Fair and Efficient Assignment via the Probabilistic Serial Mechanism*

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Abstract

This paper studies the problem of assigning a set of indivisible objects to a set of agents when monetary transfers are not allowed. We offer two characterizations of the prominent lottery assignment mechanism called the *probabilistic serial* (Bogomolnaia and Moulin, 2001). We show that it is the only mechanism satisfying *non-wastefulness* and *ordinal fairness* and the only mechanism satisfying *sd-efficiency*, *sd-envy-freeness*, and *upper invariance* (where "sd" stands for first order stochastic dominance).

Keywords: Random assignment; probabilistic serial; ordinal fairness; sd-efficiency; sd-envy-freeness

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1 Introduction

A wide range of real-life resource allocation problems involves the assignment of indivisible objects without the use of monetary transfers. Important applications include student placement in public schools, organ transplantation through live or deceased donors, on-campus housing allocation, and course allocation at business schools. Most of these markets rely on *ordinal* mechanisms, where participants reveal only their preference rankings over given choices to the central authority rather than their cardinal preferences. In such applications, ensuring fairness in an ex-post sense can entail significant efficiency losses.[1] Therefore, it has become commonplace to use random mechanisms as a way to restore fairness ex ante.[2]

We consider generalized discrete resource allocation problems, or simply problems, with multiple supplies of objects and strict preferences and voluntary participation of agents.[3] A *mechanism* is a rule that specifies how to randomly assign objects to agents based on their preferences.

In a seminal paper, Bogomolnaia and Moulin (2001) (BM hereafter) proposed the *probabilistic serial* mechanism (PS) as an attractive contender to the widely used *random serial dictatorship* mechanism (RSD).[4] Unlike RSD, PS is sd-efficient [5] and sd-envy-free.[6] This surprising observation in turn led to a rapidly growing body of literature on this problem in general and on PS in particular.[7]

[1]See, for example, Kesten and Yazici (2009).

[2]For example, the assignment mechanisms used in the context of student placement operate through a collection of strict priority orders of schools over students. In practice, determining these orders often involves randomization (Abdulkadiroğlu and Sönmez, 2003; Erdil and Ergin, 2008; Kesten and Ünver, 2010; Pathak and Sethuraman, 2011). Similarly, in the exchange of live-donor kidneys among kidney patients for transplantation, the egalitarian approach requires the design of a random mechanism (Roth, Sönmez, and Ünver, 2005).

[3]Our framework is a generalization of the model studied by Bogomolnaia and Moulin (2001) that allowed only for assignments with single-supplies.

[4]RSD works as follows: Draw a random ordering from the uniform distribution and let agents succesively pick their favorite objects among available ones.

[5]A mechanism is *sd-efficient* if its outcome is not (first-order) stochastically dominated by an alternative random assignment, or equivalently if no group of agents can all be made better off, irrespective of their vNM utilities, by exchanging assignment probabilities of some of the objects among themselves. Hence "sd" stands for first order stochastic dominance.

[6]However, RSD is sd-strategy-proof, unlike PS, which is sd-strategy-proof only in a weak sense. Nevertheless, Kojima and Manea (2010) show that in large but finite problems where each object has a sufficiently large supply, PS regains sd-strategy-proofness. Relatedly, Che and Kojima (2010) show that in the limit of discrete economies with finite object types, PS converges to RSD.

[7]There are very few papers discussing the random assignment problem prior to the new millennium; Hylland and Zeckhauser (1979) and Zhou (1990) are two of these. Following BM's ground-breaking work, many papers, such as McLennan (2002); Bogomolnaia and Moulin (2002); Abdulkadiroğlu and Sönmez (2003); Katta and Sethuraman (2006); Athanassoglou and Sethuraman (2007); Manea (2008, 2009); Kojima (2009); Yilmaz (2009, 2010); and Budish, Che,

The outcome of PS is computed via the *simultaneous eating algorithm* (SEA): Consider each object as a continuum of probability shares. Agents simultaneously "eat away" from their favorite objects at the same speed; once the favorite object of an agent is gone, she turns to her next favorite object, and so on. The amount of an object eaten away by an agent throughout the process is interpreted as the probability with which she is assigned this object by PS.

In this paper, we provide two normative axiomatic characterizations of PS. Our first result states that a mechanism is non-wasteful and ordinally fair if and only if it is PS (Theorem 1). Non-wastefulness is a very mild efficiency property that is standard in the literature. Ordinal fairness is a new property, which we introduce. Let $F_i(a) \in [0, 1]$, called the *surplus* of agent i at object a, be the total amount of objects, at least as good as a, that agent i gets. *Ordinal fairness* requires that whenever an agent i consumes an object a with some positive probability, the surplus of agent i at a should not exceed the surplus of any other agent at a, i.e., $F_i(a) \leq F_j(a)$ for any agent j. This property takes the normative viewpoint that all agents a priori have equal claims to all objects, and allows them to redistribute the assignment probabilities among themselves in any way so long as no agent is "disadvantaged" by the redistribution.

For our second characterization, we introduce an auxiliary robustness axiom, closely related to incentive properties, called upper invariance. A mechanism is *upper invariant* if whenever an agent improves the ranking of a particular object by demoting any object that she consumes with zero probability, every agent still consumes this object with the same probability as before. This is a natural monotonicity property, as it diminishes the roles of objects that have no chance of being assigned to an agent. Upper invariance is also satisfied by RSD as well as by a wide class of deterministic strategy-proof and Pareto-efficient mechanisms such as *hierarchical exchange rules* (Pápai, 2000). Our second result states that a mechanism is sd-efficient, sd-envy-free, and upper invariant if and only if it is PS (Theorem 2).[8]

Kojima, and Milgrom (2011), studied various properties of sd-efficient assignments and mechanisms, and extensions of the PS for various setups. Others, such as Knuth (1996); Abdulkadiroğlu and Sönmez (1998); Sönmez and Ünver (2005); Kesten (2009); Che and Kojima (2010); Budish and Cantillon (2010); Carroll (2010); and Ekici (2010), studied RSD-like random assignment mechanisms.

[8]Upon completion of this work, we became aware of two other contemporaneous studies that provide different axiomatic characterizations of PS for different models. Heo (2010) characterizes PS for the case when each object and each agent can have an arbitrary quota, through sd-efficiency, proportional division lower-bound, limited invariance, and probabilistic consistency axioms. She has a second characterization that replaces proportional division lower-bound with normalized sd-envy-freeness and probabilistic consistency with probabilistic converse consistency. Hashimoto and Hirata (2011) offer three characterizations of PS, with the added requirement that a null object always exists. Our domain is more general than theirs, and thus our characterizations hold in different domains including theirs. Their first chatacterization is based on sd-efficiency, sd-envy-freeness, and truncation robustness. The second one characterizes

2 Model

A discrete resource allocation problem (cf. Hylland and Zeckhauser, 1979; Shapley and Scarf, 1974), or simply a **problem**, is a list $(N, A, q, \succ)$ where $N = \{1, \ldots, n\}$ is a finite set of agents; A is a finite set of objects; $q = (q_a)_{a \in A}$ is a positive integer vector where q_a denotes the **quota** of object a such that $\sum_{a \in A} q_a \geq |N|$; and $\succ = (\succ_i)_{i \in N}$ is a preference profile where $\succ_i$ is the strict preference relation of agent i on A. Let $\mathbf{P}$ be the set of preferences for any agent. Let $\succeq_i$ denote the weak preference relation induced by $\succ_i$. We assume that preferences are linear orders on A, i.e., for all $a, b \in A$ and all $i \in N$, $a \succeq_i b \Leftrightarrow a = b$ or $a \succ_i b$. We sometimes represent $\succ_i$ as an ordered list beginning with the most preferred object of agent i and continuing to her least. For example, given $A = \{a, b, c\}$, we interpret $\succ_i = (b, c, a)$ as $b \succ_i c \succ_i a$. A centralized authority shall assign objects to agents such that each agent is entitled to receive exactly one object.

