A Theory of School-Choice Lotteries*

Onur Kesten† M. Utku Ünver‡
Carnegie Mellon University *Boston College*

June 29, 2012

Abstract

A new centralized mechanism was introduced in New York City and Boston to assign students to public schools in district school-choice programs. This mechanism was advocated for its superior fairness property, besides others, over the mechanisms it replaced. In this paper, we introduce a new framework for investigating school-choice matching problems and two ex-ante notions of fairness in lottery design, *strong ex-ante stability* and *ex-ante stability*. This framework generalizes known one-to-many two-sided and one-sided matching models. We show that the new NYC/Boston mechanism fails to satisfy these fairness properties. We then propose two new mechanisms, the *fractional deferred-acceptance mechanism*, which is ordinally Pareto dominant within the class of strongly ex-ante stable mechanisms, and the *fractional deferred-acceptance and trading mechanism*, which satisfies equal treatment of equals and constrained ordinal Pareto efficiency within the class of ex-ante stable mechanisms.

JEL Classification Numbers: C71, C78, D71, D78
Keywords: Matching, School Choice, Deferred Acceptance, Stability, Ordinal Efficiency, Market Design.

*We thank Atila Abdulkadiroğlu, Yeon-Koo Che, Lars Ehlers, Aytek Erdil, Bettina Klaus, Fuhito Kojima, Justin Leroux, Szilvia Papái, Parag Pathak, Alvin E. Roth, and Tayfun Sönmez as well as participants at the Stanford ITE Workshop, Conference on Economic Design, Econometric Society Winter Meetings, SAET Conference in Ischia, Roth-Sotomayor Matching Book 20th Year Conference, HBS, UCLA, Duke, Concordia, Pittsburgh, and Bonn for useful comments, discussions, and suggestions. We would like to also acknowledge the excellent research assistance of Ersin Körpeoğlu. We would like to thank Parag Pathak for providing us data statistics regarding the Boston school admissions.

†**Address:** Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213. **E-mail:** okesten@andrew.cmu.edu.

‡**Address:** Department of Economics, Boston College, Chestnut Hill, MA, 02467. **E-mail:** unver@bc.edu.

1 Introduction

Following the 1987 decision of the U.S. Court of Appeals, the Boston school district introduced a possibility of "choice" for public schools by relaxing the mandatory zoning policy. For this purpose the district introduced priority classes for students for each school, based on how far away a student lives and whether a sibling of the student attends the school. In 1989, a centralized clearinghouse, now commonly referred to as the *Boston mechanism* (Abdulkadiroğlu and Sönmez, 2003) was adopted by the district. Each year since then Boston school district collects preference rankings of students over schools, and determines a matching of students to schools based on students' priorities. Since there are typically several students tied for priority at schools, *random tie-breaking* has been the common practice for obtaining a strict priority ranking among students within equal priority classes. Today many U.S. school districts employ clearinghouses that operate on the random tie-breaking practice.

Abdulkadiroğlu and Sönmez (2003) pointed out that the Boston mechanism is flawed in many ways (also see Chen and Sönmez, 2006; Ergin and Sönmez, 2006; Pais and Pinter, 2007; Pathak and Sönmez, 2008). They showed, for example, that student priorities are not necessarily respected by this mechanism. Moreover, the Boston mechanism is susceptible to strategic manipulation in a very obvious manner. They proposed two alternatives to this mechanism from the mechanism design literature on indivisible good allocation and two-sided matching.[1] Eventually one of these mechanisms, the Gale-Shapley student-optimal stable mechanism, replaced the Boston mechanism in 2005 due to the collaborative efforts of Abdulkadiroğlu and Sönmez with economists Pathak and Roth (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005, 2006; Pathak and Sönmez, 2008). A version of the same mechanism was also adopted by the New York City public school system in 2004 via the efforts of economists (Abdulkadiroğlu, Pathak, and Roth, 2005, 2009).

This new mechanism relies on the idea of producing a "stable matching" first introduced by Gale and Shapley (1962). This approach has been widely and successfully used in several two-sided matching applications. Probably the most well-known of these applications is the National Resident Matching Program (cf. Roth, 1984; Roth and Peranson, 1999) that was designed to match hospital residency programs with graduating medical doctors.

An important difference between two-sided matching markets and school choice is that in the former participants' choices are elicited as strict preferences, and there are no indifferences within preferences.[2] In school choice, however, only students' choices are elicited as strict preferences,

[1] See for example Gale and Shapley (1962); Shapley and Scarf (1974); Roth (1984, 1991); Balinski and Sönmez (1999); Abdulkadiroğlu and Sönmez (1999); Pápai (2000); Ergin (2002); Kesten (2006, 2009b); Pycia and Ünver (2009).

[2] Another difference concerns the definition of the mechanism design problem. Both hospital residency programs and doctors are active "agents" in a two-sided market, and thus, both state preferences over agents on the other side of the market. On the other hand, in school choice, schools are passive in most cases and viewed only as indivisible

whereas students' priorities are coarse. That is, there are often many students who belong to the same priority class. For example, there are only four priority classes in Boston, although there are thousands of students applying each year. Consequently, to adapt the deterministic two-sided matching approach, ties among equal-priority students need to be explicitly broken in an endogenous or exogenous fashion. All previous mechanism design efforts have thus far relied on the deterministic approach upon breaking ties in priorities randomly[3] or using additional strategic tools.[4] Our approach in this paper does not rely on any form of tie-breaking and enables us to work with a general school choice model allowing for a rich set of stochastic mechanisms, which in principle can lead to superior levels of welfare and fairness than their deterministic counterparts.[5]

In school choice student priorities constitute the basis for fairness considerations. Abdulkadiroğlu and Sönmez (2003) discuss two plausible interpretations of fairness. The first interpretation, which assigns a subtle and weak role to priorities, is based on an idea of endowing each student with the ability to trade his priority at a school for a higher priority at a more desirable school. The second and arguably more popular interpretation draws parallels between school choice and two-sided matching. At a *stable* matching, there does not exist any student i who prefers a seat at a different school c than the one he is assigned to such that either (1) school c has not filled its quota, or (2) school c has an enrolled student who has strictly lower priority than i. A mechanism is *ex-post stable* if it induces a lottery over stable matchings (i.e., an ex-post stable lottery).[6] Thus, the sibling priority should be honored over an otherwise equal student only if the sibling of a student is already attending the school while the other student's is not. Based on this interpretation, the Boston school district adopted ex-post stability as the proper notion of fairness. As some of the New York schools were also active players (i.e., they withheld capacities of the schools) and had preferences instead of priorities, the "endowment" interpretation for a school was not accepted in New York, either. They also adopted the stability interpretation (Abdulkadiroğlu, Pathak, and Roth, 2005). Thus, the newly adopted NYC/Boston mechanism is ex-post stable.

Although ex-post stability is a meaningful interpretation of fairness for deterministic outcomes,

objects to be consumed. In other words, a student's priority is irrelevant to schools' preferences.

[3]See, for example, Pathak (2006); Abdulkadiroğlu, Pathak, and Roth (2009); Erdil and Ergin (2008); Abdulkadiroğlu, Che, and Yasuda (2008); Ehlers and Westkamp (2011).

[4]See, for example, Abdulkadiroğlu, Che, and Yasuda (2008); Sönmez and Ünver (2010).

[5]A related problem is the so-called *random assignment problem*. A random assignment problem can be viewed as a special school-choice problem where each school has unit quota and all students have equal priority at all schools. The seminal work of Bogomolnaia and Moulin (2001) has revealed that such a richer setup can allow one to obtain a much stronger welfare criterion than ex-post efficiency. Our approach, inspired in part by Bogomolnaia and Moulin (2001), also rests on the idea of avoiding welfare losses emanating from random tie-breaking.

[6]The Boston school district voiced concern about the first interpretation, and stated that "[...] certain priorities – e.g., sibling priority – apply only to students for particular schools and should not be traded away." From the memo of the [then] Superintendent Thomas W. Payzant on May, 25, 2005 to the Boston school committee.

for lottery mechanisms, such as the ones used for school choice, it is much less clear that it is the most suitable fairness property for this setup. To begin with, ex-post stability is not defined over random outcomes, but rather over deterministic matchings obtained post tie-breaking. And while random tie-breaking conveniently makes the deterministic approach still applicable, unfortunately it precludes broader views of ex ante fairness and can potentially cause significant welfare loss (Erdil and Ergin, 2008; Abdulkadiroğlu, Pathak, and Roth, 2009).

In this paper we present a general model of school choice in which (1) school priorities can be coarse as in real life, and (2) matchings can be random. Over random matchings, we propose two powerful notions of fairness that are stronger than ex-post stability. We say that a random matching causes *ex-ante school-wise justified envy* if there are two students i and j and a school c such that student i has strictly higher priority than j for school c but student j can be assigned to school c with positive probability while i can be assigned to a less desirable school for him than c with positive probability (i.e., i has ex-ante school-wise justified envy toward j). We refer to a random matching as *ex-ante stable* if it eliminates ex-ante school-wise justified envy. We show that (cf. Example 1) the new NYC/Boston mechanism, despite its ex-post stability, is not ex-ante stable.

Besides its normative support, our ex-ante approach has an important practical appeal. Even if one considers ex-post stability as the normatively more appealing fairness concept, the set of ex-post stable lotteries in highly nontractable for the random matching setup. Indeed, it is difficult to characterize the probability assignment matrix of a generic ex-post stable lottery since an ex-post unstable lottery may also induce the same matrix as an ex-post stable lottery (as demonstrated in our Example 1).[7] In this case, one possible practical solution is approximating ex-post stability through ex-ante stability. We show through simulations that ex-ante school-wise justified envy violations of the NYC/Boston mechanism are quite rare in realistic environments (cf. Section 6). Hence, the approximation of ex-post stability via ex-ante stability is a viable option.

Coarse priority structures also give rise to natural fairness considerations concerning students who belong to the same priority group for some school. We say that a random matching causes *ex-ante school-wise discrimination (among equal-priority students)* if there are two students i and j with equal priority for a school c such that j enjoys a higher probability of being assigned to school c than student i even though i suffers from a positive probability of being assigned to a less desirable school for him than school c. The new NYC/Boston mechanism (cf. Example 2) also induces ex-ante school-wise discrimination between equal-priority students. [8]

[7] A similar difficulty lies in the identification of ex-post efficient lotteries as illustrated by Abdulkadiroğlu and Sönmez (2003).

[8] The elimination of ex-ante school-wise discrimination is more than a normative concept and is rooted in equal treatment laws. Abdulkadiroğlu and Sönmez (2003) cite a lawsuit filed by a student against the state of Wisconsin, because the student was denied by the Superintendent from a school of a district where he did not live in due to limited space, while "similar" students were admitted. The Circuit Court reversed the decision as it considered

We refer to a random matching as *strongly ex-ante stable* if it eliminates both ex-ante school-wise justified envy and ex-ante school-wise discrimination. Both ex-ante stability and strong ex-ante stability imply ex-post stability. The latter also implies equal treatment of equals. We propose two new mechanisms that select "special" ex-ante stable and strongly ex-ante stable random matchings.

Our first proposal, the *fractional deferred-acceptance (FDA) mechanism,* selects *the* strongly ex-ante stable random matching that is ordinally Pareto dominant among all strongly ex-ante stable random matchings (Theorems 2 and 3). The algorithm it employs is in the spirit of the deferred-acceptance algorithm of Gale and Shapley (1962), with students applying to schools in an order of decreasing preference and schools tentatively retaining students based on priority. Unlike previous mechanisms, however, the FDA mechanism does not rely on tie-breaking. Loosely, schools always reject lower-priority students in favor of higher ones (if the need arises) as in the deferred-acceptance algorithm. However, whenever there are multiple equal-priority students being considered for assignment to a school, for which there is insufficient quota, the procedure tentatively assigns an equal fraction of these students and rejects the rest of the fractions. These rejected "fractions of students" continue to apply to their next-preferred schools in the usual deferred-acceptance fashion as if they were individual students. The procedure iteratively continues to make tentative assignments, until one full fraction of each student is assigned to some school. We interpret the assigned fractions of a student at the end of the procedure as his assignment probability to each corresponding school by the FDA mechanism.[9]

Our second proposal, the *fractional deferred-acceptance and trading (FDAT) mechanism* selects an ex-ante stable random matching that (1) treats equals equally and (2) is ordinally Pareto undominated within the set of ex-ante stable random matchings (Theorems 4 and 5). It employs a two-stage algorithm that stochastically improves upon the FDA matching. The FDAT mechanism starts from the random-matching outcome of the FDA algorithm and creates a trading market for school-assignment probabilities. In this market, the assignment probability of a student to a school can be traded for an equal amount of probabilities at better schools for the student so long as the trade does not result in ex-ante school-wise justified envy of some other student. Such trading opportunities are characterized by *stochastic ex-ante stable improvement cycles*, i.e., the list of students who can trade fractions of schools among each other without violating any ex-ante stability constraints. We show that a random matching is constrained ordinally efficient among ex-ante stable random matchings if and only if there is no stochastic ex-ante stable improvement cycle (Proposition

the Superintendent's decision "arbitrary" and the Appeals Court affirmed (cf. Michael E. McMorrow, Petitioner-Respondent, v. State Super Intendant of Public Instructions, John T. Benson, Respondent-Appellant. No. 99-1288. Court of Appeals of Wisconsin. Decided on July 25, 2000).

[9]In contrast with the deferred-acceptance algorithm of Gale and Shapley (1962), the above described procedure may involve rejection cycles that prevent the procedure from terminating in a finite number of steps. Therefore, to obtain a convergent algorithm we also couple this procedure with a "cycle resolution phase."

5). However many stochastic ex-ante stable improvement cycles can co-exist and intersect with each other. In order to resolve these cycles in a procedurally fair way that preserves equal treatment of equals, in the second stage of FDAT mechanism, we adapt a combinatorial network-flow algorithm originally proposed by Athanassoglou and Sethuraman (2011) for a problem domain without any priorities, in which ordinal efficiency can be improved by trading fractions of indivisible goods when agents have probabilistic endowments. In our case endowments correspond to the FDA assignment probabilities.[10]

To estimate the actual performance of FDAT and contrast it with popular assignment mechanisms from practice and theory, we also report some simulation results based on the aggregate statistics of the Boston school-choice data from Abdulkad.iroğlu et al., 2006. Specifically, we compare the overall efficiency of FDAT mechanism with those of the NYC/Boston mechanism and ex-post trading approach of Erdil and Ergin (2008) (see Footnote 11). The latter two mechanisms are only ex-post stable while FDAT is also ex-ante stable. Erdil-Ergin mechanism is constrained ex-post efficient within the ex-post stable class, whereas FDAT is constrained ordinally-efficient within in the ex-ante stable class. In the simulations, we observe that FDAT mechanism almost first-order stochastically dominates the NYC/Boston mechanism. Morover, around 32% of the students unambiguously prefer the FDAT outcome to Erdil-Ergin outcome, while only 18% have opposite preference. We also observe that the NYC/Boston and Erdil-Ergin mechanisms produce very little ex-ante school-wise justified envy, hence ex-ante stability is a fairly good approximation of ex-post stability in such an environment.[11]

[10]Yılmaz (2009, 2010) generalize the ex-ante probabilistic allocation methods for indivisible objects without exogenous priorities introduced in Katta and Sethuraman (2006) and Bogomolnaia and Moulin (2001) to an indivisible good assignment problem, where some agents have initial property rights of some of the goods for the strict and weak preference domains, respectively. Athanassoglou and Sethuraman (2011) extend these models to a framework where the initial property rights could be over fractions of goods.

[11]In addition to the papers already mentioned, in the two-sided matching literature, a version of the random matching problem with strict school preferences was analyzed by Roth, Rothblum, and Vande Vate (1993). Our analysis and results are independent and unrelated to theirs, as weak priorities change the analysis dramatically. Alkan and Gale (2003) consider a deterministic two-sided matching model in which the two sides are referred to as firms and workers. In their model, a worker can work for one hour in total, but he can share his time between different firms. A firm can hire fractions of workers that sum up to a certain quota of hours. Both firms and workers have preferences over these fractions. Although the setups seem related, our solution concepts are different from theirs and Alkan and Gale do not propose any well-defined algorithm. Previously, Erdil and Ergin (2008) (and then Abdulkadiroğlu, Pathak, and Roth, 2009) have pointed out that the new NYC/Boston mechanism may be subject to welfare losses when ties in priorities are broken randomly. Erdil and Ergin (2007, 2008) propose methods for improved efficiency without violating exogenous stability constraints for school-choice and two-sided matching problems, respectively. All these papers emphasize that random tie-breaking may entail an ex-post efficiency loss. We, on the other hand, argue that it may also entail an ex-ante stability loss both among students with different priorities (ex-ante school-wise justified envy) and among students with equal priorities (ex-ante school-wise discrimination). Erdil and Kojima (2007) independently develop

We also inspect strategic properties in the environments we introduced. We show that there will be no strategy-proof, ex-ante stable, constrained efficient, and

The rest of the paper is organized as follows. Section 3 formally introduces a general model of school choice. Section 4 discusses desirable properties of mechanisms and introduces the new ex-ante stability criteria. Section 5 presents our first proposal, the fractional deferred-acceptance mechanism, and the related results. Section 6 presents our second proposal, the fractional deferred-acceptance and trading mechanism, and the related results. Section 7 inspects the strategic properties of the mechanisms we proposed. Section 8 concludes. The proofs of our main results are relegated to the Appendices.

2 The Model

We start by introducing a general model for school choice. A *school-choice problem* is a five-tuple $[I, C, q, P, \succsim]$ where:

- I is a finite set of *students* each of whom is seeking a seat at a school.
- C is a finite set of *schools*.
- $q = (q_c)_{c \in C}$ is a *quota vector* of schools such that $q_c \in \mathbb{Z}_{++}$ is the maximum number of students who can be assigned to school c. We assume that there is enough quota for all students, that is $\sum_{c \in C} q_c = |I|$.[12]
- $P = (P_i)_{i \in I}$ is a *strict preference profile* for students such that P_i is the *strict preference relation of student i over the schools*.[13] Let R_i refer to the associated weak preference relation with P_i. Formally, we assume that R_i is a linear order, i.e. a complete, transitive, and antisymmetric binary relation. That is, for any $c, a \in C$, cR_ia if and only if $c = a$ or cP_ia.

a school-choice framework and concepts similar to ours. They do not pursue mechanisms satisfying their proposed properties. Echenique, Lee, Shum, and Yenmez (2012) more recently study observable implications of stability for aggregate matchings over populations of several "similar" individual types, and propose a condition, similar to our ex-ante stability notion, that is used for empirical rationalizability of observed matchings. However, their model and interpretation of this condition are substantially different from ours.

