On the Likelihood of Cyclic Comparisons[*]

Ariel Rubinstein[†]   Uzi Segal[‡]

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Abstract

One problem caused by cycles of choice functions is indecisiveness
— decision makers will be paralyzed when they face choice sets with
more than two options. We investigate the procedure of "random sam-
pling" where the alternatives are random variables. When comparing
any two alternatives, the decision maker samples each of the alterna-
tives once and ranks them according to the comparison between the
two realizations. We show that while this procedure may lead to vio-
lations of transitivity, the probability of such cycles is bounded from
above by $\frac{8}{27}$. Even lower bounds are obtained for some other related
procedures.

[†]University of Tel Aviv Cafes, University of Tel Aviv, and New York University
[‡]Department of Economics, Boston College (`segalu@bc.edu`)

1 Introduction

The *indecisiveness* argument is used to justify the transitivity assumption in decision theory. Suppose that $A \succ B$, $B \succ C$, and $C \succ A$. If the decision maker has to choose from the set $\{A, B, C\}$ he will be frozen: for each alternative he may choose he will find a better one. This may bring him to re-evaluate his preferences and probably to change them so that the cycles will be eliminated.

This argument might be applied to procedures of choice where the decision maker is using a random procedure to determine his attitude to each pair of alternatives. Such a procedure may yield indecisiveness. The higher is the probability that a random procedure of choice yields indecisiveness, the more likely it is that the decision maker will conclude that he should avoid this procedure.

In this paper we focus on a nondeterministic procedure of preference formation which we call *Random Sampling procedure*. When comparing two lotteries, the decision maker samples once from each lottery and ranks them according to the two realizations. (This concept is related to the S-1 procedure proposed in Osborne and Rubinstein (1998)).

The main message of our paper is that when applied to random preferences, the scope of the indecisiveness argument is limited. Whereas the argument is always applicable for deterministic procedures which yield cycles, the random procedures which we study would be less vulnerable to the indecisiveness argument. Our formal analysis provides a characterization of the upper bound on the probability that the random procedure we study yields indecisiveness and shows that this bound is quite low.

Our first result refers to the case where when choosing from three random

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variables the decision maker independently compares each pair of them. He starts by comparing some alternatives A and B, continues into comparing B and C, and then finally compares A and C. In each of the three stages he draws new samples from the relevant pair of random variables and does not use the values he observed before. We find that the bound on the probability that this random sampling procedure yields a cycle is 8/27 (Claim 1). This is somewhat lower than 1/3, which we show in Claim 2 to be the bound for the Block and Marschak's (1960) *Random Ordering* procedure. According to this alternative procedure the decision maker has in mind a set of orderings. When comparing two alternatives, he randomly samples one of the orderings and ranks the two alternatives according to that ordering.[1]

We then turn to the case where the decision maker activates the three comparisons in a pre-determined order, starting by comparing the alternatives A and B and continues with comparing B and C and then C and A, but unlike the random sampling procedure, he partially recalls past observations. In the second comparison the decision maker remembers the value of the observed sampling from B which he got in the first comparison, and in the last comparison he recalls the value of C which was used in the second comparison. However, in the third comparison he samples afresh from A. In other words, this decision maker remembers the outcomes of the last comparisons, but not what he has seen two stages ago.

This procedure will reduce the bound on the probability of a cycle only slightly to 1/4. But we then show that the probability of indecisiveness can be reduced significantly if someone else (for example, an agent who wants the consumer to make a quick choice) can control the order at which the de-

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[1]For a recent discussion of how the random ordering procedure can explain data which exhibits intransitivity, see Regenwetter, Dana, and Davis-Stober (2011).

cision maker compares the alternatives. We show that the upper bound on the probability of a cycle is reduced to $1/16(= 0.0625)$ for the case of binary lotteries (Claim 3) and to 0.091 for lotteries with at most three outcomes. Moreover, if the external agent's choice of order could depend on the realizations in the first comparison, then he can eliminate cycles altogether for binary lotteries (Claim 4) and he can reduce the bound on the probability of a cycle for lotteries with three outcomes to $1/32$.[2]

Thus, we show that nondeterministic procedures of choice, applied to three alternatives, yield transitivity with a fairly high probability. Therefore, the mere fact that choice is "almost" well behaved and only a small number of cycles is observed does not necessarily prove that decision makers are using deterministic transitive preferences (while making occasional mistakes). Such behavior can also emerge when choice is based on some variants of random sampling where decision makers do not employ preference relations and certainly do not change them to avoid indecisiveness.