Observe that this model is general enough to contain different interesting special cases:

1. ***Unacceptable objects:*** There is a specific object referred to as the *null object* and assigned a quota of $|N|$. By interpretation, agents who are assigned the null object are viewed as taking their outside options, or using the matching jargon, they *remain unassigned*. The objects ranked below the null object are called *unacceptable*. This case models assignment under voluntary participation.[9]

2. ***Perfect supply with unit quotas:*** Each object has a quota of 1 and there are exactly $|N|$ objects. This is the original setting of BM.[10]

We assume that probabilistic assignments are possible. A **random allocation** for agent i is a vector $P_i = (p_{i,a})_{a \in A}$ where $p_{i,a} \in [0, 1]$ denotes the probability that agent i receives object a, and $\sum_{a \in A} p_{i,a} = 1$. A **random assignment**, denoted as $P = [P_i]_{i \in N} = [p_{i,a}]_{i \in N, a \in A}$, is a substochastic matrix, the rows of which correspond to the random allocations of agents such that the probabilities along each row sum to one and the sum of probabilities along the column corresponding to each

the mechanism by sd-efficiency, independence of unassigned objects, and the Rawlsian Criterion. The third one is a weakening of their first result through a characterization with 2-sd-efficiency (non-existence of profitable bilateral trades plus non-wastefulness), weak sd-envy-freeness, and truncation robustness. Also there is a recent follow-up study to ours and Hashimoto and Hirata (2011): Bogomolnaia and Heo (2011) strengthen our second characterization using a weaker axiom than upper invariance called bounded invariance and relate our second result to Hashimoto and Hirata (2011)'s first result.

[9] In this setting, the standard *individual rationality* requirement, i.e., that no agent be assigned an unacceptable object with some positive probability, is implied by either efficiency property to be subsequently introduced; namely, by either non-wastefulness or sd-efficiency.

[10] In this latter setting, one of our properties, non-wastefulness, to be subsequently introduced, is satisfied vacuously.

object a does not exceed q_a, i.e., $\sum_{a\in A} p_{i,a} = 1$ for all $i \in N$ and $\sum_{i\in N} p_{i,a} \leq q_a$ for all $a \in A$. Let P_a denote the column vector of P corresponding to any $a \in A$, i.e., $P_a = (p_{i,a})_{i\in N}$. Let $\mathcal{R}$ be the set of random assignments.

Observe that a random assignment gives only the marginal probability distribution according to which each agent will be assigned an object. It does not specify the distribution according to which objects should jointly be assigned to agents. To define this joint probability distribution, we first need to define (deterministic) assignments and probability distributions over them. An **assignment** is a $P \in \mathcal{R}$ such that $p_{i,a} \in \{0, 1\}$ for all $i \in N$ and all $a \in A$. Let $\mathcal{A}$ be the set of assignments. A **lottery** $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}}$ is a probability distribution over assignments, i.e., $\lambda_\alpha \in [0, 1]$ for all $\alpha \in \mathcal{A}$ and $\sum_{\alpha\in\mathcal{A}} \lambda_\alpha = 1$.

Clearly, each lottery induces a random assignment. Let $P^\lambda \in \mathcal{R}$ be the random assignment induced by lottery λ, i.e., $p^\lambda_{i,a} = \sum_{\alpha\in\mathcal{A}:\alpha_{i,a}=1} \lambda_\alpha$ for all $i \in N$ and all $a \in A$. It turns out that the converse statement is also true: For each $P \in \mathcal{R}$ there exists a lottery λ that induces it, i.e., $P^\lambda = P$ (Birkhoff, 1946; von Neumann, 1953; Kojima and Manea, 2010). Thus, the centralized authority can simply restrict attention to random assignments rather than lotteries.[11] Throughout the paper, whenever it is not ambiguous, we shall suppress N, A, q and denote a problem by a preference profile.

Given $a \in A$ and $\succ_i \in \mathbf{P}$ for some $i \in N$, let $U(\succ_i, a) = \{b \in A \mid b \succeq_i a\}$ be the **upper contour set of object** a **at** $\succ_i$. Given a random allocation P_i, let $F(\succ_i, a, P_i) = \sum_{b\in U(\succ_i,a)} p_{i,b}$ be the probability that i is assigned an object at least as good as a under P_i; we simply refer to it as i **'s surplus at** a **under** P_i. Given $i \in N$, $\succ \in \mathbf{P}^N$, and $P, R \in \mathcal{R}$, P_i **stochastically dominates** R_i at $\succ_i$ if $F(\succ_i, a, P_i) \geq F(\succ_i, a, R_i)$ for all $a \in A$. In addition, P **stochastically dominates** R at $\succ$ if P_i stochastically dominates R_i at $\succ_i$ for all $i \in N$.

We are now ready to introduce a powerful efficiency notion. A random assignment is **sd-efficient** if it is not stochastically dominated by another random assignment.[12]

Next is a much weaker efficiency property. A random assignment is non-wasteful if the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object. Formally, given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is **non-wasteful** at $\succ$ if for all $i \in N$ and all $a \in A$ such that $p_{i,a} > 0$, we have $\sum_{j\in N} p_{j,b} = q_b$ for all $b \in A$ with $b \succ_i a$.

Our first fairness property is a fundamental principle in mechanism design theory originally proposed by Foley (1967). A random assignment is sd-envy-free if each agent, regardless of her vNM utilities, prefers her random allocation to that of any other agent. Formally, given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$

[11]Once a random assignment is determined, finding a lottery that induces it is computationally easy.

[12]Equivalently, under any alternative random assignment the surplus of some agent at some object is smaller than that under the original assignment.

is **sd-envy-free** at $\succ$ if for all $i \in N$, P_i stochastically dominates P_j for all $j \in N$ at $\succ_i$.[13]

A mechanism is a systematic way of finding a random assignment for a given problem. Formally, a **mechanism** is an allocation rule $\phi : \mathbf{P}^N \to \mathcal{R}$. A mechanism is said to satisfy a property if its outcome, for any problem, satisfies that property.

