functionals V as well.

Proposition 1 *Let $V \in \mathcal{V}$ and $x \in \Re$. The following conditions are equivalent:*

1. *For every F, $u(x;F) \geqslant \frac{1}{2}u(x-\ell;F) + \frac{1}{2}u(x+g;F)$.*
2. *For every F and $\varepsilon > 0$, $V\left((1-\varepsilon)F + \varepsilon\delta_x\right) \geqslant V\left((1-\varepsilon)F + \varepsilon H_x\right)$.*

This Proposition leads to the following behavioral conclusion:

[5] The concept of local utilities was introduced by Machina [17]. Machina assumed the stronger notion of Fréchet differentiability.

Theorem 1 *Let $V \in \mathcal{V}$, $g > \ell > 0$, and $G > b - a$, and let*

$$L > \left[(\ell + g) \frac{1 - \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}}}{1 - \frac{\ell}{g}} + (G + a - b) \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \right] \frac{(1-p)}{p} \tag{4}$$

If for every F with support in $[a - L, a + G]$, $x \in [a, b]$, and $\varepsilon > 0$,

$$V((1-\varepsilon)F + \varepsilon\delta_x) \geqslant V((1-\varepsilon)F + \varepsilon H_x)$$

then

$$V(a, 1) \geqslant V(a - L, p; a + G, 1 - p).$$

Table 1 offers the minimal values of L for different levels of G, $b - a$, and g when $\ell = 100$ and $p = \frac{1}{2}$. For example, if the decision maker rejects $(-100, \frac{1}{2}; 110, \frac{1}{2})$ on a range of 40,000, then he also rejects the lottery $(-2{,}310, \frac{1}{2}; 10{,}000{,}000, \frac{1}{2})$. If $p \neq \frac{1}{2}$, the values should be multiplied by $\frac{1-p}{p}$. For $1 - p = \frac{1}{100{,}000}$, we obtain that this decision maker will refuse to pay even three cents for a 1:100,000 chance of winning 10 million dollars!

The "if" condition of this theorem is stronger then the one used by Rabin (which is formally obtained when $\varepsilon = 1$), although within expected utility theory they are equivalent, as by the independence axiom, $V(\delta_x) \geqslant V(H_x)$ iff for all F and $\varepsilon > 0$, $V((1-\varepsilon)F + \varepsilon\delta_x) \geqslant V((1-\varepsilon)F + \varepsilon H_x)$. This condition is quite strong and behaviorally questionable. The common ratio effect (Allais [1]) shows that preferences may be reversed when the choice is conditioned on a small probability. For example, let $F = \delta_0$, $x = 100$, $\ell = 100$, and $g = 110$. The fact that $V(100, 1) > V(0, \frac{1}{2}; 210, \frac{1}{2})$ does not imply, for preferences exhibiting this effect, that $V(100, 0.1; 0, 0.9) > V(210, 0.05; 0, 0.95)$. Although the "then" part of Theorem 1 is as strong as Rabin's original claim, we consider weaker hypothesis at the cost of obtaining weaker conclusion. We offer such results in the next two sections.

G	$b - a$	$g = 101$	$g = 105$	$g = 110$	$g = 125$
1,000,000	20,000	376,873	12,662	2,421	1,125
5,000,000	30,000	1,141,280	8,241	2,316	1,125
10,000,000	40,000	1,392,440	5,035	2,310	1,125

Table 1: If the decision maker rejects $(-100, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels between a and b, then at a he also rejects $(-L, \frac{1}{2}; G, \frac{1}{2})$, values of L entered in the table.

3 Hypothesis 2

Machina [17] introduced the following notion.

Definition 1 *Suppose all local utilities are twice differentiable. The functional $V \in \mathcal{V}$ satisfies Hypothesis 2 if for all F and H such that F dominates H by first order stochastic dominance and for all x*

$$-\frac{u''(x;F)}{u'(x;F)} \geqslant -\frac{u''(x;H)}{u'(x;H)}$$

Machina [17, 18] shows that Hypothesis 2 conforms with many violations of expected utility like the Allais paradox, the common ratio effect [1], and the mutual purchase of insurance policies and lottery tickets. Hypothesis 2 implies that for given $x > y > z$, indifference curves in the probability triangle $\{(x, p; y, 1 - p - q; z, q) : p + q \leqslant 1\}$ become steeper as one moves from δ_z to δ_x.[6] Note that expected utility preferences satisfy Hypothesis 2 with equality, as all local utility functions are identical and equal to the vNM utility u.

Assuming Hypothesis 2 we get the following result.

Theorem 2 *Let $V \in \mathcal{V}$ and assume that it satisfies Hypothesis 2. Let $0 < \ell < g < L$ and let $b - a = L + g$. Then there exists $\hat{\varepsilon} > 0$ such that for $p \leqslant \hat{\varepsilon}$ and for all G satisfying*

$$G < \frac{p}{1-p}(\ell + g)\frac{\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} - 1}{\frac{g}{\ell} - 1}, \tag{5}$$

if for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then

$$V(b, 1) \geqslant V(b - g - L, p; b - g + G, 1 - p).$$

[6] The experimental evidence concerning this hypothesis, even on the probability triangle, is inconclusive. Battalio, Kagel, and Jiranyakul [2] and Conlisk [6] suggest that indifference curves become less steep as one moves closer to either δ_x or δ_z. But Conlisk's lotteries do not satisfy the requirements of Hypothesis 2. Battalio, Kagel, and Jiranyakul [2] did find some violations of Hypothesis 2, but as most of their subjects were consistent with expected utility theory, only a small minority of them violated this hypothesis. For a further discussion of violations of Hypothesis 2, see Starmer [29, Sec. 5.1.1].

Observe that if for all $x \in [a,b]$, $V(x,1) > V(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$, then, by continuity, there exists $\hat{\varepsilon} > 0$ such that for all $p \in (0, \hat{\varepsilon}]$ and for all $x \in [a,b]$,

$$V(a,p;x,1-p) \geqslant V(a,p;x-\ell,\tfrac{1-p}{2};x+g,\tfrac{1-p}{2})$$

This is the value of $\hat{\varepsilon}$ used in the theorem. To understand the implication of the theorem, first consider expected utility preferences. By the independence axiom $\hat{\varepsilon} = 1$ and, as is explained in the beginning of the theorem's proof, the lottery rejected is $(-L, p; G, 1-p)$ (a rightward shift by g of the lottery $(-g-L, p; -g+G, 1-p)$ rejected by general preferences).