2   Random Sampling

The main procedure we discuss in this paper is *random sampling*: To compare two random variables the decision maker draws a fresh sample from each and ranks them according to the sampled values.

Throughout the paper, all triples of random variables have finite and disjoint supports. Denote by $s(A)$ the support of the lottery A and by $\Pr(A > B)$ the probability that the realization of A is higher than the realization of

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[2]This manipulator is helping the decision maker avoiding cycles, unlike the Dutch bookie (discussed in Yaari (1998)) who is using the cycle to pump out the decision maker's resources.

B. By the disjoint supports assumption, $\Pr(A > B) + \Pr(B > A) = 1$. Let $\Pi(A, B, C)$ be the probability of a cycle being created by the decision maker's procedure. Applied to the random sampling procedure we have:

$$\Pi(A, B, C) = \Pr(A > B)\Pr(B > C)\Pr(C > A) + $$
$$\Pr(A > C)\Pr(C > B)\Pr(B > A)$$

Claim 1 *The maximal probability that the procedure of random sampling yields a cycle is* $\frac{8}{27}$.

Proof: Consider the three random variables presented in the following table:

Value	A	B	C
4	1/3		
3		2/3	
2			1
1	2/3		
0		1/3	

In this case, $\Pr(A > B) = \frac{5}{9}$, $\Pr(B > C) = \frac{2}{3}$, $\Pr(C > A) = \frac{2}{3}$ and the probability of a cycle $\Pi(A, B, C)$ is $\frac{20}{81} + \frac{4}{81} = \frac{8}{27}$.

In order to prove that this is the upper bound, let $x_1 > x_2 > \ldots > x_n$ be the values in the supports of the three random variables A, B and C. Denote by $X_i \in \{A, B, C\}$ the random variable that contains x_i in its support. Let $\pi_i = \Pr(X_i = x_i) > 0$.

First, we assume without loss of generality that for all i, $X_i \neq X_{i+1}$; otherwise, if $X_i = X_{i+1} = A$, let A' be the random variable which differs from A by $\Pr(A' = x_i) = \pi_i + \pi_{i+1}$ and $\Pr(A' = x_{i+1}) = 0$. Then, $\Pi(A', B, C) = \Pi(A, B, C)$.

Next, assume that for some i, $X_i = X_{i+2} \neq X_{i+1}$ (without loss of generality $X_i = A$ and $X_{i+1} = B$). Then we can (weakly) increase the probability of a cycle by replacing A with A_ε, a random variable which differs from A by either moving a probability mass $\varepsilon > 0$ from x_i to x_{i+2} or from x_{i+2} to x_i. Clearly, $\Pr(C > A_\varepsilon) = \Pr(C > A)$ and $\Pr(A_\varepsilon > B)$ is linear in ε. Since

$$
\begin{aligned}
\Pi(A_\varepsilon, B, C) \;=\; & \Pr(A_\varepsilon > B)[\Pr(B > C)\Pr(C > A)] + \\
& (1 - \Pr(A_\varepsilon > B))\Pr(A > C)\Pr(C > B)
\end{aligned}
$$

shifting probability mass from x_{i+2} to x_i or the other way around (according to the sign of $\Pr(B > C)\Pr(C > A) - \Pr(A > C)\Pr(C > B)$) will (weakly) increase the probability of a cycle.

Thus, without loss of generality we can assume that the sequence $\{X_i\}$ is of the form $\dots A, B, C, A, B, C, \dots$ ending with $X_{n-2} = A$, $X_{n-1} = B$ and $X_n = C$.

