

School Admissions Reform in Chicago and England: Comparing Mechanisms by their Vulnerability to Manipulation*

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Abstract

In Fall 2009, officials from Chicago Public Schools changed their assignment mechanism for coveted spots at selective college preparatory high schools midstream. After asking about 14,000 applicants to submit their preferences for schools under one mechanism, the district asked them re-submit their preferences under a new mechanism. Officials were concerned that “high-scoring kids were being rejected simply because of the order in which they listed their college prep preferences” under the abandoned mechanism. What is somewhat puzzling is that the new mechanism is also manipulable. This paper introduces a method to compare mechanisms based on their vulnerability to manipulation. Under our notion, the old mechanism is more manipulable than the new Chicago mechanism. Indeed, the old Chicago mechanism is at least as manipulable as any other plausible mechanism. A number of similar transitions between mechanisms took place in England after the widely popular Boston mechanism was ruled illegal in 2007. Our approach provides support for these and other recent policy changes involving matching mechanisms.

KEYWORDS: student assignment, Boston mechanism, matching, strategy-proofness

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1 Introduction

In the last few years, there have been dramatic changes in the way students are placed into publicly-funded schools worldwide. Two of the most recent developments come from education authorities in England and in Chicago, Illinois, the third largest U.S. school district. These changes were based in part on the desire to simplify the strategic aspects of the admissions process for participants. Unlike other reforms in Boston and New York City, they did not involve the direct intervention of economists as far as we know. As a result, they also provide some indication as to how policymakers and the public perceive particular mechanisms.

In England, forms of school choice have been available for at least three decades. The nationwide 2003 School Admissions Code mandated that Local Authorities, an operating body similar to a U.S. school district, coordinate their admissions practices. This reform provided families with a single application form and established a common admissions timeline, leading to a March announcement of placements for anxious 10 and 11 year-olds on “National Offer Day.” The next nationwide reform came with the 2007 School Admissions Code. While strengthening the enforcement of admissions rules, this legal code also prohibited authorities from using unfair oversubscription criteria, as described in Section 2.13:

In setting oversubscription criteria the admission authorities for all maintained schools must not:

give priority to children according to the order of other schools named as preferences by their parents, including 'first preference first' arrangements.

More specifically, a first preference first system is any “oversubscription criterion that gives priority to children according to the order of other schools named as a preference by their parents, or only considers applications stated as a first preference” (School Admissions Code, 2007, Glossary, p. 118). The 2007 Admissions Code outlaws use of this system at more than 150 Local Authorities across the country, and this ban continues with the 2010 Code.

The best known first preference first system is the Boston mechanism, employed by the Boston Public Schools until it was abandoned in 2005 (Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu, Pathak, Roth and Sönmez 2005). To obtain a school place in England, a family must submit an application to the Local Authority in their region. Oversubscription

criteria at schools depend both on the type of school and the type of student.¹ Under a first preference first system, the priority order of a school is modified based how a student ranked it. Just as in the Boston mechanism, this feature leads to potential strategic issues for parents deciding how to rank schools, which apparently were behind the nationwide ban. The rationale stated by England’s Department for Education and Skills is that “the ‘first preference first’ criterion made the system unnecessarily complex to parents” (School Code 2007, Foreword, p. 7). A story in the *Guardian*, a British newspaper, emphasizes that “the new School Admissions Code will end the practice called ‘first preference first’ which forces many parents to play an ‘admissions game’ with their children’s future, and unnecessarily complicates the admissions system” (Smith 2007).

Prior to the 2007 law, many Local Authorities experimented with their admissions procedures. The Pan-London Admissions scheme, which coordinated placements for Greater London adopted an “equal preference system.” According to Pennell, West, and Hind (2006), in this system “Local Authorities consider all preferences without reference to the rank order made by parents.... However, if there is more than one potential offer available to an applicant the highest ranked preference is used.” Pupils in London are allowed to rank up to six school choices, even though there are many more schools in Greater London.

The best known equal preference system is the student-optimal stable mechanism (Gale and Shapley 1962) which is currently used to place students in Boston and New York City. In England, with equal preference, schools may need to forecast enrollment if they wish to avoid vacancies when offered students also obtain a more preferred choice.² Some schools will not admit any unqualified students and may keep seats vacant (Coldron 2011). The report of the Pan London Board and London Inter-Authority Admissions Group states that equal preference scheme was designed to “make the admissions system fairer” and “create a simpler system for

¹In England, school types include community schools (which are similar to U.S. neighborhood schools), faith schools, grammar schools (which rely on examinations for entrance), and voluntary aided and foundation schools (which are neighborhood schools where the school’s land and building are granted by another organization.) An example of a common school priority structure is from Newcastle where students are ordered as follows: students in public care, students in feeder schools, siblings, students with particular medical conditions, and students who live closest to the school.

²The Department of Children, Schools, and Families provides advice for this purpose. See, e.g., “Guide to forecasting pupil numbers in school place planning” issued in January 2010.

parents” (Association of London Government 2005).

In Newcastle, policy discussions about first preference first versus equal preferences date back to 2003. At that time, Newcastle used a version of the Boston mechanism that allowed families to list three schools. Following the Newcastle Admissions Forum’s recommendation that “the equal preference system was more parent-friendly as it would reduce anxiety among parents as they can set out their ranked preferences without having to calculate the chances of their getting a place,” the Boston mechanism was abandoned in favor of a version of the student-optimal stable mechanism where applicants can rank 3 choices in 2005 (Young 2003). By 2010, Newcastle was using a version of student-optimal stable mechanism that allows applicants to rank four choices among 97 schools.³

Truth-telling is a weakly dominant strategy for applicants in the student-optimal stable mechanism when there is no constraint on the number of choices a student can rank. However, when only a limited number of choices are allowed by the mechanism, this result no longer holds. The logic is straightforward: when students cannot rank as many schools as they wish, they should only rank the subset of choices where they can potentially obtain an offer. According to Coldron, et. al (2008)’s comprehensive survey of Local Authorities, 101 used an equal preference system, while 47 used first preferences first in 2006. Over half of the systems using equal preference allow for more than three choices, while less than ten percent of authorities with the first preference first system allowed more than three choices.

With the 2007 law change, all 47 authorities had to change their admissions policy. We have been able to find documentation on what happened in some regions. Brighton and Hove moved from the Boston mechanism that allows three choices to student-optimal stable mechanism that also allows three choices, even though there are at least nine choice schools (Allen, Burgess, McKenna, 2010). Authorities stated it “will hopefully eliminate the need for tactical preferences” (Brighton & Hove City Council 2007). In Kent, the U.K. Schools adjudicator overruled the Boston mechanism that allowed for three choices and the district now uses a student-optimal stable mechanism where four choices are allowed (Office of Schools Adjudicator, 2006).⁴ In both school districts, and no doubt many others, even though there was a switch

³The current details on the admissions system are available at http://www.newcastle.gov.uk/core.nsf/a/adm_admissionsonline, Last accessed January 30, 2011.

⁴There are 99 secondary schools in Kent (Turner and Hohler 2010).

to a mechanism that is strategy-proof when unconstrained version of the mechanism is used, the district constrained the number of choices considered. As a result, these new mechanisms are still vulnerable to strategic manipulation.

