The Talmud On Transitivity

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Abstract

Transitivity is a fundamental requirement for consistency. Legal systems, especially when composed over time and by different agencies, may encounter nontransitive cycles. This paper shows that treatments of violations of transitivity already appear in the rabbinic literature, starting with the Mishnah and the Talmud (1st–5th c CE). This literature offers several solutions that are similar to those used in the modern economic literature, as well as some other solutions that may be adopted in modern situations.

We analyze several examples. One where nontransitive relations are acceptable; one where a violation of transitivity leads to problems with extended choice functions; and a third where a nontransitive cycle is deliberately created in order to prevent injustice.

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1 Introduction

Legal system are created over time and by different agencies and it is therefore hardly surprising that occasionally inconsistencies are discovered and need to be resolved. Most systems have some built-in resolution mechanisms, for example, legislative agencies are ranked as are courts. Such mechanisms are suitable for solving conflicts where two rules contradict each other — for example, when laws are declared unconstitutional or when higher courts nullify decisions made by lower courts.

There is another type of inconsistency, where any pair of two legislations is consistent, but two laws in conjunction imply a conclusion that is opposed by another law. Such is the case of nontransitive cycles, where by one law A is ranked above B, by another law B is ranked higher than C, but a third law ranks C higher than A. Such cycles may appear, for example, when rights of creditors are ranked, or where legislatures must be ranked.

In this paper we describe how the legal system of the Talmud†[1] copes with this challenge. Our purpose is twofold. We show that modern attitudes to violations of transitivity have a long history and more importantly, we use this ancient literature to offer new perspectives, interpretations, and solutions for such situations.

There are several theoretical objections to violations of transitivity. The first relates to the requirement that judges use logical reasoning in their decisions (see Levi [10] and Dworkin [2]). If three rules create a cycle as above and a dispute between A and C needs to be resolved, then the logical conclusion that ranks A higher than C clashes with the direct ruling concerning these two options, leaving the judge with contradicting instructions.

Transitivity is also a key axiom in social choice theory (see e.g. Harsanyi [8] and Sen [20]). The best support for transitivity in this context, or in general choice situations, comes from the requirement that choice should be consistent (see e.g. Sugden [24] or Rubinstein [**?**]). Let X be a set of alternatives, and let c be a choice function defined on all nonempty subsets of X. Suppose that for three alternatives x, y, and z, the choice function c satisfies $c(x, y) = x$, $c(y, z) = y$, but $c(x, z) = z$. What should $c(x, y, z)$ be? If the outcome of c is a singleton (say x), then there is another outcome that is even better. And if the outcome of $c(x, y, z)$ is not a singleton, then one of

[1] See Appendix 1 for a glossary and short bios of items marked with a dagger†. In this paper we refer by the term "Talmud" to the whole corpus of the rabbinic literature.

these outcomes is dominated by the other.[2]

The Talmud accepts the logic of transitivity and use it in many different ways. For example, since the ritual reading of the book of Esther is so important that it suspends temple service, and since temple service is more important than (religious) study, the Talmud [25, Meg. 3a] concludes, in line with modern reasoning, that schools are canceled for the ritual reading of the book of Esther.[3]

Because of the wide acceptance of transitive reasoning, the Talmud is sensitive to violations of this rule. In this paper we discuss three examples of such violations. The first is a set of three different activities x, y, and z, for which the Rabbis offer reasons why x overrides y, y overrides z, and z overrides x. As this nontransitive cycle does not create any operational inconsistency, the Talmud ignores the theoretical problem, very much like some of the solutions offered in the modern economic literature for individual violations of transitivity (see e.g. Fishburn and LaValle [4]). We expand on this in Section 2 below.

In Section 3 we analyze a problem from the Babylonian Talmud where a choice function defined over two-item sets cannot extend to include three-item sets. The Talmud offers two solutions. One suggests a violation of the independence of irrelevant alternatives (IIA) axiom. The other is based on ranking *priority rules* rather than options. Solutions that are similar to the

[2]In the decision theoretic literature there is another argument against violations of transitivity which is based on 'Dutch book' (or 'money pump') manipulations (see Ramsey [18, p. 78], Tversky [29, p. 45], and Yaari [30, p. 36]). Suppose that a decision maker holds x, and prefers x to y, y to z, but z to x. Offer him first to switch from x to z. As he prefers z to x, he will accept. Then offer him to switch from z to y, which once again he will accept. Finally, for a small commission, offer him to switch from y to x (charge the commission only at the last step to avoid the issue of income-dependent preferences). If preferences are continuous, he will switch once more, to be back where he started (holding x), minus the commission he paid. Obviously this behavior is foolish — why pay for nothing?

[3]Transitive chains can be used by the Talmud in nonstandard ways. When searching for the meaning of a certain biblical phrase, the Rabbis may compare it to an identical phrase in a different context, claiming that both occurrences must have the same meaning. In several cases they offer a sequence of such citations, each has a common word with its direct predecessor. The last verse then determines the meaning of the first one. See for example [25, Eruv. 51a, Pes. 7b, Ket. 45b,46a], [26, Shab. 2:3, 4d]. Another example for a non-trivial application of transitivity is the ruling that if person A owes B money, and B owes C, then A is considered a debtor of C, with the implication that B cannot waive A's debt (see [25, Ket. 19a]).

former exist in the economic literature, but the latter solution is new, and as we show, can be formalized into a procedural solution to nontransitive cycles.

Our last case is probably the most interesting — it involves medieval interpretations of a difficult Talmudic text, where legal requirements clash with considerations for justice. One of the interpretations suggests preserving a nontransitive cycle in order to force conflicting parties into a compromise. We discuss some possible applications of this idea in Section 4.