In Table 2, $\ell = 100$ and the wealth level is b. The table presents, for different combinations of $L = b-a$, G, and g, values of p such that a rejection of $(-100, \frac{1}{2}; g, \frac{1}{2})$ at all $x \in [b-L, b]$ by an expected utility decision maker leads to a rejection at b of the lottery $(-L, p; G, 1-p)$. For example, if the decision maker rejects $(-100, \frac{1}{2}; 110, \frac{1}{2})$ on a range of 30,000, then he also rejects the lottery $(-30{,}000, \frac{1}{1700}; 1{,}000{,}000, \frac{1699}{1700})$.

L	G	$g = 101$	$g = 105$	$g = 110$	$g = 125$
20,000	100,000	0.7462	0.1740	0.0054	$2.7 \cdot 10^{-7}$
30,000	1,000,000	0.9357	0.1621	$5.8 \cdot 10^{-4}$	$1.3 \cdot 10^{-10}$
50,000	100,000,000	0.9978	0.1421	$6.6 \cdot 10^{-6}$	$3.2 \cdot 10^{-17}$

Table 2: If the expected utility decision maker rejects $(-100, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels between $b-L$ and b, then at b he also rejects $(-L, p; G, 1-p)$, values of p entered in the Table.

For general preferences, Theorem 2 implies, for example, the following behavior (where the numbers are taken from Table 2). Let $g = 110$ and $L = 30{,}000$. If for all $x \in [a,b]$, $V(x,1) > V(x-100, \frac{1}{2}; x+110, \frac{1}{2})$ and if $\hat{\varepsilon} = 0.006$, then

$$V(b,1) > V(b-30{,}110, 0.006; b+10^6-110, 0.994)$$

We provide here an outline of the proof of the Theorem. In the first part of the proof we use the fact that the rejection of the small lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at x implies, by differentiability, that at some distribution on the line segment connecting δ_x and H_x, the local utility function at this distribution prefers $(x,1)$ to the lottery $(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$. Observe that if $x < b-g$ then δ_b

dominates by first-order stochastic dominance all distributions on the line-segment connecting δ_x and H_x. Hence, if $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is rejected at all $x \in [a, b-g]$, then, by Hypothesis 2, the local utility function $u(\cdot;\delta_b)$ prefers $(x, 1)$ to the lottery $(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$ for all such x. Hence $u(\cdot;\delta_b)$ rejects some of the attractive large lotteries of Table 2. Next we show that a similar property holds along the line segment connecting $(b, 1)$ and $(b-g-L, \varepsilon; b-g+G, 1-\varepsilon)$. By Gâteaux differentiability, the decision maker rejects the large lottery as well.

By definition, Hypothesis 2 needs differentiability of the local utility functions. This is not a trivial assumption, and it is closely associated with orders of risk aversion: The functional V represents first [second] order risk aversion if the risk premium the decision maker is willing to pay to avoid playing $t \otimes X := (tx_1, p_1; \ldots; tx_n, p_n)$ for $\mathrm{E}[X] = 0$ converges to zero at the same rate as t $[t^2]$. Indifference curves of preferences satisfying first order risk aversion are not differentiable at δ_x, and the local utilities $u(\cdot;\delta_x)$ are not differentiable at x (see Segal and Spivak [27, 28], Epstein and Zin [10], and Safra and Segal [26]). The next section deals with general functionals, including those not satisfying Hypothesis 2.

4 Stochastic B3

The previous section dealt with preferences satisfying the restrictive Hypothesis 2. Even when local utilities are differentiable, not all functionals satisfy this assumption (for example, some versions of Chew's [3] weighted utility theory and Gul's [13] disappointment aversion theory). And there are many models where local utilities are not differentiable, among them rank-dependent utility (Quiggin [21]), the most popular alternative to expected utility theory. This functional is given by $V(F) = \int u(x) df(F)$. For finite lotteries with $x_1 \leqslant \ldots \leqslant x_n$, the value of this functional is given by

$$V(x_1, p_1; \ldots; x_n, p_n) = u(x_1)f(p_1) + \sum_{i=2}^{n} u(x_i) \left[f\left(\sum_{j=1}^{i} p_j\right) - f\left(\sum_{j=1}^{i-1} p_j\right) \right]$$

Risk aversion implies concave f and the local utilities of this functional are given by $u(x;F) = \int^x u'(z) f'(F(z)) dz$ (see Chew, Karni, and Safra [5]).

Hence for δ_a we have

$$u(z;\delta_a) = \begin{cases} u(z)f'(0) & z \le a \\ \\ u(a)f'(0) + (u(z) - u(a))f'(1) & z \ge a \end{cases}$$

This local utility is not differentiable at a unless $f'(0) = f'(1)$. Concavity now implies $f' \equiv 1$ and V is reduced to expected utility. Hypothesis 2 is violated because at a, $u(a;\delta_a)$ represents an infinite level of risk aversion, while for $a' > a$, $u(a;\delta_{a'})$ represents a finite level of risk aversion.

A special case of the rank-dependent utility family is Yaari's [31] dual theory, given by $V(F) = \int x df(F)$. This functional seems to solve the problem raised by Rabin's analysis. Assume that f is concave (hence risk aversion) and that $f(\frac{1}{2}) = \frac{11}{21}$. Clearly, the decision maker rejects $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels for all $g < 1.1\ell$, but accepts $(-L, \frac{1}{2}; G, \frac{1}{2})$ at all wealth levels for all $G > 1.1L$. Properties D1 and D2 are satisfied and although small lotteries are rejected (B3), attractive large lotteries are accepted even for modest gains (B4). Moreover, the dual theory is a member of the larger class of constant risk aversion preferences (that is, constant absolute and constant relative risk aversion) which display similar behavior.

Definition 2 *The functional V satisfies constant risk aversion (CRA) if for all $\alpha > 0$, β, F, and H,*

$$V(F) \geqslant V(H) \Longleftrightarrow V(\alpha \otimes F \oplus \beta) \geqslant V(\alpha \otimes H \oplus \beta)$$

where $F \oplus \beta$ is obtained from F by adding β to all its outcomes and as before, $\alpha \otimes F$ is obtained from F by multiplying all outcomes by α.

If for some wealth level the CRA decision maker is indifferent between accepting and rejecting the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$, then for all $\varepsilon > 0$: *1.* he rejects the lottery $(-\ell, \frac{1}{2}; g - \varepsilon, \frac{1}{2})$ at all wealth levels, and *2.* he accepts the lottery $(-K\ell, \frac{1}{2}; Kg + \varepsilon, \frac{1}{2})$, $K > 0$ at all wealth levels. The four properties D1–B4 are thus satisfied.

Like risk-averse rank-dependent preferences, CRA preferences have differentiable local utilities only when they are reduced to expected utility (see Safra and Segal [25]. In the case of CRA, expected utility means expected value). To analyze general preferences — preferences with nondifferentiable local utilities and preferences that violate Hypothesis 2 — we consider a stochastic version of B3 where the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at both deterministic and stochastic wealth levels.