While Local Authorities were given some time to adjust their admissions rules in England, the adoption of a new mechanism was considerably more abrupt in Chicago. The district abandoned their mechanism for placing students into selective high schools halfway through running it in 2009. That is, after participants had submitted preferences under one mechanism, but before announcing placements, Chicago Public Schools asked participants to resubmit their preferences under another mechanism a few months later.

This high profile change is the only case of a midstream change of an assignment mechanism we are aware of, and is stunning to us given the high-stakes involved. The abandoned mechanism prioritized applicants based on how schools were ranked and is also a form of the Boston mechanism. Under it, Chicago authorities argued that “high-scoring kids were being rejected simply because of the order in which they listed their college prep preferences.” The new mechanism does not prioritize applicants in this way and is a special case of the student-optimal stable mechanism. Both mechanisms place constraints on the number of choices they consider. Hence, as in England, Chicago moved from one manipulable mechanism to another. These changes are seen as improvements by the communities that adopted them, suggesting perceptions of differing degrees of vulnerability to manipulation.

In this paper, we introduce a methodology to compare two manipulable mechanisms based on their vulnerability to manipulation. Our approach is simple. Let ψ and φ be two direct mechanisms. We say mechanism ψ is at least as manipulable as mechanism φ if whenever mechanism φ is manipulable, mechanism ψ is manipulable as well. Among other applications, we show that the recent changes in England and Chicago involve abandoning more manipulable mechanisms, providing support for these reforms.

The next section provides the general framework and our definitions. Section 3 provides more details about Chicago and illustrates how their old mechanism is at least as manipulable as any plausible mechanism. In Section 4 we turn our attention to student-optimal stable mechanism and show that the fewer the number of choices a student can make, the more vulnerable the mechanism is to manipulation. In Section 5 we return to Boston mechanism

and analyze policy changes in England. In Section 6 we illustrate the methodology for two-sided matching models of labor market clearinghouses, and in the last section we conclude.

2 General Framework

2.1 Primitives

There are a finite number of players indexed by $i = 1, \dots, N$ and a finite set of outcomes A . Each player has a preference relation R_i defined over the set of outcomes, where P_i is the strict counterpart of R_i . Let $R = (R_i)$ and $P = (P_i)$ denote the profile of weak and strict preferences, respectively. We adopt the convention that R_{-i} are the preferences of players other than player i , and define P_{-i} similarly. We sometimes refer to a preference profile R (or P) as a problem, fixing the set of players and outcomes.

A direct mechanism is a function, φ , that is a single-valued mapping of a preference profile to an element in A . Let $\varphi(R)$ denote the outcome produced by mechanism φ under R . Of course, we cannot always expect players to be truthful when reporting their preferences. This motivates the following definition.

Definition 1. *A mechanism φ is **manipulable by player i** at problem R if there exists a preference R'_i such that $\varphi(R'_i, R_{-i}) P_i \varphi(R)$.*

A mechanism is manipulable by a player at a problem if he can profit by misrepresenting his preferences. Observe that each mechanism induces a natural game form where the strategy space is the set of preferences for each player and the outcome is determined by the mechanism. A mechanism is **strategy-proof** if truthful preference revelation is a dominant strategy of this game for any player. Equivalently, a mechanism is strategy-proof if it is not manipulable by any player at any problem.

We next present a notion to compare mechanisms by their vulnerability to manipulation.

Definition 2. *A mechanism ψ is **at least as manipulable** as mechanism φ if for any problem where mechanism φ is manipulable, mechanism ψ is also manipulable.*

Two mechanisms can be equally manipulable if they are manipulable for exactly the same set of problems. We next consider the situations where the set of problems a mechanism is manipulable is a strict subset of the set of problems another mechanism is manipulable.

Definition 3. *A mechanism ψ is **more manipulable** than mechanism φ if*

- i) ψ is at least as manipulable as φ , and*
- ii) there is at least one problem where ψ is manipulable although φ is not.*

If mechanism φ is strategy-proof while mechanism ψ is not, then mechanism ψ is more manipulable than mechanism φ . Our main interest is the case where neither ψ nor φ are strategy-proof. Our notion is somewhat conservative in the sense that we deem a mechanism to be more manipulable than another only if there is strict inclusion of profiles where they can be manipulated. For example, it is more demanding to compare mechanism with this notion than an alternative notion that simply counts the number of profiles where the mechanisms are manipulable. However, this fact also means that any comparison we can make under our notion provides a stronger result.

While our notion makes no explicit reference to an equilibrium concept, it is possible to provide an equilibrium interpretation of this notion. Consider the preference revelation game induced by a direct mechanism. The contrapositive of the first part of the definition implies that for a problem, if ψ is not manipulable, then φ is not manipulable. This means that if at any problem, truth-telling is a Nash equilibrium of the preference revelation game induced by mechanism φ , it is also a Nash equilibrium of the preference revelation game induced by mechanism ψ (even though the converse does not hold). Recall that if truth-telling is a Nash equilibrium of the preference revelation game induced by mechanism φ for all problems, then φ is strategy-proof (see, e.g., Austen-Smith and Banks 2005).

While these definitions are general, in the applications in this paper, we focus on assignment or matching problems. In such problems, A is the set of possible assignments, each player has strict preferences, and we assume that each only cares about her own assignment. We let $\varphi_i(R)$ denote the assignment obtained by player i under report R .

2.2 Related literature

There is a large literature interested in studying how vulnerable mechanisms are to manipulation, so we only briefly mention two related contributions. First, there are papers which characterize the domains under which a particular mechanism is not manipulable (see, for instance, Barberá (2010) for a recent survey on strategy-proof social choice rules.) When interpreted as a comparison of the sets of problems where the preference revelation game has a Nash equilibrium in truthful strategies, the definition of weakly more manipulable involves a comparison of domains. Many papers in this earlier literature characterize non-manipulable domains for specific mechanisms, while our aim is to make comparisons across mechanisms.

Next, there is a literature which investigates vulnerability to manipulation in social choice problems. The idea of making comparisons across mechanisms is related to the comparison of voting rules in Dasgupta and Maskin (2008). They show that if a voting rule satisfies various axioms for a set of preferences, then simple majority voting rule also satisfies those axioms on the same set of preferences. Other than their interest in voting rules, another major difference is that we compare mechanisms based on the extent to which they encourage manipulation, while Dasgupta and Maskin focus on non-strategic properties.

3 Reform at Chicago's Public Schools in 2009

To describe the assignment problem for Chicago's selective high schools, we begin by introducing some notation. Suppose there are I students and N schools. Each school s has capacity q_s , so total capacity is $Q = \sum_{s=1}^N q_s$. We assume that $I > Q$ so the seats are in short supply. In 2009, there were over 14,000 applicants for the 9 selective CPS high schools, consisting of 3,040 seats.⁵

Each student i has a strict preference ordering P_i over schools and being unassigned. Since

⁵In practice, Chicago Public Schools splits selective high schools into five parts. The first 'ranked' part is reserved for all applicants. The other four groups are reserved for students from particular neighborhoods, where students are ordered by their test scores within their neighborhood group. To implement this the district simply modifies the rank order list of participants to accommodate this neighborhood constraint. That is, a student who ranks a school is interpreted by the assignment algorithm to rank both the 'ranked' part and the part in their neighborhood tier in that order. We abstract away from this modification because it does not affect our analysis.

each student must take an admissions test as part of their application, each student also has a composite score. We assume that no two students have the same composite score. In practice, if two students have the same test scores, the younger student is coded by CPS as having a higher composite score. The outcome of the admissions process is a matching μ , a function which maps each student either to her assigned school or to being unassigned.⁶ Let $\mu(i)$ denote the assignment of student i .