[12] If originally $\sum_{c \in C} q_c > |I|$, then we introduce $|I| - \sum_{c \in C} q_c$ additional virtual students, who have the lowest priorities at each school (say, a uniform priority ranking is available among virtual students for all schools and all virtual students have common strict preferences over schools). If originally $\sum_{c \in C} q_c < |I|$, then we introduce a virtual school with a quota $|I| - \sum_{c \in C} q_c$, which is the worst choice of each student, such that all students have equal priority for this school.

[13] For simplicity of exposition, we assume that all schools are acceptable for all students. All of our results are easy to generalize to the setting with unacceptable schools using a null school with quota ∞ and sub-stochastic matrices instead of bi-stochastic matrices.

- $\succsim = (\succsim_c)_{c \in C}$ is a *weak priority structure for schools* such that $\succsim_c$ is the *weak priority order of school c over the students*. That is, $\succsim_c$ is a reflexive, complete, and transitive binary relation on I. Let $\succ_c$ be the acyclic portion and $\sim_c$ be the cyclic portion of $\succsim_c$. That is, $i \succsim_c j$ means that student i has at least as high priority as student j at school c, $i \succ_c j$ means that i has strictly higher priority than j at c, and $i \sim_c j$ means that i and j have equal priority at c.

Occasionally, we will fix I, C, q and refer to a problem by the strict preference profile of the students and weak priorities of schools, $[P, \succsim]$.

We are seeking matchings such that each student is assigned a seat at a single school and the quota of no school is exceeded. We also allow random (or probabilistic) matchings.

A *random matching* $\rho = [\rho_{i,c}]_{i \in I,\, c \in C}$ is a real stochastic matrix, i.e., it satisfies (1) $0 \leq \rho_{i,c} \leq 1$ for all $i \in I$ and $c \in C$; (2) $\sum_{c \in C} \rho_{i,c} = 1$ for all $i \in I$; and (3) $\sum_{i \in I} \rho_{i,c} = q_c$ for all $c \in C$. Here $\rho_{i,c}$ represents the probability that student i is being matched with school c. Moreover, the stochastic row vector $\rho_i = (\rho_{i,c})_{c \in C}$ denotes the *random matching (vector) of student i at ρ*, and the stochastic column vector $\rho_c = (\rho_{i,c})_{i \in I}$ denotes the *random matching (vector) of school c at ρ*. A random matching ρ is a *(deterministic) matching* if $\rho_{i,c} \in \{0, 1\}$ for all $i \in I$ and $c \in C$. Let $\mathcal{X}$ be the set of random matchings and $\mathcal{M} \subseteq \mathcal{X}$ be the set of matchings. We also represent a matching $\mu \in \mathcal{M}$ as the unique non-zero diagonal vector of matrix μ, i.e., as a list $\mu = \left(\begin{smallmatrix} i_1 & i_2 & \dots & i_{|I|} \\ c_1 & c_2 & \dots & c_{|I|} \end{smallmatrix}\right)$ such that for each ℓ, $\mu_{i_\ell, c_\ell} = 1$. We interpret each student i_ℓ as matched with school c_ℓ in this list and, with a slight abuse of notation, use μ_{i_ℓ} to denote the match of student i_ℓ.

A *lottery* λ is a probability distribution over matchings. That is, $\lambda = (\lambda_\mu)_{\mu \in \mathcal{M}}$ such that for all $\mu \in \mathcal{M}$, $0 \leq \lambda_\mu \leq 1$ and $\sum\limits_{\mu \in \mathcal{M}} \lambda_\mu = 1$. Let $\Delta\mathcal{M}$ denote the set of lotteries. For any $\lambda \in \Delta\mathcal{M}$, let ρ^λ be the *random matching of lottery λ*. That is, $\rho^\lambda = \left[\rho^\lambda_{i,c}\right]_{i \in I, c \in C} \in \mathcal{X}$ is such that $\rho^\lambda_{i,c} = \sum\limits_{\mu \in \mathcal{M} \,:\, \mu_i = c} \lambda_\mu$ for all $i \in I$ and $c \in C$. In this case, we say that lottery λ *induces* random matching ρ^λ. Observe that $\rho^\lambda_{i,c}$ is the probability that student i will be assigned to school c under λ. Let $Supp\,(\lambda) \subseteq \mathcal{M}$ be the support of λ, i.e., $Supp\,(\lambda) = \{\mu \in \mathcal{M} : \lambda_\mu > 0\}$.

We state the following theorem whose proof is an extension of the proof of the standard Birkhoff (1946) - von Neumann (1953) Theorem (also see Kojima and Manea (2010)):

Theorem 1 ***(School-Choice Birkhoff - Von Neumann Theorem)****For any random matching $\rho \in \mathcal{X}$, there exists a lottery $\lambda \in \Delta\mathcal{M}$ that induces ρ, i.e., $\rho = \rho^\lambda$.*

Through this theorem's constructive proof and related algorithms in combinational optimization theory, such as the Edmonds (1965) algorithm, one can find a lottery implementing ρ in polynomial time. Thus, without loss of generality, we will focus on random matchings rather than lotteries. A *(school-choice) mechanism* selects a random matching for a given school-choice problem. For problem

$[P, \succsim]$, we denote the random matching of a mechanism φ by $\varphi[P, \succsim]$ and the random matching vector of a student i by $\varphi_i[P, \succsim]$.

3 Properties

3.1 Previous Notions of Fairness

We first define two previously studied notions that are satisfied by many mechanisms in the literature and real life. Throughout this section, we fix a problem $[P, \succsim]$.

We start with the most standard fairness property in school-choice problems as well as other allocation problems. This weakest notion of fairness is related to the treatment of equal students, i.e., students with the same preferences and priorities. We refer to two students $i, j \in I$ as *equal* if $P_i = P_j$ and $i \sim_c j$ for all $c \in C$. A random matching ρ *treats equals equally* if for any equal student pair $i, j \in I$, we have $\rho_i = \rho_j$, that is: two students with exactly the same preferences and equal priorities at all schools should be guaranteed the same enrollment chance at every school at a matching that treats equals equally. The real-life school-choice mechanism used earlier in Boston as well as the new NYC/Boston mechanism treat equals equally.

Before introducing the second probabilistic fairness property, we define a deterministic fairness notion. A (deterministic) matching μ is *stable* if there is no student pair i, j such that $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$.[14] That is: a matching is stable if there is no student who envies the assignment of a student who has lower priority than he does for that school. Whenever such a student pair exists at a matching, we say that there is *justified envy*. Let $\mathcal{S} \subseteq \mathcal{M}$ be the set of stable matchings. A stable matching always exists (Gale and Shapley, 1962).

The second probabilistic fairness property is a direct extension of stability to lottery mechanisms: A random matching ρ is *ex-post stable* if it is induced by a lottery whose support includes only stable matchings, i.e., there exists some $\lambda \in \triangle\mathcal{M}$ such that $Supp(\lambda) \subseteq \mathcal{S}$ and $\rho = \rho^{\lambda}$.

Since recently introduced real-life mechanisms are ex-post stable (and the implemented matchings are stable), ex-post stability has been seen as a key property in previous literature. A characterization of ex-post stability exists for strict priorities (Roth, Rothblum, and Vande Vate, 1993), yet such a characterization is unknown for weak priorities.

[14] The early literature on college admissions and school choice (e.g. Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003)) used the term *fair* instead of *stable*. Subsequent studies have used the term *stable* more often based on the connection of their models with the two-sided model of Gale and Shapley (1962). Since we already have several fairness concepts, we have adopted this terminology to avoid confusion.

3.2 A New Notion: Ex-ante Stability

We now formalize the two fairness notions over random matchings that were informally discussed in the Introduction.

We say that a random matching $\rho \in \mathcal{X}$ causes *ex-ante school-wise justified envy* of $i \in I$ *toward (lower-priority student)* $j \in I \setminus \{i\}$, with $i \succ_c j$, for $c \in C$ if $\rho_{i,a} > 0$ for some $a \prec_i c$ and $\rho_{j,c} > 0$. A random matching is *ex-ante stable* if it does not cause any ex-ante school-wise justified envy.

Observe that ex-ante stability and stability are equivalent concepts for deterministic matchings. Although ex-ante stability is appealing, it does not impose any restrictions when dealing with fairness issues regarding students with equal priorities.

We say that a random matching $\rho \in \mathcal{X}$ induces *ex-ante school-wise discrimination (between equal-priority students)* $i, j \in I$, with $i \sim_c j$, *for* $c \in C$, if $\rho_{i,a} > 0$ for some $a \prec_i c$ and $\rho_{i,c} < \rho_{j,c}$. A random matching is *strongly ex-ante stable* if it eliminates both ex-ante school-wise justified envy and ex-ante school-wise discrimination.

The elimination of ex-ante school-wise discrimination implies equal treatment of equals. Thus, a strongly ex-ante stable random matching satisfies equal treatment of equals. Strong ex-ante stability implies ex-ante stability, but the converse is not true. Theorem 2 (below) shows that a strongly ex-ante stable random matching always exists. For deterministic matchings, elimination of ex-ante school-wise discrimination between equal-priority students is equivalent to a *no-envy*[15] requirement among students with equal priority (due to a school for which equal priority is shared) and thus may not always be guaranteed.

We compare ex-ante (and strong ex-ante) stability with the earlier notion, ex-post stability. It turns out that ex-post stability is weaker than ex-ante stability (and strong ex-ante stability), while the converse is not true:

Proposition 1 *If a random matching is ex-ante stable then it is also ex-post stable. Moreover, any lottery that induces an ex-ante stable random matching has a support that includes only stable matchings.*

Proof. We prove the contrapositive of the second part of the proposition. The first part of the proposition follows from the second part. Let a random matching $\rho \in \mathcal{X}$ and a lottery $\lambda \in \Delta\mathcal{M}$ that induces it be given, i.e. $\rho^\lambda = \rho$. Suppose there exists some unstable matching $\mu \in \mathcal{M} \setminus \mathcal{S}$ such that $\lambda_\mu > 0$. Then there exists a blocking pair $(i, c) \in I \times C$ such that $cP_i\mu_i$ while for some $j \in I$, $\mu_j = c$ with $i \succ_c j$. Since $\lambda_\mu > 0$, we have $\rho_{j,c} > 0$ while $\rho_{i,\mu_i} > 0$, $i \succ_c j$, and $cP_i\mu_i$, i.e., ρ is not ex-ante stable. ■

[15] Given a deterministic matching $\mu \in \mathcal{M}$, there exists *no-envy* between a pair of students $i, j \in I$ if $\mu_i \; P_i \; \mu_j$ and $\mu_j \; P_j \; \mu_i$.

On the other hand, the following example shows that the new NYC/Boston mechanism is not ex-ante stable, and hence, an ex-post stable lottery can be ex-ante unstable, i.e., the converse of the first part of above proposition does not hold:

Example 1 Consider the following problem with five students $\{1, 2, 3, 4, 5\}$ and four schools $\{a, b, c, d\}$ where each of schools a, b, and c has one seat, and d has two seats. The priority orders and student preferences are as follows:

$\succsim_a$	$\succsim_b$	$\succsim_c$	$\succsim_d$
5	4, 5	1, 3	$\vdots$
1	$\vdots$	$\vdots$	
2			
$\vdots$			

P_1	P_2	P_3	P_4	P_5
c	a	c	b	b
a	d	d	d	a
d	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$				

Consider the new NYC/Boston mechanism, which uniformly randomly chooses a single tie-breaking order for equal-priority students at each school and then employs the student-proposing deferred-acceptance algorithm using the modified priority structure. It is straightforward to compute that this mechanism implements the following lottery:

$$\lambda = \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & d & c & b & a \end{pmatrix}}_{\mu_1} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix}}_{\mu_2} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix}}_{\mu_3} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & a & d & d & b \end{pmatrix}}_{\mu_4}$$

The above four deterministic matchings in the support of λ are stable since they are obtained by the student-proposing deferred acceptance algorithm for tie-breakers $3 \succ_c 1$ and $4 \succ_b 5$; $3 \succ_c 1$ and $5 \succ_b 4$; $1 \succ_c 3$ and $4 \succ_b 5$; $1 \succ_c 3$ and $5 \succ_b 4$, respectively. Thus λ is ex-post stable. However, the random matching that lottery λ induces is not ex-ante stable because student 1 has ex-ante school-wise justified envy toward student 2 for school a: Matching μ_1 implies that student 1 suffers from a positive probability of being assigned to school d, while matching μ_4 implies that student 2 enjoys a positive probability of being assigned to school a, for which he has strictly lower priority than 1.

Interestingly, one can find an alternative lottery that, despite being equivalent to λ, is ex-post unstable:

$$\lambda' = \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & a & c & d & b \end{pmatrix}}_{\mu'_1} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix}}_{\mu'_2} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix}}_{\mu'_3} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix}}_{\mu'_4}$$

The support of λ' contains an unstable matching, namely μ'_1, since student 1 has school-wise justified envy toward student 2 at this matching. Lottery λ' exacerbates the justified school-wise envy situation under λ by transforming it from ex-ante to ex-post. $\Diamond$

Worse still, the new NYC/Boston mechanism may also induce ex-ante school-wise discrimination:

Example 2 Consider the following problem with three students $\{1, 2, 3\}$ and three schools $\{a, b, c\}$ each with a quota of one. The priority orders and student preferences are as follows:

$\succsim_a$	$\succsim_b$	$\succsim_c$
3	2	2
1, 2	1	1
	3	3

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

The tie-breaking lottery assigns the second priority at school a to equal-priority students 1 and 2 with equal chances. Then the new NYC/Boston mechanism (which operates on the student-proposing deferred-acceptance algorithm coupled with either strict priority structure) implements the following lottery:

$$\lambda = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix}}_{\mu_1} + \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}}_{\mu_2}$$

Observe that random matching ρ^λ induces ex-ante school-wise discrimination between students 1 and 2 for school a since matching μ_1 implies that student 1 is given a positive probability of being assigned to school a while student 2 who, despite having equal priority for a, always ends up at school c which she finds worse than a. In particular, the ex-post observation that student 2 has been assigned to school c by this mechanism cannot be attributed to an unlucky lottery draw to determine the priority order at school a. $\Diamond$

3.3 Pareto Efficiency

We define and work with two Pareto efficiency concepts defined over ordinal preferences.

For student $i \in I$, random matching vector π_i *ordinally (Pareto) dominates* random matching vector ρ_i, if $\sum_{aR_ic} \pi_{i,a} \geq \sum_{aR_ic} \rho_{i,a}$ for all $c \in C$ and $\sum_{aR_ib} \pi_{i,a} > \sum_{aR_ib} \rho_{i,a}$ for some $b \in C$, i.e., π_i first-order stochastically dominates ρ_i with respect to P_i. A random matching $\pi \in \mathcal{X}$ *ordinally (Pareto) dominates* $\rho \in \mathcal{X}$, if for all $i \in I$, either π_i ordinally dominates ρ_i or $\pi_i = \rho_i$, and there exists at least one student $j \in I$ such that π_j ordinally dominates ρ_j. We say that a random matching is *ordinally (Pareto) efficient* if there is no random matching that ordinally dominates it.

We refer to ordinally efficient deterministic matchings as *Pareto efficient*. A random matching is *ex-post (Pareto) efficient* if there exists a lottery that induces this random matching and has its support only over Pareto efficient matchings.

Ordinal efficiency implies ex-post efficiency, while the converse is not true for random matchings (Bogomolnaia and Moulin, 2001). It is well known that even with strict school priorities, ex-post stability and ex-post efficiency are not compatible.

Proposition 2 ***(Roth, 1982)*** *There does not exist any ex-post stable and ex-post efficient mechanism.*

Since we take fairness notions as given, we will focus on constrained ordinal efficiency and constrained ordinal dominance as the proper efficiency concepts for mechanisms that belong to a particular class.

4 Strong Ex-ante Stable School Choice

4.1 Fractional Deferred-Acceptance Mechanism

Strong ex-ante stability is an appealing stability property since (1) it guarantees all the enrollment chances to a higher-priority student at his preferred school before all lower-priority students (i.e., by *elimination of ex-ante school-wise justified envy*) thereby also ensuring ex-post stability, and (2) it treats equal-priority students – not only equal students – fairly by giving them equal enrollment chance at "competed"[16] schools (i.e., by *elimination of ex-ante school-wise discrimination*). We now introduce the central mechanism in the theory of strongly ex-ante stable lotteries. This mechanism employs a *fractional deferred-acceptance (FDA) algorithm*.

The FDA algorithm is in the spirit of the classical student-proposing deferred-acceptance algorithm of Gale and Shapley (1962). In this algorithm, we talk about a *fraction of a student* applying to, being tentatively assigned to, or being rejected by a school. In using such language, we have in mind that upon termination of the algorithm, the fraction of a student permanently assigned to some school will be interpreted as the assignment probability of the student to that school. Hence, fractions in fact represent enrollment chances. In the FDA algorithm, a student fraction, by applying to a school, may seek a certain fraction of one seat at that school. As a result, depending on its quota and the priorities of other applicants, the school may *tentatively* assign a certain fraction (possibly less than the fraction the student is seeking) of a seat to the student and reject any remaining fraction

[16]To be precise, we would call a school such as c in the definition of *ex-ante school-wise discrimination* as a *competed* school. That is, it is not student i's least preferred school among those for which his enrollment chance is positive, i.e., student i is *competing* with student j for school c.

of the student. In the algorithm's description below, when we say *fraction w of student i applies to school c*, this means that at most a fraction w of a seat at school c can be assigned to student i. As an example, suppose fraction $\frac{1}{3}$ of student 1 applies to school c at some step of the algorithm. School c then may, for example, admit $\frac{1}{4}$ of student i and reject the remaining $\frac{1}{12}$ of him. We next give a more precise description.