A few remarks are needed here on the nature of the rabbinic materials used in this article. Essentially, Talmudic Law encompasses all aspects of [Jewish] life, legal as well as ritual. The Rabbis treat all of them with the same seriousness and meticulousness. Moreover, they use the same analytical tools in their discussions of the different materials, be it criminal law, civil law, temple rituals or other religious commandments, public or individual. This aspect of rabbinic Law characterizes the examples we use in this paper. Another important aspect of these materials is their interpretive nature. Typically, the Talmudic passage is built through interpretation and counter-interpretation of previous sources. In this article, we give special attention to the interpretive principles and processes, explicit or implicit, of the texts.

Some, if not all, of the examples discussed below may seem contrived. Indeed they are. The way of the Talmud is to create such examples in order to isolate or examine the fine details of an idea or a rule (see Moskovitz [16]).

Throughout the paper, we use many Hebrew and Aramaic sources. Whenever an English translation exists, we make a reference to it as well. In Appendix 1, we give brief descriptions of the different sources cited and the rabbinic terms used in the article.

2 Inconsistent Rules

The Law does not cover all possible cases, and courts may need to extrapolate from existing laws. This is certainly the case with casuistic Law, like the Talmudic[†] Law. Transitive deduction is a simple, yet powerful tool for such extrapolations and is indeed used in the rabbinic literature. But what happens when such deductions contradict other rulings or deductions?

Such situations create two types of difficulties. One is philosophical — we expect a legal system to be free of logical contradictions. The other one is practical — which logical deduction should we follow? It is important to

observe that although in many cases cyclic rules need to be resolved because they lead to contradictory rulings, there are situation where such rules may coexist. Although they form a philosophical or a logical problem, they do not lead to any operational conflicts. As we show in this section, the Talmud is not bothered by such cycles.

Consider the following observations which are based on the analysis of biblical verses.

1. The laws of Sabbath are suspended in order to perform temple services. Such services are performed on the Sabbath even though the Jewish Law forbids many of these acts (e.g. killing animals) during that day.

2. Executions do not suspend Sabbath.[4]

The Talmud deliberates: If Sabbath, which is suspended for temple service, is not suspended for execution, then surely temple service itself is not suspended for execution. Therefore, a priest who was sentenced to death and escaped to the temple should be able to suspend his execution by performing at the temple until after the end of the service.

This conclusion fits well into the logic of transitivity, but eventually, this is not the rabbinic Law. From another verse, "But if a man come presumptuously upon his neighbor to slay him with guile, thou shalt take him from mine altar, that he may die" (Ex. 21:14), the Mekilta† concludes a third rule:

3. Temple service is suspended in order to take the condemned priest to execution.[5]

The Mekilta does not reject the logic of transitive deduction. On the contrary, the whole argument is built upon this logic. As is obvious from the rest of the passage, the Mekilta is not at all ignorant of the cycle that is thus inevitably created, but it is not bothered by it. Acting as legislators, the Rabbis are not bound by the logic of transitivity as the Law stands above its logical manipulations. Obviously, this procedure leads to a nontransitive set of rules. But unlike the analysis of the Introduction and of Section 3 below,

[4]For Rule 1, see Num. 28:9–10; [13] Shab., III p. 209; [27] Tzav 2:10, p. 30c; 3:5 p. 31b; Emor 13:7, p. 103d. For Rule 2, see [22] San. 4:1 and [13] Shab. III p. 209–210.

[5][13] Nez. 4, III pp. 37–38.

where a resolution is needed for cases in which the choice set consists of all three elements, here there is no such choice problem.[6]

The lesson from the analysis of this section is simple. Transitivity might be used to obtain a legal ruling, but this ruling is weaker than other explicit rules. In the absence of such legislation, courts should use transitive deduction to obtain new rules. Legislators, however, need not be bound by this logic. This approach offers legislators some needed flexibility. There are situations where A is deemed more important than B and there may be other situations where B is more important than C. Short of any counter arguments, transitivity should be used to deduce the priority of A over C. But as the original priorities of A over B and of B over C are not global the Law may find that in situations where A and C are compared the attributes of C supersede those of A.

3 Inconsistent Choice

The connection between transitive rankings and consistent choice is well known (see e.g. Rubinstein [19]). In this section we show how the Talmud[†] and its commentators deal with a case in which a choice function cannot be supported by a transitive relation and yet needs to be extended from pairs to larger sets. The example we discuss is of two inconsistent laws regarding the precedences of private offerings in the Jerusalem temple.[7] Initially, the two sets of laws were created in two different rabbinic compositions. Typically, the Talmud solves such conflicts by applying the opposing rulings to different situations. But in the case discussed below this natural distinction was blurred and the Talmud was thus faced with a nontransitive cycle. The need to solve it eventually led medieval commentators to develop strategies for untangling such cycles. This section analyzes the hermeneutical development

[6]The combination of the three elements, where a condemned priest is performing a temple service on Sabbath, is irrelevant, as this priest will not be executed on that day even if he is not performing at the temple. Observe that a similar reasoning supports the famous "Rock, Paper, Scissors" game used by children all over the world for a conflict resolution. It works *because* it is applied to two-item sets only. Three children (or more) will not be able to use this mechanism for conflict resolution.

[7]The analysis of the Talmud is purely theoretical as by the time the issue was discussed the Jerusalem temple was destroyed for more than 200 years. This does not prevent the Talmud from analyzing the question as the Rabbis maintained that the study of the old laws is culturally and spiritually important, regardless of their actual relevance.

of the text, the cycle, and its possible solutions.