The mechanism that was abandoned in Fall 2009 works as follows:

Step 1: In the first round, only the first choices of students are considered. At each school, students who rank the school as their first choice are assigned one at a time according to their composite score until either there are no students who have ranked the school as their first choice left or there are no additional seats at the school.

Step ℓ : In round ℓ , each student who is not yet assigned is considered at her ℓ^{th} choice school. At each school with remaining seats, these students are assigned one at a time according to their composite score until either there are no students who have ranked the school as their ℓ^{th} choice left or there are no additional seats at the school.

Let CHI^k be the version of this mechanism that stops after k rounds. At CPS in Fall 2009, the district employed CHI^4 , with only 4 rounds. After eliciting preferences from applicants throughout the city, CPS officials computed assignments internally for discussion. The Chicago Sun-Times reported on November 12, 2009:

Poring over data about eighth-graders who applied to the city's elite college preps, Chicago Public Schools officials discovered an alarming pattern.

High-scoring kids were being rejected simply because of the order in which they listed their college prep preferences.

"I couldn't believe it," schools CEO Ron Huberman said. "It's terrible."

⁶If a student is unassigned to one of Chicago's selective high schools, she typically later enrolls in a neighborhood school, pursues other public school options such as charter and magnet schools, or leaves the public school system for either private or parochial schools.

CPS officials said Wednesday they have decided to let any eighth-grader who applied to a college prep for fall 2010 admission re-rank their preferences to better conform with a new selection system.

To help understand this quote, let us consider the situation for an applicant who is interested in applying to both Northside and Whitney Young, two of Chicago's most competitive college preps. Under CHI^k , it is possible that a student who ranks Northside and Whitney Young in that order ends up unassigned, while had she only ranked Whitney Young, she would have been assigned. If the student does not have a high enough composite score to obtain a placement at Northside, then when she ranks Northside and Whitney Young, she will only obtain a seat at Whitney Young if there seats left over after the first round. This scenario is highly unlikely given the popularity of that school, so the student ends up unassigned. Had the student only ranked Whitney Young, she would be considered alongside first choice applicants and her score may be high enough to obtain an offer of admissions there. Hence, it is possible for a high-scoring applicant to be rejected from a school because of the order in which preferences are listed.

The Chicago Sun-Times article continues:

Previously, some eighth-graders were listing the most competitive college preps as their top choice, forgoing their chances of getting into other schools that would have accepted them if they had ranked those schools higher, an official said.

Under the new policy, Huberman said, a computer will assign applicants to the highest-ranked school they qualify for on their new list.

"It's the fairest way to do it." Huberman told the Chicago Sun-Times editorial board Wednesday.

After eliciting preferences under mechanism CHI^4 but not reporting assignments to applicants, CPS officials announced new selection system that works as follows:

The student with the highest composite score is placed into her top choice. The student with the next highest score obtains her top choice among those she ranked with remaining capacity. If there are no schools left with remaining capacity, then the student is unassigned. The mechanism continues with the student with the next highest composite score until either all schools are filled or each student is processed.

Let SD^k be the version of the mechanism where only the first k choices of a student's rank order list are considered. When all choices on a student's rank order list are considered, it is well known that this **serial-dictatorship** mechanism is strategy-proof. Indeed, in the letter sent from CPS to all students who submitted an application under CHI⁴, the district explains:

... the original application deadline is being extended to allow applicants an opportunity to review and re-rank their Selection Enrollment High School choices, if they wish. It is recommended that applicants rank their school choices honestly, listing schools in the order of their preference, while also identifying schools where they have a reasonable chance of acceptance.

It would be unnecessary for students to consider what schools they have a reasonable chance of acceptance at if all choices were considered in this mechanism because the serial-dictatorship is strategy-proof. But when only a subset of choices are considered, a student's likelihood of acceptance becomes an important consideration, and a student may obtain a more preferred assignment by manipulating her preferences. Like the old Chicago mechanism, SD^k is also manipulable.

These two mechanisms are versions of widely studied assignment mechanisms for assigning students to schools. As we have already mentioned the new mechanism adopted in Chicago is a variant of a serial-dictatorship, where only the first four choices are considered. The old Chicago mechanism is a variant of the Boston mechanism that was used by Boston Public Schools until June 2005, with two important differences. First, although there are nine selective high schools in Chicago, the mechanism considers only the top four choices on a student's application form. This was not a feature of Boston's old school choice system, where all of a student's choices are potentially considered. Second, in Chicago the priority ranking of applicants is the same at

all schools and it is based on student composite scores. Under the Boston mechanism priority rankings of applicants potentially differ across schools. (In the case of Boston Public Schools, these rankings depend on sibling and walk zone priority.)

Any version of the Boston mechanism, including the version that is abandoned in Chicago, is manipulable. This feature is apparently the reason it was abandoned in Chicago. What is striking is that the new mechanism in Chicago is also manipulable; moreover, the school district appears to be aware of this fact since it explicitly suggests that applicants list schools where they have a reasonable chance of acceptance. Chicago Public Schools officials must have felt that the old mechanism is more vulnerable to manipulation. Our first result justifies this point of view.

Proposition 1. *Suppose there are at least k schools and let $k > 1$. The old Chicago mechanism (CHI^k) is more manipulable than truncated serial-dictatorship (SD^k) Chicago adopted in 2009.*

The proof of this result follows from a more general result we present in Section 5. It is remarkable to us that one of the largest public school districts abandoned a mechanism after about 14,000 participants submitted their preferences citing reasons like those in the newspaper article.⁷ The outrage expressed in the quotes from the Chicago Sun-Times suggests that the old mechanism was considered quite undesirable. Our next result allows to formalize the sense in which the old mechanism stands out among other reasonable mechanisms.

A potentially desirable goal of a student assignment mechanism is to produce an assignment which is fair according to some criteria. One basic notion in the context of priority-based student placement was proposed by Balinski and Sönmez (1999) and it is based on the well-known stability notion for two-sided matching markets: If student i prefers school s to her assignment $\mu(i)$ and under matching μ , either school s has a vacant seat or is assigned another student with lower composite score, then student i may have a legitimate objection to her assignment. An individually rational matching that cannot be blocked by such a pair (i, s) is a **stable** matching.

⁷We only became aware of the policy change in Chicago after this newspaper article. Since then, we have corresponded with CPS officials.