The FDA Algorithm:

Step 1: *Each student applies to his favorite school. Each school c considers its applicants. If the total number of applicants is greater than q_c, then applicants are tentatively assigned to school c one by one starting from the highest priority applicants such that equal-priority students, if assigned a fraction of a seat at this school at all, are assigned an equal fraction. Unassigned applicants (possibly some being a fraction of a student) are rejected.*

$\vdots$

In general,

Step s: *Each student who has a rejected fraction from the previous step, applies to his next-favorite school that has not yet rejected any fraction of him. Each school c considers its tentatively assigned applicants together with the new applicants. Applicant fractions are tentatively assigned to school c starting from the highest priority applicants as follows: For all applicants of the highest priority level, increase the tentatively assigned shares from 0 at an equal rate until there is an applicant who has been assigned all of his fraction. In such a case continue with the rest of the applicants of this priority level by increasing the tentatively assigned shares at an equal rate until there is another applicant who has been assigned all of his fraction. When all applicant fractions of this priority level are served, continue with the next priority level in a similar fashion. If at some point during the process, the whole quota of school c has been assigned, then reject all outstanding fractions of all applicants.*

The algorithm terminates when no unassigned fraction of a student remains. At this point, the procedure is concluded by making all tentative random assignments permanent. We next give a detailed example to illustrate the FDA algorithm.

Example 3 ***How does the FDA algorithm work?*** Consider the following problem with six students $\{1, 2, 3, 4, 5, 6\}$ and four schools $\{a, b, c, d\}$, two, b and d, with a quota of one, and the other two, a and c, with a quota of two:

$\succsim_a$	$\succsim_b$	$\succsim_c$	$\succsim_d$
$\vdots$	6	4	5
$\vdots$	1, 3	2, 3, 5	3, 6
$\vdots$	5	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

P_1	P_2	P_3	P_4	P_5	P_6
b	c	d	b	c	d
a	a	c	c	d	c
$\vdots$	b	b	$\vdots$	$\vdots$	b
$\vdots$	d	a	$\vdots$	$\vdots$	a

Step 1: Students 1 and 4 apply to school b (with quota one), which tentatively admits student 1 and rejects student 4. Students 2 and 5 apply to school c (with quota two), which does not reject any of their fractions. Students 3 and 6 apply to school d (with quota one), which tentatively admits $\frac{1}{2}$ of 6 and $\frac{1}{2}$ of 3, and rejects the remaining halves.

Step 2: Having been rejected by school d, each outstanding half-fraction of students 3 and 6 applies to the next-favorite school, which is school c. Having been rejected by school b, student 4 applies to his next choice, which is also school c. This means school c considers half-fraction of each of 3 and 6 and one whole of 4 together with one whole of 2 and 5. Among the five students, 4 has the highest priority, and hence, is tentatively placed at school c. Next in priority are students 2, 3, and 5 with equal priority, and thus $\frac{1}{3}$ of each is tentatively admitted at school c, which exhausts its quota of two. As a consequence, $\frac{1}{2}$ of student 6, $\frac{1}{6}$ of 3, and $\frac{2}{3}$ of each of 2 and 5 are rejected by c.

Step 3: The next choice of 2 is a, and hence the rejected $\frac{2}{3}$ of him applies to a, and is tentatively admitted there. The next choice of 3 and 6 is b, and hence $\frac{1}{2}$ of 6 and $\frac{1}{6}$ of 3 apply to b, which is currently full and holding the whole of student 1. Since 6 has higher priority than both 1 and 3, the entire applying fraction of 6 is tentatively admitted. Since 1 and 3 share equal priority at b, we gradually increase assigned shares of both students from 0 at an equal rate. This implies that $\frac{1}{6}$ of 3 and $\frac{1}{3}$ of 1 are to be tentatively admitted and the remaining $\frac{2}{3}$ of 1 is to be rejected. The next choice of student 5 is d, and hence $\frac{2}{3}$ of him applies to d, which is currently holding $\frac{1}{2}$ of both of 3 and 6. Since 5 has higher priority than 3 and 6 both of whom have equal priority, the whole $\frac{2}{3}$ of 5 is tentatively admitted, whereas $\frac{1}{6}$ of each of 3 and 6 is tentatively admitted, causing the remaining $\frac{1}{3}$ of each student to be rejected.

Step 4: The next choice of student 1 is a, hence the rejected $\frac{2}{3}$ of him applies to a, and is tentatively admitted there. For students 3 and 6, the best choice that hasn't rejected either is b, and hence $\frac{1}{3}$ of each student applies to b. School b is currently full and holding $\frac{1}{2}$ of 6, $\frac{1}{6}$ of 3, and $\frac{1}{3}$ of 1. Since 1 and 3 have equal but lower priority than 6 at b, the school holds on to all of the $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ fraction of 6, and only $\frac{1}{12}$ of each of 1 and 3 are tentatively admitted by b; while the remaining $\frac{1}{4}$ of 1 and $\frac{5}{12}$ of 3 are rejected.

Step 5: The next choice of 1 and 3 after b is a, and hence $\frac{1}{4}$ of 1 and $\frac{5}{12}$ of 3 apply to a, which is not filled yet and can accommodate all of these fractions: It is currently holding $\frac{11}{12}$ of 1, $\frac{5}{12}$ of 3,

and $\frac{2}{3}$ of 2. Since there are no further rejections, the algorithm terminates and returns the following random matching outcome:

	a	**b**	**c**	**d**
1	$\frac{11}{12}$	$\frac{1}{12}$	0	0
2	$\frac{2}{3}$	0	$\frac{1}{3}$	0
3	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$
4	0	0	1	0
5	0	0	$\frac{1}{3}$	$\frac{2}{3}$
6	0	$\frac{5}{6}$	0	$\frac{1}{6}$

$\Diamond$

While the FDA algorithm is intuitive, the computation of its outcome poses a new challenge that did not exist for its deterministic analogue (i.e., the version proposed by Gale and Shapley). It turns out that in the FDA algorithm a student may end up applying to the same school infinitely many times. Thus, we next observe that the FDA algorithm as explained above may not converge in a finite number of steps. We illustrate this with a simple example.

Example 4 ***The FDA algorithm may not terminate in a finite number of steps:*** Consider the following simple problem with three students and three schools each with a quota of one:

$\succsim_a$	$\succsim_b$	$\succsim_c$
3	1, 2	$\vdots$
1, 2	3	$\vdots$
		$\vdots$

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

Step 1: Students 1 and 2 apply to school a, and $\frac{1}{2}$ of each is tentatively admitted (while $\frac{1}{2}$ of each is rejected), since they have the same priority. Student 3 applies to b and is tentatively admitted.

Step 2: Rejected $\frac{1}{2}$ of student 2 next applies to school c and is tentatively admitted. Rejected $\frac{1}{2}$ of student 1 applies to school b, where he has higher priority than the currently admitted student 3. Now $\frac{1}{2}$ of student 3 is rejected and $\frac{1}{2}$ of 1 is tentatively admitted.

Step 3: Rejected $\frac{1}{2}$ of student 3 applies to a, where he has higher priority than both 1 and 2. As a result $\frac{1}{2}$ of 3 is tentatively admitted whereas $\frac{1}{4}$ of each of 1 and 2 is rejected.

Step 4: Rejected $\frac{1}{4}$ of 2 next applies to school c and is tentatively admitted (in addition to the previously admitted $\frac{1}{2}$ of him). Rejected $\frac{1}{4}$ of student 1 applies to school b, where he has higher priority than the currently admitted $\frac{1}{2}$ of student 3. Now $\frac{1}{4}$ of student 3 is rejected and $\frac{1}{4}$ of 1 is tentatively admitted.

$\vdots$

As the procedure goes on, rejected fractions of student 3 by school b continue to apply to school a in turn, leading to fractions of student 3 to accumulate at a, and at the same time, causing (a smaller fraction of) student 1 to be rejected by school a at each application. This, in turn, leads student 1 to apply to school c and cause (the same fraction of) 3 to be further rejected. Consequently, all fractions of student 3 accumulate at school a and all those of student 1 at school b:

Step ∞ : The sum of the admitted fractions of student 3 at school a is 1, which is the sum of the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. The sum of the admitted fractions of student 1 at school b is 1. The sum of the admitted fractions of student 2 at school c is 1. $\Diamond$

Even though the FDA algorithm may not terminate in finite time, the above example suggests that its outcome can be computed without getting lost in infinite loops by examining the rejection cycles that might form throughout the steps of the algorithm.

To define the *finite* version of the FDA algorithm, we need to define a few new concepts. We first define a binary relation between students. Let $i, j \in I$ and $c \in C$. Suppose that at some step s of the FDA algorithm, some fraction of student i gets rejected by school c, while he still has some fraction not rejected by c at this step. On the other hand, suppose also that at step s school c temporarily holds some fraction of some other student j who has not been rejected by c until step s (i.e., not rejected before or at step s). Then, we say that i is *partially rejected by c in favor of j*, and denote it by $j \hookrightarrow_c i$. At a later step r> s in the algorithm, if either some fraction of j gets rejected by c or all fractions of i get rejected by c, then the above relationship does not hold at step r or at any later step. In this case we say that $j \hookrightarrow_c i$ is no longer *current*.

A *rejection cycle* is a list of distinct students and schools $(i_1, c_1, ..., i_m, c_m)$ such that at a step of the algorithm, we have

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1$$

and all partial rejection relations are current.

Observe that at the moment the cycle occurs, student i_1 gets partially rejected by school c_m in favor of student i_m. We know that school c_1 has not rejected student i_1 at any fraction; thus, the next available choice for student i_1 is c_1. Therefore, student i_1 applies "again" to school c_1. As a result student i_2 gets partially rejected again, and the same sequence of partial rejections reoccur. That is, the algorithm cycles. We refer to this cycle as a *current* rejection cycle as long as all partial rejection relations are current, and we say that i_1 *induces* this rejection cycle.

Nonetheless, this cycle either converges to a tentative random matching in the limit or, sometimes, in a finite number of steps when some partial rejections turn into full rejections. Thus, once a cycle is detected, it can be solved as a system of linear equations.

We make the following observation, which will be crucial in the definition of the "formal" FDA algorithm:

Observation 1 *If a rejection cycle*

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1$$

is current in the FDA algorithm, then for each student i_ℓ, the best school that has not rejected a fraction of him is school c_ℓ; that is, whenever a fraction of i_ℓ gets rejected, he next makes an offer to school c_ℓ.

The outcome of the FDA algorithm described above converges (as the number of steps approaches infinity) to the the outcome of the following *finite* FDA algorithm:

The FDA Algorithm:

Step s. *Fix some student $i_1 \in I$ who has an unassigned fraction from the previous step. He applies to the next best school that has not yet rejected any fraction of him. Let c_1 be this school. Two cases are possible:*

(a) If the student i_1 induces a rejection cycle

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1,$$

then we resolve it as follows: For $i_{m+1} \equiv i_1$ and $c_0 \equiv c_m$, c_1 tentatively accepts the maximum possible fraction of i_1 such that each school c_ℓ tentatively accepts

- *all fractions of applicants tentatively accepted in the previous step except the ones belonging to the lowest-priority level,*
- *the total rejected fraction of student i_ℓ from school $c_{\ell-1}$, and*
- *an equal fraction (if possible) among the lowest-priority applicants tentatively accepted in the previous step (including student $i_{\ell+1}$)*

  so that it does not exceed its quota q_{c_ℓ}.

(b) If i_1 does not induce a rejection cycle, school c_1 considers its tentatively assigned applicants from the previous step together with the new fraction of i_1. It tentatively accepts these fractions starting from the highest priority. In case its quota is filled in this process, it tentatively accepts an equal fraction (if possible) of all applicants belonging to the lowest accepted priority level. It rejects all outstanding fractions.

We continue until no fraction of a student remains unassigned. At this point, we terminate the algorithm by making all tentative random assignments permanent.$\Diamond$

We resolve part (a) of the algorithm reducing the infinite convergence problem demonstrated in Example 4 to a linear equation system. This is demonstrated in Appendix A- Example 7. We explain this resolution in Appendix B, the proof of Proposition 3, for the general case.

Since we have defined the algorithm in a sequential fashion, it is not clear whether the procedure is independent of the order of the proposing students or which cycle is chosen to be resolved. Corollary 1 (below) shows that this statement is true, and thus, its outcome is unique.[17] We refer to the mechanism whose outcome is found through the above FDA algorithm as the *FDA mechanism*.

4.2 Properties of the FDA Mechanism

We next present some desirable properties of the FDA mechanism and its iterative algorithm:

Proposition 3 *The FDA algorithm is well defined and converges to a random matching in a finite number of steps.*

The proof of Proposition 3 is given in Appendix B.

Theorem 2 *An FDA outcome is strongly ex-ante stable.*

Proof. Let π be the FDA algorithm's outcome according to some proposal order of students. We first show that this outcome is a well-defined random matching. Suppose not. Then, there exists a student i who is not matched with probability one at π. Thus, π is sub-stochastic and there exists some school c that is *undermatched* at π, i.e., $\sum_j \pi_{j,c} \leq q_c$. At some step, student i makes an offer to c and some fraction of him gets rejected by it, as he ends up with some rejected probability at the end of the algorithm. However, school c only rejects a student if its quota is tentatively filled. Moreover, once it is tentatively filled, it never gets undermatched. However, this contradicts the earlier conclusion that its quota was not filled at π. Thus, π is a bi-stochastic matrix, i.e., it is a random matching. Next we show that π is strongly ex-ante stable. Since in the algorithm, (i) a student fraction always applies to the best school that has not yet rejected him, and (ii) when its quota is filled, a school always prefers higher priority students to the lower-priority ones, a student cannot have ex-ante school-wise justified envy toward a lower-priority student. If π is not strongly ex-ante stable, then it should be the case that there is ex-ante school-wise discrimination between equal-priority students, i.e., there are $i \sim_c j$ for some school c such that $cP_i a$, $\pi_{i,a} > 0$, and yet $\pi_{i,c} < \pi_{j,c}$. Consider the first step after which the (tentative) random matching vector of school c does not change. At this step, some students apply to school c and in return some fractions of some students with equal priority i' and j' are tentatively accepted and some are rejected. The only way $\pi_{i',c} < \pi_{j',c}$ is if no fraction of i' ever gets rejected by school c. Thus, $\pi_{i',a'} = 0$ for all $a' \prec_{i'} c$. This contradicts the claim that such a student i exists. ■

[17] This result is analogous to the result regarding the deferred-acceptance algorithm of Gale and Shapley, which can also be executed by students making offers sequentially instead of simultaneously (McVitie and Wilson, 1971).

Our next result states that from a welfare perspective the FDA outcome is the most appealing strongly ex-ante stable matching. This finding can also be interpreted as the random analogue of Gale and Shapley's celebrated result on the constrained Pareto optimality of the student-proposing deferred-acceptance outcome (among stable matchings) for the deterministic two-sided matching context.

Theorem 3 *An FDA outcome ordinally dominates all other strongly ex-ante stable random matchings.*

The proof of Theorem 3 is also given in Appendix B. Theorem 3 implies that the FDA mechanism is well defined, i.e., its outcome is unique and independent of the order of students making applications in the algorithm.

Corollary 1 *The FDA algorithm's outcome is independent of the order of students making offers or the rejection cycle chosen to be resolved if more than one is encountered simultaneously; and thus, it is unique.*

5 Ex-ante Stable School Choice

The FDA mechanism satisfies ex-ante stability but sacrifices some efficiency at the expense of finding a random matching that treats equal-priority students fairly. Therefore, we next address how we can achieve more efficient outcomes without sacrificing fairness "too much", i.e. by giving up equal treatment of equal-priority students, but maintaining ex-ante stability and equal treatment of equals.

By Proposition 2 we know that there is no mechanism that satisfies ordinal efficiency and ex-ante stability. Thus, we define the following constrained efficiency concept: A mechanism φ is *constrained ordinally efficient within its class* if there exist no mechanism ψ in the same class as φ and no problem $[P, \succsim]$ such that $\psi[P, \succsim]$ ordinally dominates $\varphi[P, \succsim]$.

We now characterize constrained ordinal efficient mechanisms within the class of ex-ante stable mechanisms. First, we restate a useful result due to Bogomolnaia and Moulin (2001) that characterizes ordinal efficiency. Fix a problem $[P, \succsim]$. For any random matching $\pi \in \mathcal{X}$, we say that i *ex-ante school-wise envies* j *for* b *due to* c, if $\pi_{j,b} > 0$, $\pi_{i,c} > 0$, and bP_ic.[18] We denote it as

$$(i, c) \succ^{\pi} (j, b).$$

[18]Under this definition, a student will ex-ante school-wise envy himself, if he is assigned fractions from two schools. This is different from the improvement relationship defined by Bogomolnaia and Moulin. Unlike them, we do not rule out this possibility and use it for the constrained efficiency characterization within ex-ante stable random matchings.

A *stochastic improvement cycle* $Cyc = (i_1, c_1, ..., i_m, c_m)$ at π is a list of distinct student-school pairs (i_ℓ, c_ℓ) such that

$$(i_1, c_1) \succcurlyeq^\pi (i_2, c_2) \succcurlyeq^\pi ... \succcurlyeq^\pi (i_m, c_m) \succcurlyeq^\pi (i_1, c_1).$$

(We use modulo m whenever it is unambiguous for subscripts, i.e., $m+1 \equiv 1$.) Let $w \leq \min_{\ell \in \{1,...,m\}} \pi_{i_\ell, c_\ell}$. Cycle Cyc is *satisfied* with fraction w at π if for all $\ell \in \{1, ..., m\}$, a fraction w of the school $c_{\ell+1}$ is assigned to student i_ℓ additionally and a fraction w of school c_ℓ is removed from his random matching, while we do not change any of the other matching probabilities at π. Formally, we obtain a new random matching $\rho \in \mathcal{X}$ such that for all $i \in I$ and $c \in C$,

$$\rho_{i,c} = \begin{cases} \pi_{i,c} + w & \text{if } i = i_\ell \text{and } c = c_{\ell+1} \text{for some } \ell \in \{1, ..., m\}, \\ \pi_{i,c} - w & \text{if } i = i_\ell \text{and } c = c_\ell \text{for some } \ell \in \{1, ..., m\}, \\ \pi_{i,c} & \text{otherwise.} \end{cases}$$

The following is a direct extension of Bogomolnaia and Moulin's result to our domain and our definition of the ex-ante school-wise envy relationship. Therefore, we skip its proof.