3 Two New Axioms

Our second fairness property, essential to our first characterization, is a natural and intuitive axiom for the random assignment setting. A random assignment is ordinally fair if whenever an agent is assigned some object with positive probability, her surplus at this object is no greater than that of any other agent at the same object. It follows that whenever an agent is assigned some object x with zero probability, she must be assigned a better object (for her) with a probability no less than any agent who is assigned object x with positive probability.

Definition 1 Given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is **ordinally fair** at $\succ$ if for all $a \in A$ and all $i, j \in N$ with $p_{i,a} > 0$, we have $F(\succ_i, a, P_i) \leq F(\succ_j, a, P_j)$.

We next introduce an auxiliary robustness axiom, essential to our second characterization. But first, we need some additional notation. Let $\succ_i |_B$ be the **restriction of** $\succ_i \in \mathbf{P}$ **to** $B \subseteq A$; that is, $\succ_i |_B$ is a preference relation over B such that for all $a, b \in B$, $a \succ_i |_B b \Leftrightarrow a \succ_i b$.

Given $P \in \mathcal{R}$ and $\succ \in \mathbf{P}^N$, preference relation $\succ_i' \in \mathbf{P}$ is an **upper invariant transformation of** $\succ_i$ for $i \in N$ at $a \in A$ under P if for some $Z \subseteq \{c \in A \mid p_{i,c} = 0\}$, $U(\succ_i', a) = U(\succ_i, a) \setminus Z$ and $\succ_i' |_{U(\succ_i', a)} = \succ_i |_{U(\succ_i', a)}$. An upper invariant transformation of a preference relation $\succ_i$ at a under P shrinks the upper contour set at a by removing from this set some of the objects that are never consumed by agent i; the relative rankings of objects weakly preferred to a stay the same at $\succ_i$ and $\succ_i'$; but the relative rankings of objects that are worse than a at $\succ_i'$ can change in any arbitrary way.

We are now ready to define our invariance concept:

Definition 2 A mechanism ϕ is **upper invariant** if for all $\succ \in \mathbf{P}^N$, all $i \in N$, all $\succ_i' \in \mathbf{P}$, and all $a \in A$ such that $\succ_i'$ is an upper invariant transformation of $\succ_i$ at a under $\phi(\succ)$, we have $\phi_a(\succ_i', \succ_{-i}) = \phi_a(\succ_i, \succ_{-i})$.

[13] Sd-envy-freeness is satisfied by many important allocation rules in different domains such as the competitive equilibrium from equal endowments (Hylland and Zeckhauser, 1979), some special VCG mechanisms, the uniform rule for single-peaked preferences over a divisible resource (Benassy, 1982), and the max-min allocation rule for quasi-linear preferences over indivisible objects and a fixed amount of money (Alkan, Demange, and Gale, 1991).

Upper invariance is similar in spirit to Maskin Monotonicity (Maskin, 1999) and has close connections with incentive properties. It implies strategy-proofness for deterministic mechanisms. Moreover, together with non-bossiness (Satterthwaite and Sonnenschein, 1981), it also implies Maskin monotonicity for deterministic mechanisms; see also Takamiya (2001).[14]

4 Probabilistic Serial Mechanism

BM introduced the **probabilistic serial mechanism** (**PS**),[15] the outcome of which can be computed via the following **simultaneous eating algorithm** (**SEA**):

Given a problem $\succ$, think of each object a as an infinitely divisible good with supply q_a.

Step 1: Each agent "eats away" from her favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted.

⋮

Step s, for $s \in \{2, \ldots, S\}$: Each agent eats away from her remaining favorite object at the same speed. Proceed to the next step when an object is completely exhausted.

The procedure terminates after $S \leqslant |N|$ steps when each agent has eaten exactly 1 total unit of objects (i.e., at time 1). The random allocation of an agent i by PS is then given by the amount of each object she has eaten until the algorithm terminates. Let $PS(\succ) \in \mathcal{R}$ denote the outcome of PS for problem $\succ$.

5 First Characterization of Probabilistic Serial

In our first result, we establish that for each problem there is a unique ordinally fair and non-wasteful random assignment and that this random assignment is the outcome of SEA. In other words, PS fully characterizes ordinal fairness with non-wastefulness, and vice versa.

Theorem 1 *A mechanism is ordinally fair and non-wasteful if and only if it is PS.*[16]

Proof of Theorem 1. Fix $\succ \in \mathbf{P}^N$. We will drop $\succ$ from all arguments below. We reinterpret the SEA such that at each step at most one object is fully exhausted: If two objects a, b are exhausted

[14]See Heo and Manjunath (2011) for a discussion of monotonicity conditions for implementation in ordinal random domains.

[15]PS was initially proposed by Crès and Moulin (2001) for a simple model where agents have the same rankings over objects.

[16]We thank Jay Sethuraman who gave us concrete ideas about shortening our original proof.

in a step according to the original definition, we will order these objects arbitrarily and say that one is exhausted first and the other one is exhausted in the next step. We redefine step S as the first step when each agent has eaten exactly 1 total unit of objects. Let $h^1, ..., h^{S-1}$ denote the objects exhausted in steps 1 to $S-1$ and the remaining ones be arbitrarily ordered as $h^S, ..., h^{|A|}$.

($\Rightarrow$) PS is *non-wasteful* as it is *sd-efficient*. We show that PS is *ordinally fair*. First, consider $s < S$: Each agent has eaten away weakly better objects than h^s until s at the same speed. Thus, for any $i \in N$ who eats away h^s at s and any $j \in N$ who eats away some $b \succeq_j h^s$, since they continue eating at the same speed and b_j is not exhausted before h^s, we have $F(\succ_i, h^s, PS_i) \leq F(\succ_j, b_j, PS_j) \leq F(\succ_j, h^s, PS_j)$. Next, consider $s \geq S$: At step S, each $j \in N$ eats away some $b_j \succeq_j h^s$. When SEA terminates after S, j's surplus at b_j is 1, and hence, $F(\succ_j, h^s, PS_j) = 1$. Thus, in either case *ordinal fairness* is satisfied for h^s.

($\Leftarrow$) Let $P \in \mathcal{R}$ be *ordinally fair* and *non-wasteful* at the fixed $\succ$. We will show that $PS= P$. Define $\pi(a) = \min_i F(\succ_i, a, P_i)$ for all $a \in A$. Relabel objects as $a^1, ..., a^{|A|}$ so that $\pi(a^s) \leq \pi(a^{s+1})$ for all $s \leq |A|-1$. Let $A^0 = \emptyset$, $A^s = \{a^1, ..., a^s\}$, and $\overline{A^s} = A \setminus A^s$ be the set complement of A^s. For all $s \geq 1$ and all $a \in \overline{A^{s-1}}$, let $N^s(a) = \{k \in N \mid a \succeq_k b \text{ for all } b \in \overline{A^{s-1}}\}$.

