Menu Auctions with Non-Transferable Utilities and Budget Constraints*

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Abstract

This paper extends Bernheim and Whinston's (1986) menu auction model under transferable utilities to a framework with non-transferable utilities and budget constraints. Under appropriate definitions of equilibria, it is shown that every truthful Nash equilibrium (TNE) is a coalition-proof Nash equilibrium (CPNE) and that the set of TNE payoffs and the set of CPNE payoffs are equivalent, as in a transferable utility framework. The existence of a CPNE is assured in contrast with the possible non-existence of Nash equilibrium under the definition by Dixit, Grossman, and Helpman (1997). Moreover, the set of CPNE payoffs is equivalent to the bidder-optimal weak core.

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1 Introduction

The menu auction model with transferable utilities introduced by Bernheim and Whinston (1986) is a complete-information principal-agent problem with multiple principals (bidders) and one agent (auctioneer), in which the auctioneer's action affects her own and bidders' payoffs. A menu auction game has two stages: in the first stage, each bidder simultaneously submits a bidding menu that is a list of contingent payments for each action to the auctioneer; in the second stage, given the submitted bidding menus, the auctioneer selects an action. Due to coordination problems among bidders, there are usually numerous Nash equilibria, many of which are implausible. Bernheim and Whinston (1986) propose truthful Nash equilibrium (TNE) as a refinement and prove that there is always a TNE in every menu auction game.[1] They show that every TNE is a coalition-proof Nash equilibrium (CPNE),[2] and that the set of TNE payoffs is equivalent to the set of CPNE payoffs and the bidder-optimal strong core.[3]

Although the menu auction game has been widely applied to political-economy models of economic influence,[4] Dixit, Grossman, and Helpman (1997) argue that assumptions of quasi-linear preferences and the absence of budget constraints in Bern-

[1] A TNE is a Nash equilibrium where each bidder submits a truthful bidding menu such that the bidder obtains the equilibrium payoff for every other action whenever possible.

[2] A CPNE is a Nash equilibrium immune to every credible joint deviation by any subset of bidders, where credibility of a coalitional deviation is recursively defined.

[3] Bernheim and Whinston (1986) do not mention the term "core" directly. However, following the auction literature, a coalitional game among the auctioneer and bidders can be defined from a menu auction game. An allocation is in the weak core if there exists no other allocation that weakly improves all members in a coalition and strictly improves some members in the coalition. The strong core is defined similarly but requires strict improvements on all members in the coalition. The bidder-optimal strong core is a strong core allocation and there is no other strong core allocation that weakly improves all bidders and strictly improves some bidders.

[4] In particular, Grossman and Helpman (1994) popularize strategic lobbying models.

heim and Whinston (1986) limit its applications in practice. Under quasi-linearity, the auctioneer does not care about the distribution of payoffs among bidders, and marginal utility of payment is always a constant.[5] Without budget constraints on bidders, it is hard to apply the model to situations with certain institutional restrictions on payments.[6] For these reasons, Dixit, Grossman, and Helpman (1997) relax the above two assumptions. Defining truthful Nash equilibrium (TNE) for generalized menu auction games, they show that every TNE is strongly Pareto efficient for the auctioneer and all bidders. However, their definition does not guarantee the existence of a TNE. Indeed, Example 1 discussed below illustrates that even Nash equilibrium may fail to exist under their definition. This paper proposes an alternative definition that guarantees the existence of equilibrium and fully characterizes the sets of TNEs and CPNEs.

One of the key consequences of imposing budget constraints on Bernheim and Whinston's (1986) definition is that when budget constraints are binding, bidders cannot provide additional incentive to induce a favorable outcome among several actions to which the auctioneer is indifferent.[7] Dixit, Grossman, and Helpman (1997) overcome this problem by implicitly assuming that budget constraints are never binding when bidders consider possible deviations. However, when the budget constraint is binding, some sort of "optimism" by the bidder is required to justify a deviation when the auctioneer is indifferent. Unfortunately, this optimism is the very reason that Nash equilibrium fails to exist. Therefore, we need an alternative defin-

[5] In a public good provision problem, the government (auctioneer) may care about how much each one contributes to the project, and the income effect is usually not independent of the level of public good provided.

[6] For example, in United States, there are legal restrictions on political contributions.

[7] This is different from standard assumptions in principal-agent models where bidders can always offer infinitesimally more to break ties.

ition of equilibrium that implies risk aversion on the part of the budget-constrained bidders: they are not willing to deviate when the new outcome, depending on the particular action eventually chosen by the auctioneer after the deviation, could be worse than the existing outcome even though there are better outcomes that could be chosen by the auctioneer.

Unlike Bernheim and Whinston (1986), in our model, the strong core might be empty (Example 1), and even if it is non-empty, Example 3 shows a TNE under our definition may not be strongly Pareto efficient, in contrast with Dixit, Grossman, and Helpman (1997). As the difference is driven by binding budget constraints, it is natural to modify the strong core, which we call the Budget-Constraint core (BC-core), and the bidder-optimality by requiring a strict improvement from a budget-unconstrained bidder.[8] Theorem 1 shows the main result of this paper that every TNE is a CPNE and the set of TNE payoffs, the set of CPNE payoffs, the bidder-optimal BC-core, and the bidder-optimal weak core are equivalent. With indispensability of private good (Mas-Collel 1977), the equivalence of the bidder-optimal weak core and the bidder-optimal strong core is reestablished, which coincides with Bernheim and Whinston (1986) (Corollary 2).