Next we show that if the three random variables (A, B, C) maximize Π and if $n > 6$, then there is a triple of random variables that maximizes Π with less than n values in their joint supports. First note that:

$$
\begin{aligned}
\Pi(A, B, C) \;=\; & \Pr(C > A)[\Pr(B > C)\Pr(A > B) - \\
& \Pr(B > A)\Pr(C > B)] + \Pr(B > A)\Pr(C > B)
\end{aligned}
$$

Changing C does not affect $\Pr(B > A)$. Consider the set of all C' with a support that is a subset of C such that $\Pr(B > C') = \Pr(B > C)$. For all such C', denote by γ_i the probability that C' yields the outcome x_i. This is the set of all vectors $(\gamma_i)_{x_i \in s(C)}$ such that $\gamma_i \geqslant 0$ for all i and the following two linear equations hold:

$$\sum_{x_i \in s(C)} \gamma_i = 1$$

$$\sum_{x_i \in s(C)} \gamma_i \times \sum_{j < i \text{ and } x_j \in s(B)} \pi_j = \Pr(B > C)$$

Since $n > 6$ and $X_n = C$, there are at least $m \geqslant 3$ points in the support of C. The set C' is therefore non empty and is given by the intersection of R_{++}^m and the above two $m-1$ dimensional hyperplanes. The two hyperplanes intersect at C, thus the set is the intersection of R_{++}^m and a linear space of dimension $m - 2 > 0$.

Replacing C with C' will increase the probability of a cycle if $\Pr(B > C)\Pr(A > B) - \Pr(B > A)\Pr(C > B)$ and $\Pr(C' > A) - \Pr(C > A)$ have the same sign. The expression

$$\Pr(C' > A) = \sum_{x_i \in s(C) \gamma_i \times \sum_{j > i \text{ and } x_j \in s(A)} \pi_j}$$

is a linear function in $(\gamma_i)_{x_i \in s(C)}$. Therefore, we can (weakly) increase $\Pr(C' > A)$ by moving in some direction until we reach the boundary where $\gamma_i = 0$ for some x_i in the support of C.

We can therefore narrow down our attention to the sequence of variables $(X_i)_{i=1\ldots6}$ which is of the form A, B, C, A, B, C. Denote by α, β, γ the probabilities that the variables A, B and C obtain the highest prize in their supports. Then,

$$\begin{aligned} \Pi(A, B, C) &= (1 - \beta + \alpha\beta)(1 - \gamma + \beta\gamma)(\gamma - \alpha\gamma) + \\ &\quad (\beta - \alpha\beta)(\gamma - \beta\gamma)(1 - \gamma + \alpha\gamma) = \\ &= \gamma^2(1 - \alpha)(\beta - 1) + \gamma(1 - \alpha)(\alpha\beta - \beta^2 + 1) \end{aligned}$$

Assuming that both $1 > \alpha$ and $\beta > 0$, the last expression is strictly increasing in γ within the interval $[0, 1]$. Thus, it attains its maximum at $\gamma = 1$. We conclude that in the optimum, one of the three variables must be degenerate and without loss of generality the sequence $(X_i)_{i=1\ldots5} = (B, C, A, B, C)$. Then,

$$\Pi = \gamma^2(\beta - 1) + \gamma(-\beta^2 + 1) = \gamma^2\beta - \gamma^2 - \gamma\beta^2 + \gamma$$

This expression has a unique maximum point at $\beta = \frac{1}{3}$ and $\gamma = \frac{2}{3}$ and a maximization value of $\Pi = \frac{8}{27}$. $\blacksquare$

Comments:

(a) In Claim 1 we obtained the upper bound on the probability that the procedure of random realizations yields one of the two possible cycles $A \succ C \succ B \succ A$ or $A \succ B \succ C \succ A$. In comparison, the highest probability that the procedure yields a particular cycle is $\frac{1}{4}$ (see Tenney and Foster (1976)).

(b) The problem we dealt with in this section is related to the so-called "paradox of nontransitive dice" (see Gardner (1970) who credits it to the statistician Bradley Efrom). This "paradox" involves three independent random variables: A, B, and C, where $Pr(A > B)$, $Pr(B > C)$, and $Pr(C > A)$ all exceed 0.5.[3] Savage (1994) further proved that $\max_{A,B,C} \min\{Pr(A > B), Pr(B > C), Pr(C > A)\} = (\sqrt{5} - 1)/2$.