We argue by induction. Fix some $s \geq$1. Assume that for all $t < s$ and all $i \in N^t(a^t)$, $F(\succ_i, a^t, P_i) = F(\succ_i, a^t, PS_i) = \pi(a^t)$; for all $k \notin N^t(a^t)$, $p_{k,a^t} = PS_{k,a^t} = 0$; and a^t is the object exhausted at step t of SEA for $t < S$. Each statement in the inductive assumption holds vacuously for $s = 1$. We prove that they also hold for step s and thus $P = PS$:

Step 1. We show that for all $k \notin N^s(a^s)$, $p_{k,a^s} = 0$: For a contradiction, suppose for some $b \in \overline{A^s}$, $k \in N^s(b)$ we have $p_{k,a^s} > 0$. For an agent j with $\pi(a^s) = F(\succ_j, a^s, P_j)$, we have $F(\succ_k, a^s, P_k) > F(\succ_k, b, P_k) \geq F(\succ_j, a^s, P_j)$ where the last inequality follows from the ordering of a^s before b through π. However, this inequality violates *ordinal fairness* of P.$\diamond$

Step 2. We show that for all $i \in N^s(a^s)$, $F(\succ_i, a^s, P_i) = \pi(a^s)$: Let $i \in N^s(a^s)$. Either $p_{i,a^s} > 0$ or $p_{i,a^s} = 0$. If $p_{i,a^s} > 0$, then by *ordinal fairness*, for all $j \in N$, $F(\succ_i, a^s, P_i) \leq F(\succ_j, a^s, P_j)$, and thus $F(\succ_i, a^s, P_i) = \pi(a^s)$ by the definition of $\pi(a^s)$. Suppose $p_{i,a^s} = 0$. Let t^* be the earliest step t such that $i \in N^t(a^s)$. If $t^* = 1$, then by $p_{i,a^s} = 0$, $F(\succ_i, a^s, P_i) = 0$, and thus $F(\succ_i, a^s, P_i) = \pi(a^s)$ by the definition of $\pi(a^s)$. Next, suppose $t^* > 1$. Then $i \in N^{t^*-1}(a^{t^*-1})$. By the inductive assumption (as $t^* \leq s$), $\pi(a^{t^*-1}) = F(\succ_i, a^{t^*-1}, P_i)$. Thus, as a^s is ranked just below a^{t^*-1} in $\succ_i$ and $p_{i,a^s} = 0$, $F(\succ_i, a^s, P_i) = \pi(a^{t^*-1})$. Moreover, since a^s is ordered after a^{t^*-1} according to π, $\pi(a^s) \geq \pi(a^{t^*-1})$. Thus, as $F(\succ_i, a^s, P_i) \geq \pi(a^s)$, $F(\succ_i, a^s, P_i) = \pi(a^s)$. $\diamond$

Step 3. We show that at step s of SEA for $s < S$, for any agent $i \in N^s(a^s)$, $F(\succ_i, a^s, PS_i) \geq \pi(a^s)$: By the inductive assumption, for each $b \in \overline{A^{s-1}}$, at the end of step $s-1$ of SEA, the amount that each $i \in N^s(b)$ has eaten away from objects in $U(\succ_i, b)$ is $x = \pi(a^{s-1})$ if $s > 1$ and $x = 0$ if $s = 1$;

and moreover, all objects in $U(\succ_i, b) \setminus \{b\}$ are also fully exhausted. For all $j \in N^s(a^s)$, all $b \in \overline{A^{s-1}}$, and all $k \in N^s(b)$, we have $\pi(a^s) \leq \pi(b)$ by the definition of a^s; and this together with Step 2 and the definition of π imply $F(\succ_j, a^s, P_j) - x = \pi(a^s) - x \leq \pi(b) - x \leq F(\succ_k, b, P_k) - x$. Thus, the remaining amount of object b is sufficiently large for each agent in $N^s(b)$ so that when each agent in $N^s(a^s)$ has eaten away $\pi(a^s) - x$ of a^s, no agent in $N^s(b)$ has yet started eating an object different from b. Therefore, each $j \in N^s(a^s)$ eats away by the end of step s at least $(\pi(a^s) - x) + x = \pi(a^s)$, the total amount from objects in $U(\succ_j, a^s)$, implying that $\pi(a^s) \leq F(\succ_j, a^s, PS_j)$.$\diamond$

Step 4. We show that for all $j \in N^s(a^s)$, $\pi(a^s) = F(\succ_j, a^s, PS_j)$ and $PS_{k,a^s} = 0$ for all $k \notin N^s(a^s)$ for $s< S$: Proving the first claim is sufficient (by Step 3). Suppose, to the contrary, for some $i \in N^s(a^s)$, $F(\succ_i, a^s, P_i) = \pi(a^s) < F(\succ_i, a^s, PS_i) \leq 1$; but then

$$\sum_j p_{j,a^s} = \sum_{j \in N^s(a^s)} \{F(\succ_j, a^s, P_j) - \sum_{b \succ_j a^s} p_{j,b}\} < \sum_{j \in N^s(a^s)} \{F(\succ_j, a^s, PS_j) - \sum_{b \succ_j a^s} PS_{j,b}\} = \sum_{j \in N^s(a^s)} PS_{j,a^s} \leq q_{a^s},$$

where $\sum_{j \in N^s(a^s)} F(\succ_j, a^s, P_j) < \sum_{j \in N^s(a^s)} F(\succ_j, a^s, PS_j)$ by Steps 2 and 3 and the supposition, and $p_{j,b} = PS_{j,b}$ for all $j \in N^s(a^s)$ and all $b \succ_j a^s$ by the inductive assumption. This violates *non-wastefulness* of P. We have showed that for all $j \in N^s(a^s)$, $F(\succ_i, a^s, PS_i) = \pi(a^s)$. Then, step s of SEA ends when a^s is fully exhausted by Step 3 of the proof. Moreover, $PS_{k,a^s} = 0$ for all $k \notin N^s(a^s)$ as none of these agents have started eating a^s before it gets fully exhausted under SEA. $\diamond$

Step 5. We show that the rest of the inductive claim holds for $s \geq S$: SEA terminates at step S when when each agent has eaten exactly 1 total unit of objects. Any agent $i \in N^S(a)$ eats away $a \in \overline{A^{S-1}}$ at step S of SEA. Thus, $F(\succ_i, a, PS_i) = 1$ and for any $k \notin N^S(a)$, $PS_{i,a} = 0$. By *non-wastefulness* of P (through the same argument in Step 4 applied to a instead of a^s), for any $i \in N^s(a)$, $F(\succ_i, a, PS_i) = \pi(a) = F(\succ_i, a, P_i)$.$\diamond$ ■

6 Second Characterization of Probabilistic Serial

Sd-efficiency and sd-envy-freeness are among the most appealing properties of mechanisms. Our second result characterizes PS through these two fundamental properties together with upper invariance:

Theorem 2 *A mechanism is sd-efficient, sd-envy-free, and upper invariant if and only if it is PS.*