The extension to non-transferable utilities and budget constraints opens the door for new applications. For example, we can now deal with lobbying models without monetary transfers. Lobbies often reward politicians not by campaign contributions but by political support during elections. Since most elections are winner-take-all, marginal payoff of political support is non-linear, which is hard to capture through quasi-linearity. Moreover, the political support provided by any lobby is often lim-

[8] This is parallel to the alternative definition of Nash equilibrium: budget-constrained bidders by themselves are unable to induce favorable outcomes.

ited, so budget constraints are needed to allow reasonable predictions.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 presents the results. Section 4 concludes. The proof of the main theorem (Theorem 1) is relegated to the appendix.

2 The Model

The model follows Dixit, Grossman, and Helpman (1997). There are N bidders and an auctioneer (denoted by 0). The auctioneer chooses an action from a finite set A.[9] Bidder $i \in N$ submits a bidding menu $T_i : A \to \mathbb{R}_+$ to the auctioneer such that $0 \leq T_i(a) \leq \omega_i(a)$ for each $a \in A$, where $\omega_i(a)$ is the highest possible amount of contingent payment for action a. An important difference from Bernheim and Whinston (1986) is that bidder i faces budget constraint $\omega_i(a)$ when the auctioneer chooses action $a \in A$. Another departure is the relaxation of quasi-linear preferences: (1) the auctioneer's payoff function $U_0(a, (T_i(a))_{i \in N})$ is continuous and strictly increasing in $T_i(a)$ for all $a \in A$ and $i \in N$, and (2) bidder i's payoff function $U_i(a, T_i(a))$ is continuous and strictly decreasing in $T_i(a)$ for all $a \in A$. A **menu auction game** $\Gamma \equiv (N, (U_i, \omega_i)_{i \in N}, (U_0, A))$ is a two-stage complete information game such that all bidders submit bidding menus simultaneously in stage 1 and the auctioneer chooses an action in stage 2. Let $\mathcal{T}_i \equiv \{T_i : 0 \leq T_i(a) \leq \omega_i(a) \text{ for all } a \in A\}$, the collection of bidding menus of bidder i, and $\mathcal{T} \equiv (\mathcal{T}_i)_{i \in N}$, the collection of bidding menus of N bidders. An **outcome** of a menu auction game Γ is (a, T) where $a \in A$ and $T \equiv (T_i)_{i \in N} \in \mathcal{T}$. Define $M(T) \equiv \arg\max_{a \in A} U_0(a, T(a))$, the auctioneer's

[9] This assumption is made for ease of exposition only. All of our results hold when A is a compact set.

best response set given bidding menus T and $m(T) \equiv \max_{a \in A} U_0(a, T(a))$, the corresponding payoff.

Definition 1. An outcome (a^*, T^*) is a **Nash equilibrium** in Γ if and only if (i) $T^* \in \mathcal{T}$, (ii) $a^* \in M(T^*)$, (iii) for all $i \in N$ there exists no $\tilde{T}_i \in \mathcal{T}_i$ and $\tilde{a} \in M(\tilde{T}_i, T^*_{-i})$ such that (a) $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a^*, T^*_i(a^*))$ and (b) $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$.[10]

Condition (iii-b) $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ deserves further explanation as this is an important difference between this paper and Dixit, Grossman, and Helpman (1997).[11] Without condition (iii-b), bidders are assumed to be *optimistic* in the sense that bidder i would deviate to $\tilde{T}_i$ when it is *possible* to gain from deviation, without worrying about whether there *might* be another unfavorable action $\hat{a} \in M(\tilde{T}_i, T^*_{-i})$ with $U_i(a^*, T^*_i(a^*)) > U_i(\hat{a}, \tilde{T}_i(\hat{a}))$ to be chosen by the auctioneer. Without budget constraints, this optimism is not restrictive because any bidder could resolve the indifference of the auctioneer by paying infinitesimally more, as in standard principal-agent models.[12] However, a budget-constrained bidder cannot pay more to persuade the auctioneer, so there is no way to ensure that the auctioneer will choose the desirable action. Therefore, omitting condition (iii-b) implicitly assumes this kind of optimism, which restricts the set of Nash equilibria. Example 1 below shows that such an optimism might lead to the non-existence of Nash equilibrium without condition (iii-b).

[10] Although a menu auction game is a two-stage game, if all bidders know which action would the auctioneer choose when there is indifference over several actions, by backward induction, the game becomes a normal-form game among all bidders. This is the standard approach in the multiple-principal-(multi)-agent literature (Bernheim and Whinston, 1987; Prat and Rustichini, 2003). Hence, we focus on Nash equilibrium without specifying the exact tie-breaking rule.

[11] Bernheim and Whinston (1986) adopt Definition 1 without condition (iii-b) since they do not have budget constraints.

[12] When the auctioneer is indifferent between a and $\tilde{a}$, a bidder can pay $\varepsilon > 0$ more to induce one of outcomes.