Definition 3 *(Stochastic B3): The functional* V *satisfies* (ℓ, g) *stochastic B3 on* $[a, b]$ *if for all* F *with support in* $[a, b]$, $V(F) > V(\frac{1}{2}[F \ominus \ell] + \frac{1}{2}[F \oplus g])$.[7]

The distribution F in the above definition serves as background risk — risk to which the binary lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is added. Since initial wealth is usually stochastic, an observed rejection of the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ indicates behavior according to the stochastic version of B3.

For functionals satisfying stochastic B3 we have the following result.

Theorem 3 *Let* $V \in \mathcal{V}$ *satisfy* (ℓ, g) *stochastic B3 on* $[a, b]$ *and let* $n = \frac{b-a}{\ell+g}$. *Then there is* F^* *with support in* $[a, b]$ *and* $\varepsilon^* > 0$ *such that for all* $\varepsilon < \varepsilon^*$,

$$V((1-\varepsilon)F^* + \varepsilon\delta_a) \geqslant V((1-\varepsilon)F^* + \varepsilon H)$$

for all $H = (a - L, \frac{1}{2}; a + G, \frac{1}{2})$ *where*

$$L \geqslant \frac{\ell(\ell+g)}{g-\ell} + 1 \text{ and } G = \frac{L-1}{\ell}[(n-1)(g-\ell)+g] \tag{6}$$

Table 3 offers some examples for values of n and G that satisfy the conditions of Theorem 3 for $\ell = 100$ and $L = 25{,}000$. For example, if the decision maker rejects the stochastic risk $(-100, \frac{1}{2}; 110, \frac{1}{2})$ at all lotteries with final outcomes between 100,000 and 325,000, then there is a distribution F^* on this support such that for a sufficiently small ε he prefers the distribution $(1-\varepsilon)F^* + \varepsilon\delta_{100{,}000}$ to $(1-\varepsilon)F^* + \varepsilon H$, where H is the distribution of the lottery $(100{,}000 - 25{,}000, \frac{1}{2}; 100{,}000 + 2{,}703{,}571, \frac{1}{2})$.

$b-a$	$g = 101$	$g = 105$	$g = 110$	$g = 125$
45,000	80,970	299,390	560,714	1,275,000
112,500	164,925	710,975	1,364,285	3,150,000
225,000	304,850	1,396,951	2,703,571	6,275,000
450,000	584,701	2,768,902	5,382,142	12,525,000

Table 3: If the decision maker rejects $\frac{1}{2}[F \ominus 100] + \frac{1}{2}[F \oplus g]$ for all distributions F with outcomes between a and b, then he also prefers $(1-\varepsilon)F^* + \varepsilon\delta_a$ to $(1-\varepsilon)F^* + \varepsilon H_{a,\,25{,}000,G}$ ($H_{a,\,25{,}000,G}$ is the distribution of $(a - 25{,}000, \frac{1}{2}; a + G, \frac{1}{2})$) for some distribution F^* and for a sufficiently small ε, values of G entered in the table.

[7] $F \ominus \ell = F \oplus (-\ell)$.

Clearly, Stochastic B3, the "if" condition of Theorem 3, is weaker than the "if" condition of Theorem 1: if, for every x, the decision maker objects to replacing the single outcome x with the distribution H_x, then, by constructing a sequence of such changes, it is evident that he objects to replacing *all* outcomes x by the corresponding lotteries H_x.

Empirical evidence seems to support stochastic B3 (see, e.g., Guiso, Jappelli and Terlizzese [12], Paiella and Guiso [19], and Hochguertel [15]).[8] Note that for expected utility functionals the stochastic version of B3 is equivalent to the deterministic one: A rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all deterministic wealth levels implies its rejection at all stochastic wealth levels. Likewise, this definition is satisfied by risk averse rank-dependent functionals with a sufficiently concave utility function u.

Rabin's logic does not imply rejection of large attractive lotteries, but that the decision maker cannot simultaneously reject small actuarially favorable lotteries and accept all large attractive ones. Theorem 3 provides similar results. It does not suggest that all functionals reject attractive lotteries — Yaari's [31] theory, for example, accepts even moderately attractive large lotteries $(-L, \frac{1}{2}; G, \frac{1}{2})$, provided $G > f\left(\frac{1}{2}\right) L / \left[1 - f\left(\frac{1}{2}\right)\right]$. But then preferences cannot satisfy stochastic B3 for small values of ℓ and g. In fact, the next proposition shows that CRA functionals do not satisfy stochastic B3 for *any* $g > \ell$.

Proposition 2 *If $V \in \mathcal{V}$ satisfies constant risk aversion and is continuously Gâteaux differentiable,*[9] *then for $g > \ell$, V cannot exhibit (ℓ, g) stochastic B3 on $[a, b]$ for a sufficiently large $b - a$.*

Consider the CRA functional $V(F) = \int x df(F(x))$ (Yaari [31]). Let $f(p) = p^\eta$ and let F be the uniform distribution over $[a, b]$. For $\eta = 0.5$, $\ell = 100$, and $g = 110$, the decision maker enjoys the additional risk $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ whenever $b - a >$ 19,461 and for $\ell = 100$, $g = 105$, and $\eta = 0.7$, whenever $b - a >$ 7,482.

[8] Rabin and Thaler [23] on the other hand seem to claim that a rejection of a small lottery is likely only when the decision maker is unaware of the fact that he is exposed to many other risks.

[9] That is, for every F and H, $\frac{\partial}{\partial \alpha} V((1-\alpha)F + \alpha H)\big|_{\alpha=0}$ exists, is linear in H, and continuous in F.

Appendix

Proof of Lemma 1 Suppose $u(\cdot;F)$ is not concave. Then there exist H^* and H such that H is a mean preserving spread of H^*, but $\int u(x;F)dH^*(x) < \int u(x;F)dH(x)$. For every ε, $(1-\varepsilon)F+\varepsilon H$ is a mean preserving spread of $(1-\varepsilon)F+\varepsilon H^*$, hence, by risk aversion, $V((1-\varepsilon)F+\varepsilon H) \leqslant V((1-\varepsilon)F+\varepsilon H^*)$. As this inequality holds for all ε, it follows that $\frac{\partial}{\partial\varepsilon}V((1-\varepsilon)F+\varepsilon H)\big|_{\varepsilon=0} \leqslant \frac{\partial}{\partial\varepsilon}V((1-\varepsilon)F+\varepsilon H^*)\big|_{\varepsilon=0}$. Hence, by equation (1), $\int u(x;F)dH(x) \leqslant \int u(x;F)dH^*(x)$, a contradiction. ■