Before proving the theorem, we introduce the following auxiliary concept and lemma. We invoke *upper invariance* only through this lemma in our proof. Given a mechanism ϕ, $\succ \in \mathbf{P}^N$, $i \in N$, $\succ'_i \in \mathbf{P}$, and $a \in A$, $\succ_i$ and $\succ'_i$ are *weakly invariant transformations of each other at* a if $\phi_{i,a}(\succ) = \phi_{i,a}(\succ'_i, \succ_{-i}) = 0$ and $\succ'_i |_{A \setminus \{a\}} = \succ_i |_{A \setminus \{a\}}$. The following lemma is immediate:

Lemma 1 *Let ϕ be an upper invariant mechanism. Then, for all $\succ \in \mathbf{P}^N$, all $i \in N$, all $\succ'_i \in \mathbf{P}$, and all $a \in A$,*

1. *if $\succ_i$ and $\succ'_i$ are weakly invariant transformations of each other at a, then $\phi(\succ) = \phi(\succ'_i, \succ_{-i})$;*

2. *if $\succ_i$ and $\succ'_i$ are upper invariant transformations of each other at a, then $\phi_b(\succ) = \phi_b(\succ'_i, \succ_{-i})$ for all $b \in U(\succ_i, a)$.*

We continue with the proof of Theorem 2:

Proof of Theorem 2. ($\Leftarrow$) The *sd-efficiency* and *sd-envy-freeness* of the PS mechanism are proved by BM in a more restricted domain. The same proofs apply to our domain. We prove its *upper invariance* below.

Let $\succ \in \mathbf{P}^N$, $i \in N$, and $a \in A$. Let $\tilde{\succ}_i \in \mathbf{P}$ be an upper invariant transformation of $\succ_i$ at a under $PS(\succ)$. Let $\tilde{\succ} = (\tilde{\succ}_i, \succ_{-i})$. Then, $U(\tilde{\succ}_i, a) = U(\succ_i, a) \setminus Z$ for some $Z \subseteq \{c \in A \,|\, PS_{i,c}(\succ) = 0\}$. We will show that $PS_a(\tilde{\succ}) = PS_a(\succ)$.

First, consider the case $Z = \emptyset$. Observe that all objects will be eaten in exactly the same amounts under both $\succ$ and $\tilde{\succ}$ until a is exhausted if a is fully exhausted under $\succ$ and until SEA terminates otherwise, as agent i has not started eating any object less preferred to a yet under either preference profile, and her preferences coincide up to object a. Thus, $PS_a(\tilde{\succ}) = PS_a(\succ)$.

Next, consider the case $Z \neq \emptyset$. Since agent i does not eat any object $b \in Z$ in $\succ$ and such an object is less preferred to a in $\tilde{\succ}$, an upper invariant transformation does not change which objects agents eat at what shares until a is exhausted if a is exhausted fully under $\succ$, and until SEA terminates otherwise. Thus, the previous conclusion still holds.

($\Rightarrow$) Let ϕ be a *sd-efficient*, *sd-envy-free*, and *upper invariant* mechanism. By *sd-efficiency*, it is *non-wasteful.* We will show that ϕ is also *ordinally fair,* and thus by Theorem 1, $\phi = PS$. Fix $\succ \in \mathbf{P}^N$.

Let $P = \phi(\succ)$. Contrary to the hypothesis, suppose there exist i and $j \in N$, and $a^* \in A$ with $p_{i,a^*} > 0$ such that $F(\succ_i, a^*, P_i) > F(\succ_j, a^*, P_j)$.

If $U(\succ_i, a^*) \subseteq U(\succ_j, a^*)$, we have $F(\succ_j, a^*, P_i) \geq F(\succ_i, a^*, P_i) > F(\succ_j, a^*, P_j)$, contradicting the *sd-envy-freeness* of P. Thus, in the rest of the proof assume that $U(\succ_i, a^*) \setminus U(\succ_j, a^*) \neq \emptyset$. If there is no object b such that $a^* \succ_j b \succeq_j a'$ and $p_{j,b} > 0$ where a' is the lowest ranked $a \in U(\succ_i, a^*) \setminus U(\succ_j, a^*)$ by j then as we have $U(\succ_i, a^*) \subseteq U(\succ_j, a')$, we still have $F(\succ_j, a', P_i) \geq F(\succ_i, a^*, P_i) > F(\succ_j, a^*, P_j) = F(\succ_j, a', P_j)$ contradicting the *sd-envy-freeness* of P.

Thus, in the rest of the proof, assume that such a b exists and assume that it is the highest ranked such b by j. Let $a \in U(\succ_i, a^*) \setminus U(\succ_j, b)$ be the highest ranked such a by j. By the *sd-efficiency* of P, $p_{j,a} = 0$, as otherwise i could trade with j some share at a^* to receive an equal share at a, and i and j both would get better off (in first-order stochastic dominance sense), while other agents would remain indifferent. Next, we upgrade a just in front of b in preferences of j, and obtain $\succ'_j$ from $\succ_j$. As $\succ_j$ and $\succ'_j$ are upper invariant transformations of each other at a^*, by By Lemma 1 Part 2, objects in $U(\succ_j, a^*) = U(\succ'_j, a^*)$ have the same assignment probabilities for any agent under $\phi(\succ'_j, \succ_{-j})$ and P, and in particular, we still have $\phi_{i,a^*}(\succ'_j, \succ_{-j}) = p_{i,a^*} > 0$. By the *sd-efficiency* of $\phi(\succ'_j, \succ_{-j})$, we have $\phi_{j,a}(\succ'_j, \succ_{-j}) = 0$ (as otherwise, agent i could trade with j some share at a^* to receive an equal share at a, and each of i and j would be better off, while the rest of the agents would remain indifferent). Then, $\succ_j$ and $\succ'_j$ are weakly invariant transformations of each other at a, and by Lemma 1 Part 1, $\phi(\succ'_j, \succ_{-j}) = P$.

We repeat this procedure of updating j's preferences for each such $a \in U(\succ_i, a^*) \setminus U(\succ_j, b)$, and hence, if $\succ''_j$ is the final preference relation of j that we obtain, then $\phi(\succ''_j, \succ_{-j}) = P$ by repeatedly invoking upper invariance through Lemma 1 as above. Then,

$$F(\succ''_j, a', P_i) \geq F(\succ_i, a^*, P_i) > F(\succ_j, a^*, P_j) = F(\succ''_j, a^*, P_j) = F(\succ''_j, a', P_j),$$

where recall that a' is the lowest ranked $a \in U(\succ_i, a^*) \setminus U(\succ_j, a^*)$ in $\succ''_j$ and hence, the first inequality follows as $U(\succ_i, a^*) \subseteq U(\succ''_j, a')$ by construction of $\succ''_j$. The overall inequality contradicts the *sd-envy-freeness* of $\phi(\succ''_j, \succ_{-j}) = P$. Thus, we showed that $F(\succ_i, a^*, P_i) \leq F(\succ_j, a^*, P_j)$. This proves the *ordinal fairness* of ϕ. ■

7 Concluding Remarks

Our study is not the first attempt at this kind of a characterization. BM gave a full characterization of these two axioms together with weak sd-strategy-proofness when there are three agents and three objects. A mechanism is *weakly sd-strategy-proof* if no agent ever stochastically gains by misreporting her preferences.[17] The following result shows that this characterization no longer holds with five or more agents:

Proposition 1 *If the number of agents is greater than or equal to five, the PS mechanism is not characterized by sd-efficiency, sd-envy-freeness, and weak sd-strategy-proofness.*

We prove this proposition through a counterexample, i.e., by providing a mechanism, different from PS, that satisfies all three properties. The mechanism we describe in the next example differs from PS only at one preference profile out of 120^5 profiles. It turns out that it is not upper invariant.