Remark. Condition (iii) is identical to requiring in a Nash equilibrium that no bidder is able to *convincingly* persuade the auctioneer to choose an action that is strictly better for the bidder. Hence, Definition 1 can be stated equivalently as follows: An outcome (a^*, T^*) is a **Nash equilibrium** in Γ if and only if (i) $T^* \in \mathcal{T}$, (ii) $a^* \in M(T^*)$, (iii) for all $i \in N$ there exists no $\tilde{T}_i \in \mathcal{T}_i$ and $\tilde{a} = M(\tilde{T}_i, T^*_{-i})$ such that $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a^*, T^*_i(a^*))$. This alternative definition illustrates that condition (iii) is related to the idea of subgame perfection: if the auctioneer chooses a^* for tie-breaking, by backward induction, a bidder can deviate only if the auctioneer can be convinced to choose another action that is favorable to the bidder. Therefore, the set of Nash equilibria under Definition 1 is the set of subgame-perfect Nash equilibria (without imposing condition (iii-b)). In the other words, the set of Nash equilibria under the definition of Dixit, Grossman, and Helpman (1997) might exclude some subgame-perfect Nash equilibrium by the optimism, as shown in Example 1.

Example 1. Consider $N = \{1, 2\}$ and $A = \{a_1, a_2\}$. Assume quasi-linear preferences such that for all $a \in A$, $U_i(a, T(a)) = V_i(a) - T_i(a)$ with $\omega_i(a) = 2$ for all $i \in N$ and $U_0(a, T(a)) = V_0(a) + \sum_{i \in N} T_i(a)$ where

	a_1	a_2
$V_1(a)$	6	1
$V_2(a)$	1	6
$V_0(a)$	0	0

There is no Nash equilibrium if condition (iii-b) is omitted, because no matter which action is chosen, one of two bidders will prefer another action.[13] However, there

[13] Here we have $\tilde{T}_i = T_i$ for all $i \in N$ though this is considered as deviation from Definition 1.

exists a Nash equilibrium with condition (iii-b). Consider T such that $T_1(a_1) = 2$, $T_1(a_2) = 0$, $T_2(a_1) = 0$ and $T_2(a_2) = 2$. Outcomes (a_1, T) and (a_2, T) are Nash equilibria.[14]

Example 2 below shows that condition (iii-b) implies risk-averse behaviors of bidders.

Example 2. Consider $N = \{1, 2\}$ and $A = \{a_1, a_2, a_3\}$. Assume quasi-linear preferences such that for all $a \in A$, $U_i(a, T(a)) = V_i(a) - T_i(a)$ with $\omega_i(a) = 2$ for all $i \in N$ and $U_0(a, T(a)) = V_0(a) + \sum_{i \in N} T_i(a)$ where

	a_1	a_2	a_3
$V_1(a)$	6	1	4
$V_2(a)$	1	6	4
$V_0(a)$	0	0	-2

Consider T such that $T_1(a_1) = 2$, $T_1(a_2) = 0$, $T_1(a_3) = 2$, $T_2(a_1) = 0$, and $T_2(a_2) = T_2(a_3) = 2$. Outcomes (a_1, T), (a_2, T), and (a_3, T) are Nash equilibria. For outcome (a_3, T), suppose bidder 1 is considering whether to deviate from T_1 to $\tilde{T}_1(a_1) = 2$, $\tilde{T}_1(a_2) = \tilde{T}_1(a_3) = 0$. If bidder 1 is risk averse, the bidder would not deviate because although a_1 is more favorable than a_3, the auctioneer might choose a_2 and a_2 is less favorable than a_3. The argument is similar for bidder 2.

To see this more clearly, one could slightly modify this example to include action a_3 that is never preferred by the auctioneer and any bidders. Then one can have $\tilde{T}_i \neq T_i$ by trivially changing $T_i(a_3)$.

[14] If the auctioneer chooses a_1 facing T, then (a_1, T) is the outcome from subgame-perfect equilibrium. Similarly, (a_2, T) is the outcome from subgame-perfect equilibrium if a_2 is chosen facing T.

Similar to Bernheim and Whinston (1986), there are usually a large number of Nash equilibria in a menu auction game due to coordination problems among bidders. They argue that not all of them are equally plausible and propose truthful Nash equilibrium (TNE) as a refinement. In this class of equilibrium, each bidder submits a truthful bidding menu relative to the equilibrium payoff. For each bidder, a truthful bidding menu relative to a payoff level restricts the bidding menu so that the bidder obtains the same level payoff for every possible action whenever possible. Clearly, bidders cannot pay more than budget constraints or extract payments from the auctioneer, so we have to accommodate those cases in the definition.

For bidder $i \in N$, a bidding menu T_i is a **truthful bidding menu relative to payoff** u_i if for all $a \in A$,

$$T_i(a) = \begin{cases} 0 & \text{if } U_i(a, 0) < u_i \\ \tau_i(a, u_i) & \text{if } U_i(a, \omega_i(a)) \leq u_i \leq U_i(a, 0) \\ \omega_i(a) & \text{if } u_i < U_i(a, \omega_i(a)) \end{cases}$$

where $\tau_i(a, u_i)$ is implicitly defined by $U_i(a, \tau_i(a, u_i)) = u_i$.[15] Denote $T_i^{u_i}$ to be the truthful bidding menu relative to payoff u_i and $T^u \equiv (T_i^{u_i})_{i \in N}$ to be the truthful bidding menus relative to payoffs $u = (u_i)_{i \in N}$. A TNE is a refinement on a Nash equilibrium such that all bidders choose truthful bidding menus relative to their equilibrium payoffs.