Proposition 4 ***(Bogomolnaia and Moulin, 2001)*** *A random matching is ordinally efficient if and only if it has no stochastic improvement cycle.*

5.1 Ex-ante Stability and Constrained Ordinal Efficiency

Proposition 4 suggests that if a random matching has a stochastic improvement cycle, then one can obtain a new random matching that ordinally dominates the initial one simply by satisfying this stochastic improvement cycle. Observe, however, that satisfying such a cycle may induce ex-ante school-wise justified envy at the new random matching. Consequently, given that our goal is to maintain ex-ante stability, to improve the efficiency of an ex-ante stable random matching, we can only work with those stochastic improvement cycles that respect the ex-ante stability constraints. For this purpose we introduce an envy relationship as follows:

We say that i *ex-ante top-priority school-wise envies* j *for* b *due to* c, and denote it as

$$(i, c) \blacktriangleright^\pi (j, b),$$

if $(i, c) \succcurlyeq^\pi (j, b)$ and $i \succsim_b k$ for all $(k, a) \in I \times C$ such that $(k, a) \succcurlyeq^\pi (j, b)$. That is, i envies j for b due to c, and i is the highest-priority student that envies j for b.[19]

[19] Like the ex-ante school-wise envy relationship, a student will ex-ante top-priority school-wise envy himself if he is assigned fractions from two schools and for the better of the two schools, he is among the highest-priority students ex-ante school-wise–envying.

An *ex-ante stable improvement cycle* $(i_1, c_1, ..., i_m, c_m)$ at π is a list of distinct student-school pairs (i_ℓ, c_ℓ) such that

$$(i_1, c_1) \blacktriangleright^\pi (i_2, c_2) \blacktriangleright^\pi ... \blacktriangleright^\pi (i_m, c_m) \blacktriangleright^\pi (i_1, c_1).$$

We state our main result of this subsection below. Although one direction of this result is easy to prove, the other direction's proof needs extra attention to detail. Our result generalizes Proposition 4 (stated for the equal priority domain by Bogomolnaia and Moulin), and a result by Erdil and Ergin (2008) (stated for the deterministic domain) to the probabilistic school-choice framework:

Proposition 5 *An ex-ante stable random matching ρ is not ordinally dominated by any other ex-ante stable random matching if and only if there is no ex-ante stable improvement cycle at ρ.*

The proof of Proposition 5 is given in Appendix B.

5.2 Ex-ante Stable Fraction Trading

Motivated by Proposition 5, we shall use the FDA outcome to obtain a constrained ordinally efficient ex-ante stable matching. Our second proposal, roughly, rests on the following intuition: Since the outcome of the FDA mechanism is ex-ante stable, if we start initially from this random matching and iteratively satisfy ex-ante stable improvement cycles, we should eventually arrive at a constrained ordinally efficient ex-ante stable random matching. Though intuitive, this approach need not guarantee equal treatment of equals. Therefore, in what follows we will also need to pay attention to the ex-ante stable improvement cycles that are to be selected.

Our second proposal, the *fractional deferred-acceptance and trading (FDAT)*, starts from the FDA outcome and satisfies *all* ex-ante stable improvement cycles simultaneously so as to preserve equal treatment of equals to obtain a new random matching. It iterates until there are no new ex-ante stable improvement cycles. Before formalizing this procedure, to fix ideas and point out some potential difficulties, we first illustrate our approach with an example:

Example 5 ***How does the FDAT algorithm work?*** We use the same problem as in Example 3.

Step 0. We have found the FDA outcome in Example 3 as

$\rho^1 =$

	a	**b**	**c**	**d**
1	$\frac{11}{12}$	$\frac{1}{12}$	0	0
2	$\frac{2}{3}$	0	$\frac{1}{3}$	0
3	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$
4	0	0	1	0
5	0	0	$\frac{1}{3}$	$\frac{2}{3}$
6	0	$\frac{5}{6}$	0	$\frac{1}{6}$

Step 1. We form top-priority school-wise envy relationships as

$$
\begin{aligned}
&(1,a) \blacktriangleright^{\rho^1} (3,b),(6,b),(1,b)\\
&(2,a) \blacktriangleright^{\rho^1} (3,c),(4,c),(5,c),(2,c)\\
&(3,f) \blacktriangleright^{\rho^1} (5,d),(6,d),(3,d) \qquad \forall f \in \{a,b,c\}\\
&(3,f) \blacktriangleright^{\rho^1} (2,c),(4,c),(5,c),(3,c) \qquad \forall f \in \{a,b\}\\
&(3,a) \blacktriangleright^{\rho^1} (1,b),(6,b),(3,b)\\
&(5,d) \blacktriangleright^{\rho^1} (2,c),(3,c),(4,c),(5,c)\\
&(6,b) \blacktriangleright^{\rho^1} (3,d),(5,d),(6,d).
\end{aligned}
$$

There is only one ex-ante stable improvement cycle:

$$(3,c) \blacktriangleright^{\rho^1} (5,d) \blacktriangleright^{\rho^1} (3,c)$$

We satisfy this cycle with the maximum possible fraction $\frac{1}{3}$ and obtain:

$\rho^2 =$

	a	**b**	**c**	**d**
1	$\frac{11}{12}$	$\frac{1}{12}$	0	0
2	$\frac{2}{3}$	0	$\frac{1}{3}$	0
3	$\frac{5}{12}$	$\frac{1}{12}$	0	$\frac{1}{2}$
4	0	0	1	0
5	0	0	$\frac{2}{3}$	$\frac{1}{3}$
6	0	$\frac{5}{6}$	0	$\frac{1}{6}$

Step 3. There are no new top-priority school-wise envy relationships at ρ^2, and no new ex-ante stable improvement cycles; thus, ρ^2 is the outcome of the FDAT algorithm. $\Diamond$

The main difficulty with this approach is determining which ex-ante stable improvement cycle to satisfy, if there are many. This choice may cause fairness violations regarding the equal treatment of

equals, or there can be many ways to find a solution respecting equal treatment of equals. Thus, the outcome of the FDAT algorithm as explained above is not uniquely determined. Furthermore, there are also legitimate computational concerns in finding more than one ex-ante stable improvement cycle at a time.[20] We overcome these fairness and computational issues by adapting to our domain a fractional trading algorithm, which was introduced in the operations research literature by Athanassoglou and Sethuraman (2011). It is referred to as the *constrained-consumption algorithm* and was introduced to obtain ordinally efficient allocations in house allocation problems with existing tenants (Abdulkadiroğlu and Sönmez, 1999). Similar algorithms were also previously introduced by Yılmaz (2009, 2010). Our version, the *ex-ante stable consumption (EASC) algorithm*, is embedded in step $s \geq 1$ of the FDAT algorithm as a way to satisfy ex-ante stable improvement cycles simultaneously and equitably. It is explained in detail in Supplementary Appendix D.

We state the FDAT algorithm formally as follows:

The FDAT Algorithm:

Step 0. *Run the FDA algorithm. Let* ρ^1 *be its random matching outcome.*

$\vdots$

Step s. *Let* $\rho^s \in \mathcal{X}$ *be found at the end of step s-1. If there is an ex-ante stable improvement cycle, run the EASC algorithm. Let* ρ^{s+1} *be the outcome and continue with Step s+1. Otherwise, terminate the algorithm with* ρ^s *as its outcome.* $\Diamond$

We refer to the mechanism whose outcome is found through this algorithm as the *FDAT mechanism.*

In Supplementary Appendix E-Example 8, we illustrate the EASC algorithm to show how the formal FDAT algorithm works for the problem in Example 5. Although the execution of the FDAT algorithm is obvious and simple in this example without the implementation of the EASC algorithm in each step, for expositional purposes we re-execute it with the embedded EASC algorithm.[21]

5.3 Properties of the FDAT Mechanism

Proposition 6 *The FDAT algorithm is well defined and converges to a random matching in a finite number of steps.*

Proof. We know that Step 0 of FDA algorithm works in finite steps (by Proposition 3), as well as Step 1, the EASC algorithm (Athanassoglou and Sethuraman, 2011).

[20] In a worst-case scenario, the number of ex-ante stable improvement cycles at an ex-ante stable matching grows exponentially with the number of students.

[21] In general, without the use of the EASC algorithm or a similar well-defined technique, step $s \geq 1$ of the FDAT algorithm may not be well defined.

Next, we prove that the number of steps in FDAT is finite. After each step t ≥ 1 of the FDAT algorithm, at least one student $i \in I$ leaves a school $c \in C$ with $\rho_{i,c}^{\text{t-1}} > 0$ with 0 fraction and gets into better schools, i.e., $\rho_{i,c}^{\text{t}} = 0$ and $\sum_{aP_ic} \rho_{i,a}^{\text{t}} > \sum_{aP_ic} \rho_{i,a}^{\text{t-1}}$. (Otherwise, the same ex-ante stable improvement cycle of $\rho_{i,c}^{\text{t-1}}$ would still exist at ρ^{t}, contradicting that the EASC algorithm has converged at step t.) Thus, the FDAT algorithm converges in no more than $|C|\,|I| + 1$ steps (including Step 0). ■

Theorem 4 *The FDAT mechanism is ex-ante stable.*

Proof. Consider each step of the FDAT algorithm.

In Step 0, the outcome of the FDA has no school-wise justified envy toward a lower-priority student by Theorem 2.

In Step 1, students in determined ex-ante stable improvement cycles are made better off (in an ordinal dominance sense), while others' welfare is unchanged. Moreover, the students who are made better off are among the highest-priority students who desire a seat at the school where they receive a larger share. That is, for any student i with $\rho_{i,c}^1 > \rho_{i,c}^0$, there is some school b with cP_ib, and $\rho_{i,b}^1 < \rho_{i,b}^0$, and there is no student $j \succ_c i$ such that $\rho_{j,a}^1 > 0$ for some school a with cP_ja. (Otherwise, i would not ex-ante top-priority school-wise envy a student k with $\rho_{k,c}^0 > 0$ for c due to b, since j would do that due to a or a worse school. Moreover, since ρ^0 is ex-ante stable, $\rho_{i,c}^0 = 0$. The last two statements would imply $(i,c) \notin \mathcal{A}(\rho^0)$, which in turn implies that $\rho_{i,c}^1 = 0$.) Hence, ρ^1 is ex-ante stable.

We repeat this argument for each step. Hence, when the algorithm is terminated, the outcome is ex-ante stable. ■

Our next result states that from a welfare perspective the FDAT outcome is among the most appealing ex-ante stable random matchings. Improving upon this matching would necessarily lead to ex-ante school-wise justified envy. This finding can also be interpreted as the random analogue of the mechanism proposed by Erdil and Ergin (2008) for a deterministic school-choice model with random tie-breaking. In that context, the outcome of the Erdil-Ergin mechanism has been shown to be constrained ex-post Pareto efficient among ex-post stable matchings.

Theorem 5 *The FDAT mechanism is constrained ordinally efficient within the ex-ante stable class.*

Proof. Suppose that the FDAT outcome ρ is ordinally dominated by an ex-ante stable random matching for some problem P. By Proposition 5, there exists an ex-ante stable improvement cycle at P. Thus, this contradicts the fact that ρ is the FDAT outcome. ■

Theorem 6 *The FDAT mechanism treats equals equally.*

Proof. The FDA mechanism treats equals equally as it is strongly ex-ante stable (by Theorem 2). Thus, two students with the same preferences and priorities have exactly the same random matching vector under the FDA outcome ρ^0. Let i, j be two equal students. Then $\rho_i^0 = \rho_j^0$ and $(i, c) \in \mathcal{A}(\rho^0)$ if and only if $(j, c) \in \mathcal{A}(\rho^0)$ for any school $c \in C$. By Athanassoglou and Sethuraman (2011), the EASC algorithm treats equals equally. The last two statements imply that outcome of Step 1, ρ^1 treats equals equally. We repeat this argument iteratively for each step, showing that the FDAT outcome treats equals equally. ■

5.4 The FDAT Mechanism vs. Probabilistic Serial Mechanism

The way the FDA and FDAT mechanisms treat equal-priority students resembles the probabilistic serial (PS) mechanism of Bogomolnaia and Moulin (2001) proposed for the "random assignment" problem where there are no exogenous student priorities. Loosely speaking, within any given step of the PS algorithm, those students who compete for the available units of the same object are allowed to consume equal fractions until the object is exhausted. Similarly, within any given step of the FDA algorithm those equal-priority students who have applied to the same school are also treated equally in very much the same way. Despite such similarity the two procedures are indeed quite different in general. The difference of the two algorithms comes from the fact that the PS algorithm makes permanent random matchings within each step, whereas the FDA algorithm always makes tentative random matchings till the last step. We can expect to have some efficiency loss due to FDA's strong ex-ante stability property, while the PS mechanism is not strongly ex-ante stable. Even if FDA and PS outcomes are different, one may think that starting from the FDA outcome, fractional trading will somehow establish the equivalence with the PS outcome. However, as the following example shows, the PS outcome does not necessarily ordinally dominate the FDA outcome, and hence, the FDAT outcome, which ordinally dominates the FDA outcome, and the PS outcome are not the same either:

Example 6 ***Neither FDA nor FDAT is equivalent to the PS mechanism when all students have the same priority:*** Assume there are four students $\{1, 2, 3, 4\}$ and four schools $\{a, b, c, d\}$ each with a quota of one. All students have equal priorities at all schools. The students' preferences are given as

P_1	P_2	P_3	P_4
d	a	d	c
c	d	c	b
a	c	b	d
b	b	a	a

The FDA, FDAT, and PS outcomes are:

$\rho^{FDA} =$

	a	**b**	**c**	**d**
1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{2}{3}$	0	0	$\frac{1}{3}$
3	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
4	0	$\frac{2}{3}$	$\frac{1}{3}$	0

$\rho^{FDAT} =$

	a	**b**	**c**	**d**
1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
2	1	0	0	0
3	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
4	0	$\frac{2}{3}$	$\frac{1}{3}$	0

$\rho^{PS} =$

	a	**b**	**c**	**d**
1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
2	$\frac{5}{6}$	$\frac{1}{6}$	0	0
3	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
4	0	$\frac{1}{3}$	$\frac{2}{3}$	0

Observe that ρ^{FDAT} and ρ^{PS} are both ordinally efficient. Moreover, ρ^{PS} does not ordinally dominate ρ^{FDA} (e.g., contrast student 2's random matching vectors under ρ_2^{FDA} vs. ρ_2^{PS}). $\Diamond$

6 Simulations

We ran simulations to estimate the performance of FDAT mechanism and contrast it with those of the NYC/Boston (DA henceforth) and Erdil and Ergin (2008) (EE henceforth) mechanisms in problems that approximately match the main characteristics of the Boston data from 2008-11 (Abdulkad.iroğlu et al., 2006) . EE dominates DA, as it starts from a deterministic DA outcome using some ex-ante tie-breaking at each instance and finds deterministic stable improvement cycles randomly and satisfies them. There is no clear theoretical efficiency comparison between FDAT and EE (nor between FDAT and DA). FDAT is a constrained ordinally efficient mechanism within the ex-ante stable class while EE is a constrained ex-post efficient mechanism within the ex-post stable class. Ordinal efficiency is a stronger efficiency notion, however the ex-post stable class is larger than the ex-ante stable class. Even if in instances the outcome of the EE is ex-ante stable, FDAT may not dominate it.

In our simulations we randomly generated 100 markets, each with $|S|$ schools and $|I|$ students, and computed the corresponding outcomes of FDAT, DA and EE, where 100 random tie-breaking priority orderings were additionally generated for the latter two mechanisms. More specifically, we assumed that students were zoned in n neighborhoods, $|S|/n$ schools per each neighborhood. Students were grouped in these neighborhood such that $|I|/n$ students were assumed to be living in each neighborhood. Also s students in each neighborhood were assumed to have elder siblings attending high school, some attending a neighborhood school and others a non-neighborhood school. As in Boston, the priorities at each school were generated to prioritize the neighborhood students with siblings attending the school first, non-neighborhood students with siblings attending the school second, neighborhood students without siblings at the school third, and non-neighborhood students without siblings attending the school last. We generated student preferences using the following randomization process: Each student had p_{sn} probability to first-rank a particular neighborhood school that a sibling is attending, p_s probability to first-rank a particular non-neighborhood school that a sibling is attending, p_n probability to first-rank a neighborhood school that a sibling is attending,

and the remaining probability was divided up equally for each non-neighborhood school to determine its probability to be ranked first. If the student did not have a sibling, then p_s and p_{sn} were ignored; if the student had a sibling attending a neighborhood school, then p_s was ignored; and if the student had a sibling attending a non-neighborhood school then p_{sn} was ignored when generating the first choice. Once the first choice school is randomly determined, the conditional probabilities for the remaining schools were updated and then second choice was randomly generated. The remaining choices were determined sequentially and randomly after updating the probabilities for remaining schools after each selection. We chose the above preference parameters roughly based on real preference summary statistics of Boston high school applicants in years 2008-11 (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2006).