(c) It follows from Claim 1 and comment (a) above that for every three distributions F, G, and H with a bounded domain and which do not have an atom in the same point:

$$\int FdG \int GdH \int HdF + \int FdH \int HdG \int GdF \leqslant \frac{8}{27}$$

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[3] See http://singingbanana.com/dice/article.htm for an entertaining demonstration of this setup.

and

$$\int F dG \int G dH \int H dF \leqslant \frac{1}{4}$$

(d) When a decision maker applies the ordering sample procedure to a set of size n, the maximum probability that his ranking is acyclic goes to zero as the number of alternatives increases to infinity. To see it consider n random variables which are uniform on the interval $[0, 1]$ (and obviously could be approximated by random variables with finite and disjoint supports). For any two of these random variables, the probability that the realization of one is higher than of the other is $\frac{1}{2}$. By Moon and Moser (1962), the probability that the realized tournament is irreducible (i.e., there are no two non-empty disjoint sets such that every node in one set "beats" every node in the other) goes to 1 as $n \to \infty$. By Moon (1966), a tournament with n nodes has a cycle of length n (and therefore is not acyclic) if and only if it is irreducible. Thus, the probability that the decision maker's comparisons of n uniform random variables yields a cycle of size n goes to 1 as $n \to \infty$.

3   The Random Ordering Procedure

In the random ordering procedure (Block and Marschak (1960)) the decision maker is characterized by π, a probability measure over the six orderings of the three alternatives A, B, and C. When comparing any pair of alternatives, the decision maker draws an ordering that will determine his ranking of these alternatives. Thus, he might apply different orderings in ranking two different pairs of alternatives. In this section we show that the bounds we obtained in the previous section are lower than the bounds on the probability of a cycle in the random ordering procedure.

Claim 2 *The maximal probability that the random ordering procedure yields a cycle is $\frac{1}{3}$.*

Proof: Consider π to be a probability measure on the orderings that assigns equal probabilities to the three orderings $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$ and $C \succ_3 A \succ_3 B$. Then, $\Pr(A \succ B) = \Pr(B \succ C) = \Pr(C \succ A) = \frac{2}{3}$ and the probability of a cycle is $\frac{8}{27} + \frac{1}{27} = \frac{1}{3}$.

To see that $\frac{1}{3}$ is indeed the bound, note that by the inequality of arithmetic and geometric means:

$$
\begin{aligned}
\Pi(A, B, C) \;=\; & \Pr(A \succ B)\Pr(B \succ C)\Pr(C \succ A) + \\
& \Pr(A \succ C)\Pr(C \succ B)\Pr(B \succ A) \;\leqslant\; \\
& \Pr(A \succ B) + \Pr(B \succ C) + \Pr(C \succ A)]^3/27 + \\
& [\Pr(A \succ C) + \Pr(C \succ B) + \Pr(B \succ A)]^3/27
\end{aligned}
$$

Since every ordering must satisfy at least one and at most two of $A \succ B$, $B \succ C$ and $C \succ A$, we obtain: $1 \leqslant [\Pr(A \succ B) + \Pr(B \succ C) + \Pr(C \succ A)] \leqslant 2$. The function $x^3 + (3-x)^3$ is convex in the interval $[1, 2]$ and obtains its maximum at $x = 1$ and $x = 2$. Thus $\Pi(A, B, C) \leqslant \frac{1}{27} + \frac{8}{27} = \frac{1}{3}$. ■

Comments:

(a) Note that the above example is the only one in which the probability of a cycle is $\frac{8}{27}$. To see this, count the six orderings: $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, $C \succ_3 A \succ_3 B$, $A \succ_4 C \succ_4 B$, and $B \succ_5 A \succ_5 C$, $C \succ_6 B \succ_6 A$. Denote by π_i the probability of $\succ_i$. Then, $\Pr(A \succ B)\Pr(B \succ C)\Pr(C \succ A) = (\pi_1 + \pi_3 + \pi_4)(\pi_1 + \pi_2 + \pi_5)(\pi_2 + \pi_3 + \pi_6)$. The maximum is attained only when $\pi_4 = \pi_5 = \pi_6 = 0$ and $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$.

(b) Similarly to comment (a) to Claim 1, the maximal probability that the procedure of random ordering yields a particular cycle is $\frac{8}{27}$. The example in

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the above proof attains the bound. To prove that the bound is $\frac{8}{27}$, note that $\Pr(A \succ B)\Pr(B \succ C)\Pr(C \succ A) \leqslant \Pr(A \succ B) + \Pr(B \succ C) + \Pr(C \succ A)]^3/27$ and that the function x^3 in the interval $[1,2]$ attains the maximum at 2.