Definition 2. An outcome (a^*, T^*) is a **truthful Nash equilibrium (TNE)** in Γ if it is a Nash equilibrium and T^* are the truthful bidding menus relative to

[15] Dixit, Grossman and Helpman (1997) define $T_i(a) = \min\{\omega_i(a), \max\{0, \tau_i(a, u_i)\}\}$ to be the truthful bidding menu relative to u_i. However, $\tau_i(a, u_i)$ may be undefined. For example, consider $A = \{0, 1\}$ and $U_i(a, T_i(a)) = a + (T_i(a) + 1)^{-1}$. It is clear that $\tau_i(1, 1)$ is unbounded.

equilibrium payoffs $u^* = (U_i(a^*, T_i^*(a^*)))_{i\in N}$.

Bernheim and Whinston (1986) argue that a TNE may be quite "focal" because truthful bidding menus are simple. A further support is that a bidder suffers no loss in using truthful bidding menus because there is always a truthful bidding menu in the set of best responses. Proposition 1 in the next section shows that this still holds.

As a truthful bidding menu mirrors the relative payoffs which the bidder attaches to various actions, one may conjecture that coordination problems in Nash equilibria are solved in TNEs. In fact, Bernheim and Whinston (1986) shows that a TNE has a strong stable property: every TNE is a coalition-proof Nash equilibrium (CPNE) and the set of TNE payoffs is the same as the set of CPNE payoffs. A CPNE is a Nash equilibrium immune to any credible joint deviation by any subset of bidders, where credibility of a coalitional deviation is defined recursively. The main result of this paper (Theorem 1) shows that this important property justifying the TNE refinement is also true in the generalized framework.

Formally, we define coalition-proof Nash equilibrium as follows. Given any non-empty subset of bidders $J \subseteq N$ and bidding menus $(T_i)_{i\in N\setminus J}$, a J-**component game** relative to $(T_i)_{i\in N\setminus J}$ is defined as $\Gamma\setminus(T_i)_{i\in N\setminus J} \equiv (J, (U_j, \omega_j)_{j\in J}, (\tilde{U}_0, A))$ where $\tilde{U}_0(a, (\tilde{T}_j(a))_{j\in J}) \equiv U_0(a, (\tilde{T}_j(a))_{j\in J}, (T_i(a))_{i\in N\setminus J})$ is the auctioneer's payoff facing bidding menus $\tilde{T}_j(a))_{j\in J}$ while bidding menus $(T_i)_{i\in N\setminus J}$ are fixed in advance.

Definition 3. (i) An outcome (a^*, T_j^*) is a **coalition-proof Nash equilibrium (CPNE)** in $\Gamma\setminus T_{-j}$ if and only if it is a Nash equilibrium in $\Gamma\setminus T_{-j}$.

(ii-a) An outcome $(a^*, (T_j^*)_{j\in J})$ is **self-enforcing** in $\Gamma\setminus(T_i)_{i\in N\setminus J}$ if for all non-empty $S \subsetneq J$, $(a^*, (T_j^*)_{j\in S})$ is a CPNE in $\Gamma \backslash ((T_i)_{i\in N\backslash J}, (T_j^*)_{j\in J\backslash S})$.

(ii-b) An outcome $(a^*, (T_j^*)_{j\in J})$ is a **CPNE** in $\Gamma \backslash (T_i)_{i\in N\backslash J}$ if it is self-enforcing in $\Gamma \backslash (T_i)_{i\in N\backslash J}$, and there exists no other self-enforcing $(\tilde{a}, (\tilde{T}_j)_{j\in J})$ in $\Gamma \backslash (T_i)_{i\in N\backslash J}$ such that (α) $U_j(\tilde{a}, \tilde{T}_j(\tilde{a})) \geq U_j(a^*, T_j^*(a^*))$ for all $j \in J$, and (β) $U_{j'}(\tilde{a}, \tilde{T}_{j'}(\tilde{a})) > U_{j'}(a^*, T_{j'}^*(a^*))$ and $\tilde{T}_{j'}(\tilde{a}) < \omega_{j'}(\tilde{a})$ for some $j' \in J$.

Comparing the above definition with the one in Bernheim and Whinston (1986), the only difference is condition (ii-b-β): a strict improvement is needed from a *budget-unconstrained bidder.* Same as the case in a Nash equilibrium, since a budget-constrained bidder cannot provide extra incentives to persuade the auctioneer to choose a favorable action, a group of budget-constrained bidders cannot provide extra incentives even if they act together.

3 Results

As suggested in the previous section, we will show that there is always a truthful bidding menu in the set of best responses.[16] Following our definition of Nash equilibrium, a bidding menu T_i is a bidder i's **best response** to other bidder's bidding menus T_{-i} if there exists $a \in M(T)$ such that there exists no $\tilde{T}_i \in \mathcal{T}_i$ such that $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a, T_i(a))$ with $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ and $\tilde{a} \in M(\tilde{T}_i, T_{-i})$.

Proposition 1. In every menu auction game Γ, for all $i \in N$, there exists a truthful

[16] Strictly speaking, without knowing how the auctioneer chooses among payoff-equivalent actions, the set of best responses for a bidder is not well defined. Bernheim and Whinston (1986) argue (in their footnote 11) that such a problem disappears if payment has some smallest unit (however small). Milgrom (2005) argues that a bidding menu can be loosely defined as a best response of a bidder if for some $\varepsilon > 0$, the bidder will not choose another bidding menu assuming that the auctioneer considers the bidder is paying ε more on the bidders' favorable action when choosing an action, but the bidder's payoff is evaluated without paying ε more.

bidding menu being a bidder i's best response.