The data suggested that 60% of the students have siblings in the system, although we do not have data on how many of them are older siblings who generate sibling priorities for the students at their schools. There were on average 26 schools and 2705 students per year applying for high school. Each school had neighborhood priority (either with or without sibling) for 208 students on average, and there were on average two schools located in each neighborhood. Based on the aggregate statistics, students ranked a non-priority school 64% of the time, a sibling's non-neighborhood school 3% of the time and a sibling's neighborhood school 3% of the time, and a neighborhood school without a sibling priority 30% of the time. Observe that the latter includes all students and does not distinguish among student types with or without siblings. Also we did not have reliable data on quotas of schools. Our simulation statistics assumed that half of the students with siblings have older siblings in high school (a total of 30%) so that 10% of the students have older in-walk-zone siblings and 20% of the students have older siblings attending non-neighborhood schools. Using a back-of-the-envelope calculation for our simulations to approximately match the preference characteristics of data with our preference generation process, we chose $p_{sn} = 0.3, p_s = 0.15$, and $p_n = 0.15$. We also chose the number of schools comparable in size to the sample: we had $|S| = 20$ and $n = 10$ neighborhoods, so that there were two schools per neighborhood. However, to have a manageable simulation (as we ran DA and EE 10,000 times and FDAT 100 times) we chose $|I| = 200$ total students, and hence, 20 students per walk zone, instead of 208. We report the results of these simulations below.[22]

The following tables show the average allocation of different types of students in the simulations. The first table compares the DA outcome with that of FDAT. Although there is no domination relationship between the two mechanisms in theory, FDAT does extremely well for almost all students with respect to DA. The first row in Table 1 shows the average proportion of students for whom FDAT first-order stochastically dominates (fosd, for short) DA, DA fosd FDAT, DA and FDAT outcomes are the same and there is no comparison with fosd among the two outcomes. A super-majority of students, 62.9% prefer FDAT over DA. Among these students, the probability a student receives his

[22]The reported results are highly robust to changes in in-walk-zone/out-of-walk-zone sibling ratios.

first choice school is 0.76, while this probability is 0.456 under DA. While only 1.5% prefer DA over FDAT, the average assignment probabilities of these students to their choices are very similar under both mechanisms and only slightly favorable for the DA. The probability for the first two choices are 0.928 for FDAT and only 0.787 for DA. On the other hand, the students who get the same outcome under FDAT and DA always get their first choices and these students comprise 30.7% of the whole population. For 5% of the population the FDAT and DA outcomes are not comparable in fosd sense, and the average allocation probabilities for their choices are similar.

The second table reports the results of similar comparisons between FDAT and EE. Note that there is no overall domination relation between these mechanisms in theory. Students seem overall much better under EE with respect to DA, still FDAT outcomes appear to be more favorable for a higher percentage of students than EE: 31.6% of the students unambiguously prefer FDAT over EE while only 18% of the students prefer EE over FDAT. 37% of the students receive the same allocation under both mechanisms, at which each of them receives his first choice with probability 1. For 13.4% of the students the outcomes are not comparable with respect to fosd.

Although neither EE nor DA is ex-ante stable, we observe very small percentage of agents having ex-ante justified envy. Hence, under these preferences and priorities both mechanisms almost behave like ex-ante stable mechanisms (the last row in both tables). Although in theory, there can be ex-post stable and more efficient mechanisms than FDAT, as the two stability concepts seem to be close under realistic simulations, FDAT's superior performance with respect to both mechanisms is not surprising.

Table 1: FDAT vs. DA		**FDAT fosd DA**		**DA fosd FDAT**		**FDAT=DA**	**not comp.**		**Overall**	
Fraction of Students		62.9%		1.5%		30.7%	5.0%		100%	
Average Prob.	**Choices**	**FDAT**	**DA**	**FDAT**	**DA**	**FDAT=DA**	**FDAT**	**DA**	**FDAT**	**DA**
	1st	0.760	0.456	0.407	0.482	1.000	0.498	0.414	0.815	0.621
	2nd	0.168	0.331	0.565	0.496		0.256	0.311	0.127	0.231
	3rd	0.055	0.131	0.026	0.020		0.179	0.177	0.044	0.091
	4th	0.013	0.049	0.002	0.001		0.055	0.066	0.011	0.034
	5th	0.003	0.020				0.010	0.020	0.002	0.013
	6th	0.001	0.008				0.002	0.007	0.000	0.005
	7th		0.003				0.001	0.003		0.002
	8th		0.001					0.001		0.001
	9th							0.001		0.000
Fraction of justifiably ex-ante envious students in DA							5.6%			

Table 2: FDAT vs. EE		**FDAT fosd EE**		**EE fosd FDAT**		**FDAT=EE**	**not comp.**		**Overall**	
Fraction of Students		31.6%		18.0%		37.0%	13.4%		100%	
Average Prob.	**Choices**	**FDAT**	**EE**	**FDAT**	**EE**	**FDAT=EE**	**FDAT**	**EE**	**FDAT**	**EE**
	1st	0.795	0.692	0.617	0.756	1.000	0.498	0.414	0.815	0.812
	2nd	0.168	0.241	0.268	0.191		0.256	0.311	0.127	0.138
	3rd	0.030	0.046	0.087	0.042		0.179	0.177	0.044	0.035
	4th	0.005	0.013	0.022	0.009		0.055	0.066	0.011	0.010
	5th	0.001	0.005	0.005	0.001		0.010	0.020	0.002	0.003
	6th		0.002	0.001			0.002	0.007	0.000	0.001
	7th		0.001				0.001	0.003	0.000	0.000
	8th							0.001		0.000
	9th							0.001		0.000
Fraction of justifiably ex-ante envious students in EE							2.0%			

7 Incentives

Strategic issues regarding lottery matching mechanisms, in general, have not been well understood. In the context of one-sided matching (i.e., the special case of our model where all students have equal priority at all schools) strategy-proofness is essentially incompatible with ordinal efficiency. A mechanism is *strategy-proof* if, for each agent, his random matching vector obtained through the mechanism via his truth-telling behavior ordinally dominates or is equal to the one obtained via his revelation of any untruthful ranking. Therefore, notwithstanding its appeal in terms of various properties including ordinal efficiency, the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) that has triggered a rapidly growing literature on the random assignment problem is not strategy-proof. In the context of school choice, due to the well-known three-way tension among stability, efficiency, and incentives, strategy-proof and stable mechanisms are necessarily inefficient (see e.g. Erdil and Ergin (2008), Abdulkadiroğlu, Pathak, and Roth (2009), and Kesten (2009a)). The current NYC/Boston mechanism, which is strategy-proof, is the most efficient stable mechanism (Gale and Shapley, 1962) when priorities are strict. However, in the school-choice problem with weak priorities, it is not even ex-post efficient within the ex-post stable class of mechanisms. Moreover, it has been shown empirically (Abdulkadiroğlu, Pathak, and Roth, 2009) and theoretically (Kesten, 2009a) to be subject to significant and large welfare losses. As a result of this observation non-strategy-proof mechanisms have been highly advocated and proposed in the recent literature on school choice (see e.g. Erdil and Ergin (2008), Kesten (2009a), and Abdulkadiroğlu, Che, and Yasuda (2008)).

Given the negative results outlined above regarding different fairness and efficiency properties it is probably not surprising that the two mechanisms proposed in this paper are not strategy-proof. This observation follows from the following two impossibility results regarding the existence of strategy-

proof mechanisms in our problem domain. We state these observations in the next two remarks. The first remark is a reformulation of a result due to Bogomolnaia and Moulin (2001) for the present context.

Remark 1 *When $|I| \geq 4$, there is no strategy-proof, ex-ante stable, and constrained ordinally efficient mechanism that also respects equal treatment of equals.*

The next remark shows the incompatibility between strategy-proofness and strong ex-ante stability. Its proof is given Appendix B.

Remark 2 *When $|I| \geq 3$, there is no strategy-proof and strongly ex-ante stable mechanism.*

However, in sufficiently large markets, non-strategy-proof mechanisms of small markets can turn out to be strategy-proof (cf. Kojima and Manea, 2010; Azevedo and Budish, 2012). Indeed, in a large market with diverse preference types of students, FDA is strategy-proof. We prove this result in the following subsection. FDAT is not readily known to be strategy-proof or not in a similar large market formulation or under further restrictions.

7.1 Incentives under FDA in a Large Market

A *continuum school-choice problem* is denoted by a six-tuple $[I, T, \tau, C, q, P, \succsim]$ where: I is a Lebesgue-measurable continuum set of *students* each of whom is seeking a seat at a school; T is a finite set of *priority types* of students; $\tau : I \to T$ is a type specification function for students; C is a finite set of *schools;* $q = (q_c)_{c \in C}$ is a *quota vector* of schools such that $q_c \in \mathbb{Z}_{++}$ is the maximum Lebesgue measure of students who can be assigned to school c; $P = (P_i)_{i \in I}$ is a *strict preference profile* for students; and $\succsim = (\succsim_c)_{c \in C}$ is a *weak priority structure for schools* over T. Let $|J|$ denote the Lebesque measure of student subset $J \subseteq I$. Let $|I| > 0$.[23] We assume that there is enough quota for all students, that is $\sum_{c \in C} q_c = |I|$. We also assume that if $t \in \tau(I)$ then $\tau^{-1}(t)$ has a positive Lebesgue measure, $|\tau^{-1}(t)| > 0$. In particular, for all possible preference relations P_j over schools, there exists a positive Lebesgue measure of students in $\tau^{-1}(t)$ with the same preference relation P_j, that is: $|\{i \in \tau^{-1}(t) \ : \ P_i = P_j\}| > 0$ for all $t \in \tau(I)$ and preference relation P_j.[24] The ordinally Pareto-dominant strongly ex-ante stable random matching still exists in this framework for each problem.[25] Let FDA be defined through the mechanism that picks this random matching. In this framework, we can state the following result:

[23] See Che and Kojima (2010) for a foundational exercise in modeling continuum matching problem.

[24] Note that we do not assume that all prioritizations of different preference types are possible in a given problem. The possible prioritizations are given through the fixed priority profile $\succsim$.

[25] Establishing this fact requires a little more formal work, but we skip it for brevity and refer our reader to the corresponding result in a finite problem.

Theorem 7 *In continuum school-choice problems as specified above, FDA is strategy-proof; that is, for any student it is a weakly ordinally dominant strategy to reveal his true preferences.*

Proof. Suppose a student i of type t, instead of revealing his true preference P_i, reveals some other student's preference P_j where j is also of type t (for every manipulation of student i, such a student j exists by the assumptions). Now, the outcome of the FDA mechanism is the same under both problems, with truthful revelation of i and with i pretending to be a student identical to$j$. This is true as the set of strongly ex-ante stable random matchings will not change without changing the measures of types of agents existing in the problem. Suppose ρ is this outcome. All we need to show is that ρ_i ordinally dominates ρ_j under P_i or $\rho_i = \rho_j$ (suppose we denote this relationship by $\rho_i \succeq_i \rho_j$). We assume that $a_1 P_i a_2 P_i .... P_i a_n$ denotes the preference relation of i. There exists some a_k with $k \geq 1$ such that $\rho_{i,a_k} > 0$. Suppose a_k is the lowest ranked school in P_i with this property. Then by elimination of ex-ante discrimination of equal priority students i and j under ρ, we have $\rho_{i,a_\ell} \geq \rho_{j,a_\ell}$ for all $\ell < k$. Therefore for all $\ell < k$, $\sum_{m=1}^{\ell} \rho_{i,a_m} \geq \sum_{m=1}^{\ell} \rho_{j,a_m}$. Moreover, we have $\sum_{m=1}^{k} \rho_{i,a_m} = 1$. Hence, $\sum_{m=1}^{\ell} \rho_{i,a_m} \geq \sum_{m=1}^{\ell} \rho_{j,a_m}$ for all ℓ, showing that $\rho_i \succeq_i \rho_j$. ■[26]

8 Concluding Comments

In this paper, we have established a framework that generalizes one-to-many two-sided and one-sided matching problems. Such a framework enables the mechanism designer to achieve strong and appealing ex-ante efficiency properties when students are endowed with ordinal preferences as exemplified in the pioneering work of Bogomolnaia and Moulin (2001). On the other hand, fairness considerations play a crucial role in the design of practical school-choice mechanisms, since school districts are vulnerable to possible legal action resulting from a violation of student priorities. We have formulated two natural and intuitive ex-ante fairness notions called strong ex-ante stability and ex-ante stability and have shown that they are violated by prominent school-choice mechanisms such as the current NYC/Boston mechanism. We have proposed two mechanisms that stand out as attractive members of their corresponding classes.

The research on school-choice lotteries is a relatively new area in market design theory and there are many remaining open questions. One important question is about the characterization of ex-post stability when matchings are allowed to be random. Similarly to the results we have established for strong ex-ante stability (Theorem 3) and ex-ante stability (Theorem 5), a characterization of constrained ordinal efficient and ex-post stable random matchings currently remains an important

[26]It can be shown that there is no strategy-proof and constrained-efficient ex-ante stable mechanism that also satisfies equal treatment of equals in the continuum school-choice problems. Hence, FDAT is also not strategy-proof in such a model.

future issue. One other important area of future investigation is the analysis of strategic properties of FDA in a large market.

A Appendix: How does the FDA algorithm work when there is a rejection cycle?

Example 7 ***How does the finite FDA algorithm work?*** Assume there are four students $\{1, 2, 3, 4\}$ and four schools, $\{a, b, c, d\}$ each with a quota of one. The priorities and preferences are given as follows:

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	1, 2	2	⋮
2	3, 4	1, 3	
⋮		4	

P_1	P_2	P_3	P_4
c	a	b	b
b	c	c	a
d	b	d	⋮
a	d	a	⋮

Students propose according to the order $1, 2, 3, 4$:

Step 1: Student 1 applies to school c and is tentatively admitted.

Step 2: Student 2 applies to school a and is tentatively admitted.

Step 3: Student 3 applies to school b and is tentatively admitted.

Step 4: Student 4 applies to school b. The applicants of school b are students 3 and 4 (who have equal priority). Since applications exceed the quota, $\frac{1}{2}$ of each of 3 and 4 are rejected by b, while $\frac{1}{2}$ of each of 3 and 4 are tentatively admitted.

Step 5: Student 3 has an outstanding fraction of $\frac{1}{2}$ and applies to his next best school, c. The applicants of c are student 1 with whole fraction and student 3 with fraction $\frac{1}{2}$. Each has equal priority at school c whose quota has been exceeded. Thus $\frac{1}{2}$ of each of students 1 and 3 are tentatively admitted at school c, while $\frac{1}{2}$ of student 1 is rejected. Since 1 is partially rejected in favor of 3 by c, we have $3 \hookrightarrow_c 1$.

Step 6: Student 1 has an outstanding fraction of $\frac{1}{2}$, and applies to his next best school, b. School b has three applicants, $\frac{1}{2}$ of 3, $\frac{1}{2}$ of 4, and $\frac{1}{2}$ of 1. Since the quota of the school is exceeded and student 1 has the highest priority among the three applicants, $\frac{1}{2}$ of 1 is tentatively admitted, while $\frac{1}{4}$ of each of 3 and 4 are tentatively admitted, and $\frac{1}{4}$ of each of 3 and 4 are rejected. We have $1 \hookrightarrow_b 4$, and $1 \hookrightarrow_b 3$, hence there is a rejection cycle $(3, c, 1, a)$. The resolution of this cycle is trivial, since once 3 applies to school c again with his outstanding fraction $\frac{1}{4}$, all of this is rejected by c since both 1 and 3 have equal priority at c and they already have $\frac{1}{2}$ fraction each at c. Thus, it is no longer true that $3 \hookrightarrow_c 1$, and the cycle is resolved.

Step 7: Student 3 has an outstanding fraction of $\frac{1}{4}$, and applies to his next best school, d. This is tentatively accepted by d.

Step 8: Student 4 has an outstanding fraction of $\frac{3}{4}$, and applies to his next best school, a. School a has two applicants, whole fraction of 2 and $\frac{3}{4}$ of 4. Since the quota of a is only one, and student 4 has higher priority than 2 at a, then $\frac{3}{4}$ of 4 and $\frac{1}{4}$ of 2 are tentatively admitted to a, while $\frac{3}{4}$ of 2 is rejected. We have $4 \hookrightarrow_a 2$.

Step 9: Student 2 has an outstanding fraction of $\frac{3}{4}$ and applies to his next best school, c. School c has three applicants, 1 with fraction $\frac{1}{2}$, 3 with fraction $\frac{1}{2}$, and 2 with fraction $\frac{3}{4}$. Since the quota of c, which is one, has been exceeded, and 2 has higher priority than each of 1 and 3 who have equal priority, $\frac{3}{4}$ of 2, $\frac{1}{8}$ of each 1 and 3 are tentatively admitted to c, while $\frac{3}{8}$ of each of 1 and 3 are rejected. We have $2 \hookrightarrow_c 1$ and $2 \hookrightarrow_c 3$. The former relation induces a new cycle $(1, b, 4, a, 2, c)$. This cycle is not trivial. We use a simple system of equations to resolve this cycle with unknowns y_1, y_4, y_2 as the eventual limit rejected fractions from c, b, a and tentatively admitted fractions to b, a, c of students of 1, 4, 2, respectively:

$$\begin{aligned} y_1 + \omega_1 &= \max\{y_4, 0\} + \max\left\{\phi_{3,b} - (\phi_{4,b} - y_4), 0\right\}, \\ y_4 &= \max\{y_2, 0\}, \\ y_2 &= \max\left\{y_1, 0\right\} + \max\left\{\phi_{3,c} - (\phi_{1,c} - y_1), 0\right\}; \end{aligned}$$

where $\omega_1 = \frac{3}{8}$ is the fraction of 1 that will be tentatively admitted to b when the cycle is initiated; $\phi_{4,b} = \phi_{3,b} = \frac{1}{4}$ are the fractions of 4 and 3 currently tentatively admitted to b when the cycle is initiated; and $\phi_{3,c} = \phi_{1,c} = \frac{1}{8}$ are the fractions of 3 and 1 currently tentatively admitted to c. Observe that these unknowns can be solved through the linear system,

$$y_1 + \frac{3}{8} = 2y_4, \;\; y_4 = y_2, \;\; y_2 = 2y_1.$$

By solving them we obtain

$$y_4 = \frac{1}{4}, \;\; y_2 = \frac{1}{4}, \;\; y_1 = \frac{1}{8}.$$

As these rejected fractions are all less than or equal to the initially admitted fractions of 4,2,1 to b, a, c respectively indeed it is possible to resolve this cycle with these fractions.[27] At this point, the tentative random matchings of students are: a whole fraction of 1 at b, a whole fraction of 4 at a, a whole fraction of 2 at c. From the previous step we also have $\frac{1}{4}$ of 3 tentatively admitted at d.