The notion of stability has long been studied in the literature on two-sided matching problems for both normative and positive reasons (see Roth and Sotomayor 1990). In the operations research literature, the stability condition is often treated a sort of feasibility requirement and two-sided matching problems are often described as the “stable matching problem.” And yet many school choice mechanisms are not stable mechanisms. That is perhaps why there is a long gap between the introduction of two-sided matching problems by Gale and Shapley (1962) and formal analysis of school choice mechanisms by Abdulkadiroğlu and Sönmez (2003). The old Chicago Public Schools mechanism (CHI^k) is one of those mechanisms that is not stable. A key reason why so many school districts use mechanisms that fail stability is that many school districts wish to pay special attention to the first choices of applicants. For instance, the currently illegal system in England is known as “first preference first.” This observation motivates the following definition.

Let matching μ be **strongly unstable** if there is a student i and school s such that student i is not assigned to s under μ , student i ’s top choice is school s , and either school s has a vacancy or there is another student assigned there with lower composite score. A matching is **weakly stable** if it is not strongly unstable. This notion is a relaxation of stability because a student is allowed to block a matching only with its top choice school. While there are quite a few school districts that use unstable mechanisms, we are unaware of any school district which prioritizes students at schools with some criteria and yet uses a mechanism that fails weak stability. In that sense weak stability is a very natural requirement in the context of priority based student admissions. In particular, both the old mechanism that is abandoned in Chicago in 2009 and its replacement are weakly stable.

We are ready to present our next result which justifies why Chicago Public Schools CEO Ron Huberman was so frustrated with the mechanism they abandoned in 2009 in the middle of the assignment process.

Theorem 1. *Suppose each student has a complete rank ordering and $k > 1$. The old Chicago Public Schools mechanism (CHI^k) is at least as manipulable as any weakly stable mechanism.*

We assume that students have complete rank orderings to keep the proof relatively simple. It is possible to state a version of this result without this assumption, but at the expense of

significant expositional complexity. This and all other proofs are contained in the appendix.

Based on Propositions 1 and Theorem 1, the new mechanism in Chicago is an improvement in terms of encouraging manipulation. That being said, the lack of efficiency in the new 2009 mechanism should be obvious to economists. Clearly any mechanism that restricts reported student preferences to only 4 choices suffers a potential efficiency loss. Moreover, it is possible to have a completely non-manipulable system (i.e a strategy-proof one) by considering all choices of applicants. These observations beg the question of what Chicago Public Schools should do in future years. For the 2010-2011 school year, Chicago Public Schools decided to consider up to 6 (out of a total of 9 choices) from applicants.

In the next section, we demonstrate that even though the new 2010 mechanism is still manipulable, its incentive properties are an improvement over the 2009 mechanism under our notion.

4 Manipulation under Constrained Versions of Student-Optimal Stable Mechanism

Understanding the properties of constrained school choice mechanisms is relevant for districts other than Chicago. To describe these issues, it is necessary to present a richer model of student assignment where students may be ordered in different ways across schools.

Vulnerability of school choice mechanisms to manipulation played a role in the adoption of new student assignment mechanisms not only in Chicago, but also in Boston and New York City (see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2005)). An important difference between Chicago and these two cities is that in Boston and New York City priority rankings of students are not the same at all schools. Abdulkadiroğlu and Sönmez (2003) first proposed using the celebrated student-optimal stable mechanism (Gale and Shapley 1962) in such a setting. For given student preferences and list of priority rankings at schools, the outcome of this mechanism can be obtained with the following **student-proposing deferred acceptance** algorithm:

Round 1: Each student applies to her first choice school. Each school rejects the lowest-ranking

students in excess of its capacity and all unacceptable students among those who applied to it, keeping the rest of students temporarily (so students not rejected at this step may be rejected in later steps.)

In general, at

Round ℓ : Each student who was rejected in Round $\ell-1$ applies to her next highest choice (if any). Each school considers these students *and* students who are temporarily held from the previous step together, and rejects the lowest-ranking students in excess of its capacity and all unacceptable students, keeping the rest of students temporarily (so students not rejected at this step may be rejected in later steps.)

The algorithm terminates either when every student is matched to a school or every unmatched student has been rejected by every acceptable school. Since there are a finite number of students and school, the algorithm terminates in a finite number of steps. Gale and Shapley (1962) show that this algorithm results in a stable matching that each student weakly prefers to any other stable matching. Moreover, Dubins and Freedman (1981) and Roth (1982) show that truth-telling is a dominant strategy for each student under this mechanism. Their result implies that student-optimal stable mechanism is strategy-proof in the context of school choice where only students are potentially strategic agents.

Interaction of matching theorists with officials at New York City and Boston lead to adoption of versions of student-optimal stable mechanism by these school districts in 2003 and 2005, respectively. In New York City, however, the version of the mechanism adopted only allows students to submit a rank order list of 12 choices. Based on the strategy-proofness of the student-optimal stable mechanism, the following advice was given to students:

You must now rank your 12 choices according to your true preferences.

For a student with more than 12 acceptable schools, truth-telling is no longer a dominant strategy under this version of the mechanism. In practice, between 20 to 30 percent of students rank 12 schools, even though there are over 500 choice options in New York City.⁸ This

⁸These details together with the entire description of the new assignment procedure is contained in Abdulkadiroğlu, Pathak and Roth (2010).

issue was first theoretically investigated by Haeringer and Klijn (2009) and experimentally by Calsamiglia, Haeringer, and Klijn (2010).

We next show that the greater the number of choices a student can make, the less vulnerable the constrained version of student-optimal stable mechanism is to manipulation. Let GS be the student-optimal stable mechanism, and GS^k be the constrained version of the student-optimal stable mechanism where only the top k choices are considered. By 2010, Newcastle England had switched from GS^3 to GS^4 . Our next result supports this change verifying the intuition that it makes the mechanism less vulnerable to manipulation.

Theorem 2. *Let $\ell > k > 0$ and suppose there are at least ℓ schools. Then GS^k is more manipulable than GS^ℓ .*

When there is a unique priority ranking across all schools (as in the case of Chicago), mechanism GS^k reduces to mechanism SD^k . Hence the following corollary to Theorem 2 is immediate:

Corollary 1. *Let $\ell > k > 0$. Mechanism SD^ℓ is more manipulable than mechanism SD^k .*

Parallel to the recent change in Newcastle England, Chicago switched from SD^4 to SD^6 in 2010. In terms of manipulation, this is a further improvement although the unconstrained version of the mechanism would completely eliminate the possibility of manipulation.

5 The Ban of the Boston Mechanism in England with the 2007 Admissions Code

The mechanism that was abandoned in Chicago midstream in 2009 is a special case of the widely studied Boston mechanism. From July 1999 to July 2005, the Boston mechanism has been used by school authorities in Boston to assign over 75,000 students to public school. Variants of the mechanism have been used in many different US school districts including: Cambridge MA, Charlotte-Mecklenburg NC, Denver CO, Miami-Dade FL, Minneapolis MN, Providence RI, Seattle, and Tampa-St. Petersburg FL.

For given student preferences and school priorities, the outcome of the **Boston mechanism** is determined with the following procedure:

Round 1: Only the first choices of students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until there are no seats left or there is no student left who has listed it as her first choice.