Ranking all possible private offerings, the Mishnah† [22, Zev. 10:4] is using two rules.

Rule 1. Type of animal: Cattle precede birds.

Rule 2. Ritual function: Within each type, offerings are ranked by their ritual importance. For example, sin-offerings† precede burnt-offerings† and burnt-offerings precede tithe.†

The Mishnah is using these rules lexicographically, first by the type of the animal and then by its ritual function. According to this procedure, the tithe, being from the cattle, precedes bird sin-offering, even though ritually the sin-offering is several degrees higher than the tithe. In the words of the Mishnah: "The Tithe precedes the Bird-offerings, since it is an animal-offering."

There may be another relevant criterion, about which the Mishnah is silent. The gathering of a group of various offerings may occur in two different circumstances.

(a) Several offerings accumulated at random — one person or more brought them simultaneously for different ritual purposes.

(b) A group of offerings is brought by one person as one sacrificial unit for a specific ritual.[8]

The Mishnah clearly intends to impose its ruling in the first situation because it ranks some offerings that can never form any of the known groups of the second. However, the Mishnah does not address the issue of sacrificial units that are composed of more than one animal.

In contrast with the rules of the Mishnah, another text from the same period, the Sifra,† offers another rule. Restricting itself to sin- and burnt-offerings, it says:

Rule 3. Sin-offerings precede burnt-offerings — "be it a bird sin-offering with a bird burnt-offering, be it a bird sin-offering with a cattle burnt-offering or a cattle sin-offering with a cattle burnt-offering — sin-offerings always precede burnt-offerings that are brought with them."[9]

[8]For example, a nazirite, when accomplishing his vow, should bring a group of three animals: a sin-offering, a burnt offering and a peace-offering† (Num. 6:16-17). A common group is the pair of birds – a sin-offering and a burnt-offering – brought by the poor person who cannot afford the regular cattle sin-offering (e.g. Lev. 5:7, 12:8, Luke 2:24).

[9]Sifra, Ḥova 18:5-6, [23, p. 189].

This ruling in the Sifra is derived from Scripture[10] and its formulation rules explicitly that the only criterion to be used is ritual function and not the type of the sacrificed animal. This seems to contradict Rule 1 of the Mishnah where the type of animal is the first criterion to apply. However, one should notice the words "that are brought with them" at the end of rule 3 of the Sifra, which imply that rule 3 must apply only to situation (b) above. Therefore, the Mishnah and the Sifra may be reconciled in the following way. If two or more offerings accumulate at random, then they are sacrificed according to rules 1 and 2. But if a "closed" set of offerings is brought as one sacrificial unit then rule 3 applies. It is only in the latter case that a bird sin-offering precedes a cattle burnt-offering.[11]

Yet, the conflict between the rulings of the Mishnah and the Sifra is not completely resolved. What if situations (a) and (b) occur simultaneously? For example, a person arrives at the temple with a tithe (say, a calf) and at the same time a woman after giving birth brings the pair of bird (sin-offering) and lamb (burnt-offering)? The internal order of the second group is governed by rule 3: First the bird and then the lamb; but which should be brought first to the altar, the tithe or the woman's offerings? On the one hand, the woman's pair contains a burnt-offering (the lamb) which according to rule 2 precedes the tithe. On the other hand, the actual choice is between the tithe and the bird and according to rule 1 the tithe should come first. This is a simple ranking problem of two items (where one of the items is composed of two animals) and the above two are the only ways in which it can be solved. This is probably the original form of the question asked by the Talmud [25, Zev. 90b]:

> A bird sin-offering, an animal burnt-offering and tithe — which of these come first?

meaning: "[A pair of] {a bird sin-offering and an animal burnt-offering} and tithe — which of these come first?" This is exactly the question we posed before. The Talmud offers two answers.

1. Here [= in Babylon] they held that an animal offering is superior.

[10] Through an exegesis to Lev. 5:8: "And [the priest] shall offer that which is for the sin-offering first."

[11] Actually, the only example for such a combination is the set of offerings brought by a mother after giving birth (Lev. 12:6). See Rashi† [25, Zev. 90a, s.v. *ḥattat ha'of*].

2. In the West [= Palestine] they say: [The superiority of] the animal burnt-offering [over tithe] enters in the bird sin-offering and elevates it over the tithe.

These two solutions are the two answers suggested above and as argued, are the only possible solutions. The Babylonian solution considers the two actual possible offerings — bird and tithe — and ranks the animal (tithe) before the bird.[12] The order is therefore [tithe, ⟨ bird, lamb ⟩]. The Palestinian solution evaluates the pair ⟨ bird, lamb ⟩ by its strongest element, hence the order is [⟨bird, lamb⟩, tithe].

This is a simple technical discussion, and this was probably the Talmud's original form. Nevertheless, in the present form of the Talmud the explanatory sentence that follows this question completely changes its meaning:

> A bird sin-offering, an animal burnt-offering and tithe — which of these come first?
>
> Shall the bird come first? There is the tithe that must precede it! Shall tithe come first? There is the animal burnt-offering that must precede it! Shall the animal burnt-offering come first? There is the bird sin-offering that must precede it!

Undoubtedly, the explanatory passage ("Shall the bird...") presents the question as a nontransitive cycle of three independent items.[13] According to this passage, the fact that the bird and the lamb are brought by one person as a pair is insignificant and the two offerings are considered two separate entities. In Appendix 2 we show the conceptual change that took place in this Talmudic text. For the present discussion it is sufficient to assert that the question is now transformed from a simple inquiry of which of two items should come first (where one of the items by itself is a pair), a question that has two simple and intuitive answers, into a contradiction created by a nontransitive cycle.