Proof. Consider T_i to be a best response to T_{-i} such that there exists $a \in M(T)$ such that there exists no $\tilde{T}_i \in \mathcal{T}_i$ such that $U_i(\tilde{a}, \tilde{T}_i(\tilde{a})) > U_i(a, T_i(a))$ with $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ and $\tilde{a} \in M(\tilde{T}_i, T_{-i})$. Consider a truthful bidding menu $T_i^{u_i}$ with $u_i = U_i(a, T_i(a))$. If $a = M(T_i^{u_i}, T_{-i})$, then $T_i^{u_i}$ is already a best response to T_{-i}. Hence, consider that there exists $\bar{a} \neq a$ with $\bar{a} \in M(T_i^{u_i}, T_{-i})$. There are two cases: (Case 1) $T_i(\bar{a}) > T_i^{u_i}(\bar{a})$: it implies $U_0(\bar{a}, T(\bar{a})) > U_0(\bar{a}, T_i^{u_i}(\bar{a}), T_{-i}(\bar{a}))$. As $\bar{a} \in M(T_i^{u_i}, T_{-i})$, we have $U_0(\bar{a}, T_i^{u_i}(\bar{a}), T_{-i}(\bar{a})) \geq U_0(a, T_i^{u_i}(a), T_{-i}(a))$. Hence, $U_0(\bar{a}, T(\bar{a})) > U_0(a, T_i^{u_i}(a), T_{-i}(a)) = U_0(a, T(a))$, which contradicts $a \in M(T)$; (Case 2) $T_i(\bar{a}) \leq T_i^{u_i}(\bar{a})$: it implies either $T_i^{u_i}(\bar{a}) = \tau_i(\bar{a}, u_i)$ or $T_i^{u_i}(\bar{a}) = \omega_i(\bar{a})$ but both imply $U_i(\bar{a}, T_i^{u_i}(\bar{a})) \geq u_i = U_i(a, T_i(a))$ so that $T_i^{u_i}$ is at least as good as T_i. Therefore, $T_i^{u_i}$ is a best response to T_{-i}.□

Following the auction literature, we can construct a coalitional game between the auctioneer and N bidders from a menu auction game Γ.[17] In Bernheim and Whinston (1986), every TNE is a CPNE and the set of TNE/CPNE payoffs is the bidder-optimal strong core. Theorem 1 will show this is still true with some modifications. As we have seen the presence of budget constraints requires alternative definitions of Nash equilibrium and CPNE, it is not surprising that we need alternative definitions of the core and bidder-optimality.[18]

[17] This is different from the menu auction literature. Bernheim and Whinston (1986) do not mention "core". Laussel and Le Breton (2001) consider transferable utility coalitional games generated from menu auction games between bidders only.

[18] Day and Milgrom (2008) discuss the importance of the core and bidder-optimality in auction mechanisms (with transferable utilities). They argue that auctions selecting core allocations have the advantages that bidders have no incentive to merge bids, submit bids under other identities, or renege after the auction is conducted. Furthermore, if the selected allocation is in the bidder-optimal core, then bidders have minimal incentives to misreport among all core-selecting auctions

In our model, a non-transferable utility coalitional game $(N \cup \{0\}, (\mathcal{U}_\Gamma(S))_{S \subseteq N \cup \{0\}})$ constructed from a menu auction game Γ is a coalitional game between the auctioneer and N bidders such that $\mathcal{U}_\Gamma(S)$ is the set of payoffs achievable by $S \subseteq N \cup \{0\}$ in Γ. Since bidders cannot generate meaningful payoffs without the auctioneer, define $\mathcal{U}_\Gamma(S) \equiv \{(u_i)_{i\in S} \in \mathbb{R}^S$: there exists $(a, T) \in A \times \mathcal{T}$ such that $u_0 = U_0(a, (T_i(a))_{i\in S})$ and $u_i = U_i(a, T_i(a))$ for all $i \in S\}$ if $\{0\} \in S$, and $\mathcal{U}_\Gamma(S) = \{(u_i)_{i\in S} \in \mathbb{R}^S : u_i = \inf_{a\in A} U_i(a, 0)$ for all $i \in S\}$ if $\{0\} \notin S$. To save notation, let $S_0 \equiv S \cup \{0\}$, a set comprising the auctioneer and all bidders in $S \subseteq N$, and $u_{S_0} \equiv (u_0, (u_i)_{i\in S})$, a list of their payoffs. A list of payoffs u_{N_0} is an **allocation** if $u_{N_0} \in \mathcal{U}_\Gamma(N_0)$.[19] An allocation u is **supported** by an outcome (a, T) if $u_0 = U_0(a, T(a))$ and $u_i = U_i(a, T_i(a))$ for all $i \in N$.

Definition 4. An allocation u is **weakly blocked** by S if there exists $\tilde{u}_S \in \mathcal{U}_\Gamma(S)$ such that (i) $\tilde{u}_i \geq u_i$ for all $i \in S$ and (ii) $\tilde{u}_i > u_i$ for some $i \in S$. An allocation u is in the **strong core** ($Score_\Gamma$) if it is not weakly blocked by any $S \subseteq N \cup \{0\}$.