[27]If we had a situation such that $y_{i_\ell} > \phi_{i_\ell, c_{\ell-1}}$ for some i_ℓ in the cycle where $\phi_{i_\ell, c_{\ell-1}}$ is the initially tentatively admitted fraction of i_ℓ at $c_{\ell-1}$, from which he is being rejected by repeated applications of $i_{\ell-1}$, then we would set $y_{i_\ell} = \phi_{i_\ell, c_{\ell-1}}$ and solve the other equations. See the proof of Proposition 3 in Appendix B for generalization of this method.

Step 10: Student 3 has an outstanding fraction of $\frac{3}{4}$, with which he applies to his best school that has not rejected him yet, d. Now, the whole fraction of 3 is applying to d, which tentatively admits him.

There are no outstanding student fractions left. The algorithm terminates with the outcome

	a	**b**	**c**	**d**
1	0	1	0	0
2	0	0	1	0
3	0	0	0	1
4	1	0	0	0

.

$\Diamond$

B Appendix: Proofs of the Results Regarding the FDA Mechanism

Proof of Proposition 3. First, we prove that a rejection cycle can be resolved in finite time. Suppose a rejection cycle occurs at a step when i_1 applies to c_1 with fraction ω_1 after this fraction is rejected from c_m:

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1,$$

At this point for each s, let ϕ_{i,c_s} be the fraction of student i tentatively assigned to school c_s. Note that among the students $i_1, ..., i_m$ only i_1 has a positive fraction, ω_1, that is tentatively unassigned at this time. Let ϕ_{i,c_s} be the fraction of student $i \in I$ tentatively held at each c_s at this point. We will place ω_1 in c_1 if we can. If not, i_1 will be rejected by c_1 and cycle will be resolved. Suppose ω_1 can be tentatively placed in c_1.

Let y_{s+1} be the rejected fraction of i_{s+1} from c_s when the cycle is resolved. This fraction will be tentatively held by school c_{s+1} at this stage. Observe that all rejected fractions from c_s belong to the students that are at the same priority level with i_{s+1}. We need to make sure that each student i at the same priority level as i_{s+1} is held at the same fraction $\phi_{i_{s+1},c_s} - y_{s+1}$ at c_s, unless i did not have that much fraction to start with. The sum of all tentatively accepted fractions at the resolution of the cycle will be q_{c_s}.

A simple way to solve these equations is as follows: Let $M_s = \{i \sim_{c_s} i_{s+1} | \phi_{i,c_s} > 0\}$ for all s. Observe that if we did have sufficient fractions already held at the school c_s of i_{s+1}, then we can iteratively solve for y_s as

$$y_s = \sum_{i \in M_s} \max\{\phi_{i,c_s} - (\phi_{i_{s+1},c_s} - y_{s+1}), 0\} \ \ \forall s \in \{2, ..., m\}, \tag{1}$$

as when y_s of i_s is admitted at c_s it causes the fraction $[\phi_{i,c_s} - (\phi_{i_{s+1},c_s} - y_{s+1})]$ to be rejected for each student $i \in M_s$, whenever this fraction is greater than 0 to start with (where $m+1=1$ in modulo m). For c_1 we have

$$y_1 + \omega_1 = \sum_{i \in M_1} \max\{\phi_{i,c_1} - (\phi_{i_2,c_1} - y_2), 0\}, \tag{2}$$

as $y_1 + \omega_1$ is the total admitted fraction of i_1 at school c_1. Then we can solve these m equations in m unknowns using a number of linear equation systems.

At the determined $\{y_t\}$ vector satisfying Equations (1) and (2), a student i_{s+1} may have $y_{s+1} > \phi_{i_{s+1},c_s}$, i.e., we cannot reject y_{s+1} fraction of i_{s+1} from c_s. Then, we try setting $y_{s+1} = \phi_{i_{s+1}}$ and otherwise the equations for the other $y_t \neq y_{s+1}$ is given as in Equations (1) and (2). We can similarly solve this system. Each y_t decreases, as y_{s+1} decreased and y_s and y_{s+2} are positively correlated with y_{s+1} and so on so forth for all the other y_t's. If we still have at the new vector $\{y_t\}$ a student i_{u+1} such that $y_{u+1} > \phi_{i_{u+1},c_u}$, we set $y_{u+1} = \phi_{u+1,c_u}$ and all other $y_t \neq y_{u+1}$ are given as in Equations (1) and (2). We solve the new system. As $\{y_t\}$ decreases again, we have $y_{s+1} < \phi_{i_{s+1},c_s}$ and hence the problem for the first student i_{s+1} is resolved. We continue iteratively as above until for all student i_{t+1}, $y_{t+1} \leq \phi_{i_{t+1},c_t}$. We are done.

If a cycle does not occur, similarly the step of the algorithm can be resolved easily.

Observe that in each step of the FDA algorithm, students get weakly worse off, since they only make proposals to a school that has not rejected a fraction of themselves. After all $|I|$ students make offers, at least one student gets rejected by one school and has an outstanding fraction, or the algorithm converges, whether or not a cycle occurs. Since there are $|C|$ schools, the algorithm converges in at most $|I|\,|C|$ steps. ■

Proof of Theorem 3. We argue by contradiction. Suppose this is not true for some school-choice problem. Fix a problem $[P, \succsim]$. Let $\pi \in \mathcal{X}$ be the FDA algorithm's outcome random matching for some order of students making offers, and $\rho \in \mathcal{X}$ be a strongly ex-ante stable random assignment that is not stochastically dominated by π. This means that

$$\begin{gathered}\text{there exist } i_0 \in I \text{ and } a_0 \in C \text{ such that } 0 \neq \rho_{i_0,a_0} > \pi_{i_0,a_0} \\ \text{where } a_0 P_{i_0} e_0 \text{ for some } e_0 \in C \text{ with } 0 \neq \pi_{i_0,e_0} > \rho_{i_0,e_0}.\end{gathered} \tag{3}$$

We will construct a finite sequence of student-school pairs as follows:

Construction of a trading cycle from π to ρ: Statement 3 implies that there exists $i_1 \in I \backslash \{i_0\}$ such that $\rho_{i_1,a_0} < \pi_{i_1,a_0} \neq 0$. Then strong ex-ante stability of the FDA outcomes implies that $i_1 \succsim_{a_0} i_0$, for otherwise π would have induced ex-ante school-wise justifiable envy of i_1 toward i_0 for a_0 (in case $i_1 \succ_{a_0} i_0$) or π would have ex-ante discriminated i_0 and i_1 at a_0 (in case $i_1 \sim_{a_0} i_0$). Then, since $\rho_{i_1,a_0} < \pi_{i_1,a_0}$, $\rho_{i_0,a_0} > \pi_{i_0,a_0}$, and ρ is strongly ex-ante stable, in order for ρ not to have ex-ante school-wise justified envy of i_1 toward i_0 for a_0 (in case $i_1 \succ_{a_0} i_0$) and ρ not to have ex-ante school-

wise discrimination between i_0 and i_1 for a_0 (in case $i_1 \sim_{a_0} i_0$), there must exist some $a_1 \in C \backslash \{a_0\}$ such that $0 \neq \rho_{i_1,a_1} > \pi_{i_1,a_1}$ where $a_1 P_{i_1} a_0$. (More precisely, there are two cases:

Case (1) $i_1 \succ_{a_0} i_0$: Suppose by contradiction that for all $b \in C$ with $bP_{i_1}a_0$, we have $\rho_{i_1,b} \leq \pi_{i_1,b}$. Then by feasibility there is $c \in C$ with $a_0 P_{i_1} c$ and $\rho_{i_1,c} > \pi_{i_1,c}$. But then i_1 would ex-ante school-wise justifiably envy i_0 for a_1 at ρ, contradicting ρ is strongly ex-ante stable.

Case (2) $i_1 \sim_{a_0} i_0$: Since $\pi_{i_0,e_0} \neq 0$ and $a_0 P_{i_0} e_0$ (by Statement 3 above), we have $0 \neq \pi_{i_1,a_0} \leq \pi_{i_0,a_0} \neq 0$. Thus, $\rho_{i_1,a_0} < \pi_{i_1,a_0}$ and $\rho_{i_0,a_0} > \pi_{i_0,a_0}$ imply that $\rho_{i_1,a_0} < \rho_{i_0,a_0}$. Then, no ex-ante school-wise discrimination at ρ between i_0 and i_1 for a_0 implies that there is no $d \in C$ where $a_0 P_{i_1} d$ with $\rho_{i_1,d} \neq 0$. Then such an a_1 should exist for i_1.)

Observe that i_1 satisfies the same Statement 3 above as i_0 does using a_1 instead of a_0, and a_0 instead of e_0, and $i_1 \succsim_{a_0} i_0$, i.e.,

$$\rho_{i_1,a_1} > \pi_{i_1,a_1} \ , \ a_1 P_{i_1} a_0 \text{ with } \rho_{i_1,a_0} < \pi_{i_1,a_0} \neq 0 \text{ and } i_1 \succsim_{a_0} i_0.$$

Thus, as we continue iteratively we obtain a finite sequence of students and schools such that each pair (a_{s-1}, i_s) (subscripts are modulo $n+1$, so that $n+1 \equiv 0$) appears only once in the sequence,

$$e_0, i_0, a_0, i_1, a_1, ..., i_n, a_n$$

and each i_s satisfies Condition 3 replacing i_s with i_0, a_s with a_1 and a_{s-1} with e_0, and additionally satisfying $i_s \succsim_{a_{s-1}} i_{s-1}$, i.e.,

$$\rho_{i_s,a_s} > \pi_{i_s,a_s} \ , \ a_s P_{i_s} a_{s-1} \text{ with } \rho_{i_s,a_{s-1}} < \pi_{i_s,a_{s-1}} \neq 0 \text{ and } i_s \succsim_{a_{s-1}} i_{s-1} \tag{4}$$

and finally, by finiteness of schools and students, we have

$$a_n \equiv e_0 \text{ and yet } i_n \neq i_0,$$

where e_0 can be chosen as defined in Condition 3. This sequence describes a special probability trading cycle from π to ρ for some better schools, so that ρ cannot be ordinally dominated by π. $\Diamond$

Observe that there can be many such cycles, some of them overlapping. And each such cycle has at least two agents and two schools. Suppose there are m^* such cycles $Cyc^1, ...Cyc^m, ..., Cyc^{m^*}$ and let $I^1, I^2, ..., I^m, ..., I^{m^*}$ be the sets of students and $C^1, C^2, ..., C^m, ..., C^{m^*}$ be the corresponding sets of schools in these cycles, respectively. Let I^* be the union of all above student sets and C^* be the union of all above school sets. We will prove some claims that will facilitate the proof of the theorem:

Claim 1: Take a cycle $Cyc^m = (i_0, a_0, ..., i_n, a_n)$. There is no $a_s \in C^m$ and no $b \in C$ such that for student i_{s+1}, we have $a_s P_{i_{s+1}} b$ and $\rho_{i_{s+1},b} \neq 0$.

Proof of Claim 1: Suppose, on the contrary, there are $a_s \in C^m$ and $b \in C$ such that $a_s P_{i_{s+1}} b$ and $\rho_{i_{s+1},b} \neq 0$. We also have, $\rho_{i_{s+1},a_s} < \pi_{i_{s+1},a_s} \neq 0$ by construction of cycle Cyc^m (see Statement 4 above). We also have by construction, $0 \neq \rho_{i_s,a_s} > \pi_{i_s,a_s}$, $a_s P_{i_s} a_{s-1}$, $\rho_{i_s,a_{s-1}} < \pi_{i_s,a_{s-1}} \neq 0$, and, finally $i_{s+1} \succsim_{a_s} i_s$ (see Statement 4 above). Consider two cases:

Case (1') $i_{s+1} \succ_{a_s} i_s$: Since $\rho_{i_s,a_s} \neq 0$, $\rho_{i_{s+1},b} \neq 0$ and $a_s P_{i_{s+1}} b$, student i_{s+1} ex-ante justifiably envies i_s for a_s at ρ, contradicting ρ is strongly ex-ante stable.

Case (2'): $i_{s+1} \sim_{a_s} i_s$: By the strong ex-ante stability of π, there is no ex-ante school-wise discrimination between i_{s+1} and i_s for a_s at π. Since $a_s P_{i_s} a_{s-1}$ and $\pi_{i_s,a_{s-1}} \neq 0$, we must have $\pi_{i_s,a_s} \geq \pi_{i_{s+1},a_s}$. Then, we have $\rho_{i_s,a_s} > \pi_{i_s,a_s} \geq \pi_{i_{s+1},a_s} > \rho_{i_{s+1},a_s}$. Recall that $\rho_{i_{s+1},b} \neq 0$ for $a_s P_{i_{s+1}} b$. The last two statements imply that ρ ex-ante discriminates between i_s and i_{s+1} at a_s, contradicting that ρ is strongly ex-ante stable. $\Diamond$

Consider the sequence of offers and rejections in the FDA algorithm that leads to π. Let $i_s \in I^m$ for a cycle Cyc^m (without loss of generality, let $(i_0, a_0, i_1, a_1, ..., i_n, a_n)$ be this cycle) be the *last* student in I^* to apply and get a positive fraction under π from the next school in his cycle (i.e., for i_s, this school is $a_s \in C^m$). Let t be this step of the algorithm. We prove the following claim:

Claim 2: The total sum of student fractions that school a_{s-1} has tentatively accepted until the beginning of step t of the FDA algorithm is equal to its quota, i.e., school a_{s-1} is filled at the beginning of step t.

Proof of Claim 2: Consider agent i_{s-1}. We have $\pi_{i_{s-1},a_{s-2}} > 0$ by construction of Cyc^m. By the choice of student i_s, student i_{s-1} should have applied to school a_{s-2} at some step p $<$ t. We also have $a_{s-1} P_{i_{s-1}} a_{s-2}$ by construction of Cyc^m. Then, in the FDA algorithm, i_{s-1} should have applied to a_{s-1} first at some step r $<$ p. This is true as he can apply to a_{s-2} in the algorithm only after having been rejected by school a_{s-1}. A school can reject a student only if it has tentatively accepted student fractions summing up to its quota. Since a_{s-1} remains to be filled after it becomes filled in the algorithm, the claim follows. $\Diamond$

Thus, by Claim 2, a_{s-1} is full at the beginning of step t just before student i_s applies. Then, there exists some student $j \in I$ with $i_s \succsim_{a_s} j$ such that some fraction of j was tentatively accepted by school a_s before step t and some fraction of j gets kicked out of school a_s at the end of step t (so that by the choice of i_s, some fraction of his gets in a_{s-1}). Since the FDA algorithm converges to a well-defined random matching, there is some $b \in C$ such that $a_{s-1} P_j b$ and $\pi_{j,b} \neq 0$. We prove the following claim:

Claim 3: We have $j \notin I^*$.

Proof of Claim 3: Suppose not, i.e., j is in some cycle. By the choice of student i_s, the ordered four-tuple $b(=a_{s-2}), j\,(=i_{s-1}), a_{s-1}, i_s$ cannot be part of Cyc^m, i.e., j cannot be accepted by b after being rejected by a_{s-1} in the FDA algorithm and yet $\rho_{j,b} < \pi_{j,b}$ (i.e., see Statement 4 for the construction of a cycle). But then by the choice of school b, $\rho_{j,b} \geq \pi_{j,b} \neq 0$. However, Claim 1 applied for school a_{s-1} and student $j(=i_{s-1})$ and the fact that $a_{s-1}P_jb$ together imply that $\rho_{j,b} = 0$, contradicting the previous statement. Thus, $j \notin I^*$. $\Diamond$

We are ready to finish the proof of the theorem. Since school a_{s-1} is full at the beginning of step t (by Claim 2), there is student $i_{s-1} \in I^m \setminus \{i_s\}$, i.e., preceding a_{s-1} in Cyc^m, with $0 \neq \rho_{i_{s-1},a_{s-1}} > \pi_{i_{s-1},a_{s-1}}$ who applied to school $a_{s-2} \prec_{i_{s-1}} a_{s-1}$ after being rejected by school a_{s-1}. Moreover, by the choice of i_s, student i_{s-1} applies to a_{s-2} before step t (for the last time), and hence, he got rejected by a_{s-1} before step t. Moreover, $i_{s-1} \neq j$ (by Claim 3). Thus, $j \succsim_{a_{s-1}} i_{s-1}$. We will establish a contradiction, and complete the proof of the theorem. Two cases are possible:

Case (1"'): $j \succ_{a_{s-1}} i_{s-1}$: Since $\rho_{j,b} \neq 0$, strong ex-ante stability of ρ implies that $\rho_{i_{s-1},a_{s-1}} = 0$ leading to a contradiction to the fact that $0 \neq \rho_{i_{s-1},a_{s-1}}$.

Case (2"'): $j \sim_{a_{s-1}} i_{s-1}$: Recall again that $\pi_{i_{s-1},a_{s-2}} > 0$ and $a_{s-1}P_{i_{s-1}}a_{s-2}$, $\pi_{j,b} > 0$, and $a_{s-1}P_jb$. But then, i_{s-1} gets rejected by a_{s-1} at the FDA algorithm at the same step as j gets rejected with some fraction, which is step t (since ρ does not ex-ante discriminate j and i_{s-1} at a_{s-1}, they should have equal fractions at a_{s-1} prior to step t), and thus i_{s-1} applies to school a_{s-2} after step t, contradicting the choice of student i_s. ■

Proof of Remark 2. Let φ be a strongly ex-ante stable mechanism. Consider the following problem with three students $1, 2, 3$, and three schools a, b, c, each with quota one:

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

$\succsim_a$	$\succsim_b$	$\succsim_c$
3	1	$\vdots$
1, 2	$\vdots$	

There is a unique strongly ex-ante stable random matching that is given as follows:

$$\rho = \begin{array}{|c|c|c|c|} \hline & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{1} & 0 & 1 & 0 \\ \hline \mathbf{2} & 0 & 0 & 1 \\ \hline \mathbf{3} & 1 & 0 & 0 \\ \hline \end{array}$$

Thus, $\varphi[P, \succsim] = \rho$.

