Inverse Elasticity Rule in a Production Efficiency Problem*

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Abstract

Diamond and Mirrlees (1971) and Dasgupta and Stiglitz (1972) show that production efficiency is achieved under the optimal commodity tax when profit income is zero. Here, we consider the simplest possible model to analyze production efficiency in the presence of profit income: a tax reform problem in an economy with a representative consumer, two goods, and two firms with decreasing returns to scale technologies. We show that differentiating a uniform producer tax according to the inverse elasticity rule, while keeping government revenue constant, reduces additional distortions caused by the presence of profit income and improves social welfare.

Keywords: production efficiency, inverse elasticity rule, pure profit, optimal commodity taxation, tax reform

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1 Introduction

The theory of second-best says that if distortions in a market cannot be removed, then removing distortions in other markets may not necessarily improve social welfare (Lipsey and Lancaster 1956). This implies that differential taxation on different producers may be desirable even for the same commodities. Nonetheless, Diamond and Mirrlees (1971) prove that if technologies exhibit constant returns to scale, production efficiency is satisfied under a mild condition when the optimal commodity taxation is imposed (see also Myles 1995). In contrast, Dasgupta and Stiglitz (1972) show that if technologies exhibit decreasing returns to scale, production efficiency is no longer desirable unless the profit tax rate is 100 percent, since profit income works as a wedge between the budget line and the feasibility constraint. Mirrlees (1972) further clarifies the relationship between profit income and production efficiency. Assuming no profit income for consumers, Weymark (1979) provides a necessary and sufficient condition for production efficiency, extending the Diamond-Mirrlees sufficient condition.

Still, it appears that the role of production inefficiency in the presence of profit income has not yet been sufficiently explored. In this note, we revisit this production efficiency problem using a tax reform approach as in Corlett and Hague (1953). We consider the simplest possible economy for the analysis — a two-good (leisure and a commodity), two-firm economy with decreasing returns to scale technologies. Starting from a uniform producer tax, we differentiate producer tax rates between the firms. Our two good assumption isolates the production efficiency problem from the optimal commodity tax problem, generating clear-cut results. Assuming that producer tax rates are not too high and that the profit tax rate is less than 100 percent, raising the producer tax on the firm with (relatively) more inelastic supply and adjusting the other firm's tax rate to keep the tax revenue constant improves welfare (Proposition 1). This result reminds us of ones from partial equilibrium analyses. However, our overall result is more subtle than this. Dasgupta and Stiglitz (1972) showed that production efficiency is optimal as long as the profit income tax is 100 percent even if there are pure profits in the economy. This important relationship between production efficiency and profit income is not considered in the partial equilibrium analysis of the inverse elasticity rule. However, we can further show that the inverse elasticity policy reform increases total output, and reduces the consumer price (Proposition 2), and reduces total profit (Proposition 3), and thus decreases profit income. In a commodity tax problem, the consumer having positive profit income is another source of distortion — a wedge between the feasible set and the budget set. Thus, the fact that the above policy reduces total profit means that it reduces one source of distortion. This is clearly the effect based on the theory of second best in the general equilibrium model. If profit income is zero then production efficiency is optimal (Diamond and Mirrlees 1971, and Dasgupta and Stiglitz 1972), while if it is positive, production inefficiency is desirable to fix additional distortion caused by a wedge caused by the presence of profit income. That is, our Proposition 3 connects the important result in partial equilibrium analysis and the one of the theory of second best.