In Bernheim and Whinston (1986), the strong core is non-empty and includes the set of TNE payoffs. However, this does not extend to our model. Example 1 shows that the strong core can be empty,[20] though it will be shown that there is always a TNE. Moreover, Example 3 below shows that even when the strong core is non-empty, there is an allocation supported by a TNE but not in the strong core.

Example 3. Consider $N = \{1, 2\}$ and $A = \{a_1, a_2, a_3\}$. Assume quasi-linear preferences such that for all $a \in A$, $U_i(a, T(a)) = V_i(a) - T_i(a)$ with $\omega_i(a) = 2$ for

and the auctioneer would not have incentive to disqualify bidders.

[19] Without confusion, we drop the subscript N_0 when a list of payoffs is an allocation.

[20] In Example 1, every allocation, except those weakly blocked by some bidders only, is weakly blocked by allocations $(2, 4, 1)$ or $(2, 1, 4)$.

all $i \in N$ and $U_0(a, T(a)) = V_0(a) + \sum_{i \in N} T_i(a)$ where

	a_1	a_2
$V_1(a)$	6	6
$V_2(a)$	6	1
$V_0(a)$	0	2

Consider T such that $T_1(a_1) = T_1(a_2) = 0$, $T_2(a_1) = 2$ and $T_2(a_2) = 0$. Outcomes (a_1, T) and (a_2, T) are TNEs. However, the allocation supported by (a_2, T) is not in the strong core.

As hinted above, it seems natural to modify the definition of weak blocking by taking budget constraints into account.

Definition 4. An allocation u is **BC-blocked** (Budget-Constraint blocked) by S if there exists $\tilde{u}_S \in \mathcal{U}_\Gamma(S)$ supported by an outcome $(\tilde{a}, \tilde{T})$ such that (i) $\tilde{u}_i \geq u_i$ for all $i \in S$, and (ii) either $\tilde{u}_0 > u_0$, or $\tilde{u}_i > u_i$ and $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ for some $i \in S \backslash \{0\}$. An allocation u is in the **BC-core** (Budget-Constraint core, $BCcore_\Gamma$) if it is not BC-blocked by any $S \subseteq N \cup \{0\}$.

Since the strong core and the weak core are equivalent in Bernheim and Whinston (1986), it is interesting to see how the BC-core is related to the weak core in our model.

Definition 5. An allocation u is **strongly blocked** by S if there exists $\tilde{u}_S \in \mathcal{U}_\Gamma(S)$ such that $\tilde{u}_i > u_i$ for all $i \in S$. An allocation u is in the **weak core** ($Wcore_\Gamma$) if it is not strongly blocked by any $S \subseteq N \cup \{0\}$.

At the first glance, one may conjecture that BC-blocking is more effective than strong blocking, but Proposition 2 shows that they are equivalent.

Proposition 2. In every menu auction game Γ, we have

$$Wcore_\Gamma = BCcore_\Gamma.$$

Proof. By definition, if there is a strong blocking deviation for $u \in \mathcal{U}_\Gamma(N_0)$, then it is also a BC-blocking deviation for u because there is a strict improvement for the auctioneer.[21] Therefore, we have $Wcore_\Gamma \supseteq BCcore_\Gamma$. Now suppose $u \in Wcore_\Gamma$ but u is BC-blocked by S_0.[22] There exists $\tilde{u}_{S_0} \in \mathcal{U}_\Gamma(S_0)$ supported by an outcome $(\tilde{a}, \tilde{T}) \in A \times \mathcal{T}$ such that $\tilde{u}_i \geq u_i$ for all $i \in S_0$, and $\tilde{u}_j > u_j$ and $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$ for some $j \in S$. Let $\tilde{S} \equiv S \backslash \{i \in S : \tilde{T}_i(\tilde{a}) = 0\}$ and $K \equiv \{j \in \tilde{S} : \tilde{u}_j > u_j$ and $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})\}$. There exists $\varepsilon_i > 0$ for all $i \in \tilde{S}$ such that $\bar{u}_{\tilde{S}_0} \in \mathcal{U}_\Gamma(\tilde{S}_0)$ supported by $(\tilde{a}, (\tilde{T}_{i'})_{i' \in N \backslash \tilde{S}}, (\bar{T}_i)_{i \in \tilde{S}}) \in A \times \mathcal{T}$ strongly blocks u by $\tilde{S}_0$ where for all $j \in K$, $\bar{T}_j(\tilde{a}) = \tilde{T}_j(\tilde{a}) + \varepsilon_j < \omega_j(\tilde{a})$ and $\bar{T}_j(a) = 0$ for all $a \in A \backslash \{\tilde{a}\}$, and for all $i \in \tilde{S} \backslash K$, $\bar{T}_i(\tilde{a}) = \tilde{T}_i(\tilde{a}) - \varepsilon_i > 0$ and $\bar{T}_i(a) = 0$ for all $a \in A \backslash \{\tilde{a}\}$.[23] Thus, $Wcore_\Gamma \subseteq BCcore_\Gamma$.$\square$

Dixit, Grossman, and Helpman (1997) prove that every TNE under their definition is strongly Pareto efficient for the auctioneer and all bidders.[24] However,

[21] If a strong deviation comes from some bidders only, then it is easy to construct a BC-blocking deviation by those bidders.

[22] If the BC-blocking deviation comes from some bidders only, then it is also a strong blocking deviation by some of those bidders. Similarly, if the BC-blocking deviation comes from the auctioneer only, it is also a strong blocking deviation by the auctioneer. Both cases contradict $u \in Wcore_\Gamma$.