[12]The Babylonian's wording alludes to the formula used by the Mishnah in rule 1 to postulate the precedence of tithe to birds: "The Tithe precedes the Bird-offerings, since it is an animal-offering" [22, Zev. 10:3].

[13]Observe that the language of the Talmud is very clear — the question relates to a choice function which is to select one out of a set of three options. The question is *not* what is the right order at which the animals will be sacrificed, although eventually this question too will be answered.

Note that the explanatory passage transformed the question, but did not touch the answers. This inevitably causes a severe incoherence in the text of the Talmud. The two approaches (the Babylonian and the Palestinian) that answered the original question no longer fit the new form of the question. The Babylonian answer "an animal offering is superior" is helpful only when the choice is between a bird and an animal, but it cannot choose between the two animals, the tithe and the burnt-offering. The Palestinian solution too is no longer clear — if the three sacrifices are three independent items, then why should one pair together the bird and the burnt-offering and not any of the other two possible pairs? Any two items paired together will be able to overcome the third.

The reader is now encountered with a serious problem — how is one to interpret the original answers as solutions to the new question? We show here how this is done by Rashi, who silently introduced two novel solutions in order to adapt the original answers to the new nontransitive puzzle. These solutions are the main points of this section. One of them fits some modern analysis of inconsistent choice. The other one does not exist in the modern literature and offers a novel, yet intuitive solution to inconsistent choice rules.

The Babylonian solution that "an animal offering is superior" justifies the precedence of the tithe over the bird, as if the lamb burnt-offering does not exist. Rashi explains the reason for the "disappearance" of the lamb. Rule 3 is derived from Scripture,[14] hence has priority over non-scriptural rules and should be applied first. Applying rule 3 to the pair ⟨bird (sin-offering), lamb (burnt-offering)⟩ leaves only the bird to compete with the tithe. In this match the tithe wins according to rule 1.

According to this interpretation, the Babylonian solution to the nontransitive cycle is to rank the *rules* leading to this cycle. In our case, rule 3 is applied first and then rule 1. Using transitivity there is no need for rule 2 as rules 3 and 1 (in that order) imply the ranking [tithe, bird, lamb]. We show in Appendix 3 how this idea can be generalized to sets with more than three items.[15]

[14] In Rashi's words, rule 3 is a "scriptural decree."

[15] The case of the Talmud is relatively simple as it is using only rules 3 and 1. One can imagine more involved situations where rule 2 too will be used. Consider the case of four offerings — the above three and an ox peace-offering. By rule 3 the bird sin-offering precedes the burnt-offering, by rule 1 the tithe and the peace-offering precede the bird sin-offering, and by rule 2 the peace-offering precede the tithe. The order of sacrifice is therefore peace-offering, tithe, bird sin-offering, burnt-offering.

The idea that rules should be ordered is both natural and normatively appealing. Many judicial decisions involve conflicting past rulings or legislations and therefore require courts and legislators to rank rules, e.g., the superiority of the constitution over other laws or the superiority of the Supreme Court over other courts. Of course, these rankings too may lead to nontransitive cycles, yet this is less likely to happen. To understand why, consider the following scenario for a nontransitive cycle. Suppose A supersedes B for reason α. This reason is relevant to the comparison of A and B only and is irrelevant to comparisons between A and C or B and C. Similarly, B supersedes C and C supersedes A for reasons β and γ which likewise do not apply to A and B, respectively. The wider is the scope of the ranked items A, B, and C, the harder it is to find reasons that apply to only two of them. But if α is relevant for the comparison of C with both A and B, then A supersedes B and B supersedes C are governed by reason α which will also imply that A supersedes C and the cycle breaks down. Finally, the fact that the scope of rules is typically wider than the scope of specific laws establish our claim.[16]

The Palestinian solution "the animal burnt-offering enters in the bird sin-offering and elevates it over the tithe" was originally based on the idea that the tight connection between the bird and the lamb makes them one sacrificial unit that has to be ranked against the tithe. The question, however, has been modified according to the Talmud's later perception of rule 3, which dissolves the connection and holds that the fact that the bird and the lamb come as a pair does not make them one closed unit. A coherent interpretation of the passage in its new conceptual form has to give some significance to the linkage between the bird and the lamb, since this is the nature of the Palestinian solution, but this significance must not imply that the two sacrifices form a firm pair. This effort is reflected in Rashi's explanation, that the lamb "lends its importance to the bird and makes it superior to the tithe." Even if the "pairness" of the bird and the lamb has no sacrificial importance, one cannot forget that they were brought together by the same person and for the same ritual. The nature of the bird is influenced by its co-existence with the lamb, a small change that is sufficient to stop the cycle. This "remote

[16]Kalai, Rubinstein, and Spiegler [9] introduced the concept of *rationalization by multiple rationales*, where the set of strict preferences $\succ_1, \ldots, \succ_k$ over a set S thus rationalize a choice function if for every $T \subseteq S$ there is $\succ_i$ such that $c(T)$ is $\succ_i$-maximal in T. The Babylonian solution does that and goes a step further by ranking the partial preferences $\{\succ_i\}$. The rules are applied one at a time until a complete and transitive order is created. Naturally, it may happen that not all rules are used.

influence" enhances the bird and ranks it above the tithe. As it is also ranked higher than the lamb (rule 3), it will be sacrificed first. The ranking of the remaining two animals, the lamb burnt-offering and the tithe, is governed by rule 2 and is not contradicted by the other two rules.