4   The Random Sampling Procedure with Partial Recall

In the procedure discussed in Section 2 each comparison is done independently of the other two comparisons. A decision maker who compares first A and B and moves to compare B and C does not recall the previous value of B. Thus the existence of a cycle did not depend on the order by which the comparisons were done. In contrast, in this section we assume that the decision maker carries out the comparisons sequentially in three stages and at each stage he remembers the realizations of the previous stage, but not those of two stages earlier. In other words, he applies the procedure of *Random sampling with partial recall*. It is applied to the sequence of three lotteries (A, B, C) in the following way:

 (i) Compare A and B by sampling each once.

 (ii) Compare B and C by sampling C once and compare the outcome with that of the previous-stage sampling of B.

(iii) Compare C and A by sampling A again and compare the outcome with that of the previous-stage sampling of C.

The probability that the procedure yields a cycle is

$$\Pi(A, B, C) = \Pr(A_1 > B > C > A_2) + \Pr(A_2 > C > B > A_1)$$

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where A_1 and A_2 are copies of A, i.e., they are i.i.d and distributed like A. Note that $\Pi(A, B, C)$ might differ from $\Pi(B, A, C)$ but that $\Pi(A, B, C) = \Pi(A, C, B)$. For example, let A be the random variable that receives the values of 3 or 0 with equal probabilities. Let $B \equiv 2$ and let $C \equiv 1$. Then $\Pi(A, B, C) = \frac{1}{4}$. In fact, the maximal probability that the random sampling procedure with partial recall yields a cycle is $\frac{1}{4}$. To see why, denote by Π_b the probability of a cycle given that the value of B is b:

$$
\begin{aligned}
\Pi_b \;&=\; \Pr(A_1 > b > C > A_2) + \Pr(A_2 > C > b > A_1) \leqslant \\
&\quad \Pr(A_1 > b > C)\Pr(b > A_2) + \Pr(A_2 > b)\Pr(C > b > A_1) = \\
&\quad \Pr(A > b)\Pr(b > C)\Pr(b > A) + \Pr(A > b)\Pr(C > b)\Pr(b > A) = \\
&\quad \Pr(b > A)\Pr(A > b)[\Pr(b > C) + \Pr(C > b)] \leqslant \frac{1}{4}
\end{aligned}
$$

Since $\Pi_b \leqslant \frac{1}{4}$ for every possible realization of B, $\Pi(A, B, C) \leqslant \frac{1}{4}$ as well.

Imagine now that the order in which the alternatives are presented to the decision maker is determined by a "master of ceremonies" (MC) who wants the decision maker having a clear ordering of the alternatives. Let $V(A, B, C) = \min\{\Pi(A, B, C), \Pi(B, C, A), \Pi(C, A, B)\}$ be the probability of a cycle given that the MC chooses the order of the comparisons of the three variables A, B and C in order to minimize the probability of the cycle. In the example used above $\Pi(A, B, C) = \frac{1}{4}$ but $\Pi(B, C, A) = 0$ and thus $V(A, B, C) = 0$. On the other hand, if A, B, C are uniformly distributed over $[0, 1]$ then $V(A, B, C) = \Pi(A, B, C) = \frac{1}{12}$ (each ordering of four identical random variables has the same probability of $\frac{1}{24}$ and therefore $\Pr(A_1 > B > C > A_2) + \Pr(A_2 > C > B > A_1) = \frac{1}{12}$). We succeeded to find the bound on V for only a limited family of random variables.

Claim 3 *The maximal $V(A, B, C)$ for three binary random variables is $\frac{1}{16}$.*

Proof: First note that for the following three variables $V(A, B, C) = \frac{1}{16}$.

Value	A	B	C
5	1/2		
4		1/2	
3			1/2
2	1/2		
1		1/2	
0			1/2

If the three variables are such that between the two values of one of the lotteries, say A, there are no values of another lottery, say C, then $\Pi(A, B, C) = 0$. Thus, we need to consider only the case in which the values of the three lotteries can be ordered as A,B,C,A,B,C. Denote by α,β,γ the probabilities of the highest value of each of the three lotteries A,B,C respectively. Then, $\Pi(A, B, C) = \alpha\beta\gamma(1 - \alpha)$, $\Pi(B, A, C) = \beta\gamma(1 - \alpha)(1 - \beta)$ and $\Pi(C, A, B) = \gamma(1 - \alpha)(1 - \beta)(1 - \gamma)$.