Proof of Proposition 1

(1) $\Longrightarrow$ (2): Let $F_\alpha = (1-\varepsilon)F + \varepsilon[(1-\alpha)\delta_x + \alpha H_x]$. For $\zeta \geqslant 0$ we obtain

$$F_{\alpha+\zeta} = \left(1 - \tfrac{\zeta}{1-\alpha}\right) F_\alpha + \tfrac{\zeta}{1-\alpha}((1-\varepsilon)F + \varepsilon H_x)$$

From eq. (1) we have[10]

$$\begin{aligned}
\frac{\partial}{\partial\alpha}V(F_\alpha) &= \lim_{\zeta\to 0}\tfrac{1}{\zeta}[V(F_{\alpha+\zeta}) - V(F_\alpha)] \\
&= \lim_{\zeta\to 0}\tfrac{1}{\zeta}\left[V\left(\left(1-\tfrac{\zeta}{1-\alpha}\right)F_\alpha + \tfrac{\zeta}{1-\alpha}((1-\varepsilon)F+\varepsilon H_x)\right) - V(F_\alpha)\right] \\
&= \lim_{\zeta\to 0}\tfrac{1}{\zeta}\left\{\tfrac{\zeta}{1-\alpha}\left[\int u(y;F_\alpha)d((1-\varepsilon)F+\varepsilon H_x) - \right.\right. \\
&\qquad\qquad \left.\left.\int u(y;F_\alpha)dF_\alpha\right] + o(\| \tfrac{\zeta}{1-\alpha} \|)\right\} \\
&= \lim_{\zeta\to 0}\tfrac{1}{\zeta}\left\{\zeta\varepsilon[\tfrac{1}{2}u(x-\ell;F_\alpha) + \tfrac{1}{2}u(x+g;F_\alpha) - u(x;F_\alpha)] + o(\zeta)\right\} \\
&= \varepsilon[\tfrac{1}{2}u(x-\ell;F_\alpha) + \tfrac{1}{2}u(x+g;F_\alpha) - u(x;F_\alpha)] \\
&\leqslant 0
\end{aligned}$$

Hence $V((1-\varepsilon)F+\varepsilon\delta_x) = V(F_0) \geqslant V(F_1) = V((1-\varepsilon)F+\varepsilon H_x)$.

(2) $\Longrightarrow$ (1): By eq. (1),

$$\begin{aligned}
& V((1-\varepsilon)F+\varepsilon\delta_x) \geqslant V((1-\varepsilon)F+\varepsilon H_x) \\
\Longleftrightarrow\quad & \varepsilon\left[\int u(y;F)d\delta_x - \int u(y;F)dF\right] + o(\|\varepsilon\|) \geqslant \\
& \qquad \varepsilon\left[\int u(y;F)dH_x - \int u(y;F)dF\right] + o(\|\varepsilon\|)
\end{aligned}$$

[10] In eq. (1) notation, $\frac{\zeta}{1-\alpha}$ is ε, F_α is F, and $(1-\varepsilon)F+\varepsilon H_x$ is H.

$$
\begin{aligned}
&\Longrightarrow \quad \int u(y;F)d\delta_x - \int u(y;F)dF \geqslant \int u(y;F)dH_x - \int u(y;F)dF \\
&\Longleftrightarrow \quad u(x;F) \geqslant \tfrac{1}{2}u(x-\ell;F) + \tfrac{1}{2}u(x+g;F)
\end{aligned}
$$

The third line follows from the second by letting $\varepsilon \to 0$. ■

Proof of Theorem 1 We first show that if an expected utility maximizer with utility u rejects at all wealth levels x between a and b the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$, then when his wealth level is a, he will also reject any lottery of the form $(-L, p; G, 1-p)$, $G > b-a$, provided that inequality (4) is satisfied.

Rejecting the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at $a+\ell$ implies $u(a+\ell) - u(a) > u(a+\ell+g) - u(a+\ell)$. By concavity, $u'(a) \geqslant [u(a+\ell) - u(a)]/\ell$ and

$$
u'(a+\ell+g) \leqslant \frac{u(a+\ell+g) - u(a+\ell)}{g} < \frac{u(a+\ell) - u(a)}{\ell}\frac{\ell}{g} \leqslant \frac{\ell}{g}u'(a)
$$

Similarly, suppose for simplicity that $b = a + k(\ell+g)$ where $k = \frac{b-a}{\ell+g}$ is an integer and obtain

$$
u'(b) < u'(a)\left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \tag{7}
$$

Concavity implies that for every c, $u(c+\ell+g) \leqslant u(c) + (\ell+g)u'(c)$, hence

$$
u(b) \leqslant u(a) + (\ell+g)u'(a)\sum_{i=1}^{\frac{b-a}{\ell+g}}\left(\frac{\ell}{g}\right)^{i-1} \tag{8}
$$

Normalizing $u(a) = 0$ and $u'(a) = 1$ we obtain from eqs. (7) and (8)

$$
u'(b) \leqslant \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \quad \text{and} \quad u(b) \leqslant (\ell+g)\frac{1-\left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}}}{1-\frac{\ell}{g}} \tag{9}
$$

For concave u we now obtain that for every $x \notin [a,b]$

$$
u(x) \leqslant \begin{cases} -(a-x) & x < a \\ \\ u(b) + (x-b)\left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} & x > b \end{cases} \tag{10}
$$

Inequalities (9) and (10) imply that if for all wealth levels x between a and b the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$, then when his wealth level is a, he will also reject any lottery of the form $(-L, p; G, 1-p)$, $G > b-a$, provided that inequality (4) is satisfied.

Let H be the distribution of the lottery $(a-L, p; a+G, 1-p)$ and denote $F_\alpha = (1-\alpha)\delta_a + \alpha H$. Similarly to the proof of Proposition 1 we obtain that

$$\frac{\partial}{\partial\alpha}V(F_\alpha) = pu(a-L; F_\alpha) + (1-p)u(a+G; F_\alpha) - u(a; F_\alpha)$$

Since for all $x \in [a,b]$, $V((1-\varepsilon)F_\alpha + \varepsilon\delta_x) \geqslant V((1-\varepsilon)F_\alpha + \varepsilon H_x)$, then similarly to the proof of (2) $\Longrightarrow$ (1) in Proposition 1, for all $x \in [a,b]$ and α, $u(x; F_\alpha) \geqslant \frac{1}{2}u(x-\ell; F_\alpha) + \frac{1}{2}u(x+g; F_\alpha)$. Therefore it follows by the statement at the beginning of this proof that the expression $pu(a-L; F_\alpha) + (1-p)u(a+G; F_\alpha) - u(a; F_\alpha)$ is nonpositive. As $F_0 = \delta_a$ and $F_1 = H$, it follows that $V(a,1) \geqslant V(H)$. ■