The justification for the Palestinian rule is surprisingly similar to modern arguments. It suggests that although when the two stand against each other tithe precedes any bird-offering, at the presence of the burnt-offering that adjoins the bird sin-offering this order is reversed, and now the the bird precedes the tithe. In other words, the Palestinian solution recognizes that the inability to extend the choice function c from two-item sets to three-item sets is probably due to a violation of the Independence of Irrelevant Alternatives axiom:

IIA If T is a subset of S and $c(S)$ is in T, then $c(T) = c(S)$.

The rational for IIA, like the rational for the Weak Axiom of Revealed Preferences (see e.g. Rubinseten [19]), is that the ranking of any two outcomes should not depend on the availability of other possible outcomes. This is exactly the intuition rejected by the Palestinian solution. Although it is true that when the choice is between tithe and a bird tithe takes precedence, the existence of the third option, the burnt-offering, changes the nature of the bird sin-offering, and now the pair {bird,lamb} is elevated above the tithe.

In a famous example, Luce and Raiffa [11, p. 288] tell the story of a diner in an unfamiliar seemingly modest restaurant. There is no menu, but the waiter tells him that he may have either broiled salmon or a steak. After his choice of salmon is reported to the kitchen, the waiter tells him that in fact they also have fried snails and frog's legs. The diner detests both, but now wants to order the steak instead of the salmon. The explanation offered by Luce and Raiffa is simple. Initially, the traveler was uncertain about the quality of the restaurant, and as he prefers a mediocre salmon to a mediocre steak, he ordered the former. The fact that the restaurant serves fancy food tells him that his choice is actually between a high quality salmon and a high quality steak, of which he prefers the latter. The existence of the other alternatives, even if not chosen, changes the nature of the chosen options. According to the sages of the west, something similar happens here. Although it is generally true that the tithe precedes bird-offerings, the existence of the burnt-offering lamb that adjoins the birth-giver's bird sin-offering enhances this bird offering and elevates it above the tithe.

The shift in the meaning of rule 3 caused a profound change in the conceptual structure of the passage, transforming the question from a simple inquiry regarding a void in the Law into an exploration of a contradiction that leads to a nontransitive cycle. The incoherence thus generated in the passage compels the reader to reinterpret the original solutions and to adapt them to their new role. From this procedure, the original intuitive solutions came out as two innovative strategies for breaking nontransitive cycles.

4 Cycles and Fairness

Standard analysis of individual and social preferences tries to avoid violations of transitivity. In this section we show how such cycles may be actively created in order to promote justice. We describe below a case where a nontransitive cycle can be eliminated, but is nevertheless welcomed and actually enforced in order to prevent depravation of some individuals' material rights.

According to the Jewish Law, when a man marries a woman he signs a contract ("Ketubah†") which lists properties she will receive upon his death or when she is divorced. In the meanwhile, the rights and fruits of these properties belong to the husband who may sell them, with the explicit understanding that if he dies or divorces his wife, the properties are to be transferred to her.[17] A man may marry more than one wife, and if he has only one property, the same property is assigned to all of them. The first wife has precedence over the second, but if she dies before their husband, the second wife inherits the property.

Suppose a man married two women and then sold his only field. The order of claims is simple and is fully determined by the order of events. The first wife has the strongest claim to the field, then the second wife, and the buyer has the weakest claim. The Mishnah† [22, Ket. 10:5] complicates matters by adding a twist in which the first wife gives a written declaration to the buyer, saying 'I have no claim whatsoever upon you.[18]' The Mishnah rules that upon their husband's death, "the second wife may distrain on the buyer, and the first wife on the second, and the buyer on the first wife, and so they go on in turn until they arrange some compromise between them."[19]

[17]This condition is presumably reflected by the price of the property.

[18]Regarding this field and I would not seize it from you. (Albeck, Commentary to Mishnah Ket. 10:6 in [22, Vol. 3, p. 125].

[19]The Mishnah is usually not shy of dictating specific solutions, as is done for example

The cycle results from the fact that although the first wife waived her rights to seize the field from the buyer, she did not forego her claims against the second wife. The latter can claim the field from the buyer, as she married her late husband *before* the field was sold. After the field is transferred to the first wife, the written declaration signed by her enables the buyer to take possession of the field, thus creating a cycle.

The Talmud† [25, Ket. 95b] analyzes another, invented, case which is somewhat similar to the story of the Mishnah, in which a different ruling is obtained. In this case a man said to a married woman 'My property shall be yours and after you it shall be given to person X.' The woman then sold the property and died. Her husband may now seize the property from the buyer because his wife received the property after their marriage and therefore he 'bought' it before the buyer.[20] The woman's successor, person X, seizes it from the husband because, as explained by the Talmud, the original owner meant to tell the woman 'After your death, only your successor shall acquire possession; your husband shall not.' Now the buyer can seize the property back from the successor because he bought it while the woman was still alive, when the successor did not yet have a claim for the property. Abbaye (Babylon, mid 4th century) rules that the property should be confirmed in the possession of the buyer.

This ruling seems problematic. As all parties appear to have the same power, why should the property stay at the hands of the buyer? Moreover, as is asked by the Talmud, "why should this case be different from the Mishnah where we learned 'and so they go on in turn until they arrange some compromise between them'?" To this question the Talmud offers the following surprising answer: "In the Mishnah they all gave up something. But here it is only the buyer who paid." [25, Ket. 95b].