[23] If $\tilde{S}$ is empty, then $\tilde{u}$ BC-blocks u by the auctioneer only so that $\tilde{u}$ also strongly blocks u.

[24] An allocation u is strongly Pareto efficient for the auctioneer and all bidders if there exists no

Example 3 shows that (a_1, T) weakly Pareto dominates (a_2, T), both of which are TNEs. This arises because budget-constrained bidders are unable to provide incentives to the auctioneer to induce a Pareto improvement. Therefore, we have to incorporate the implication of budget constraints into the bidder-optimality.[25]

Definition 6. An allocation u is in the **bidder-optimal BC-core** ($\overline{BCcore_\Gamma}$) if there exists no $\tilde{u} \in BCcore_\Gamma$ supported by an outcome $(\tilde{a}, \tilde{T})$ such that $\tilde{u}_i \geq u_i$ for all $i \in N$, and $\tilde{u}_i > u_i$ and $\tilde{T}_i(\tilde{a}) < \omega_i(\tilde{a})$ for some $i \in N$. The **bidder-optimal strong core** ($\overline{Score_\Gamma}$) and the **bidder-optimal weak core** ($\overline{Wcore_\Gamma}$) are defined similarly.

While $\overline{Score_\Gamma}$ may be empty, Proposition 3 below shows that $\overline{BCcore_\Gamma}$ and $\overline{Wcore_\Gamma}$ are always non-empty. As Theorem 1 shows that $\overline{BCcore_\Gamma}$ is equivalent to the set of TNE/CPNE payoffs, the existence of a TNE/CPNE is assured.

Proposition 3. In every menu auction game Γ, the bidder-optimal BC-core is non-empty.

Proof. Scarf (1967) proves that in a coalitional game $(N \cup \{0\}, (\mathcal{U}_\Gamma(S))_{S \subseteq N \cup \{0\}})$ if for all $S \subseteq N \cup \{0\}$, $\mathcal{U}_\Gamma(S)$ is comprehensive and closed, and satisfies balancedness, and $\{u_S \in \mathcal{U}_\Gamma(S) : u_i \geq \sup \mathcal{U}_\Gamma(\{i\}) \text{ for all } i \in S\}$ is non-empty and bounded, then $Wcore_\Gamma \neq \emptyset$. It is easy to check that all conditions are satisfied. As $Wcore_\Gamma$ is compact and dominance relationship in the bidder-optimality is weaker than strongly Pareto efficiency, we have $\overline{Wcore_\Gamma} \neq \emptyset$, and hence $\overline{BCcore_\Gamma} \neq \emptyset$ by Proposition 2.□

$\tilde{u} \in \mathcal{U}_\Gamma(N_0)$ such that $\tilde{u}_i \geq u_i$ for all $i \in N_0$, and $\tilde{u}_i > u_i$ for some $i \in N_0$.

[25]The standard definition of the bidder-optimality is strongly Pareto efficiency for all bidders without taking budget constraints into account.

The following theorem is the main result of this paper.

Theorem 1. In every menu auction game Γ, every truthful Nash equilibrium (TNE) is a coalition-proof Nash equilibrium (CPNE) and

$$\overline{Score_\Gamma} \subsetneq \overline{Wcore_\Gamma} = \overline{BCcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}$$

where $\mathcal{U}_\Gamma^{TNE}$ and $\mathcal{U}_\Gamma^{CPNE}$ are the sets of TNE payoffs and CPNE payoffs in Γ.

The proof of this theorem is complex and we defer it to the appendix. It is interesting to compare this result with the existing literature. Dixit, Grossman, and Helpman (1997) show that under similar settings as this paper, the set of TNE payoffs under their definition is included in the set of strongly Pareto efficient allocations with respect to all bidders and the auctioneer.[26] Bernheim and Whinston (1986), under the assumptions of quasi-linear preferences and the absence of budget constraints, show that every TNE is a CPNE and

$$\overline{Score_\Gamma} = \overline{Wcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}.^{27}$$

Therefore, Theorem 1 almost completely extends the results by Bernheim and Whinston (1986) to the generalized framework. The only difference is that in our framework $\overline{Score_\Gamma}$ does not coincide with $\overline{Wcore_\Gamma}$. This is unavoidable since $\overline{Wcore_\Gamma}$ is non-empty, whereas $\overline{Score_\Gamma}$ can be empty. However, it is possible to reconcile our result with Bernheim and Whinston (1986) with one of the following two ad-

[26] Note that condition (iii-b) is absent in their definition of Nash equilibrium.

[27] Though our bidder-optimality takes budget constraints into account, it is the same as the standard definition of the bidder-optimality when bidders have no budget constraints.

ditional assumptions. Bidders' preferences satisfy **indispensability of private good** if for all $i \in N$, $U_i(a, \omega_i(a)) = U_i(\tilde{a}, \omega_i(\tilde{a}))$ for all $a, \tilde{a} \in A$ (Mas-Colell 1977). Alternatively, bidders are deep-pocketed if for all $i \in N$, for all $a \in A$, $U_i(a, \omega_i(a)) < \min_{\tilde{a} \in A} U_i(\tilde{a}, 0)$. Either assumption implies that BC-blocking and weak blocking are the same, so we have $\overline{BCcore_\Gamma} = \overline{Score_\Gamma}$. Thus, Theorem 1 implies the following result.