In general, at

Round ℓ : Consider the remaining students. In Round ℓ , only the ℓ^{th} choices of these students are considered. For each school with still available seats, consider the students who have listed it as their ℓ^{th} choice and assign the remaining seats to these students one at a time following their priority order until there are no seats left or there is no student left who has listed it as her ℓ^{th} choice.

The procedure terminates when each student is assigned a seat at a school.

The fact the Boston mechanism is vulnerable to preference manipulation seems to be well understood by some participants. For instance, some families have developed rules of thumb for submitting preferences strategically. See, for instance, the description of the strategies employed by the West Zone Parents Group in Boston in Pathak and Sönmez (2008). Similar heuristics have developed in other school districts as well (see Ergin and Sönmez 2006 for more examples). Finally, in controlled experiments, Chen and Sönmez (2006) show that more than 70% of participants in their experiment do not reveal their preferences truthfully under the Boston mechanism. Of course, the Boston mechanism is more manipulable than the student-optimal stable mechanism, which is strategy-proof.

As we have discussed, many school districts using mechanisms based on the Boston mechanism limit the number of schools that participants may rank. In Providence Rhode Island, students may only list four schools (out of 28 schools), while in Cambridge Massachusetts, students may only list three schools (out of 9 schools).⁹ Let β be the Boston mechanism and β^k be the Boston mechanism when only the top k choices of students are considered. It will

⁹See Parent Handbook, Providence Public Schools and Controlled Choice Plan, Cambridge Public Schools.

be convenient to let a matching in this and the next section indicate not only which school a student is assigned, but also what students are assigned to a school. In the later case $\beta_s(P)$ are the set of students assigned to school s .

The U.S. is not the only country where Boston mechanism and its versions are used to assign students to public schools. As we discussed in detail in the Introduction, a large number of Local Authorities had been using what they referred to as “first preference first” systems in England until it became illegal in 2007. The Boston mechanism is one of the most widely used examples of such systems. One of the key reasons for the ban of first preference first systems (including the Boston mechanism) was the strong incentives it gives parents to distort their submitted preferences. Even before the ban in 2007, this issue was central in several debates comparing first preference first systems with equal preference systems (such as the student-optimal stable mechanism). The following statement from the Coldron, et. al (2008) report prepared for Department for Children, Schools and Families summarizes what is at the heart of the debate:

Further, the difference between the two systems in the numbers of parents gaining their first preferences should not be interpreted as necessarily meaning that equal preference systems lead to less parental satisfaction overall. In a first preference first area, if the schools a parent puts as first, second or third are oversubscribed they risk not getting in to their first preference school and are also likely not to get their second or third choice because they do not fit the first preference over-subscription criterion of those schools. This means that the first preference system to some extent restricts parents’ room for manoeuvre, reduces their options and constrains them to put preferences for schools that are not their real preferred choice.

According to the report, at least 47 Local Authorities in England abandoned a first preference first system as a result of the 2007 ban. Due to lack of rigorous documentation, we do not know the exact details of many of these systems. However at least in four occasions the Local Authorities switched from a constrained version of the Boston mechanism to a constrained version of the student-optimal stable mechanism: Newcastle moved from β^3 to GS^3 in 2005 (and to GS^4 by 2010), Brighton-Hove moved from β^3 to GS^3 in 2007, East Sussex moved from β^3 to GS^3 after the 2007 ban, and Kent moved from β^3 to GS^4 after the 2007 ban. As in the case

of Chicago, the vulnerability of the Boston mechanism to manipulation resulted in its removal throughout England while ironically several Local Authorities adopted a constrained version of the student-optimal stable mechanism.

Our next result shows that not only is the Boston mechanism more manipulable than the student-optimal stable mechanism, its constrained version is more manipulable than the constrained version of the student-optimal stable mechanism. This result indicates that recent reforms in Newcastle, Brighton-Hove, East Sussex, and Kent involve adopting less manipulable mechanisms.

Theorem 3. *Suppose there are at least k schools where $k > 1$. Then β^k is more manipulable than GS^k .*

The following result that immediately follows from Theorem 2 and Theorem 3 is of interest based on the reforms in Newcastle and Kent.

Corollary 2. *Let $\ell > k > 0$ and suppose there are at least ℓ schools. Then β^k is more manipulable than GS^ℓ .*

When each school orders applicants using the same criteria, the old Chicago mechanism CHI^k is a special case of the β^k and the new Chicago mechanism SD^k is a special case of GS^k . As a result, Proposition 1 is a corollary of Theorem 3.

6 Stable Labor Market Clearinghouses

So far, each application has focused on comparing mechanisms applying the Definitions 2 (weakly more manipulable than) and 3 (at least as manipulable as). The last application involves a strong comparison in the study of stable matching mechanisms in the original college admissions model of Gale and Shapley (1962). Here, both sides of the market are active players, in that both submit preference lists over the other side of the market. In the college admissions model, we have students as before and colleges with potentially many seats. Following most of the literature, we assume that each college's preferences are responsive (Roth 1985). That is, the ranking of a student is independent of her colleagues, and any set of students exceeding

quota is unacceptable.¹⁰ Given this assumption, we sometimes abuse notation and let P_c be the preference list of college c defined over singleton sets and the empty set. (To avoid confusion, in this section S is the set of students with element s and C is the set of colleges with element c .)

Since no mechanism is strategy-proof for all players, researchers have focused on the incentives for one side of the market holding fixed the behavior of the other side of the market. This perspective has led to possibility results such as the case of the student-proposing deferred acceptance algorithm which is strategy-proof for students. Denote this mechanism as GS^S . It is also possible to define a college-proposing variant of the deferred acceptance algorithm, which yields the most preferred stable matching for colleges. We refer to this variant of the mechanism as GS^C , the college-optimal stable mechanism.

While truth-telling is a dominant strategy for each student under GS^S , an analogous result does not hold for colleges under GS^C . Indeed, there is no stable mechanism where truth-telling is a dominant strategy for colleges in the college admissions model (Roth 1985). The following example illustrates this possibility.

Example 1. *There are two students, s_1 and s_2 , and two colleges, c_1 and c_2 , where c_1 has two seats and c_2 has one seat. The preferences are:*

$$\begin{aligned} R_{s_1} &: c_1, c_2, s_1 & R_{c_1} &: \{s_1, s_2\}, \{s_2\}, \{s_1\}, \emptyset \\ R_{s_2} &: c_2, c_1, s_2 & R_{c_2} &: \{s_1\}, \{s_2\}, \emptyset. \end{aligned}$$

The only stable matching for this problem is:

$$\begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix},$$

which means that student s_1 is matched to college c_1 and student s_2 is matched to college c_2 .

Now suppose college c_1 submits the manipulated preference R'_{c_1} where only student s_2 is

¹⁰The preference relation over sets of students is responsive if, whenever $S' = S'' \cup \{s\} \setminus \{s''\}$ for some $s'' \in S''$ and $s \notin S''$, college c prefers S' to S'' if and only if college c prefers s to s'' .

acceptable. With this report, the only stable matching is:

$$\begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix}.$$

Hence college c_1 benefits by manipulating its preferences under any stable mechanism (including the college-optimal stable mechanism).