[17] Formally, ϕ is *weakly sd-strategy-proof* if for all $\succ$ and all $i \in N$, there is no $\succ'_i$ such that $\phi_i(\succ'_i, \succ_{-i})$ stochastically dominates $\phi_i(\succ_i, \succ_{-i})$ and $\phi_i(\succ'_i, \succ_{-i}) \neq \phi_i(\succ_i, \succ_{-i})$.

Example 1 Suppose that there are five agents $N = \{1, 2, ..., 5\}$ and five objects $A = \{a, b, c, d, e\}$ each with unit quota. Let $\succ^*$ be defined as follows:

$$\begin{aligned} &a \succ_i^* c \succ_i^* d \succ_i^* e \succ_i^* b \quad \text{for } i = 1, 2, 3 \\ &b \succ_4^* c \succ_4^* d \succ_4^* e \succ_4^* a \\ &b \succ_5^* a \succ_5^* c \succ_5^* e \succ_5^* d \end{aligned}$$

The PS outcome for this problem is

$$PS(\succ^*) = \frac{1}{720} \begin{pmatrix} 240 & 0 & 192 & 180 & 108 \\ 240 & 0 & 192 & 180 & 108 \\ 240 & 0 & 192 & 180 & 108 \\ 0 & 360 & 72 & 180 & 108 \\ 0 & 360 & 72 & 0 & 288 \end{pmatrix}.$$

Define

$$P^* = \frac{1}{720} \begin{pmatrix} 220 & 0 & 210 & 185 & 105 \\ 220 & 0 & 210 & 185 & 105 \\ 220 & 0 & 210 & 185 & 105 \\ 0 & 360 & 75 & 165 & 120 \\ 60 & 360 & 15 & 0 & 285 \end{pmatrix}.$$

Construct the mechanism ϕ as follows:

$$\phi(\succ) = \begin{cases} P^* & \text{if } \succ = \succ^*, \\ PS(\succ) & \text{otherwise.} \end{cases}$$

This mechanism is sd-efficient, sd-envy-free, and weakly sd-strategy-proof, but not upper invariant (see the appendix). $\diamond$

Finally, we establish the logical independence of the axioms in Theorems 1 and 2. We start with Theorem 1. An ordinally fair but wasteful mechanism is the following: When the total quota of objects exceeds the number of agents,[18] consider the following: Fix $q'_a \leq q_a$ for all $a \in A$ such that $\sum_{a \in A} q'_a = |N|$. The PS mechanism that assigns objects according to the artificial quota vector $(q'_a)_{a \in A}$ is ordinally fair but wasteful. On the other hand, RSD is a non-wasteful but ordinally unfair mechanism.

[18] If the total quota of objects is equal to the number of agents, we have an assignment problem with perfect supply. Thus, non-wastefulness holds vacuously.

The independence of the axioms for Theorem 2 can be shown as follows. The mechanism in Example 1 is sd-efficient and sd-envy-free, but not upper invariant. A serial dictatorship is a sd-efficient and upper invariant mechanism that induces sd-envy. A sd-envy-free and upper invariant mechanism that is not sd-efficient can be constructed as follows: Choose a quota $q'_a \leq q_a$ for each $a \in A$ so that $\sum_{a \in A} q'_a = |N|$. For any problem assign each agent each $a \in A$ with probability $\frac{q'_a}{|N|}$.

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Appendix: Proof of Proposition 1

We show that the mechanism in Example 1 is sd-efficient, sd-envy-free, and weakly sd-strategy-proof.

First, note that P^* is sd-efficient at $\succ^*$, and the PS mechanism is sd-efficient (BM). Thus, the mechanism ϕ is sd-efficient.

Second, we show that the mechanism ϕ is sd-envy-free. Since the PS mechanism is sd-envy-free (BM), we need to show that P^* is sd-envy-free at $\succ^*$. Let $sd(\succ_i)$ be the stochastic dominance relation induced by preference $\succ_i$. For example, we need to show, for agent 1, $P_1^* sd(\succ_1^*) P_4^*$ and $P_1^* sd(\succ_1^*) P_5^*$. We have similar conditions for agents 4 and 5. To this end, we use the following table:

$\succ_1^*$	a	c	d	e	b
P_1^*	220	430	615	720	720
P_4^*	0	75	240	360	720
P_5^*	60	75	75	360	720

$(\times \frac{1}{720})$

The first row indicates the houses in order of the preference $\succ_1^*$ of agent 1. The second row calculates $F(\succ_1^*, a', P_1^*)$ for each corresponding house a' in the first row. The third row calculates $F(\succ_1^*, a', P_4^*)$ for each corresponding house a' in the first row. Similarly defined is the fourth row. To have $P_1^* sd(\succ_1^*) P_4^*$, we need to compare the row of P_1^* with that of P_4^*. That is, for *each* column, the number in the second row must be greater than or equal to the number in the third row. The above table actually shows $P_1^* sd(\succ_1^*) P_4^*$ and $P_1^* sd(\succ_1^*) P_5^*$. Similarly, we have tables for agents 4 and 5:

$\succ_4^*$	b	c	d	e	a
P_4^*	360	435	600	720	720
P_1^*	0	210	395	500	720
P_5^*	360	375	375	660	720

$\succ_5^*$	b	a	c	e	d
P_5^*	360	420	435	720	720
P_1^*	0	220	430	535	720
P_4^*	360	360	435	555	720

$(\times \frac{1}{720})$

Looking at the above tables, we conclude that P^* is sd-envy-free at $\succ^*$.

Finally, we show that the mechanism ϕ is weakly sd-strategy-proof. For notational simplicity, we use $P = \phi(\succ_i, \succ_{-i}^*)$ for a preference $\succ_i \neq \succ_i^*$ of an agent i. Recall $P^* = \phi(\succ^*)$. First, because ϕ consists of the PS mechanism that is weakly sd-strategy-proof (BM), we need to show for all $i \in N$ $, \succ_i$, if $P_i \neq P_i^*$, then

it is not possible that $P_i \, sd(\succ_i^*) \, P_i^*$, and

it is not possible that $P_i^* \, sd(\succ_i) \, P_i$.

By symmetry, we show this for $i = 1, 4, 5$. Before checking the above two conditions, we introduce two kinds of tables. For example, consider the case where $i = 1$ and her preference is $\succ_1 = (e, b, a, c, c) \neq \succ_1^*$. We denote $P = \phi(\succ_1, \succ_{-1}^*) \equiv PS(\succ_1, \succ_{-1}^*)$, and recall $P^* \equiv \phi(\succ_1^*, \succ_{-1}^*)$. To examine the first condition in the above, we use the following table.