2 The Model

The economy consists of a representative consumer, two goods (leisure as the numeraire good and a commodity), and two firms. The consumer's utility function is written as

$$U = U(x^0, x),$$

where x^0 and x are net leisure consumption and commodity consumption, respectively. Net leisure consumption is negative if the representative consumer supplies labor: in fact, hours of labor supply is exactly the same as negative net leisure consumption, $-x^0$. It is convenient for us to use the expenditure function (see Hatta 1986):

$$E(q, u) \equiv \min_{(x^0, x)} x^0 + qx \quad \text{subject to} \quad U(x^0, x) = u,$$

where q is the consumer price of the commodity. We denote compensated demand functions of (net) leisure and the commodity by $x^0(q, u)$ and $x(q, u)$, respectively. We assume that both leisure and the commodity are normal: $x^0_u > 0$ and $x_u > 0$, where a subscript denotes the derivative with respect to the variable. Clearly, we have

$$E(q, u) = x^0(q, u) + qx(q, u).$$

There are two firms, j and k. Each firm uses labor as its input and produces the commodity with decreasing returns to scale technology. Firm j's profit function is written as

$$\pi^j(p^j) = z^{j0}(p^j) + p^j z^j(p^j),$$

where p^j is firm j's producer price (which is different from the consumer price with producer tax distortions), and $z^{j0}(\cdot)$ and $z^j(\cdot)$ are (net) supply functions of labor and the commodity, respectively. Firm k's profit function can be written similarly. The consumer's budget constraint is written as

$$E(q,u) = (1-\tau)\left(\pi^j(p^j) + \pi^k(p^k)\right), \tag{1}$$

where $0 \leq \tau < 1$ is an exogenous profit tax rate. If $\tau = 1$, then this is a 100% profit tax case. The government needs a certain amount of tax revenue $T > 0$ measured by labor, and it is financed by the producer and profit taxes. Thus, market clearing conditions are

$$x^0(q,u) + T = z^{j0}(p^j) + z^{k0}(p^k), \tag{2}$$

and

$$x(q,u) = z^j(p^j) + z^k(p^k). \tag{3}$$

Firm-specific producer taxes t^j and t^k are described by

$$q = p^j + t^j = p^k + t^k. \tag{4}$$

There is no additional commodity tax. The government's budget constraint,

$$t^j z^j(p^j) + t^k z^k(p^k) + \tau\left(\pi^j(p^j) + \pi^k(p^k)\right) = T$$

can be derived by the Walras Law from the other four equations.

3 The Results

We will consider a policy that raises t^j while slightly adjusting t^k in order to keep the tax revenue constant at T, starting from a uniform producer tax ($t^j = t^k$). If this policy improves the social welfare (the representative consumer's utility), then it means that production inefficiency is desirable since $p^j \neq p^k$ implies that production is done in the interior of the aggregate production set.

Totally differentiating the system of equations (1), (2), and (3) with respect to u, q, t^k, and t^j, we obtain

$$\begin{pmatrix} E_u & \tau x & (1-\tau)\,z^k \\ x_u^0 & x_q^0 - z_p^0 & z_p^{k0} \\ x_u & x_q - z_p & z_p^k \end{pmatrix} \begin{pmatrix} du \\ dq \\ dt^k \end{pmatrix} = \begin{pmatrix} -(1-\tau)\,z^j \\ -z_p^{j0} \\ -z_p^j \end{pmatrix} dt^j,$$

where subscripts denote derivatives and $z_p^0 = z_p^{j0} + z_p^{k0}$ and $z_p = z_p^j + z_p^k$. In order to conduct comparative static exercises, we need the sign of the determinant of the matrix in the LHS, D. Following the approach by Corlett and Hague (1953) and Hatta (1986), we can determine the sign of D.

Lemma. *Suppose that a marginal increase in t^k increases the government tax revenue. Then, $D < 0$ holds.*

Proof. Totally differentiating equations (1), (2), and (3) with respect to u, q, T, and t^k, we have

$$\begin{pmatrix} E_u & \tau x & 0 \\ x_u^0 & x_q^0 - z_p^0 & 1 \\ x_u & x_q - z_p & 0 \end{pmatrix} \begin{pmatrix} du \\ dq \\ dT \end{pmatrix} = \begin{pmatrix} -(1-\tau) z^k \\ -z_p^{k0} \\ -z_p^k \end{pmatrix} dt^k.$$