Given that no stable mechanism is strategy-proof for colleges, our next result still allows us to compare stable mechanisms for colleges by their vulnerability to manipulation. Indeed we can make a stronger comparison between student-optimal stable mechanism and college-optimal stable mechanism using the following more demanding notion.

Definition 4. A mechanism ψ is **strongly more manipulable** than mechanism φ if

- i) for any problem where φ is manipulable, ψ is manipulable by any player who can manipulate φ , and
- ii) there is at least one problem where ψ is manipulable although φ is not.

Clearly if mechanism ψ is strongly more manipulable than mechanism φ , then mechanism ψ is also more manipulable than mechanism φ .

Theorem 4. The student-optimal stable mechanism (GS^S) is strongly more manipulable than the college-optimal stable mechanism (GS^C) for colleges.

A natural question is if it is possible to order stable mechanisms when both students and colleges are able to manipulate. Unfortunately, no comparison is possible because of the well-known conflict of interest between the two sides of the market. This tension is apparent in the following generalizations of Theorem 4.

Let φ be an arbitrary stable mechanism. Then

- a) φ is at least as manipulable as GS^C for colleges,
- b) GS^S is at least as more manipulable as φ for colleges, and

c) GS^c is at least as more manipulable than φ for students.

While we make no distinction between whether it is the same player or different players who manipulate a mechanism for our definition of “at least as manipulable,” in each of these comparisons it is the same player who can manipulate for each problem. The proofs of these results are almost identical to the proof of Theorem 4 and hence are omitted.

This result is related to the recent policy discussion about the reforms of the National Resident Matching Program (NRMP), the job market clearinghouse that annually fills more than 25,000 jobs for new physicians in the United States. Prior to 1998, the mechanism was inspired by the college-proposing deferred acceptance algorithm. As we have discussed, in the college-optimal stable mechanism truth-telling is not a dominant strategy for students or colleges. In the mid-1990s, the NRMP came under increased scrutiny by students and their advisors who believed that the NRMP did not function in the best interest of students and was open to the possibility of different kinds of strategic behavior (Roth and Rothblum 1999). The mechanism was changed to one based on the student-proposing deferred acceptance algorithm (Roth and Peranson 1999).¹¹ One reason for this change was that truth-telling is a dominant strategy for students. For instance, one statement is from the minutes of the Committee of the American Medical Student Association (AMSA) and the Public Citizen Health Research Group (cited in Ma 2010):

...Since it is impossible to remove all incentives for hospitals to misrepresent, it would be best to choose the student-optimal algorithm to remove incentives, at least for students. In other words, within the set of stable algorithms, you either have incentives for both the hospitals and the students to misrepresent their true preferences or only for the hospitals.

Theorem 4 implies that by choosing the stable mechanism which removes incentives for manipulation among students, the market organizer is also choosing the mechanism which is most manipulable for colleges.

¹¹This reform was mimicked in a number of other clearinghouses. A comprehensive list of 43 clearinghouses is presented in Table 1 in Roth (2008).

Finally, let us mention that our strong definition of manipulability can easily be extended to a more general environment where participants report their types, not only their preference list. For instance, suppose in the college admissions model, colleges report both their preferences and their capacities to the market organizer as in Sönmez (1997) and let this denote their type. Since whenever a college can manipulate with a combination of preferences and capacity reports, the college can do at least as well with only a preference manipulation (see Kojima and Pathak 2009), it is straightforward to see that in a model with a larger message space, all of the results of this section continue to hold.

7 Conclusion

Recent school admissions reforms have been motivated in part by the desire to minimize strategic considerations among participants, yet many new mechanisms are still not immune to this possibility. This motivates the development of a method to compare mechanisms by their vulnerability to manipulation. In Chicago, the mechanism abandoned midstream is at least as manipulable as another other plausible mechanism. In England, the 2007 School Code banned systems using first preference first and numerous districts have adopted an equal preference system. Our results imply that changes in many English districts involved doing away with a more manipulable mechanism. The other results are also related to recent policy discussions involving matching mechanisms used in practice.

It is fascinating to observe such widespread condemnation of the Boston mechanism without the direct intervention of economists. Our methodology provides a way to formalize some concerns about the Boston mechanism, even relative to other manipulable mechanisms. Following Boston Public Schools' abandonment of the mechanism in 2005, there has been a renewed interest in understanding its properties. Some researchers have cautioned against a hasty rejection of the Boston mechanism in favor of the student-optimal stable mechanism (Miralles 2008, Abdulkadiroglu, Che and Yasuda 2010), while others have used laboratory experiments to show the Boston mechanism can have desirable properties in certain environments (Featherstone and Niederle 2009).

The two case studies may provide an indication of the revealed preferences of the public and policymakers about the mechanism and its variants. Coldron (2005) surveyed over 1,400 families in the Calderon Local Authority about the two mechanisms. He reports that 72% of parents “wanted the system changed to equal preference” in 2005 and over 90% of parents answered that the issue “mattered a great deal to them.” A survey of admissions officers in Greater London by Pennel, West, and Hind (2006) indicates that 82% of officers are satisfied with equal preference, while 15% are not. Another interesting aspect of these case studies is that participants themselves (and not matching theorists) advocated re-organizing market designs, in a manner analogous to the change of marketplace rules for medical residencies in the early 1950s as documented by Roth (1984).

Despite our focus on particular assignment and matching problems, the definitions we propose may have additional applications. We certainly have not exhausted the possibilities for matching problems. For instance, following our paper, Chen and Kesten (2011) compare student assignment mechanisms in China using our notion, which employs a hybrid of the Boston and student-optimal stable mechanism. Closely related work in progress by Dasgupta and Maskin (2010) explores a similar idea in social choice problems, when comparing Condorcet and Borda rules, and similar ideas have been studied in problem of fair division with indivisible objects (see, e.g., Andersson, Ehlers, and Svensson (2010)).

Finally, it is important to emphasize that vulnerability to manipulation is not the only criterion one might consider when comparing mechanisms. Still this seems to have been a critical reason for the 2009 policy change in Chicago and changes throughout England. Of course, when deciding whether to change a mechanism, it is important to consider many different properties of a mechanism and its alternative as well as political and practical issues. In situations where strategy-proof mechanisms do not have obvious drawbacks, as one might argue for eliminating restrictions on the number of choices allowed in school choice, an interesting question for future work is to understand the reasons they are not used.

A Proofs

Theorem 1. *Suppose each student has a complete rank ordering and $k > 1$. The old Chicago Public Schools mechanism (CHI^k) is at least as manipulable as any weakly stable mechanism.*

Proof. Fix a problem P and let φ be an arbitrary mechanism that is weakly stable. Suppose that CHI^k is not manipulable for problem P .

Claim 1: Any student assigned under $\text{CHI}^k(P)$ receives her top choice.

Proof. If not, since each student has a complete rank order list, $I > Q$, $k > 1$, there must be a student that is assigned to a school s he has not ranked first. Consider the highest composite score student i who is unassigned. Student i can rank school s first and will be assigned a seat there in the first round of CHI^k mechanism instead of some student who has not ranked school s first. That contradicts CHI^k is not manipulable for problem P .