Proof of Theorem 2 Similarly to the proof of Theorem 1, it can be shown that if an expected utility maximizer with utility u rejects at all wealth levels the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$, then when his wealth level is b, he also rejects the lottery $(-L, p; G, 1-p)$, provided inequality (5) is satisfied. By concavity, for every c, $u(c-\ell-g) \leqslant u(c) - (\ell+g)u'(c)$, hence

$$u(a) \leqslant u(b) - (\ell+g)u'(b)\sum_{i=1}^{\frac{b-a}{\ell+g}}\left(\frac{g}{\ell}\right)^{i-1} \tag{11}$$

Normalizing $u(b)=0$ and $u'(b)=1$ we obtain by (7) and (11)

$$u'(a) \geqslant \left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} \quad \text{and} \quad u(a) \leqslant -(\ell+g)\frac{1-\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}}}{1-\frac{g}{\ell}} \tag{12}$$

and hence, for every $x \notin [a,b]$

$$u(x) \leqslant \begin{cases} u(a) - (a-x)\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} & x < a \\ \\ x - b & x > b \end{cases} \tag{13}$$

Inequalities (12) and (13) imply that if for all wealth levels x between a and b the decision maker rejects the lottery $(-\ell, \frac{1}{2};\ g, \frac{1}{2})$, then

$$u(b) > pu(b-L) + (1-p)u(b+G) \tag{14}$$

provided that p and G satisfy inequality (5) and $L \geqslant b-a$. Examples of numbers satisfying this inequality are given in Table 2. Similarly, the decision maker rejects the same lottery at $b-g$, a fact that is used in inequality (16) below.

Let $\hat{\varepsilon}$ be as in the discussion following Theorem 2 and let L, G, and p satisfy the requirements of the theorem. Consider $x \in [a, b-g]$. As $V(x,1) > V(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$, it follows by Gâteaux differentiability that there is $q^* \in (0,1)$ such that $\frac{\partial}{\partial q} V(x, 1-q; x-\ell, \frac{q}{2}; x+g, \frac{q}{2})$ is strictly negative at q^*. Hence

$$u(x; K_x) > \tfrac{1}{2}u(x-\ell; K_x) + \tfrac{1}{2}u(x+g; K_x) \tag{15}$$

where K_x is the distribution of $(x, 1-q^*; x-\ell, \frac{q^*}{2}; x+g, \frac{q^*}{2})$.

Obviously, δ_b dominates K_x by first order stochastic dominance whenever $b \geqslant x+g$. Hence, by Hypothesis 2, $u(x;\delta_b) > \frac{1}{2}u(x-\ell;\delta_b) + \frac{1}{2}u(x+g;\delta_b)$ for all $x \in [a, b-g]$. The increasing monotonicity of $u(\cdot;\delta_b)$ and inequality (14) now imply

$$\begin{aligned} u(b;\delta_b) > u(b-g;\delta_b) > \\ pu(b-g-L;\delta_b) + (1-p)u(b-g+G;\delta_b) \end{aligned} \tag{16}$$

By Gâteaux differentiability and continuity, this implies that for sufficiently small μ, the decision maker with wealth level b prefers not to participate in the lottery $X(\mu) = (-g-L, \mu p; 0, 1-\mu; -g+G, \mu(1-p))$. We now show that $\mu = 1$, which is the claim of the theorem.

Let $\bar{\mu} = \max\{\mu : V(b,1) \geqslant V(b+X(\mu))\}$ and suppose that $\bar{\mu} < 1$. Denote by $\bar{F}$ the distribution of $b + X(\bar{\mu})$. Our next step is to show that for all $x \in [b-g-L, b-g]$,

$$u(x;\bar{F}) > \tfrac{1}{2}u(x-\ell;\bar{F}) + \tfrac{1}{2}u(x+g;\bar{F}) \tag{17}$$

We defined $a = b-g-L$, therefore, as $\bar{\mu}p < p$ and by the construction of $\hat{\varepsilon}$, $V(\hat{X}_x) > V(\tilde{X}_x)$, where $\hat{X}_x = (b-g-L, \bar{\mu}p; x, 1-\bar{\mu}p)$ and $\tilde{X}_x = (b-g-L, \bar{\mu}p; x-\ell, \frac{1-\bar{\mu}p}{2}; x+g, \frac{1-\bar{\mu}p}{2})$. Let $\hat{F}_x$ and $\tilde{F}_x$ denote the distributions of $\hat{X}_x$ and $\tilde{X}_x$, respectively. Similarly to the derivation of eq. (15), it follows by Gâteaux differentiability that there exists F on the line segment connecting $\hat{F}_x$ and $\tilde{F}_x$ for which

$$u(x;F) > \tfrac{1}{2}u(x-\ell;F) + \tfrac{1}{2}u(x+g;F)$$

As $\bar{F}$ dominates both $\hat{F}_x$ and $\tilde{F}_x$ by first order stochastic dominance it dominates F as well and eq. (17) follows by Hypothesis 2.

Similarly to the derivation of eq. (16), the local utility at $\bar{F}$ satisfies

$$u(b;\bar{F}) > pu(b-g-L;\bar{F}) + (1-p)u(b-g+G;\bar{F}) \tag{18}$$

Let H denote the cumulative distribution function of $(b-g-L,p;b-g+G,1-p)$. Then, by Gâteaux differentiability and eq. (18),

$$\frac{\partial}{\partial t}V((1-t)\bar{F}+tH) = \\ (1-\bar{\mu})[pu(b-g-L;\bar{F}) + (1-p)u(b-g+G;\bar{F}) - u(b;\bar{F})] < 0$$

But this means that $\exists\mu \in (\bar{\mu},1)$ such that $V(b-g-L,\mu p;b,1-\mu;b-g+G,\mu(1-p)) < V(b-g-L,\bar{\mu}p;b,1-\bar{\mu};b-g+G,\bar{\mu}(1-p)) \leqslant V(b,1)$; a contradiction. Hence $\bar{\mu}=1$ and

$$V(b,1) \geqslant V(b-g-L,p;b-g+G,1-p) \qquad \blacksquare$$

Proof of Theorem 3 Eq. (6) is equivalent to

$$G \geqslant (n-1)(\ell+g) + \frac{g(\ell+g)}{g-\ell} \quad \text{and} \quad L = \frac{\ell G}{(n-1)(g-\ell)+g} + 1 \tag{19}$$

In the proof of the theorem we will use the following lemma. Its proof appears after the proof of Theorem 3.