The Cramer rule tells us

$$\begin{aligned} \frac{dT}{dt^k} &= \frac{\begin{vmatrix} E_u & \tau x & -(1-\tau) z^k \\ x_u^0 & x_q^0 - z_p^0 & -z_p^{k0} \\ x_u & x_q - z_p & -z_p^k \end{vmatrix}}{\begin{vmatrix} E_u & \tau x & 0 \\ x_u^0 & x_q^0 - z_p^0 & 1 \\ x_u & x_q - z_p & 0 \end{vmatrix}} \\ &= \frac{-D}{\tau x x_u - E_u (x_q - z_p)} \\ &= \frac{D}{E_u (x_q - z_p) - \tau x x_u}. \end{aligned}$$

Since the commodity is normal, $x_u > 0$. Moreover, concavity of the expenditure function and convexity of the profit functions imply $x_q < 0$ and $z_p > 0$. Thus, we conclude $\frac{dT}{dt^k} > 0$ if and only if $D < 0$.□

Unless the economy is in the Laffer situation, the condition of Lemma is satisfied.[1] We will state our main comparative static result by using each firm's supply elasticity: $\epsilon^j = \frac{p^j z_p^j}{z^j}$ and $\epsilon^k = \frac{p^k z_p^k}{z^k}$.

[1] Another necessary and sufficient condition for $D < 0$ when $t^j = t^k$ is $\frac{t}{q} < \frac{\epsilon^k(\eta + \tau\alpha) + (1-\tau)(\eta + \alpha\epsilon)}{\epsilon^k(\eta+\epsilon) + \tau\epsilon^k}$, where $\eta = \frac{-q x_q}{x}$, $\epsilon = \frac{p z_p}{z}$, and $\alpha = \frac{q x_u}{E_u}$ are compensated price elasticity of demand, price elasticity of suppy, and marginal propensity for the commodity, respectively. Thus, unless t is too high, the inequality tends to be satisfied and $D < 0$ holds.

Proposition 1 *Suppose that a marginal increase in t^k increases the government tax revenue. Suppose further that $t^j = t^k$ holds initially. Then, raising t^j accompanied by adjusting t^k to keep the revenue constant improves the social welfare if and only if firm j's supply curve is relatively more inelastic: $\epsilon^j < \epsilon^k$.*

Proof. Since the compensated demand and supply functions are homogeneous of degree zero with price, we obtain

$$x_q^0 + qx_q = 0,$$

$$z_p^{j0} + p^j z_p^j = z_p^{k0} + p^k z_p^k = 0.$$

Using the Cramer rule with the above relationships and $p^j = p^k = p$, we obtain

$$\frac{du}{dt^j} = \frac{N^u}{D},$$

where

$$\begin{aligned}
N^u &= \begin{vmatrix} -(1-\tau) z^j & \tau x & (1-\tau) z^k \\ -z_p^{j0} & x_q^0 - z_p^0 & z_p^{k0} \\ -z_p^j & x_q - z_p & z_p^k \end{vmatrix} \\
&= \begin{vmatrix} -(1-\tau) z^j & \tau x & (1-\tau) z^k \\ p^j z_p^j & -qx_q + pz_p & p^k z_p^k \\ -z_p^j & x_q - z_p & z_p^k \end{vmatrix} \\
&= \begin{vmatrix} -(1-\tau) z^j & \tau x & (1-\tau) z^k \\ 0 & -tx_q & 0 \\ -z_p^j & x_q - z_p & z_p^k \end{vmatrix} \\
&= tx_q (1-\tau) \left(z^j z_p^k - z^k z_p^j \right) \\
&= -tx_q (1-\tau) \frac{z^j z^k}{p} \left(\epsilon^j - \epsilon^k \right).
\end{aligned}$$

Since $D < 0$ and $x_q < 0$, we conclude that $\epsilon^j < \epsilon^k$ holds if and only if $\frac{du}{dt^j} > 0$. ■

This proposition corresponds to the deadweight loss minimization arguments in the partial equilibrium analysis, asserting the desirability of production inefficiency. This proposition holds unless $\tau = 1$ or $t = 0$. The former case corresponds to the production efficiency result in the absence of pure profit in Diamond and Mirrlees (1971), Dasgupta and Stiglitz (1972), and Weymark (1979), and the latter corresponds to the first welfare theorem.