Claim 2. The set of students who are assigned a seat under $\text{CHI}^k(P)$ is equal to the set of top Q composite score students.

Proof. If not, there is a school seat assigned to a student j who does not have a top Q score. Let student i be the highest scoring top Q student who is not assigned. Since student i has a complete rank order list, she can manipulate CHI^k by ranking student j 's assignment as her top choice again contradicting CHI^k is not manipulable for problem P .

Since each of the top Q students is matched to her top choice in matching $\text{CHI}^k(P)$, all other students are unassigned.

Claim 3. In problem P , matching $\text{CHI}^k(P)$ is the unique weakly stable matching.

Proof. By Claims 1 and 2 it is possible to assign each one of the top Q students a seat at their top choice school under P and $\text{CHI}^k(P)$ picks that matching. Let $\mu \neq \text{CHI}^k(P)$. That means under μ there exists a top Q student i who is not assigned to her top choice s . Pick the highest composite score such student i . Since all higher score students are assigned to their top choices, either there is a vacant seat at her top choice s or it admitted a student with lower composite score. In either case the pair (i, s) strongly blocks matching μ . Hence $\text{CHI}^k(P)$ is the unique weakly stable matching under P .

We are now ready to complete the proof. By Claim 3, $\varphi(P) = \text{CHI}^k(P)$ and hence mechanism φ assigns all top Q students a seat at their top choices. None of the top Q students has an incentive to manipulate φ since each receives her top choice. Moreover no other student can manipulate φ because regardless of their stated preferences, $\varphi(P) = \text{CHI}^k(P)$ remains the unique weakly stable matching and hence φ picks the same matching for the manipulated economy. Hence, any other weakly stable mechanism is also not manipulable under P . $\square$

Theorem 2. *Let $\ell > k > 0$ and suppose there are at least ℓ schools. Then GS^k is more manipulable than GS^ℓ .*

Proof. Suppose there is a student i and preference $\hat{P}_i$ such that

$$GS_i^\ell(\hat{P}_i, P_{-i}) \succ P_i \succ GS_i^\ell(P). \quad (1)$$

For any student j , let P_j^ℓ be the truncation of P_j after the ℓ^{th} choice. This means that in P_j^ℓ any choice after the top ℓ in P_j are unacceptable, and choices among the top ℓ are ordered according to P_j . Observe that relation (1) implies that

$$GS_i(\hat{P}_i^\ell, P_{-i}^\ell) \succ P_i \succ GS_i(P^\ell). \quad (2)$$

Since GS is strategy-proof, relation (2) implies that student i does not receive one of her top ℓ choices from the GS mechanism under profile P^ℓ . Hence, $GS_i(P^\ell) = GS_i^\ell(P) = i$.

For $k < \ell$, there are two cases to consider.

Case 1: $GS_i^k(P) = i$.

Let $GS_i^\ell(\hat{P}_i, P_{-i}) = s$ and let $\tilde{P}_i$ be such that s is the only acceptable school.

Claim: $GS_i^k(\tilde{P}_i, P_{-i}) = s$.

Proof: First note that $GS_i^\ell(\tilde{P}_i, P_{-i}) = s$. Moreover, by definition

$$GS^\ell(\tilde{P}_i, P_{-i}) = GS(\tilde{P}_i, P_{-i}^\ell) \quad \text{and} \quad GS^k(\tilde{P}_i, P_{-i}) = GS(\tilde{P}_i, P_{-i}^k).$$

Gale and Sotomayor (1985) (see also Theorem 5.34 of Roth and Sotomayor 1990) implies that

$$GS_i(\tilde{P}_i, P_{-i}^k) R_i GS_i(\tilde{P}_i, P_{-i}^\ell).$$

Substituting the definitions,

$$GS_i^k(\tilde{P}_i, P_{-i}) R_i \underbrace{GS_i^\ell(\tilde{P}_i, P_{-i})}_{=s}.$$

Since c is the only acceptable school in $\tilde{P}_i$, the claim follows. $\diamond$

Thus, in the first case, student i can manipulate GS^k :

$$\underbrace{GS_i^k(\tilde{P}_i, P_{-i})}_{=s} P_i \underbrace{GS_i^k(P)}_{=i}.$$

Case 2: $GS_i^k(P) \neq i$.

Claim 1: $\exists j \in I$ such that $GS_j^k(P) = j$ although $GS_j^\ell(P) \neq j$.

Proof: Suppose not. Then, since $GS_i^\ell(P) = i$ and $GS_i^k(P) \neq i$, there is a school that is assigned strictly more students under $GS^k(P)$ than $GS^\ell(P)$. This is a contradiction to Gale and Sotomayor (1985), which requires that each school is weakly worse off under GS^k (since profile P^k is a truncation of profile P^ℓ). $\diamond$

Pick any $j \in I$ such that $GS_j^k(P) = j$ although $GS_j^\ell(P) \neq j$. Let $GS_j^\ell(P) = s$ and let $\tilde{P}_j$ be such that s is the only acceptable school.

Claim 2: $GS_j^k(\tilde{P}_j, P_{-j}) = s$.

Proof: Since $GS_j^\ell(P) = s$, we have $GS_j^\ell(\tilde{P}_j, P_{-j}) = c$ as well. Moreover, by definition

$$GS^\ell(\tilde{P}_j, P_{-j}) = GS(\tilde{P}_j, P_{-j}^\ell) \quad \text{and} \quad GS^k(\tilde{P}_j, P_{-j}) = GS(\tilde{P}_j, P_{-j}^k).$$

Gale and Sotomayor (1985) implies that

$$GS_j(\tilde{P}_j, P_{-j}^k) R_j GS_j(\tilde{P}_j, P_{-j}^\ell).$$

Substituting the definitions,

$$GS_j^k(\tilde{P}_j, P_{-j}) \ R_j \ \underbrace{GS_j^\ell(\tilde{P}_j, P_{-j})}_{=s}.$$

Since s is the only acceptable school in $\tilde{P}_j$,

$$GS_j^k(\tilde{P}_j, P_{-j}) = ss,$$

which establishes the claim. ◇

Thus, for the second case, student j can manipulate GS^k :

$$\underbrace{GS_j^k(\tilde{P}_j, P_{-j})}_{=s} \ P_j \ \underbrace{GS_j^k(P)}_{=j}.$$

Finally, we describe a problem where GS^ℓ is not manipulable by any students, but GS^k is manipulable by some student. Suppose there are two students, i_1 and i_2 , and two schools, s_1 and s_2 , each with one seat. The students have identical preferences which rank i_1 ahead of s_2 and both schools have identical priority orderings: i_1 is ordered ahead of i_2 . Under GS^2 , no student can manipulate because each obtains her top or second choice and GS is strategy-proof. Under GS^1 , i_2 is unassigned, and can benefit from ranking s_2 as her top choice. This example can be generalized to the case of GS^k and GS^ℓ . This completes the proof.¹² □

Theorem 3. *Suppose there are at least k schools where $k > 1$. Then β^k is more manipulable than GS^k .*

Proof. For any student j , let P_j^k be the truncation of P_j after the k^{th} choice. By definition,

$$\beta^k(P) = \beta(P^k) \quad \text{and} \quad GS^k(P) = GS(P^k).$$

¹²It is also possible to provide an alternative, indirect proof of this result using the equilibrium interpretation of the definition of weakly more manipulable than together with the characterization of the set of Nash equilibria in the preference revelation game induced by GS^k in Theorem 6.5 of Haeringer and Klijn (2009).