$\succ_1^*$	a	c	b	d	e
P_1^*	$F(\succ_1^*, a, P_1^*)$	$F(\succ_1^*, c, P_1^*)$	$F(\succ_1^*, b, P_1^*)$	$F(\succ_1^*, d, P_1^*)$	$F(\succ_1^*, e, P_1^*)$
P_1	$F(\succ_1^*, a, P_1)$	$F(\succ_1^*, c, P_1)$	$F(\succ_1^*, b, P_1)$	$F(\succ_1^*, d, P_1)$	$F(\succ_1^*, e, P_1)$

Here the first row indicates the houses in order of preference $\succ_1^*$. To verify the first condition (i.e., we cannot have $P_1 sd(\succ_1^*) P_1^*$, it suffices to have that, at *some* column, the number in the second row is *strictly greater* than the one in the third row. Thus, we will list houses until the column with this condition.

Similarly, to examine the second condition in the above, we use the following table.

$\succ_1$	e	b	a	c	d
P_1	$F(\succ_1, e, P_1)$	$F(\succ_1, b, P_1)$	$F(\succ_1, a, P_1)$	$F(\succ_1, c, P_1)$	$F(\succ_1, d, P_1)$
P_1^*	$F(\succ_1, e, P_1^*)$	$F(\succ_1, b, P_1^*)$	$F(\succ_1, a, P_1^*)$	$F(\succ_1, c, P_1^*)$	$F(\succ_1, d, P_1^*)$

Here the first row indicates the houses in order of preference $\succ_1$. To verify the condition 2, it suffices to have that, at *some* column, the number in the second row is *strictly greater* than the one in the third row. Thus, we will list houses until the column with this condition.

Now we start checking each case.

First, consider agent 1. Take any preference $\succ_1$.

Case 1-1: $\succ_1 = (a, c, d, b, e)$ or (a, c, b, d, e).
Then, $P = PS(\succ^*)$ (Recall $P = PS(\succ_1, \succ^*_{-1})$) and thus $P_1 = \frac{1}{720}(240, 0, 192, 180, 108)$. Hence,

$\succ^*_1$	a	c	d
P^*_1	220	430	615
P_1	240	432	612

$\succ_1$	a
P_1	240
P^*_1	220

Case 1-2: $\succ_1 = (a, c, b, e, d)$ or $(a, c, e, \cdots)$.
Then, $P_1 = \frac{1}{720}(240, 0, 192, 0, 288)$. Hence,

$\succ^*_1$	a	c	d
P^*_1	220	430	615
P_1	240	432	432

$\succ_1$	a
P_1	240
P^*_1	220

Case 1-3: $\succ_1 = (a, b, c, \cdot, \cdot)$.
Then, $P_1 = \frac{1}{720}(240, 80, 112, \cdots)$. Hence,

$\succ^*_1$	a	c
P^*_1	220	430
P_1	240	352

$\succ_1$	a
P_1	240
P^*_1	220

Case 1-4: $\succ_1 = (a, b, d, \cdot, \cdot)$ or $(a, b, e, \cdot, \cdot)$.
Then, $P_1 = \frac{1}{720}(240, 80, 0, \cdots)$. Hence,

$\succ^*_1$	a	c
P^*_1	220	430
P_1	240	240

$\succ_1$	a
P_1	240
P^*_1	220

Case 1-5: $\succ_1 = (a, d, \cdots)$ or $(a, e, \cdots)$.
Then, $P_1 = \frac{1}{720}(240, 0, 0, \cdots)$. Hence,

$\succ^*_1$	a	c
P^*_1	220	430
P_1	240	240

$\succ_1$	a
P_1	240
P^*_1	220

Case 1-6: $\succ_1 = (b, \cdots)$.
Obviously, $p_{1b} = 1/3 = 240/720$. Hence,

$\succ_1$	b
P_1	240
P^*_1	0

$(\times \frac{1}{720})$

Note that p_{1a} is the largest if $\succ_1 = (b, a, \cdots)$. Suppose $\succ_1 = (b, a, \cdots)$. Then, $P_1 = \frac{1}{720}(60, 240, \cdots)$. Hence, the other table is

$\succ^*_1$	a
P^*_1	220
P_1	60

Thus, for any preference $\succ_1$, we have the desired result.

Case 1-7: $\succ_1 = (c, \cdots)$.
Then, $P_1 = \frac{1}{720}(0, 0, 432, \cdots)$. Hence,

$\succ^*_1$	a
P^*_1	220
P_1	60

$\succ_1$	c
P_1	432
P^*_1	210

Case 1-8: $\succ_1 = (d, \cdots)$.
Then, $P_1 = \frac{1}{720}(0, 0, 0, 585, 135)$. Hence,

$\succ^*_1$	a
P^*_1	220
P_1	0

$\succ_1$	d
P_1	585
P^*_1	185

Case 1-9: $\succ_1 = (e, \cdots)$.
Then, $P_1 = \frac{1}{720}(0, 0, 0, 90, 630)$.

$\succ^*_1$	a
P^*_1	220
P_1	0

$\succ_1$	e
P_1	630
P^*_1	105

Next, consider agent 4. Take any preference $\succ_4$ ($\neq \succ^*_4$). We denote $P = \phi(\succ_4, \succ^*_{-4}) \equiv PS(\succ_4, \succ^*_{-4})$, and recall $P^* \equiv \phi(\succ^*_4, \succ^*_{-4})$.

Case 4-1: $\succ_4 = (b, c, d, a, e)$, (b, c, a, d, e), or (b, a, c, d, e).

Then, $P = PS(\succ^*)$. Thus,

$\succ_4^*$	b	c
P_4^*	360	435
P_4	360	432

$\succ_4$	b	(a)	c	(a)	d
P_4	360	(360)	432	(432)	612
P_4^*	360	(360)	435	(435)	600

Case 4-2: $\succ_4 = (b, c, a, e, d)$, $(b, c, e, \cdots)$, or (b, a, c, e, d).

Then, $P_4 = \frac{1}{720}(0, 360, 72, 0, 288)$. Hence,

$\succ_4^*$	b	c
P_4^*	360	435
P_4	360	432

$\succ_4$	b	(a)	c	(a)	e
P_4	360	(360)	432	(432)	720
P_4^*	360	(360)	435	(435)	555

Case 4-3: $\succ_4 = (b, a, d, \cdots)$ or $(b, d, \cdots)$.

Then, $P_4 = \frac{1}{720}(0, 360, 0, 247.5, 112.5)$. Hence,

$\succ_4^*$	b	c
P_4^*	360	435
P_4	360	360

$\succ_4$	b	(a)	d
P_4	360	(360)	607.5
P_4^*	360	(360)	525

Case 4-4: $\succ_4 = (b, a, e, \cdots)$ or $(b, e, \cdots)$.