Note that by the continuity of Π, at a maximum point of $V(A, B, C)$ it must be that two of the terms $\Pi(A, B, C)$, $\Pi(B, C, A)$, and $\Pi(C, A, B)$ are equal and are weakly less than the third. If $\Pi(B, A, C)$ is minimal then $\Pi(B, C, A) = \beta\gamma(1 - \alpha)(1 - \beta) = \min\{\alpha\beta\gamma(1 - \alpha), \ \gamma(1 - \alpha)(1 - \beta)(1 - \gamma)\}$. It follows that $1 - \beta \leqslant \alpha$ and $\beta \leqslant 1 - \gamma$ and thus, $\Pi(B, C, A) \leqslant \beta(1 - \beta)(1 - \alpha)\alpha \leqslant \frac{1}{16}$. If $\Pi(B, C, A)$ is not minimal then at the maximum point of V, $\beta\gamma(1 - \alpha)(1 - \beta) > \alpha\beta\gamma(1 - \alpha) = \gamma(1 - \alpha)(1 - \beta)(1 - \gamma)$, hence $1 - \alpha > \beta$ and $\beta > 1 - \gamma$. The maximum with respect to β of the function $\alpha\beta\gamma(1 - \alpha)$ (which is linear in β) given the linear constraints $\alpha\beta = (1 - \beta)(1 - \gamma)$ and $(1 - \alpha) \geqslant \beta \geqslant (1 - \gamma)$ must be obtained where either $\beta = 1 - \alpha$ or $\beta = 1 - \gamma$. In the former case $\alpha\beta\gamma(1 - \alpha) = (1 - \beta)(1 - \gamma)\gamma\beta \leqslant \frac{1}{16}$ while in the latter $\alpha\beta\gamma(1 - \alpha) = \alpha(1 - \gamma)\gamma(1 - \alpha) \leqslant \frac{1}{16}$. ∎

When the support of each of the random variables has at most three points, numerical methods prove that the maximum of $V(A, B, C)$ is roughly 0.0910 and is attained near the triple of random variables:

Value	A	B	C
6	0.19		
5		0.37	
4			0.63
3		0.63	
2	0.62		
1			0.37
0	0.19		

We do not know what are the upper bounds for the cases of at most n outcomes in each lottery for $n > 3$. It is worthwhile mentioning, though, that almost all experiments in the literature utilize lotteries with no more than three different outcomes each as lotteries with more prizes are often difficult to absorb. (For a survey of this literature see Starmer (2000)).

The probability of a cycle can be reduced even further if the MC can choose the first couple of alternatives and only after he observes their realizations he determines which of the two alternatives will be compared with the third one at the second stage. Using numerical methods we conclude that for any triple of lotteries with no more than three outcomes the MC can present the comparisons such that the probability of a cycle is not greater than $\frac{1}{32}$. Moreover, if each lottery has at most two outcomes cycles can be eliminated:

Claim 4 *Let A, B, and C be three binary random variables. If the decision maker follows the Random Sampling procedure with partial recall then the*

MC who observes the realizations can arrange the order of comparisons so that no cycles emerge.

Proof: Suppose that between the outcomes of one lottery, say A, there are no outcomes of another lottery, say B. Then the MC will ask the decision maker to compare A and B and then B and C. Assume $B \succ A$. If $B \succ C$ then there is no cycle. If $C \succ B$ then the fresh realization of C is higher than both values of A and at the third stage $C \succ A$. The case that $A \succ B$ is similar.

Suppose that the outcomes are ordered $a_1 > b_1 > c_1 > a_2 > b_2 > c_2$. The MC's instructions could be the following: Start by comparing A and B. Then,

1. If the realization of A is a_1 continue with comparing A and C. Whatever is the realization of C, $A \succ C$ and hence no cycle.

2. If the realizations are a_2 and b_1 ($B \succ A$) then continue by comparing B and C. Whatever is the realization of C, $B \succ C$, hence no cycle.

3. If the realizations are a_2 and b_2 then $A \succ B$. Proceed to compare B and C. If the realization is c_1 then $C \succ B$, hence no cycle. If the realization is c_2 then $B \succ C$ and when A and C are compared (using c_2) then $A \succ C$ and there is no cycle. ∎

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