Suppose that no student can manipulate β^k . We will show that no student can manipulate GS^k either. Consider two cases:

Case 1: $\beta^k(P) = \beta(P^k)$ is stable under profile P .

Since $\beta(P^k)$ is stable under P , it is stable under P^k as well. Moreover, $GS(P^k)$ is stable for P^k by definition. Since the set of unmatched students across stable matchings is the same (McVitie and Wilson 1970), for all students i ,

$$GS_i(P^k) = i \Leftrightarrow \beta_i(P^k) = i. \quad (3)$$

Pick some student i . If $GS_i^k(P^k) \neq i$, then student i receives one of her top k choices. This implies that i receives one of her top k choices under GS . Since GS is strategy-proof, student i cannot manipulate GS^k .

Suppose $GS_i^k(P^k) = i$ and s can manipulate. We derive a contradiction. Since i can manipulate, there exists some school s and preference $\hat{P}_i$ such that

$$\underbrace{GS_i^k(\hat{P}_i, P_{-i}^k)}_{=s} P_i i.$$

Observe that s is not one of the top k choices of student i under P_i for otherwise student i could manipulate GS . Construct $\tilde{P}_i$ which lists s as the only acceptable school.

Matching $GS^k(\hat{P}_i, P_{-i}^k)$ remains stable under $(\tilde{P}_i, P_{-i}^k)$ and therefore

$$GS_i^k(\tilde{P}_i, P_{-i}^k) = s.$$

Since $GS(P^k)$ is stable under P^k and $GS_i^k(P^k) = i$ by assumption, relation (3) implies

$$\beta_i(P^k) = i.$$

By Roth (1984), matching $\beta(P^k)$ is not stable under $(\tilde{P}_i, P_{-i}^k)$ since student i remains single under $\beta(P^k)$ although not under stable matching $GS^k(\hat{P}_i, P_{-i}^k)$. Since matching $\beta(P^k)$ is not stable under $(\tilde{P}_i, P_{-i}^k)$, but it is stable for P^k , the only possible blocking pair of $\beta(P^k)$ in $(\tilde{P}_i, P_{-i}^k)$

is (i, s) . But since $\beta_i(P^k) = i$, this implies that (i, s) also blocks $\beta(P^k)$ under P^k , which is the desired contradiction. Thus, in case 1, no student can manipulate GS^k .

Case 2: $\beta(P^k)$ is not stable for profile P .

In this case, some pair (i, s) blocks $\beta(P^k)$, so that there exists $j \in \beta_s(P^k)$ such that i obtains higher priority than j at school s and $sP_i\beta_s(P^k)$.

Construct $\tilde{P}_i$ so that school s is the only acceptable school for student i . Since $j \in \beta_s(P^k)$ and student i has higher priority than student j at school s , we must have $i \in \beta_s(\tilde{P}_i, P_{-i}^k)$. But this means that

$$\underbrace{\beta_i(\tilde{P}_i, P_{-i}^k)}_{=s} P_i \beta_i(P^k),$$

contradicting the assumption that no student can manipulate β at P^k .

Finally, the following example describes a problem where the constrained version of the Boston mechanism is manipulable although the constrained version of the student-optimal stable mechanism is not. There are three students and three schools each with one seat. The student preferences and school priorities are:

$$\begin{array}{ll} R_{i_1} : s_1, s_2, s_3, i_1 & \pi_{s_1} : i_1, i_3, i_2 \\ R_{i_2} : s_2, s_3, s_1, i_2 & \pi_{s_2} : i_3, i_2, i_1 \\ R_{i_3} : s_1, s_2, s_3, i_3 & \pi_{s_3} : i_3, i_1, i_2. \end{array}$$

The matchings produced by β^2 and GS^2 are:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix},$$

respectively. Since no student receives an outcome worse than her second choice from GS^2 , no student can manipulate GS^2 by the strategy-proofness of GS . On the other hand, student i_3 can manipulate β^2 by declaring that s_2 is her only acceptable school. This example can be generalized to the case of GS^k and β^k , completing the proof. $\square$

Theorem 4. *The student-optimal stable mechanism (GS^S) is strongly more manipulable than the college-optimal stable mechanism (GS^C) for colleges.*

Proof. Fix student preferences, let P denote college preferences, and let P_{-c} denote the preferences of colleges other than college c . Suppose there is some college c and preference $\hat{P}_c$ such that

$$GS_c^C(\hat{P}_c, P_{-c}) \succeq_c GS_c^C(P). \quad (4)$$

First, we want to show that there exists some $\tilde{P}_c$ such that

$$GS_c^S(\tilde{P}_c, P_{-c}) \succeq_c GS_c^S(P).$$

By Gale and Shapley (1962), the college-optimal stable matching is weakly more preferred by colleges than the student-optimal stable matching:

$$GS_c^C(P) \succeq_c GS_c^S(P). \quad (5)$$

Construct $\tilde{P}_c$ as follows: for any $s \in S$,

$$s\tilde{P}_c\emptyset \Leftrightarrow s \in GS_c^C(\hat{P}_c, P_{-c}).$$

That is, only students in $GS_c^C(\hat{P}_c, P_{-c})$ are acceptable to college c under $\tilde{P}_c$.

Since matching $GS^C(\hat{P}_c, P_{-c})$ is stable under $(\hat{P}_c, P_{-c})$, it is also stable under $(\tilde{P}_c, P_{-c})$. Moreover by Roth (1984), college c is assigned the same number of students at any stable matching under profile $(\tilde{P}_c, P_{-c})$. Since only students in $GS_c^C(\hat{P}_c, P_{-c})$ are acceptable to college c under $\tilde{P}_c$, we have

$$GS_c^S(\tilde{P}_c, P_{-c}) = GS_c^C(\hat{P}_c, P_{-c}). \quad (6)$$

Hence, by (4), (5), and (6), we have

$$\underbrace{GS_c^C(\hat{P}_c, P_{-c})}_{=GS_c^S(\tilde{P}_c, P_{-c})} \succeq_c GS_c^C(P) \succeq_c GS_c^S(P),$$

which shows that college c can manipulate GS^S with report $\tilde{P}_c$.

Finally, we describe a problem where GS^C is not manipulable by any college, while some college can manipulate GS^S . Suppose there are two students, s_1 and s_2 , and two colleges, c_1 and c_2 , each with one seat. The student and college preferences are

$$\begin{aligned} R_{s_1} : c_1, c_2, s_1 & & R_{c_1} : \{s_2\}, \{s_1\}, \emptyset \\ R_{s_2} : c_2, c_1, s_2 & & R_{c_2} : \{s_1\}, \{s_2\}, \emptyset. \end{aligned}$$

Since each college obtains her top choice under GS^C , no college can manipulate. However, if college c_1 declares that only s_2 is acceptable, it can manipulate GS^S . This completes the proof. $\square$

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