In the Mishnah, each of the three agents (buyer, first wife, and second wife) gave up something for the disputed asset. The buyer obviously paid for it, and the two women entered into a contract with their late husband in which they gave him the right to use some of their assets in return for rights

in an earlier Mishnah in the same chapter, where exact divisions are laid out for various scenarios (see analysis in Aumann and Maschler [1]). Since nothing is said here about the nature of the compromise, it seems that the Mishnah deliberately refrains from doing so. Medieval commentators offer all possible combinations: between all three, any set of two or specific sets of two.

[20]Marriage is considered an implicit sale (to the husband) of all the properties a woman acquires during the tenure of the couple joint life. See [22, Ket. 8:1], [25, Ket. 50a].

to the field after his death. But in Abbaye's story of the buyer, husband, and successor it is only the buyer who paid anything and he is the only one to suffer a loss. The husband was never intended to receive anything[21] and for the successor too it was like a windfall. Justice therefore requires to leave the property in the hands of the buyer.[22]

But now a new legal question needs to be answered — how can the legal infinite cycle be stopped? This questions is asked by Rashi† who answers it by offering the following explanation. The first time the husband approaches the buyer he has a just claim for the property. The buyer obtained it from the husband's late wife and he, the husband, "bought" it first and with respect to the woman's assets the husband has a priority over the buyer. But after the buyer seizes the property from the successor the court can refuse the husband's claim as now the buyer holds a property that was taken from someone else and not from the wife while the superiority of the husband is only with respect to his wife's property.[23]

Rashi shows how to stop after one round — but exactly the same distinction between the first and second round exists in the Mishnah. When the second wife first approached the buyer, her claim was based on the fact that she implicitly bought the field from her husband before the field was sold to the buyer. But in the second round the buyer holds the same field after re-

[21] And would have been able to seize the field only if the successor had died before the woman. But as this is not the case, the husband suffers no loss (Ginzburg [5, p. 92b]).

[22] The crux of Abbaye's ruling is that the situation of the three parties (buyer, husband, and successor) is not symmetric and therefore they should receive differential treatment. A similar attitude appears in the social choice literature. In Condorcet's voting paradox three voters α, β, and γ vote on three issues A, B, and C. Their rankings are ABC, BCA, and CAB, hence two out of three prefer A to B, B to C, and C to A and majority rule leads to a nontransitive cycle.

This procedure ignores possible differences in the intensity of preferences. Social welfare functions solve this paradox by maximizing a function of individual utilities, for example, by taking a (weighted) sum of these utilities (Harsanyi [7, 8]). If the utility of α from the three options is 10, 8, 6, the utility of β is $6, 10, 8$, and the utility of γ is 8, 6, 20, then the sum of utilities is maximized at option C and society prefers this option to the other two (between which it is indifferent). Utilitarianism breaks the spell of infinite cycles by paying attention to asymmetries between voters. Abbaye is doing something similar here — the situation is not symmetric because only the buyer suffers a real loss.

[23] Rashi [25, Ket. 95b] s.v. *Umuqminan*. Rashi's language is somewhat ambiguous regarding the actual existence of a first round. But the Tosafist† R. Isaiah di Trani (Italy, d. circ. 1240), essentially offering Rashi's interpretation and wording, explicitly suggests one round. For an alternative explanation of Abbaye see Tosafot [25, Ket. 95b s.v. *Umuqminan*], [28, p. 272]. See also Maimonides [12, Zechiya Umatana 12:12].

ceiving it from the first wife and not from the husband. Rashi's explanation should lead us to the conclusion that the field should stay at the hands of the buyer, but the Mishnah orders the cycle to continue until a compromise is reached.

This observation forces us to understand the Talmud according to Rashi in the following way. There is an infinite nontransitive cycle which occurs when a certain person is changing the default rules (the first wife in the Mishnah, by declaring 'I have no claim whatsoever upon you,' and the original owner in the story of Abbaye by saying 'after you it shall be given to person X'). Although there is a legal way to break the resulting infinite cycle, the Mishnah prefers not to use it, but rather to keep the cycle going. Not only does the Mishnah not stop the cycle, it actually rules that it must go on until the parties compromise.

According to the Talmud, both rulings, the one of the Mishnah and that of Abbaye, follow justice considerations. Abbaye used the legal distinction to do justice to the buyer. But in the Mishnah, where all parties are equal, it would be unjust to leave the field in the hands of the buyer. A just solution that takes care of the interests of all parties is preferred to the legal loophole that can stop the cycle.[24] Typically, infinite cycles are the problem, but here such cycles serve as a solution to an unjust allocation. The idea that compromises may solve a locked political situation is obvious. The idea that a nontransitive cycle is deliberately kept in order to force the disputing parties into a compromise is, to the best of our knowledge, novel.

5 A Tale of Two Cycles

Case 1: Sometime around the year 1000 CE, a question [17, §175] was sent to R. Ḥanoch (b. Moshe, of Spain, d. 1025). Jacob[25] married Leah and they had several daughters. Later he also married Rachel. Jacob then sold the fields brought to him by Rachel, and gave her instead a bill on one of his own

[24] We want to emphasize that Rashi does not explicitly offer this interpretation of the Mishnah — in fact, he offers no commentary to that part of the Mishnah at all. But our suggestion seems to be the only possible interpretation following Rashi's comments. There is almost no traditional commentary on this passage of Rashi. This may be due to the fact that although it appears in medieval manuscripts, it is missing from early printed editions of the Talmud and did not reappear until 1714.