However, if student 1 submits the preferences $P_1' = (acb)$ instead of P_i, then the unique strongly ex-ante stable random matching will be

$$\rho' = \begin{array}{|c|c|c|c|} \hline & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{1} & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline \mathbf{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline \mathbf{3} & 0 & 1 & 0 \\ \hline \end{array}$$

Hence, $\varphi[(P_1', P_{-1}), \succsim] = \rho'$. Observe that there can be von Neumann - Morgenstern utility functions of Student 1 that may make ρ_1' more desirable than ρ_1.

If $|I| > 3$, we can make the example hold by embedding it in a problem with $I = \{1, 2, ..., |I|\}$ and $C = \{a, b, c, d_4, ..., d_{|I|}\}$ where each student $i \in \{4, ..., |I|\}$ is ranking school d_i as his first choice, and each student $i \in \{1, 2, 3\}$ is ranking each school d_i lower than schools a, b, c. Under any ex-ante strongly stable matching, each $i \in \{4, 5, ..., |I|\}$ will be matched with d_i and $\{1, 2, 3\}$ will be mapped with $\{a, b, c\}$. ■

C Appendix: Proof of Proposition 5

Proof of Proposition 5. "Only if" part: Let ρ be an ex-ante stable random matching with an ex-ante stable improvement cycle $Cyc = (i_1, a_1, ..., i_m, a_m)$. Let $i_{m+1} \equiv i_1$ and $a_{m+1} \equiv a_1$. Let π be the random matching obtained by satisfying this cycle with some feasible fraction. Then π ordinally dominates ρ. Since each student i_s envies student i_{s+1} for a_{s+1} due to a_s at ρ, and i_s is a highest a_l-priority student ex-ante school-wise–envying a student with a positive probability at school a_{s+1}. Thus, either (1) i_s is at the same priority level with i_{s+1} for a_{s+1}, or (2) i_s is at a lower-priority level than i_{s+1} for a_{s+1} but any $i \sim_{a_{s+1}} i_{s+1}$ does not ex-ante school-wise envy himself or i_{s+1} for a_{s+1} at ρ, that is: i is not assigned with a positive probability to a worse school than a_{s+1} at ρ. Thus, when we satisfy the cycle Cyc, there will be no ex-ante school-wise justified envy toward a lower-priority student and π is ex-ante stable.

"If" part: Let ρ be an ex-ante stable random matching. Let $\pi \neq \rho$ be an ex-ante stable random matching that ordinally dominates ρ. We will construct a particular ex-ante stable improvement cycle at ρ.

Let $I' = \{i \in I : \rho_i \neq \pi_i\}$. Clearly, $I' \neq \emptyset$. Note that for all $i' \in I'$, $\pi_{i'}$ stochastically dominates $\rho_{i'}$. Thus, whenever $\pi_{i',a} > \rho_{i',a}$ for some $i' \in I'$ and $a \in C$, then there is $j' \in I'$ with $\pi_{j',a} < \rho_{j',a}$; moreover, since $\pi_{j'}$ stochastically dominates $\rho_{j'}$, there is $b \in C$ with $bP_{j'}a$ and $\pi_{j',b} > \rho_{j',b}$. Let $C' = \{c \in C : \pi_{i,c} > \rho_{i,c}$ for some $i \in I'\}$. Clearly, $C' \neq \emptyset$.

Consider the following directed graph: Each pair student-school pair $(i, c) \in I' \times C'$ with $\rho_{i,c} \neq 0$ is represented by a node. Fix a school $c \in C'$. Let each student-school pair (i, c) in this graph containing

school c be pointed to by every student-school pair containing a student school-wise–envying student i for school c and has the highest priority among such school-wise–envying students in I'. We repeat this for each $c \in C'$.

Note that no student-school pair in the resulting graph points to itself, and each student-school pair in this graph is pointed to by at least one other student-school pair. Moreover, each student-school pair (i, c) in this graph can only be pointed to by a student-school pair that contains a different school than c. Then there is at least one cycle of student-school pairs $Cyc = (i_1, a_1, i_2, a_2, ..., i_m, a_m)$ with $(i_m, a_m) \equiv (i_0, a_0)$ and $m \geq 2$. By construction, we have $(i_s, a_s) \trianglerighteq^{\rho} (i_{s+1}, a_{s+1})$ for $s = 0, \ldots, m-1$. Note also that cycle Cyc contains at least two distinct students. Then cycle Cyc is a stochastic improvement cycle.

Now consider school a_{s+1} of the pair (i_{s+1}, a_{s+1}) in cycle Cyc. Suppose, for a contradiction, that student i_s does not ex-ante top-priority school-wise envy i_{s+1} for a_{s+1} due to a_s. Then there is a student-school pair (j, d) with $j \notin I'$, which is not represented in our graph, such that $(j, d) \blacktriangleright^{\rho} (i_{s+1}, a_{s+1})$. In particular, $j \succ_{a_{s+1}} i$ for any $i \in I'$ such that $(i, d) \trianglerighteq^{\rho} (i_{s+1}, a_{s+1})$ for any $d \in C'$. Let $k \in I'$ such that $\pi_{k,a_{s+1}} > \rho_{k,a_{s+1}}$. Since $j \succ_{a_{s+1}} k$ and $\rho_{j,d} = \pi_{j,d}$, student j justifiably ex-ante envies k at π. This contradicts the ex-ante stability of π. ■

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D Supplementary Appendix : The EASC Algorithm

The description of the algorithm mostly follows the constrained consumption algorithm of Athanassoglou and Sethuraman (2011) with a few modifications:

Given an ex-ante stable matching ρ, we first define the following set:

$$\mathcal{A}(\rho) = \{(i,c) \in I \times C : \rho_{i,c} > 0 \text{ or } (i,a) \blacktriangleright^{\rho} (j,c) \text{ for some } j \in I \text{ and } a \in C\}.$$

Given an initial ex-ante stable random matching ρ, our adaptation of the constrained consumption algorithm finds a random matching π such that (1) π ordinally dominates or is equal to ρ and (2) $(i,c) \notin \mathcal{A}(\rho) \Longrightarrow \pi_{i,c} = 0$.

It is executed through a series of *flow networks*, each of which is a directed graph from an artificial *source node* to an artificial *sink node*, denoted as σ and τ, respectively. We will carry the assignment probabilities from source to sink over this flow network, so that the eventual flow will always determine a feasible random matching. The initial network is constructed as follows:

The *nodes* of the network are (1) source σ and sink τ; (2) each school $c \in C$; and (3) for each $i \in I$ and $\ell \in \{1, ..., |C|\}$, $i_{(\ell)} \in I \times \{1, ..., |C|\}$; i.e., the ℓ^{th} node of student i is a node, where this node corresponds to the ℓ^{th} choice of student i among the schools.

Let $N = I \times \{1, ..., |C|\} \cup C \cup \{\sigma, \tau\}$ be the set of nodes of the network.

An *arc* from node x to node y is represented as $x \to y$. Let $\omega_{x \to y}$ be the *capacity* of arc $x \to y$.[28] The arcs have the following load capacities:

(1) Each arc $\sigma \to i_{(\ell)}$ has the capacity $\rho_{i,c}$, where school c is the ℓ^{th} choice of student i.

(2) Each arc $i_{(\ell)} \to c$ has the capacity ∞, if $(i,c) \in \mathcal{A}(\rho)$, and c is ranked ℓ^{th} or better at the student i's preferences; and 0, otherwise.

(3) Each arc $c \to \tau$ has the capacity q_c, the quota of school c.

(4) Any arc between any other two nodes has capacity zero.

Thus, the arcs with positive load capacities are directed from the source σ to the student nodes, from the student nodes to feasible school nodes with respect to $\mathcal{A}(\rho)$, and from school nodes to the sink τ.

Let $\Gamma = \langle N, \omega \rangle$ denote this network. We define additional concepts for such a network.

A *cut* of the network is a subset of nodes $K \subseteq N$ such that $\sigma \in K$ and $\tau \in N \backslash K$. The *capacity* of a cut K is the sum of the capacities of the arcs that are directed from nodes in K to nodes in $N \backslash K$,

[28] Without loss of generality, we focus on rational numbers as load capacities.

and it is denoted as $\Omega(K)$, that is: $\Omega(K) = \sum_{x\in K, y\in N\setminus K} \omega_{x\to y}$. A *minimum cut* K^* is a minimum capacity cut, i.e., $K^* \in \arg\min_{\{\sigma\}\subseteq K\subseteq N\setminus\{\tau\}} \Omega(K)$. A *flow* of the network is a list $\phi = (\phi_{x\to y})_{x,y\in N}$ such that (1) for each $x, y \in N$, $\phi_{x\to y} \leq \omega_{x\to y}$, i.e., the flow cannot exceed the capacity, and (2) for all $x \in N\setminus\{\sigma,\tau\}$, $\sum_{y\in N} \phi_{y\to x} = \sum_{y\in N} \phi_{x\to y}$, i.e., total incoming flow to a node should be equal to the total outgoing flow. Let Φ be the set of flows. The *value* of a flow ϕ is the total outgoing flow from the source, i.e., $\Omega(\phi) = \sum_{y\in N} \phi_{\sigma\to y}$. A *maximum flow* ϕ^* is a flow with the highest value, i.e., $\phi^* \in \arg\max_{\phi\in\Phi} \Omega(\phi)$. Observe that in our network Γ, the maximum flow value is equal to $|I|$.

The algorithm solves iterative *maximum flow-minimum cut problems*, a powerful tool in graph theory and linear programming. The corresponding duality theorem is stated as follows:

Theorem 8 ***(Ford and Fulkerson, 1956), (Maximum Flow-Minimum Cut Theorem)*** *The value of the maximum flow is equal to the capacity of a minimum cut.*

There are various polynomial-time algorithms, such as the *Edmonds and Karp (1972) algorithm*, which can determine a minimum cut and maximum flow.

The ex-ante stable consumption algorithm updates the network starting from Γ by updating the capacity of some of the source arcs $\omega_{\sigma\to i_{(\ell)}}$ over *time*, which is a continuous parameter $t \in [0,1]$. It starts from $t = 0$ and increases up to $t = 1$. Thus, let's relabel the source arc weights as a function of time t as $\omega^t_{\sigma\to i_{(\ell)}}$ by setting $\omega^0_{\sigma\to i_{(\ell)}} \equiv \omega_{\sigma\to i_{(\ell)}}$ for each arc $\sigma \to i_{(\ell)}$. No other arc capacity is updated. Let Γ^t be the corresponding flow network at time t.

There will also be iterative steps in the algorithm with start times $t^1 = 0 \leq t^2 \leq ... \leq t^{\mathrm{n}} \leq 1 = t^{\mathrm{n}+1}$, for steps 1,...,n, respectively. All assignment activity in step m occurs in the time interval $(t^{\mathrm{m}}, t^{\mathrm{m}+1}]$.

This algorithm is in the class of eating algorithms introduced by Bogomolnaia and Moulin (2001), and t also represents the assigned fraction of each student, since each student is assumed to be assigned at a uniform speed of 1. This activity is referred to as *eating a school*. Each school is assumed to be a perfectly divisible object with q_c copies.

We update the feasible assignment set $\mathcal{A}(\rho)$ in each step. Let $\mathcal{A}^{\mathrm{m}}(\rho)$ be the feasible student-school pairs at step m=1,...,n. We have $\mathcal{A}^1(\rho) = \mathcal{A}(\rho) \supseteq \mathcal{A}^2(\rho) \supseteq ... \supseteq \mathcal{A}^{\mathrm{n}}(\rho)$.

At each step m, let $b_i \in C$ be the *best feasible* school for student i, that is, $(i, b_i) \in \mathcal{A}^{\mathrm{m}}(\rho)$, and $b_i R_i c$ for all c with $(i, c) \in \mathcal{A}^{\mathrm{m}}(\rho)$. Also, let $e_i \in C$ be the *endowment* school of student i, that is, if $R_i(c)$ is the rank of school c for i, then $\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(e_i))}} > 0$ and $b_i P_i e_i R_i c$ for all c with $\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(c))}} > 0$. Observe that e_i may not exist for a student i, which case is denoted as $e_i = \emptyset$. In the algorithm we describe here, each student consumes the best school feasible for him at t while his endowment of a worse school decreases.

We are ready to state the algorithm, a slightly modified version of the Athanassoglou and Sethuraman (2011) algorithm:

The EASC Algorithm:

Suppose that until step $m \geq 1$, we determined t^m, $\left\{\omega^{t^m}_{\sigma\to i_{(\ell)}}\right\}_{i\in I,\ell\in\{1,....|C|\}}$, and $\mathcal{A}^m(\rho)$.

Step m: *We determine t^{m+1}, $\left\{\omega^{t}_{\sigma\to i_{(\ell)}}\right\}_{i\in I,\ell\in\{1,....|C|\}}$ for all $t\in(t^m,t^{m+1}]$, and $\mathcal{A}^{m+1}(\rho)$ as follows: Initially time satisfies $t=t^m$. Let $\{b_i,e_i\}_{i\in I}$ be determined given $\mathcal{A}^m(\rho)$ and $\left\{\omega^{t^m}_{\sigma\to i_{(\ell)}}\right\}$.*

Then t continuously increases. At t the arc capacities $\omega^{(t)}\left(\sigma\to i_{(\ell)}\right)$ are updated as follows for each $i\in I$ and $c\in C$:

$$\omega^{t}_{\sigma\to i_{(R_i(c))}} := \begin{cases} \max\left\{t-\sum_{\ell=1}^{R_i(b_i)-1}\omega^{t^m}_{\sigma\to i_{(\ell)}},\ \omega^{t^m}_{\sigma\to i_{(R_i(b_i))}}\right\} & \text{if } c=b_i \text{ and } e_i\neq\emptyset,\\ \min\left\{\sum_{\ell=1}^{R_i(b_i)}\omega^{t^m}_{\sigma\to i_{(\ell)}}+\omega^{t^m}_{\sigma\to i_{(R_i(e_i))}}-t,\ \omega^{t^m}_{\sigma\to i_{(R_i(e_i))}}\right\} & \text{if } c=e_i,\\ \omega^{t^m}_{\sigma\to i_{(R_i(c))}} & \text{otherwise.}\end{cases}$$

That is, each student i consumes his best feasible school b_i with uniform speed by trading away fractions from his endowment school e_i, if it exists and the consumption fraction of the best school exceeds his initial consumption of $\omega^{t^m}_{\sigma\to i_{(R_i(b_i))}}$.

Time t increases until one of the following two events occurs:

- *$t<1$, and yet*
  - *the endowment school fraction endowed to some student reaches to zero, i.e., $\omega^{(t)}_{\sigma\to e_i}=0$ for some $i\in I$: We update*

$$t^{m+1} := t,$$
$$\mathcal{A}^{m+1}(\rho) := \mathcal{A}^m(\rho);$$

  or

  - *any further increase in t will cause the maximum flow capacity in the network to fall, i.e., for $t^\varepsilon>t$ and arbitrarily close to t, the network $\Gamma^{(t^\varepsilon)}$ has a maximum flow capacity less than $|I|$ (which can be determined by an algorithm such as Edmonds-Karp): This means that if some student were to consume his best feasible school anymore, some ex-ante stability constraint will be violated. Let K be a minimum cut of Γ^{t^ε}. Any student i with $i_{(R_i(b_i))}\in K$ and $i_{(R_i(e_i))}\notin K$, i is one of such students. Thus, we update*

$$t^{m+1} := t,$$
$$\mathcal{A}^{m+1}(\rho) := \mathcal{A}^m(\rho)\setminus\left\{(i,b_i): e_i\neq\emptyset,\ i_{(R_i(b_i))}\in K, \text{ and } i_{(R_i(e_i))}\notin K\right\}.$$

We continue with Step m+1.

- $t = 1$*: The algorithm terminates. The outcome of the algorithm* $\pi \in \mathcal{X}$ *is found as follows: Let* ϕ *be a maximum flow of the network* Γ^1 *(i.e., the final network at time t=1). Then, we set for all* $i \in I$ *and* $c \in C$,

$$\pi_{i,c} := \sum_{\ell=1}^{|C|} \phi_{i_{(\ell)} \to c},$$

i.e., the total flow from student i *to school* c. $\Diamond$

Athanassoglou and Sethuraman (2011) proved that this algorithm with $\mathcal{A}(\rho) = I \times C$ (i.e., the case in which all schools are feasible to be assigned to each student) converges to a unique ordinally efficient random matching such that it treats equals equally whenever ρ treats equals equally; and it Pareto dominates, or is equal to ρ. Their statements can be generalized to the case in which $\mathcal{A}(\rho) \subseteq I \times C$ such that the outcome of the above algorithm π is constrained ordinally efficient in the class of random matchings $\chi \in \mathcal{X}$ satisfying $\chi_{i,c} > 0 \Rightarrow (i,c) \in I \times C$. Moreover, π is also ex-ante stable whenever ρ is ex-ante stable; it ordinally dominates, or is equal to ρ; and it treats equals equally whenever ρ treats equals equally. We skip these proofs for brevity.

E Supplementary Appendix: How is the EASC algorithm embedded in the FDAT algorithm?

Example 8 We illustrate the functioning of the FDAT algorithm with the EASC algorithm using the same problem in Example 3 (and Example 5):

Step 0. We found the FDA outcome in Example 3 as

$\rho^1 =$

	a	**b**	**c**	**d**
1	$\frac{11}{12}$	$\frac{1}{12}$	0	0
2	$\frac{2}{3}$	0	$\frac{1}{3}$	0
3	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$
4	0	0	1	0
5	0	0	$\frac{1}{3}$	$\frac{2}{3}$
6	0	$\frac{5}{6}$	0	$\frac{1}{6}$

Step 1. We form the feasible student-school pairs for matching as

$$\mathcal{A}^{1.1}\left(\rho^1\right) = \{(1,a),(1,b),(2,a),(2,c),(3,a),(3,b),(3,c),(3,d),(4,c),(5,c),(5,d),(6,a),(6,c)\}.$$

We execute the EASC algorithm as follows:

Step 1.1. Time is set as $t^{1.1} = 0$: Given that $i_{(\ell)}$ represents the ℓ^{th} choice school of student i, we form the flow network with the positive weights obtained from the endowment random matching ρ^1 as for all $i \in I$ and for all schools $f \in C$, we set the arc capacities

$$\omega^0_{\sigma \to i_{(R_i(f))}} = \rho^1_{i,f},$$

where $R_i(f)$ is the ranking of school f in i's preferences. Next, for all $i \in I$ and $f \in C$, if $(i, f) \in A^1(\rho^1)$, we set the arc capacities of the flow network as

$$\omega^0_{i_{(\ell)} \to f} = \infty,$$

for all ranks $\ell \leq R_i(f)$. Finally, for all $f \in C$, we set the arc capacities

$$\omega^0_{f \to \tau} = q_f.$$

Figure 1 shows this network for $t \in \left[0, \frac{1}{12}\right]$.