An intuitive reason that the policy described in Proposition 1 improves the social welfare can be found in the inverse elasticity rule in the partial equilibrium analysis. If firm j has a more inelastic supply curve, imposing a higher producer tax on firm j by reducing t^k reduces the deadweight loss. We can also find another reason for the result from a somewhat more general equilibrium point of view. With the government's tax distortions, the firms' total commodity output is suppressed. If a tax reform can increase the production level, that would diminish the distortions caused by taxes. The following proposition confirms this.

Proposition 2 *Suppose that a marginal increase in t^k increases the government tax revenue. Suppose further that $t^j = t^k$ holds initially. Then, raising t^j accompanied by adjusting t^k to keep the revenue constant decreases the consumer price q and increases the total commodity output, if and only if firm j's supply curve is relatively more inelastic: $\epsilon^j < \epsilon^k$.*

Proof. Using the Cramer rule, we obtain

$$\frac{dq}{dt^j} = \frac{N^q}{D} \quad \text{and} \quad \frac{dt^k}{dt^j} = \frac{N^k}{D}$$

where

$$\begin{aligned} N^q &= \begin{vmatrix} E_u & -(1-\tau) z^j & (1-\tau) z^k \\ x_u^0 & -z_p^{j0} & z_p^{k0} \\ x_u & -z_p^j & z_p^k \end{vmatrix} = \begin{vmatrix} E_u & -(1-\tau) z^j & (1-\tau) z^k \\ x_u^0 + p x_u & 0 & 0 \\ x_u & -z_p^j & z_p^k \end{vmatrix} \\ &= \left(x_u^0 + p x_u\right)(1-\tau)\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right) \end{aligned}$$

$$N^k = \begin{vmatrix} E_u & \tau x & -(1-\tau) z^j \\ x_u^0 & x_q^0 - z_p^0 & -z_p^{j0} \\ x_u & x_q - z_p & -z_p^j \end{vmatrix}.$$

Thus, we have

$$\frac{dq}{dt^j} = \frac{1}{D}\left(x_u^0 + p x_u\right)(1-\tau)\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right),$$

and the consumer price q decreases if and only if $\epsilon^j < \epsilon^k$ under our assumptions. Now, totally differentiating total output with t^j, t^k, and q, we obtain

$$dz = -z_p^j dt^j - z_p^k dt^k + z_p dq.$$

Thus, $\frac{dz}{dt^j}$ can be written as

$$\begin{aligned}
\frac{dz}{dt^j} &= -z_p^j - z_p^k \frac{N^k}{D} + z_p \frac{N^q}{D} \\
&= \frac{1}{D}\left[-z_p^j D - z_p^k N^k + (1-\tau)\left(z_p^j + z_p^k\right)\left(x_u^0 + p x_u\right)\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right)\right].
\end{aligned}$$

Calculating $-z_p^j D - z_p^k N^k$, we obtain:

$$\begin{aligned}
&-z_p^j D - z_p^k N^k \\
= & -z_p^j \begin{vmatrix} E_u & \tau x & (1-\tau) z^k \\ x_u^0 & x_q^0 - z_p^0 & z_p^{k0} \\ x_u & x_q - z_p & z_p^k \end{vmatrix} - z_p^k \begin{vmatrix} E_u & \tau x & -(1-\tau) z^j \\ x_u^0 & x_q^0 - z_p^0 & -z_p^{j0} \\ x_u & x_q - z_p & -z_p^j \end{vmatrix} \\
= & -z_p^j \begin{vmatrix} E_u & \tau x & (1-\tau) z^k \\ x_u^0 + p x_u & x_q^0 + p x_q & 0 \\ x_u & x_q - z_p & z_p^k \end{vmatrix} - z_p^k \begin{vmatrix} E_u & \tau x & -(1-\tau) z^j \\ x_u^0 + p x_u & x_q^0 + p x_q & 0 \\ x_u & x_q - z_p & -z_p^j \end{vmatrix} \\
= & \begin{vmatrix} E_u & \tau x & (1-\tau)\left(-z_p^j z^k + z_p^k z^j\right) \\ x_u^0 + p x_u & x_q^0 + p x_q & 0 \\ x_u & x_q - z_p & -z_p^j z_p^k + z_p^k z_p^j \end{vmatrix} \\
= & \begin{vmatrix} E_u & \tau x & (1-\tau)\left(-z_p^j z^k + z_p^k z^j\right) \\ x_u^0 + p x_u & x_q^0 + p x_q & 0 \\ x_u & x_q - z_p & 0 \end{vmatrix} \\
= & (1-\tau)\left(-z_p^j z^k + z_p^k z^j\right)\left[\left(x_u^0 + p x_u\right)\left(x_q - z_p\right) - x_u\left(x_q^0 + p x_q\right)\right] \\
= & (1-\tau)\left[\left(x_u^0 + p x_u\right)\left(x_q - z_p\right) + x_u t x_q\right]\frac{z^k z^j}{p}\left(\epsilon^k - \epsilon^j\right).
\end{aligned}$$

Putting them together, we have

$$\begin{aligned}
\frac{dz}{dt^j} &= -z_p^j - z_p^k \frac{N^k}{D} + z_p \frac{N^q}{D} \\
&= \frac{(1-\tau)}{D} \left[\left(x_u^0 + p x_u \right) (x_q - z_p) + x_u t x_q + z_p \left(x_u^0 + p x_u \right) \right] \frac{z^k z^j}{p} \left(\epsilon^k - \epsilon^j \right) \\
&= \frac{(1-\tau)}{D} \left[\left(x_u^0 + p x_u \right) x_q + x_u t x_q \right] \frac{z^k z^j}{p} \left(\epsilon^k - \epsilon^j \right) \\
&= \frac{(1-\tau) E_u x_q}{D} \frac{z^k z^j}{p} \left(\epsilon^k - \epsilon^j \right).
\end{aligned}$$

Since $D < 0$ and $x_q < 0$, we have $\frac{dz}{dt^j} > 0$ if and only if $\epsilon^k > \epsilon^j$. ■

Finally but not the least, we check to see how the consumer's profit income is affected by the policy. The presence of profit income works as a wedge between the consumer's budget constraint and the feasibility condition, so it is natural to conjecture that a welfare-improving policy reform reduces profit income. The following proposition confirming this intuition is our main result that bridges the theory of second best and the inverse elasticity rule. It requires an additional mild regularity condition: $1 > \theta(\alpha + \eta)$, where $\theta = \frac{t}{q}$ (ad valorem producer tax rate), $\alpha = \frac{q x_u}{E_u}$ (marginal propensity for the commodity), and $\eta = -\frac{q x_q}{x}$ (price elasticity of commodity demand). This condition holds if θ is not too high.