[25] As is common in this literature, generic names are used in actual cases.

fields. Later, when Leah's daughter married, Jacob, with the full consent of Leah, gave the daughter the fields he pledged to Rachel.[26] He died, and now Rachel wants to seize the fields from Leah's son-in-law, as unlike Leah, she never agreed to this present.

R. Ḥanoch analyzes this case by using the Mishnah† of Ketubot (see Section 4 above), and concludes that Rachel can indeed seize the fields, but will not be able to keep them against a claim by Leah, as the latter married Jacob first. Leah too will not be able to keep the fields but will have to surrender them to her daughter. This cycle should continue until they reach a compromise.

Case 2: About 300 years later, a somewhat similar question [21, 63:4] was sent to R. Asher (b. Yeḥiel, "Rosh," Germany and Spain, d. 1327). A couple sold a house and wrote to the buyer that they do not leave themselves any rights to this property. The buyer found out, after the death of the husband, that the house was mortgaged by the sellers against a loan, and the lender now wants to seize the house from the buyer. But the woman who (with her late husband) sold the house shows her Ketubah which is earlier to the loan, and informs the lender that she will seize the house from him and transfer it back to the buyer. Citing the above Mishnah, R. Asher ordered the house to remain in the hands of the buyer.

How could two courts reach such different conclusions from the same Mishnah? Apparently, the difference is in the (implicit) coalition structure of the two stories. In the case of R. Ḥanoch, each of the parties would like to keep the fields. This is obviously true for Rachel and Leah's daughters, but is also the case with Leah, who may want her daughters to prosper, but will not necessarily like to transfer to her sons-in-law assets that are supposed to support her through her years of widowhood. Like the conflict of the Mishnah, the three parties have to reach a compromise.

The story of R. Asher is different, as the remaining seller makes it clear that she intends to transfer the house back to the buyer. In his answer, R. Asher considers this to be the compromise ordered by the Mishnah and concludes that the buyer now occupies two positions: that of a buyer, but also that of the woman whose Ketubah dates earlier to the loan. Although R. Asher does not say so, his argument is in line with the Palestinian solution to the sacrifices cycle (see Section 3 above). The Ketubah of the seller elevates

[26] Leah's consent is needed because her Ketubah† is dated earlier to the gift to her daughter (see Appendix 1).

the buyer above the lender, similarly to the way in which the superiority of the cattle burnt-offering over tithe elevates the bird sin-offering over the tithe.

Even if not formal, this may be the most practical solution to nontransitive cycles. One should check for natural connections between elements of the set of options, thus reducing the size of this set. In Section 3, there is an obvious connection between the burnt-offering and the sin-offering, as both are brought, as a pair, by the birth-giving woman. Likewise, the female seller and the buyer of Case 2 above form a natural pair (the seller wants to seize the house so that she will be able to transfer it to the buyer, as was her initial intention according to the letter she and her husband wrote to him). When such natural pairs do not exist, other solutions should be used, like prioritizing rules or prioritizing options. Sometimes, it is even better not to offer any solution.

Appendix 1: Glossary

Burnt-offering An animal sacrifice, which is entirely consumed by fire on the altar. Burnt-offering may be brought either as public or individual sacrifice. In the latter case, it can be obligatory, for certain rituals, or voluntary, as an act of piety.

Ketubah The Jewish marriage contract. The ketubah lists the dowry items given by the wife's family as well as the husband's obligations and suspended debts to his wife, which she may collect if he dies before her or in a case of divorce. From the date the ketubah is signed, the whole property of the husband (primarily his real estate) is pledged to the contract, and the widow, or divorcee, can seize any part of it form her husband, his heirs, or anyone else, including other wives of her husband, who got holding on the property after that date.

Mishnah A collection of old rabbinic oral traditions and instructions edited circ. 220 CE. The Mishnah is composed of sixty tractates, ordered in six "orders" (sedarim). Each single paragraph of the Mishnah is also called a Mishnah.

Mekilta The early (3rd c CE) rabbinic commentary to Exodus, attributed to the school of R. Ishmael.

Peace-offering A voluntary individual offering, usually brought as thanksgiving offering or for the purpose of a feast. Except for some inner parts of the animal, the flesh of the Peace-offering is eaten by the priest, the donor, and his family.

Rashi R. Shlomo b. Yitzhak (northern France, d. 1105), preeminent commentator to most parts of the Bible and the Babylonian Talmud. In this article, reference is made to Rashi's commentary printed on the inner margins of the traditional editions of the Babylonian Talmud.

Sifra The old (3rd c CE) rabbinic commentary to Leviticus, attributed to the school of R. Akiva.

Sin-offering A sacrifice made for the atonement of an unintentional sin or ritual fault. As the Hebrew name *ḥattat* means also "purification" (see Milgrom [11, pp. 758–9]), sin-offering is obligatory in specific rituals even when no sin is committed, like after childbirth or after recovery from leprosy.

Talmud Babylonian (3rd—5th c CE) and Palestinian (3rd—4th c CE). The collection of rabbinic materials of various genres — rulings, discussions, interpretations, stories, morals, and other literary forms — following the order of the Mishnah.

Tithe A tenth of the newly born cattle. Animal tithe is given each year to the priests, who has to sacrifice and eat it according to laws of Peace-offering.

Tosafot A fragmentary, critical commentary to the Babylonian Talmud, by and large the product of the French and German Talmudic schools of 12th–13th centuries. In this article, reference is made to the Toasafot printed on the outer margins of the traditional editions of the Babylonian Talmud.