Moreover, given these constraints, the best available schools and endowment schools are

students (i)	best school (b_i)	endow. school (e_i)
1	b	a
2	c	a
3	d	c
4	c	$\emptyset$
5	c	d
6	d	b

We start increasing time t starting from $t^1 = 0$, thus, each student starts consuming his best available school by trading away from his endowment school (whenever $e_i \neq \emptyset$): that is, the capacity of each arc $\sigma \to i_{(R_i(b_i))}$ is updated as

$$\omega^t_{\sigma \to i_{(R_i(b_i))}} = \max \left\{ t - \sum_{\ell=1}^{R_i(b_i)-1} \omega^0_{\sigma \to i_{(\ell)}},\ \omega^0_{\sigma \to i_{(R_i(b_i))}} \right\},$$

and the capacity of each arc $\sigma \to i_{(R_i(e_i))}$ is updated as

$$\omega^t_{\sigma \to i_{(R_i(e_i))}} = \min \left\{ \sum_{\ell=1}^{R_i(b_i)} \omega^0_{\sigma \to i_{(\ell)}} + \omega^0_{\sigma \to i_{(R_i(e_i))}} - t,\ \omega^0_{\sigma \to i_{(R_i(e_i))}} \right\},$$

as long as a feasible random assignment can be obtained in the network, i.e., the value of the maximum flow of the network is $|I| = 6$ or the capacity of the endowment school arc does not go to zero.

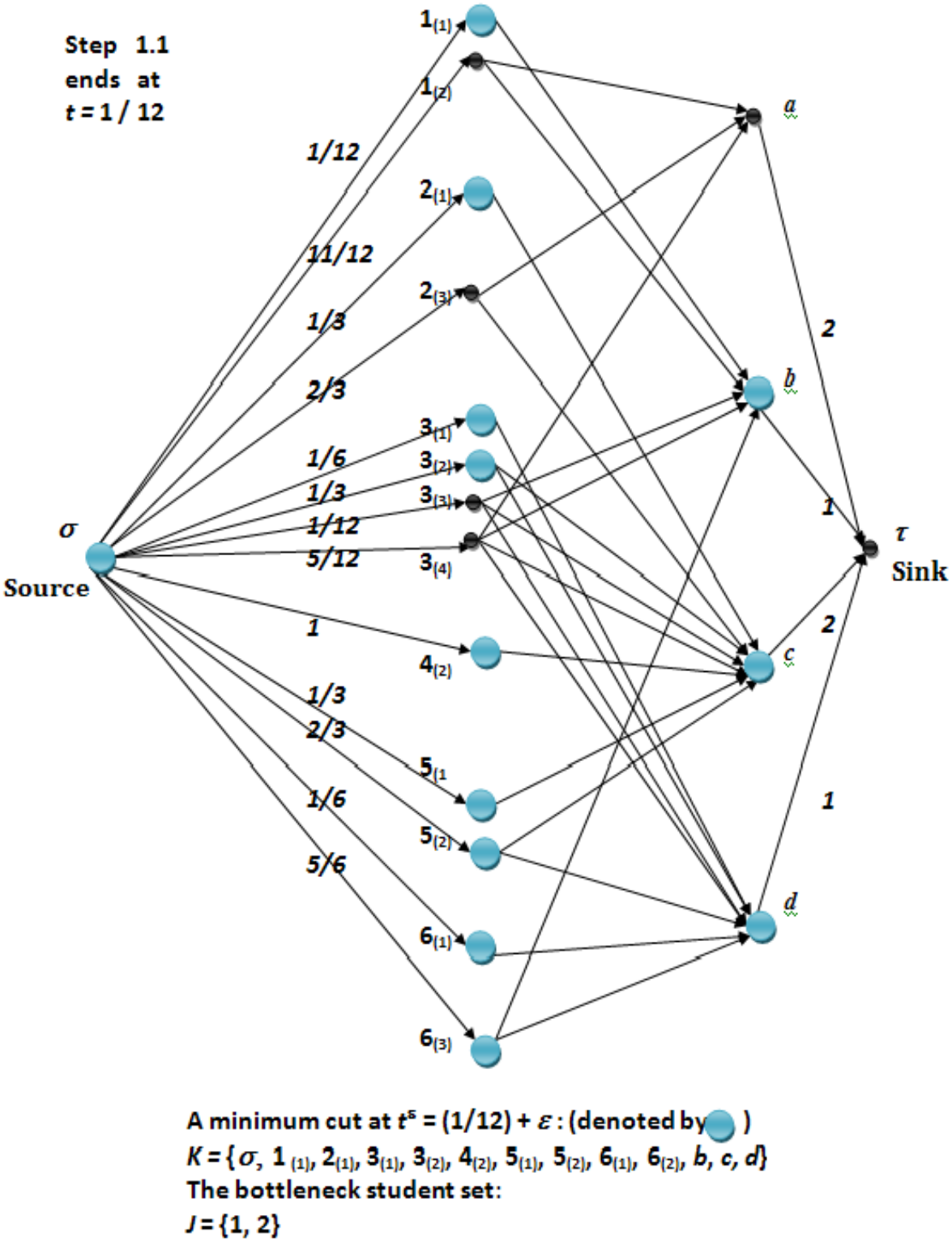

Figure 1: The consumption network for Example 8 at Step 1.1 for times $t \in \left[0, \frac{1}{12}\right]$.

The first condition is satisfied at $t = \frac{1}{12}$: If t increases above $\frac{1}{12}$, the value of the maximum flow falls below 6, because of the bottleneck set of agents $J = \{1, 2\}$. At this t, there is an excess demand for 1 and 2's best schools, but other agents do not demand 1 and 2's endowment school. Thus, 1 and 2 can no longer trade their endowment school in exchange for a fraction of their best schools. To see that {1,2} is a bottleneck set, we find a minimum cut K as seen in Figure 1 for network at $t = \frac{1}{12}$. Each student's representative nodes for his best school and his endowment school are in K, except for students 1 and 2. Their nodes for best schools are in K, but not their nodes for endowment schools. Also their endowment school a is not in K. Thus, Step 1.1 ends, and students 1 and 2 can no longer consume their best schools b and c, respectively. (Observe that the network at $t = \frac{1}{12}$ is identical to the network at $t = 0$.)We set:

$$t^{1.2} = \frac{1}{12},$$
$$\mathcal{A}^{1.2}\left(\rho^1\right) = \mathcal{A}^{1.1}\left(\rho^1\right) \setminus \{(1, b), (2, c)\}.$$

Step 1.2. Time is set as $t^{1.2} = \frac{1}{12}$. The best and endowment schools are updated as

students (i)	best school (b_i)	endow. school (e_i)
1	a	$\emptyset$
2	a	$\emptyset$
3	d	c
4	c	$\emptyset$
5	c	d
6	d	b

Time increases until $t = \frac{1}{6}$, when there is a new bottleneck set of students with minimum cut

$$K = \left\{\sigma, 2_{(1)}, 3_{(1)}, 3_{(2)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\right\}.$$

Since $6_{(R_6(b_6))} = 6_{(R_6(d))} = 6_{(1)} \in K$ and $6_{(R_6(e_6))} = 6_{(R_6(b))} = 6_{(3)} \notin K$, and there is no other student such that his node for his best (available) school is in K while his node for his endowment school is not, we determine the new bottleneck set as

$$J = \{6\}.$$

Thus, we update

$$t^{1.3} = \frac{1}{6},$$
$$\mathcal{A}^{1.3}\left(\rho^1\right) = \mathcal{A}^{1.2}\left(\rho^1\right) \setminus \{(6, d)\}.$$

At this point the capacities of the source-agent nodes are set still as their initial values at ω^0 (seen in Figure 1).

Step 1.3. Time is set as $t^{1.3} = \frac{1}{6}$. The best and endowment schools are updated as

students (i)	best school (b_i)	endow. school (e_i)
1	a	$\emptyset$
2	a	$\emptyset$
3	d	c
4	c	$\emptyset$
5	c	d
6	b	$\emptyset$

At this step, we observe actual trading of fractions of schools c and d between students 3 and 5, since all other students have no endowment schools to trade: time t increases until $\frac{1}{2}$ at which point only the following arc capacities are changing, while the others are still at ω^0 level:

$$\begin{aligned}
\omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(b_3))}} &= \omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(d))}} = \omega^{\frac{1}{2}}_{\left(\sigma\to 3_{(1)}\right)} = \\
&= \max\left\{ t - \sum_{\ell=1}^{R_3(b_3)-1} \omega^{\frac{1}{6}}_{\sigma\to 3_{(\ell)}},\ \omega^{\frac{1}{6}}_{\sigma\to 3_{(R_3(b_3))}} \right\} \\
&= \max\left\{ \frac{1}{2} - 0, \frac{1}{6} \right\} = \frac{1}{2};
\end{aligned}$$

$$\begin{aligned}
\omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(e_3))}} &= \omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(c))}} = \omega^{\frac{1}{2}}_{\sigma\to 3_{(2)}} = \\
&= \min\left\{ \sum_{\ell=1}^{R_3(b_3)} \omega^{\frac{1}{6}}_{\sigma\to 3_{(\ell)}} + \omega^{\frac{1}{6}}_{\sigma\to 3_{(R_3(e_3))}} - t,\ \omega^{\frac{1}{6}}_{\sigma\to 3_{(R_3(e_3))}} \right\} \\
&= \min\left\{ \frac{1}{6} + \frac{1}{3} - \frac{1}{2}, \frac{1}{3} \right\} = 0;
\end{aligned}$$

$$\begin{aligned}
\omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(b_5))}} &= \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(c))}} = \omega^{\frac{1}{2}}_{\sigma\to 5_{(1)}} = \\
&= \max\left\{ t - \sum_{\ell=1}^{R_5(b_5)-1} \omega^{\frac{1}{6}}_{\sigma\to 5_{(\ell)}},\ \omega^{\frac{1}{6}}_{\sigma\to 5_{(R_3(b_3))}} \right\} \\
&= \max\left\{ \frac{1}{2} - 0, \frac{1}{3} \right\} = \frac{1}{2};
\end{aligned}$$

$$\begin{aligned}\omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(e_5))}} &= \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(d))}} = \omega^{\frac{1}{2}}_{\sigma\to 5_{(2)}} = \\ &= \min\left\{\sum_{\ell=1}^{R_5(b_5)} \omega^{\frac{1}{6}}_{\sigma\to 5_{(\ell)}} + \omega^{\frac{1}{6}}_{\sigma\to 5_{(R_5(e_5))}} - t,\ \omega^{\frac{1}{6}}_{\sigma\to 5_{(R_5(e_5))}}\right\} \\ &= \min\left\{\frac{1}{3}+\frac{2}{3}-\frac{1}{2}, \frac{2}{3}\right\} = \frac{1}{2}.\end{aligned}$$

Since the endowment school's matching probability reaches zero for student 3, the step ends and we update:

$$\begin{aligned} t^{1.4} &= \frac{1}{2}, \\ \mathcal{A}^{1.4}\left(\rho^1\right) &= \mathcal{A}^{1.3}\left(\rho^1\right). \end{aligned}$$

Step 1.4. Time is set to $t^{1.4} = \frac{1}{2}$, only student 3's endowment school changed as $e_3 = b$. But at this time there is a minimum cut

$$K = \left\{\sigma, 2_{(1)}, 3_{(1)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\right\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(d))} = 3_{(1)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(b))} = 3_{(3)} \notin K$, and there is no other student with this property, the bottleneck set is

$$J = \{3\}.$$

Thus, we set

$$\begin{aligned} t^{1.5} &= \frac{1}{2}, \\ \mathcal{A}^{1.5}\left(\rho^1\right) &= \mathcal{A}^{1.4}\left(\rho^1\right) \setminus \{(3,d)\}. \end{aligned}$$

Step 1.5. Time is set to $t^{1.5} = \frac{1}{2}$, only student 3's best school changed as $e_3 = c$. But at this time there is a minimum cut

$$K = \left\{\sigma, 2_{(1)}, 3_{(1)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\right\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(c))} = 3_{(1)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(b))} = 3_{(3)} \notin K$, and there is no other student with this property, the bottleneck set is

$$J = \{3\}.$$

Thus, we set

$$\begin{aligned} t^{1.5} &= \frac{1}{2}, \\ \mathcal{A}^{1.5}\left(\rho^1\right) &= \mathcal{A}^{1.4}\left(\rho^1\right) \setminus \{(3,c)\}. \end{aligned}$$

Step 1.6. Time is set to $t^{1.6} = \frac{1}{2}$, student 3's best school changed as $b_3 = b$ and his endowment school changed as $e_3 = a$. Time t increases until $\frac{7}{12}$, when further increasing t would create a bottleneck set of students with minimum cut

$$K = \left\{\sigma, 1_{(1)}, 2_{(1)}, 3_{(1)}, 3_{(3)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, 6_{(3)}, b, c, d\right\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(b))} = 3_{(3)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(a))} = 3_{(4)} \notin K$, and there is no other student with this property, we have the bottleneck set at

$$J = \{3\}.$$

Observe that in the interval $t \in (\frac{1}{2}, \frac{7}{12}]$, student 3 does not consume his best school more than his capacity. This interval serves as the continuation of the trading between student 3 and 5 regarding schools c and d that has started at Step 1.3. Although 3 has already traded all his endowment of $\frac{1}{3}c$ away in return to get $\frac{1}{3}d$, student 5 has not fully gotten $\frac{1}{3}d$ and traded away $\frac{1}{3}c$. Thus, the market has not cleared yet. Increase in t helps the market to clear, since now we have

$$\begin{aligned}
\omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(b_5))}} &= \omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(c))}} = \omega^{\frac{7}{12}}_{\sigma\to 5_{(1)}} = \\
&= \max\left\{t - \sum_{\ell=1}^{R_5(b_5)-1} \omega^{\frac{1}{2}}_{\sigma\to 5_{(\ell)}},\ \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_3(b_3))}}\right\} \\
&= \max\left\{\frac{7}{12} - 0, \frac{1}{2}\right\} = \frac{7}{12};
\end{aligned}$$

$$\begin{aligned}
\omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(e_5))}} &= \omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(d))}} = \omega^{\frac{7}{12}}_{\sigma\to 5_{(2)}} = \\
&= \min\left\{\sum_{\ell=1}^{R_5(b_5)} \omega^{\frac{1}{2}}_{\sigma\to 5_{(\ell)}} + \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(e_5))}} - t,\ \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(e_5))}}\right\} \\
&= \min\left\{\frac{1}{2} + \frac{1}{2} - \frac{7}{12}, \frac{1}{2}\right\} = \frac{5}{12}.
\end{aligned}$$

while all other arc capacities remain the same. We update as

$$\begin{aligned}
t^{1.7} &= \frac{7}{12}, \\
\mathcal{A}^{1.7}\left(\rho^1\right) &= \mathcal{A}^{1.6}\left(\rho^1\right) \setminus \{(3, b)\}.
\end{aligned}$$

Step 1.7. Time is set to $t^{1.7} = \frac{7}{12}$, student 3's best school changed as $b_3 = a$ and he no longer has an endowment school, i.e., $e_3 = \emptyset$. Time t increases until $\frac{2}{3}$, when further increasing t would create a bottleneck set of students with minimum cut

$$K = \left\{\sigma, 4_{(2)}, 5_{(1)}, c\right\}.$$

Since $5_{(R_5(b_5))} = 5_{(R_3(c))} = 5_{(1)} \in K$ and $5_{(R_5(e_5))} = 5_{(R_5(d))} = 5_{(2)} \notin K$, and there is no other student with this property, we have the bottleneck set as

$$J = \{5\}.$$

Similar to Step 1.6, trade of c from 3 to 5 has continued at this step in return of d, and it can be verified that the only updated arc capacities are as follows:

$$\omega^{\frac{2}{3}}_{\sigma \to 5_{(1)}} = \frac{2}{3},$$
$$\omega^{\frac{2}{3}}_{\sigma \to 5_{(2)}} = \frac{1}{3}.$$

We update as

$$t^{1.8} = \frac{2}{3},$$
$$\mathcal{A}^{1.8}\left(\rho^1\right) = \mathcal{A}^{1.7}\left(\rho^1\right) \setminus \{(5, c)\}.$$

Step 1.8. Time is set to $t^{1.8} = \frac{2}{3}$, and student 5's best school is updated as $b_5 = d$ and he no longer has an endowment school, i.e., $e_5 = \emptyset$. Since no student has any endowment school, no more trade takes place in this step, time t increases to 1, and the ex-ante stable consumption algorithm terminates with

$\rho^2 =$

	a	**b**	**c**	**d**
1	$\frac{11}{12}$	$\frac{1}{12}$	0	0
2	$\frac{2}{3}$	0	$\frac{1}{3}$	0
3	$\frac{5}{12}$	$\frac{1}{12}$	0	$\frac{1}{2}$
4	0	0	1	0
5	0	0	$\frac{1}{6}$	$\frac{1}{3}$
6	0	$\frac{5}{6}$	0	$\frac{1}{6}$

Step 2. We have the feasible student-school set

$$\begin{aligned}\mathcal{A}^{2.1}\left(\rho^2\right) &= \{(1,a),(1,b),(2,a),(2,c),(3,a),(3,b),(3,c),(3,d),(4,c),(5,c),(5,d),(6,b),(6,d)\} \\ &= \mathcal{A}^{1.1}\left(\rho^1\right).\end{aligned}$$

It is easy to check that there are no feasible ex-ante stable improvement cycles and the FDAT algorithm terminates with outcome ρ^2. $\Diamond$