Lemma 2 *Let u be a concave vNM function such that $u(a-\ell)=0$ and $u(a)=\ell$. Let $X=(a,\frac{1}{n};\ldots;a+(n-1)(\ell+g),\frac{1}{n})$. If $\mathrm{E}[u(X)] \geqslant \mathrm{E}[u(X-\ell,\frac{1}{2};X+g,\frac{1}{2})]$, then for G satisfying inequality (19) we obtain*

$$u(a+G) \leqslant \frac{\ell G}{(n-1)(g-\ell)+g} + \ell \qquad \square$$

Let F be the distribution of X of the Lemma and let $F' = \frac{1}{2}[F\ominus\ell]+\frac{1}{2}[F\oplus g]$ be the distribution of $(X-\ell,\frac{1}{2};X+g,\frac{1}{2})$. By stochastic B3, $V(F) > V(F')$. There is therefore $F^* = (1-\alpha)F+\alpha F'$ such that $V((1-\alpha)F+\alpha F')$ is strictly decreasing in α at F^*, hence

$$\mathrm{E}[u(F;F^*)] > \mathrm{E}[u(F';F^*)]$$

Normalize $u(\cdot;F^*)$ such that $u(a-\ell;F^*)=0$ and $u(a;F^*)=\ell$. As $u(\cdot;F^*)$ is concave (see Lemma 1), it follows that $u(a-L;F^*)\leqslant \ell-L$. By Lemma 2 and eq. (19),

$$\begin{aligned} \mathrm{E}[u(H;F^*)] &= \frac{1}{2}u(a-L;F^*)+\frac{1}{2}u(a+G;F^*) \\ &\leqslant \frac{1}{2}(\ell-L)+\frac{1}{2}\left[\frac{\ell G}{(n-1)(g-\ell)+g}+\ell\right] \\ &= \ell-\frac{1}{2}<u(a;F^*) \end{aligned}$$

For sufficiently small ε it thus follows by Gâteaux derivative that $V((1-\varepsilon)F^*+\varepsilon\delta_a)\geqslant V((1-\varepsilon)F^*+\varepsilon H)$, hence the theorem. ∎

Proof of Lemma 2 Observe first that

$$\begin{aligned} &(X-\ell,\tfrac{1}{2};X+g,\tfrac{1}{2})= \\ &\quad\big(a-\ell,\tfrac{1}{2n};a+g,\tfrac{1}{n};\ldots;a+(n-1)(\ell+g)-\ell,\tfrac{1}{n}; \\ &\qquad a+(n-1)(\ell+g)+g,\tfrac{1}{2n}\big) \end{aligned}$$

Denote $a_i=a+(i-1)(\ell+g)$, $i=1,\ldots,n$, $b_i=a_i-\ell$, $i=1,\ldots,n$, and $b_{n+1}=a_n+g$. Let $c_i=u(a_i)$ and $d_i=u(b_i)$. We assumed that $d_1=0$ and $c_1=\ell$, hence

$$\frac{c_1-d_1}{\ell}=1$$

As u is concave it has at each point x right and left derivatives denoted $u'_-(x)$ and $u'_+(x)$. By concavity, $u'_-\geqslant u'_+(b_1)\geqslant 1$. Also,

$$u'_+(b_{n+1})\leqslant u'_-(b_{n+1})\leqslant \frac{d_{n+1}-c_n}{g}$$

and

$$u(a+G)\leqslant d_{n+1}+(a+G-b_{n+1})\frac{d_{n+1}-c_n}{g}$$

(see Fig. (1)). Our aim is to solve

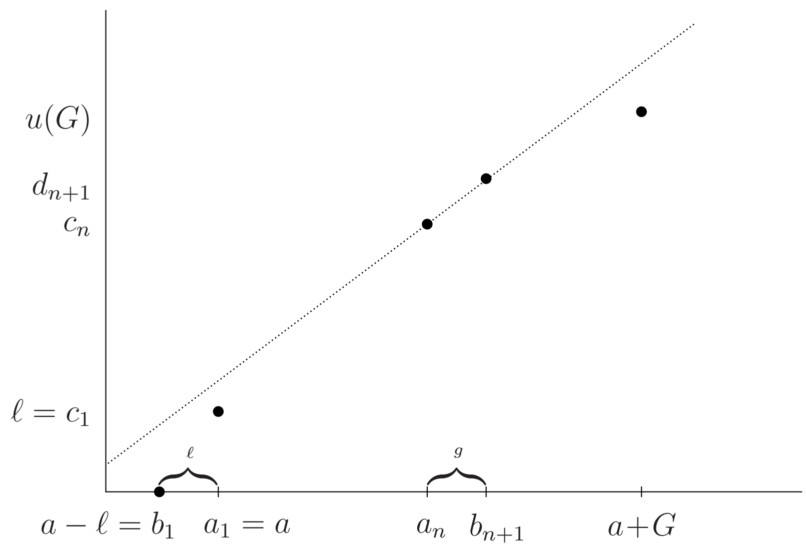

Figure 1: The function u and its value at $a+G$

$$
\begin{aligned}
&\max_{c_2,\ldots,c_n,d_2,\ldots,d_{n+1}} && d_{n+1} + (a+G-b_{n+1})\frac{d_{n+1}-c_n}{g} && (20)\\
&s.t. && \frac{1}{n}\sum_{i=1}^{n} c_i \geqslant \frac{1}{2n}d_1 + \frac{1}{n}\sum_{i=2}^{n} d_i + \frac{1}{2n}d_{n+1} && (21)\\
& && d_1 \leqslant c_1 \leqslant \ldots \leqslant c_n \leqslant d_{n+1} && (22)\\
& && \frac{c_1-d_1}{\ell} \geqslant \ldots \geqslant \frac{d_{n+1}-c_n}{g} && (23)
\end{aligned}
$$

Constraint (21) represents the rejection of the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ that is added to the original lottery. Constraints (22) follow by the monotonicity of u, while constraints (23) represent the concavity of u.

The target function is linear and there are $2n-1$ variables. As $d_1 = 0$ and $c_1 = \ell$, constraints (22) and (23) consist of $2n-1$ inequalities each, hence, together with (21), there are $4n-1$ linear constraints. At least one of the inequalities of line (23) must be strict, otherwise u is linear and inequality (21) is not satisfied. Since no subgroup of the other constraints follows from any other subgroup of these constraints, at least $2n-1$ of them must be satisfied with equality. In other words, no more than $2n$ of the constraints are satisfied with strict inequality, and one of them belongs to (23).

Suppose that the constraint (21) is satisfied with a strict inequality. One can then multiply by $\gamma > 1$ all the values of c and d from a strict inequality of (23) on without violating any of the constraints, thus increasing d_{n+1}, $d_{n+1} - c_n$, and ultimately the value of the target function. Therefore the first constraint must be satisfied with equality. Let $\hat{u}$ be an increasing concave function that obtains the values of c_i and d_i solving the optimization problem (20) at the points a_i and b_i.