Then, $P_4 = \frac{1}{720}(0, 360, 0, 0, 360)$. Hence,

$\succ_4^*$	b	c
P_4^*	360	435
P_4	360	360

$\succ_4$	b	(a)	e
P_4	360	(360)	720
P_4^*	360	(360)	480

Case 4-5: $\succ_4 = (a, \cdots)$.

Obviously, $p_{4a} = 1/4 = 180/720$. Thus,

$\succ_4$	a
P_4	180
P_4^*	0

Note that p_{4b} is the largest if $\succ_4 = (a, b, \cdots)$. Suppose $\succ_4 = (a, b, \cdots)$. Then, $P_4 = \frac{1}{720}(180, 270, 0, \cdots)$. And the other table is

$\succ_4^*$	b
P_4^*	360
P_4	270

Thus, for any preference, we have the desired result.

Case 4-6: $\succ_4 = (c, \cdots)$.

We can calculate as $p_{4c} = 1/2 = 360/720$. Thus,

$\succ_4$	c
P_4	360
P_4^*	75

Note that p_{4b} is the largest if $\succ_4 = (c, b, \cdots)$. Suppose $\succ_4 = (c, b, \cdots)$. Then, $P_4 = \frac{1}{720}(0, 180, 360, \cdots)$. And the other table is

$\succ_4^*$	b
P_4^*	360
P_4	180

Thus, for any preference, we have the desired result.

Case 4-7: $\succ_4 = (d, \cdots)$.

Obviously, $p_{4d} \geq 1/3 = 240/720$. Then,

$\succ_4$	d
P_4	at least 240
P_4^*	165

Note that p_{4b} is the largest if $\succ_4 = (d, b, \cdots)$. Suppose $\succ_4 = (d, b, \cdots)$. Then, $P_4 = \frac{1}{720}(0, 90, \cdots)$. And the other table is

$\succ_4^*$	b
P_4^*	360
P_4	90

Thus, for any preference, we have the desired result.

Case 4-8: $\succ_4 = (e, \cdots)$.
Then, $P_4 = \frac{1}{720}(0, 0, 0, 0, 720)$.

$\succ_4^*$	b
P_4^*	360
P_4	0

$\succ_4$	e
P_4	720
P_4^*	120

Finally, we consider agent 5. Take any preference $\succ_5$ $(\neq \succ_5^*)$. We denote $P = \phi(\succ_5, \succ_{-5}^*) \equiv PS(\succ_5, \succ_{-5}^*)$, and recall $P^* \equiv \phi(\succ_5^*, \succ_{-5}^*)$.

Case 5-1: $\succ_5 = (b, a, c, d, e)$.
Then, $P_5 = \frac{1}{720}(0, 360, 72, 144, 144)$. Hence,

$\succ_5^*$	b	a
P_5^*	360	420
P_5	360	360

$\succ_5$	b	a	c	d
P_5	360	360	432	576
P_5^*	360	420	435	435

Case 5-2: $\succ_5 = (b, a, d, \cdots)$.
Then, $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence,

$\succ_5^*$	b	a
P_5^*	360	420
P_5	360	360

$\succ_5$	b	a	d
P_5	360	360	576
P_5^*	360	420	420

Case 5-3: $\succ_5 = (b, a, e, \cdots)$.
Then, $P_5 = \frac{1}{720}(0, 360, 0, 0, 360)$. Hence,

$\succ_5^*$	b	a
P_5^*	360	420
P_5	360	360

$\succ_5$	b	a	e
P_5	360	360	720
P_5^*	360	420	705

Case 5-4: $\succ_5 = (b, c, \cdots)$.
Then, $P_5 = \frac{1}{720}(0, 360, 72, \cdots)$. Hence,

$\succ_5^*$	b	a
P_5^*	360	420
P_5	360	360

$\succ_5$	b	c
P_5	360	432
P_5^*	360	375

Case 5-5: $\succ_5 = (b, d, \cdots)$.
P coincides with the one in Case 5-2, i.e., $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence,

$\succ_5^*$	b	a
P_5^*	360	420
P_5	360	360

$\succ_5$	b	d
P_5	360	576
P_5^*	360	360

Case 5-6: $\succ_5 = (b, e, \cdots)$.
P coincides with the one in Case 5-3, i.e., $P_5 = \frac{1}{720}(0, 360, 0, 0, 360)$. Hence,

$\succ_5^*$	b	a
P_5^*	360	420
P_5	360	360

$\succ_5$	b	e
P_5	360	720
P_5^*	360	645

Case 5-7: $\succ_5 = (a, \cdots)$.
Obviously, $p_{5a} = 1/4 = 180/720$. Thus,

$\succ_5$	a
P_5	180
P_5^*	60

Note that p_{5b} is the largest if $\succ_5 = (a, b, \cdots)$. Suppose $\succ_5 = (a, b, \cdots)$. Then, $P_5 = \frac{1}{720}(180, 270, 0, \cdots)$.

$\succ_5^*$	b
P_5^*	360
P_5	270

Thus, for any preference, we have the desired result.

Case 5-8: $\succ_5 = (c, \cdots)$.
Obviously, $p_{5c} \geq 1/3 = 240/720$. Thus,

$\succ_5$	c
P_5	at least 240
P_5^*	15

Note that p_{5b} is the largest if $\succ_5 = (c, b, \cdots)$. Suppose $\succ_5 = (c, b, \cdots)$. Then, $P_5 = \frac{1}{720}(0, 180, 360, \cdots)$.

$\succ_5^*$	b
P_5^*	360
P_5	180

Thus, for any preference, we have the desired result.

Case 5-9: $\succ_5 = (d, \cdots)$.
Obviously, $p_{5d} \geq 1/3 = 240/720$. Thus,

$\succ_5$	d
P_5	at least 240
P_5^*	0

Note that p_{5b} is the largest if $\succ_5 = (d, b, \cdots)$. Suppose $\succ_5 = (d, b, \cdots)$. Then, $P_5 = \frac{1}{720}(0, 90, 0, 540, 90)$.

$\succ_5^*$	b
P_5^*	360
P_5	90

Thus, for any preference, we have the desired result.

Case 5-10: $\succ_5 = (e, \cdots)$.
Then, $P_5 = \frac{1}{720}(0, 0, 0, 0, 720)$. Hence,

$\succ_5^*$	b
P_5^*	360
P_5	0

$\succ_5$	e
P_5	720
P_5^*	285

We next show that mechanism ϕ is not upper invariant: Consider the preference of agent 1, $\succ_1' = (a, c, d, b, e)$. Then, $\phi(\succ_1', \succ_{-1}^*) = PS(\succ_1', \succ_{-1}^*) = PS(\succ^*)$. In particular, $\phi_{1b}(\succ_1', \succ_{-1}^*) = 0$. Thus, $\succ_1^* = (a, c, d, e, b)$ is an upper invariant transformation of $\succ_i'$ at e under $\phi(\succ_1', \succ_{-1}^*)$. However, $\phi_{je}(\succ_1^*, \succ_{-1}^*) \neq \phi_{je}(\succ_1^*, \succ_{-1}^*)$ for all $j = 1, 2, \cdots, 5$. Hence, ϕ is not upper invariant.