Proposition 3 *Suppose that a marginal increase in t^k increases the government tax revenue, and that $1 > \theta(\alpha + \eta)$ is satisfied. Suppose further that $t^j = t^k$ holds initially. Then, raising t^j accompanied by adjusting t^k to keep the revenue constant decreases the consumer's profit income, if and only if firm j's supply curve is relatively more inelastic: $\epsilon^j < \epsilon^k$.*

Proof. By Hotelling's lemma, we have $\frac{d\pi^j}{dp^j} = z^j$ and $\frac{d\pi^k}{dp^k} = z^k$. Totally differentiating the total profit produces

$$\begin{aligned}
d\pi &= d(\pi^j + \pi^k) \\
&= z^j dp^j + z^k dp^k \\
&= z^j \left(dq - dt^j \right) + z^k \left(dq - dt^k \right).
\end{aligned}$$

Thus, the result of the comparative static exercise is

$$\begin{aligned}\frac{d\pi}{dt^j} &= \left(z^j+z^k\right)\frac{dq}{dt^j}-z^j-z^k\frac{dt^k}{dt^j}\\ &= x\frac{dq}{dt^j}-z^j-z^k\frac{dt^k}{dt^j}.\end{aligned}$$

The latter two terms can be written as

$$\begin{aligned}&-z^j-z^k\frac{dt^k}{dt^j}\\ =\ & \frac{1}{D}\left[-z^jD-z^kN^k\right]\\ =\ & \frac{1}{D}\left[-z^j\begin{vmatrix}E_u & \tau x & (1-\tau)\,z^k\\ x_u^0 & x_q^0-z_p^0 & z_p^{k0}\\ x_u & x_q-z_p & z_p^k\end{vmatrix}-z^k\begin{vmatrix}E_u & \tau x & -(1-\tau)\,z^j\\ x_u^0 & x_q^0-z_p^0 & -z_p^{j0}\\ x_u & x_q-z_p & -z_p^j\end{vmatrix}\right]\\ =\ & \frac{1}{D}\left[-z^j\begin{vmatrix}E_u & \tau x & (1-\tau)\,z^k\\ x_u^0+px_u & x_q^0+px_q & 0\\ x_u & x_q-z_p & z_p^k\end{vmatrix}-z^k\begin{vmatrix}E_u & \tau x & -(1-\tau)\,z^j\\ x_u^0+px_u & x_q^0+px_q & 0\\ x_u & x_q-z_p & -z_p^j\end{vmatrix}\right]\\ =\ & \frac{1}{D}\begin{vmatrix}E_u & \tau x & 0\\ x_u^0+px_u & x_q^0+px_q & 0\\ x_u & x_q-z_p & -z^jz_p^k+z^kz_p^j\end{vmatrix}\\ =\ & \frac{1}{D}\left[\left(E_u\left(x_q^0+px_q\right)-\tau x\left(x_u^0+px_u\right)\right)\frac{z^jz^k}{p}\left(\epsilon^j-\epsilon^k\right)\right].\end{aligned}$$

Putting things together, we obtain:

$$
\begin{aligned}
\frac{d\pi}{dt^j} &= x\frac{dq}{dt^j} - z^j - z^k\frac{dt^k}{dt^j} \\
&= \frac{1}{D}\Bigg[x\left(x_u^0 + px_u\right)(1-\tau)\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right) \\
&\quad + \left(-E_u\left(x_q^0 + px_q\right) + \tau x\left(x_u^0 + px_u\right)\right)\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right)\Bigg] \\
&= \frac{1}{D}\left[x\left(x_u^0 + px_u\right) - E_u\left(x_q^0 + px_q\right)\right]\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right) \\
&= \frac{xE_u}{D}\left[\frac{x_u^0 + px_u}{E_u} - \frac{x_q^0 + px_q}{x}\right]\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right) \\
&= \frac{xE_u}{D}\left[1 - \frac{tx_u}{E_u} + \frac{tx_q}{x}\right]\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right) \\
&= \frac{xE_u}{D}\left[1 - \theta\alpha - \theta\eta\right]\frac{z^j z^k}{p}\left(\epsilon^k - \epsilon^j\right).
\end{aligned}
$$

The sign of the contents of the above bracket tends to be positive (this term is exactly the same term that is assumed to be positive in Corlett-Hague 1953 and Hatta 1986), especially if t is not too high. Since $D < 0$, we have the desired result. ■

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