Case 1: Assume first that the function $\hat{u}$ is strictly increasing. Then the $2n-1$ constraints (22) are satisfied with strict inequalities, and therefore no more than one of the constraints (23) is strict. In other words, the function $\hat{u}$ on $[b_1, a+G]$ is linear on both sides of one of the points a_i or b_i. Its slope is first 1 and then s, such that constraint (21) is satisfied with equality. Obviously such a kink cannot happen at a point b_i, or the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is not rejected. When the kink is at the point a_j we obtain

- $c_i = \ell + (i-1)(\ell + g), \quad i = 1, \ldots, j$
- $c_i = \ell + (j-1)(\ell + g) + s(i-j)(\ell + g), \quad i = j+1, \ldots, n$
- $d_i = (i-1)(\ell + g), \quad i = 1, \ldots, j$
- $d_i = \ell + (j-1)(\ell + g) - s\ell + s(i-j)(\ell + g), \quad i = j+1, \ldots, n+1$

The equality in line (21) now yields

$$
\begin{aligned}
&2\sum_{i=1}^{n} c_i = d_1 + 2\sum_{i=2}^{n} d_i + d_{n+1} \Longrightarrow \\
&2n\ell + \left[(2n-j)(j-1) + s(n-j)(n-j+1)\right](\ell + g) = \\
&\quad \left[(2n-j+1)(j-1) + s(n-j+1)^2\right](\ell + g) + \\
&\qquad \left[2(n-j)+1\right](1-s)\ell \Longrightarrow \\
&2n\ell = \left[(j-1) + s(n-j+1)\right](\ell + g) + \left[2(n-j)+1\right](1-s)\ell \Longrightarrow \\
&s = \frac{2n\ell - (j-1)(\ell+g) - [2(n-j)+1]\ell}{(n-j+1)(\ell+g) - [2(n-j)+1]\ell} = \\
&\quad \frac{j\ell - jg + g}{-n\ell + j\ell + ng - jg + g} = \\
&\quad \frac{j\ell - jg + g}{(n-j)(g-\ell) + g}
\end{aligned}
$$

We thus obtain that

$$
\begin{aligned}
&\hat{u}(a+G) = \\
&j\ell + (j-1)g + s(a+G-a_j) = \\
&j\ell + (j-1)g + \left[\frac{j\ell - jg + g}{(n-j)(g-\ell)+g}\right](G-(j-1)(\ell+g))
\end{aligned} \tag{24}
$$

Differentiate this last expression with respect to j to obtain

$$
\begin{aligned}
&\ell + g - \left[\frac{\begin{array}{c}(g-\ell)\,[(n-j)(g-\ell)+g]+\\ [j(g-\ell)-g](g-\ell)\end{array}}{[(n-j)(g-\ell)+g]^2}\right] \times [G-(j-1)(\ell+g)] - \\
&\qquad \frac{j\ell - jg + g}{(n-j)(g-\ell)+g} \times (\ell+g) = \\
&\frac{n(g-\ell)\times\left\{\begin{array}{c}-(g-\ell)[G-(j-1)(\ell+g)]+\\ (\ell+g)[(n-j)(g-\ell)+g]\end{array}\right\}}{[(n-j)(g-\ell)+g]^2} = \\
&\frac{n(g-\ell)\times\left\{\begin{array}{c}-(g-\ell)[G+\ell+g]+\\ (\ell+g)[n(g-\ell)+g]\end{array}\right\}}{[(n-j)(g-\ell)+g]^2} \leqslant 0
\end{aligned}
$$

where the last inequality follows from

$$
G \geqslant (n-1)(\ell+g) + \frac{g(\ell+g)}{g-\ell}
$$

(which was assumed by the Lemma). It follows that $\hat{u}(G)$ is decreasing in j. For $j=1$ we obtain from eq. (24) that

$$
\hat{u}(a+G) = \frac{\ell G}{(n-1)(g-\ell)+g} + \ell
$$

hence the claim of the Lemma.

Case 2: Assume now that the function $\hat{u}$ is not strictly increasing, and one of the constraints (22) is satisfied with equality. As the function $\hat{u}$ is increasing, it must be flat from that point on. By constraint (21), this point must be one of the a_i points. Suppose not, that is, suppose that there is a point b_i such that $c_{i-1} < d_i = c_i$. Define

$$\tilde{u}(y) = \begin{cases} \hat{u}(y) & y \leqslant b_i \\ d_i + \frac{1}{g}(y - b_i)(d_i - c_{i-1}) & b_i < y < a_i \\ d_i + \frac{1}{g}(a_i - b_i)(d_i - c_{i-1}) & a_i \leqslant y \end{cases}$$

As the function $\hat{u}$ satisfies constraint (21), so does the function $\tilde{u}$, and obviously $\tilde{u}(a+G) > \hat{u}(a+G)$.

As before, we must assume that constraint (21) is satisfied with equality, otherwise $\hat{u}$ can be replaced with a higher function still satisfying this constraint. Therefore we now have to solve the optimization problem under the constraints

$$\max_{c_2,\ldots,c_{i^*},d_2,\ldots,d_{i^*}} \quad c_{i^*}$$

$$s.t. \quad \frac{1}{n}\sum_{i=1}^{i^*-1} c_i + \frac{1}{2n}c_{i^*} = \frac{1}{2n}d_1 + \frac{1}{n}\sum_{i=2}^{i^*} d_i \tag{25}$$

$$d_1 < c_1 < \ldots < d_{i^*} < c_{i^*} \tag{26}$$

$$\frac{c_1 - d_1}{\ell} \geqslant \ldots \geqslant \frac{c_{i^*} - d_{i^*}}{\ell} \tag{27}$$

This problem has $2(i^* - 1)$ variables (recall that $d_1 = 0$ and $c_1 = \ell$) and $4(i^* - 1) + 1$ constraints. Of these, the $2(i^* - 1)$ constraints of (26) are satisfied with strict inequalities. Therefore at most one of the constraints of (27) is strict. In other words, $\hat{u}$ has slope 1 up to either a point b_i or a point a_i, then slope s up to point a_{i^*}, and slope zero thereafter. It is easy to verify that the highest value of c_{i^*} is obtained when $s = 1$, in which case, by constraint (25), c_{i^*} is bounded by $\frac{g}{g-\ell}$.