Appendix 2: The Transformation of Rule 3

In order to clarify our analysis of the Talmudic text, a short methodological introduction is needed. In general, a typical Talmudic dialectical passage is composed of two different sorts of textual components. Old, and relatively

short, materials ("sources") are interwoven in a complex dialectic structure. The dialectical structure, usually made by an anonymous redactor, is always later than the sources themselves (See Goodblatt [6, pp. 314–318]). Therefore, in critical analysis of the Talmudic passage one should be aware of the possibility that the meaning of the sources themselves, as autonomous textual units, may be different from the meaning ascribed to them by the redactor of the dialectical structure.

In our case, the fact that the answers to the question are offered by the sages of "here" and "the West" shows that this kernel, the question and its two answers, is not a mere speculation of the redactor, but a historical source. This source underwent a conceptual modification and its form in the final passage is significantly different from what it was in the original state.

Based on rule 3 in its form in the Sifra (which also appears in the beginning of the passage at the Talmud), the question must have been of the form suggested above — a simple two-item choice problem.[27] However, at a certain stage of the Talmudic redaction the understanding of rule 3 has been essentially changed. In another part of the broader passage, as well as in a parallel discussion elsewhere in the Talmud,[28] rule 3 is applied to a case of situation (a), i.e. to offerings that are not members of a closed group,[29] in sharp contrast to its original form in the Sifra.[30] Whoever did so understood rule 3 as a global rule: Any sin-offering, in any circumstance, precedes any burnt-offering.

We do not know why or how this change happened, but the new formation of rule 3 is undeniable. Once the distinction between situations (a) and (b) disappears, it is no longer possible to settle the Mishnah and the Sifra so that they will not contradict each other. If the two contradictory systems are to be held together, a nontransitive cycle emerges.

[27] A similar question, based on the same rules with a different set of sacrifices, appears in the Palestinian Talmud [26, Hor. 3:7, 48a]. Only the first answer (the Babylonian) is offered there.

[28] See [26, Hor. 13a].

[29] The offerings mentioned in Lev. 4:14 and Num. 15:24.

[30] See Minkowski [15, p. 104].

Appendix 3: Ranking Rules

This appendix explains how to create a complete ranking from ordered pairwise comparisons.

Let $N = \{1, \ldots, n\}$ and let c be a choice function such that for all $i \neq j$, $c(i,j)$ exists and is a singleton. Let R be a linear ranking of the $n(n-1)/2$ pairs $\{i,j\}$, $i \neq j$, that is,

$$\{i_1, j_1\} \, R \, \{i_2, j_2\} \, R \, \ldots \, R \, \{i_{n(n-1)/2}, j_{n(n-1)/2}\}$$

and for no $k > 1$, $\{i_k, j_k\} \, R \, \{i_{k-1}, j_{k-1}\}$. Let $\varnothing \neq T \subseteq N$ and construct inductively partial linear orders $\succ_0^T, \ldots, \succ_{n(n-1)/2}^T$ on T as follows.

1. $\succ_0^T = \varnothing$ (that is, no elements of T are compared by $\succ_0^T$). Set $k = 1$ and move to the next step.

2. If $k = \frac{n(n-1)}{2} + 1$ set $\succ^T = \succ_{n(n-1)/2}^T$ and the construction is complete. Otherwise, move to the next step.

3. If $\{i_k, j_k\} \not\subseteq T$, set $\succ_k^T = \succ_{k-1}^T$, increase the value of k by 1, and move to step 2. Otherwise, move to the next step.

4. If i_k and j_k are comparable by $\succ_{k-1}^T$, set $\succ_k^T = \succ_{k-1}^T$,[31] increase the value of k by 1, and move to step 2. Otherwise, move to the next step.

5. Define $\succcurlyeq_k^T$ as follows:

   - For all $\{i,j\} \neq \{i_k, j_k\}$, $i \succcurlyeq_k^T j$ iff $i \succ_{k-1}^T j$.
   - If $c(i_k, j_k) = i_k$, then $i_k \succcurlyeq_k^T j_k$. If $c(i_k, j_k) = j_k$, then $j_k \succcurlyeq_k^T i_k$.

   And let $\succ_k^T$ be the transitive closure of $\succcurlyeq_k^T$. Increase k by 1 and move to step 2.

By construction, the ranking $\succ^T$ is transitive. It is also complete. Let $i, j \in T$ and let k such that $\{i_k, j_k\} = \{i, j\}$. Either i_k and j_k were comparable by $\succ_{k-1}^T$, and then they are also comparable by $\succ_k^T$, or $\succ_k^T$ is constructed to compare them. As $\succ_1^T \subseteq \ldots \subseteq \succ_{n(n-1)/2}^T = \succ^T$, the ranking $\succ^T$ is complete.

This method may not work if we drop the requirement that for no $k > 1$, $\{i_k, j_k\} \, R \, \{i_{k-1}, j_{k-1}\}$. Let $N = \{a_1, \ldots, a_4\}$, and suppose that $c(a_1, a_2) =$

[31] Even if $c(i_k, j_k)$ disagrees with $\succ_{k-1}^T$. In such a case c is overruled by $\succ_{k-1}^T$.

a_1, $c(a_2, a_3) = a_2$, $c(a_3, a_4) = a_3$, and $c(a_1, a_4) = a_4$. Suppose further that the top four pairs are $\{a_1, a_2\}R\{a_3, a_4\}R\{a_2, a_3\}R\{a_1, a_4\}$ but also $\{a_1, a_4\}R\{a_2, a_3\}$. the two orders $a_1 \succ a_2 \succ a_3 \succ a_4$ and $a_3 \succ a_4 \succ a_1 \succ a_2$ are consistent with the above algorithm.

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