For G satisfying inequality (19) it is easy to verify that for $\ell > 1$,

$$\frac{\ell G}{(n-1)(g-\ell)+g} + \ell > \frac{g}{g-\ell}$$

hence the lemma. ■

Proof of Proposition 2 Let F be the uniform distribution over $[a,b]$, that is,

$$F(x)=\begin{cases}0 & x<a\\ \frac{x-a}{b-a} & a\leqslant x<b\\ 1 & b\leqslant x\end{cases}$$

Let $\tilde{F}=\frac{1}{2}(F\oplus(-\ell))+\frac{1}{2}(F\oplus g)$. Then

$$\tilde{F}(x)=\begin{cases}0 & x<a-\ell\\ \frac{x-a+\ell}{2(b-a)} & a-\ell\leqslant x<a+g\\ \frac{x-a-g}{b-a}+\frac{\ell+g}{2(b-a)} & a+g\leqslant x<b-\ell\\ \frac{x}{2(b-a)}+1-\frac{b+g}{2(b-a)} & b-\ell\leqslant x<b+g\\ 1 & b+g\leqslant x\end{cases}$$

Also define H and H_b by

$$H(x)=\begin{cases}0 & x<a\\ x-a & a\leqslant x<a+1\\ 1 & a+1\leqslant x\end{cases}$$

$$H_b(x)=\begin{cases}0 & x<a-\frac{\ell}{b-a}\\ \frac{x}{2}-\frac{1}{2}(a-\frac{\ell}{b-a}) & a-\frac{\ell}{b-a}\leqslant x<a+\frac{g}{b-a}\\ x-a-\frac{g-\ell}{2(b-a)} & a+\frac{g}{b-a}\leqslant x<a+1-\frac{\ell}{b-a}\\ \frac{x+1-a}{2}-\frac{g}{2(b-a)} & a+1-\frac{\ell}{b-a}\leqslant x<a+1+\frac{g}{b-a}\\ 1 & a+1+\frac{g}{b-a}\leqslant x\end{cases}$$

(See Fig. 2). By CRA, $V(F)\geqslant V(\tilde{F})$ iff $V(H)\geqslant V(H_b)$, for all b. We now show that it is impossible to have $V(H)\geqslant V(H_b)$ for all b.

Suppose $V(H)\geqslant V(H_b)$. Then there is λ_b such that the local utility at $K_b:=(1-\lambda_b)H+\lambda_b H_b$ prefers H to H_b. We obtain

$$\int_a^{a+1}u(x;K_b)dH(x)\geqslant\int_{a-\frac{\ell}{b-a}}^{a+1+\frac{g}{b-a}}u(x;K_b)dH_b(x)\iff$$

$$\int_{a-\frac{\ell}{b-a}}^{a+\frac{\ell}{b-a}}u(x;K_b)d[H-H_b](x)\geqslant\int_{a+\frac{\ell}{b-a}}^{a+1+\frac{g}{b-a}}u(x;K_b)d[H_b-H](x)\iff$$

$$\int_{a-\frac{\ell}{b-a}}^{a+\frac{\ell}{b-a}}[H_b-H](x)du(x;K_b)\geqslant\int_{a+\frac{\ell}{b-a}}^{a+1+\frac{g}{b-a}}[H-H_b](x)du(x;K_b)$$

where the last inequality is obtained by integration by parts of the second line. Since $u(\cdot;K_b)$ is concave, this last inequality implies that

$$u'_+\left(a-\frac{\ell}{b-a};K_b\right)D_1 \geqslant u'_-\left(a+1+\frac{g}{b-a};K_b\right)D_2 \tag{28}$$

where D_1 and D_2 are the sizes of the areas S_1 and S_2 marked in Fig. 2. We obtain

$$\begin{aligned} D_1 &= \frac{1}{2}\left(\frac{\ell}{b-a}\right)^2 \\ D_2 &= \frac{g-\ell}{2(b-a)}\left(1-\frac{\ell+g}{b-a}\right) \end{aligned}$$

Clearly, as $b\to\infty$, $D_1/D_2\to 0$, hence, by inequality (28),

$$\lim_{b\to\infty}\frac{u'_+\left(a-\frac{\ell}{b-a};K_b\right)}{u'_-\left(a+1+\frac{g}{b-a};K_b\right)}=\infty$$

By concavity of the local utility functions, for a given b^* where $u'(a-\frac{\ell}{b^*-a};H)$ and $u'(a+1+\frac{g}{b^*-a});H)$ both exist,[11]

$$\begin{aligned} \infty &= \lim_{b\to\infty}\frac{u'_+\left(a-\frac{\ell}{b-a};K_b\right)}{u'_-\left(a+1+\frac{g}{b-a};K_b\right)} \leqslant \\ & \lim_{b\to\infty}\frac{u'_+\left(a-\frac{\ell}{b^*-a};K_b\right)}{u'_-\left(a+1+\frac{g}{b^*-a};K_b\right)} = \\ & \frac{u'\left(a-\frac{\ell}{b^*-a};H\right)}{u'\left(a+1+\frac{g}{b^*-a};H\right)} \end{aligned} \tag{29}$$

The last equation sign follows by the continuity of the Gâteaux derivative and by the fact that all local utilities are concave. Since $a-\frac{\ell}{b^*-a}$ is not the left end of the domain of $u(\cdot;H)$, concavity implies that $u'\left(a-\frac{\ell}{b^*-a};H\right)<\infty$. On the other hand, as $a+1+\frac{g}{b^*-a}$ is not the right end of the domain on that function, first order stochastic dominance and concavity imply that $u'\left(a+1+\frac{g}{b^*-a};H\right)>0$, a contradiction to the fact that in eq. (29) the limit in ∞. Hence $V(H_b)>V(H)$ and $V(\tilde{F})>V(F)$ ■

[11] As $u(\cdot;H)$ is concave, $u'(\cdot;H)$ exists almost everywhere.

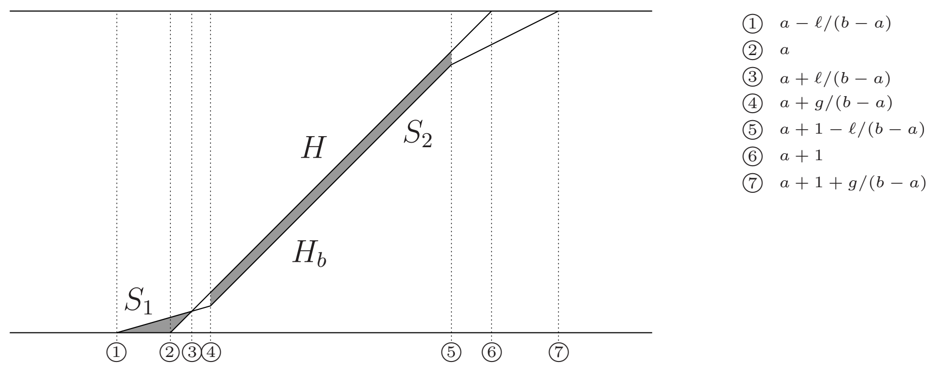

Figure 2: H and H_b

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