Corollary 1. In every menu auction game Γ, if bidders' preferences satisfy indispensability of private good or bidders are deep-pocketed, then every TNE is a CPNE and

$$\overline{Score_\Gamma} = \overline{Wcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}.$$

Corollary 1 implies the main result of Bernheim and Whinston (1986).

Corollary 2. In every menu auction game Γ, if the auctioneer and all bidders have quasi-linear preferences ($U_0(a) = V_0(a) + \sum_{i \in N} T_i(a)$ and $U_i(a) = V_i(a) - T_i(a)$ for all $i \in N$), and bidders have no budget constraint, then every TNE is a CPNE, the auctioneer chooses $a^* \in \max_{a \in A} V_0(a) + \sum_{i \in N} V_i(a)$ in every TNE/CPNE, and $\mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE} = \overline{Score_\Gamma} = \overline{Wcore_\Gamma} = \{u \in \mathcal{U}_\Gamma(N_0) : \sum_{i \in S} u_i \leq W(N) - W(N \backslash S)$ for all $S \subseteq N\}$ where $W(S) \equiv \max_{a \in A} V_0(a) + \sum_{i \in S} V_i(a)$ for all $S \subseteq N$.

4 Concluding Remarks

In this paper, we generalize Bernheim and Whinston's (1986) menu auction game to the class of non-transferable utility game with budget constraints. This extension is useful since it allows more applications, as discussed in Section 1. However, there is another reason to study this extension. The efficiency result in menu auctions under a restricted domain has been used as a benchmark in general package/combinatorial auction designs. For example, Ausubel and Milgrom (2002) propose a generalized ascending package auction, which allows non-quasi-linear preferences and budget constraints. After bidders report their preferences, the auction mechanism uses an algorithm to determine an allocation that is shown to be in the weak core with respect to *reported* preferences.[28] This paper provides the theoretical basis for a comparison: *every* allocation in the *bidder-optimal* weak core with respect to *actual* preferences is implemented by a generalized menu auction game in CPNEs, irrespective of reported preferences.[29]

[28] There might be equilibria where bidders do not report their actual preferences.

[29] Note that bidder-optimality is slightly modified for budget constraints as defined in section 3.

Appendix: Proof of Theorem 1

First, Proposition 5 shows every TNE is a CPNE. Second, we establish $\overline{BCcore_\Gamma} = \mathcal{U}_\Gamma^{TNE} = \mathcal{U}_\Gamma^{CPNE}$ by $\overline{BCcore_\Gamma} \subseteq \mathcal{U}_\Gamma^{TNE}$ (Proposition 4), $\mathcal{U}_\Gamma^{TNE} \subseteq \mathcal{U}_\Gamma^{CPNE}$ (Proposition 5) and $\mathcal{U}_\Gamma^{CPNE} \subseteq \overline{BCcore_\Gamma}$ (Proposition 6). By Proposition 2, $\overline{Wcore_\Gamma} = \overline{BCcore_\Gamma}$. By definition, we have $Score_\Gamma \subseteq Wcore_\Gamma$ but Example 3 shows it is possible to have $Score_\Gamma = \emptyset$ and $Wcore_\Gamma \neq \emptyset$, so in general $\overline{Score_\Gamma} \subsetneq \overline{Wcore_\Gamma}$, which completes the proof.

Proposition 4. In every menu auction game Γ, every allocation $u^* \in \overline{BCcore_\Gamma}$ can be supported by a truthful Nash equilibrium.

Proof. Consider $u^* \in \overline{BCcore_\Gamma}$ supported by $(a^*, T) \in A \times \mathcal{T}$. Suppose that for all $i \in N$, bidder i chooses the truthful bidding menu T_i^* relative to u_i^*, that is, $T_i^* = T_i^{u_i^*}$. It suffices to check (a^*, T^*) is a Nash equilibrium since the resulting allocation is u^* by construction. Clearly, $T^* \in \mathcal{T}$. Suppose $a^* \notin M(T^*)$. There exists $\tilde{a} \in A$ such that $U_0(\tilde{a}, T^*(\tilde{a})) > U_0(a^*, T^*(a^*))$. Let $K \equiv \{i \in N : U_i(\tilde{a}, 0) < u_i^*\}$ so that $U_0(\tilde{a}, (T_i^*(\tilde{a}))_{i \in N \backslash K}) > U_0(a^*, T^*(a^*))$. Truthful bidding menus imply $U_i(\tilde{a}, T_i^*(\tilde{a})) \geq U_i(a^*, T_i^*(a^*))$ for all $i \in N \backslash K$. Then u^* is BC-blocked by $N_0 \backslash K$, which contradicts $u^* \in \overline{BCcore_\Gamma}$. Hence, $a^* \in M(T^*)$.

Suppose for some $j \in N$, there exists $(\bar{a}, \bar{T}_j) \in A \times \mathcal{T}_j$ such that $\bar{a} \in M(\bar{T}_j, T_{-j}^*)$, $U_j(\bar{a}, \bar{T}_j(\bar{a})) > U_j(a^*, T_j^*(a^*))$ and $\bar{T}_j(\bar{a}) < \omega_j(\bar{a})$. There are two cases. (Case 1): $m(T^*) \leq m(\bar{T}_j, T_{-j}^*)$. Let $\bar{K} \equiv \{i \in N : U_i(\bar{a}, 0) < u_i^*\}$, $\bar{u}_0 = U_0(\bar{a}, (T_i^*(\bar{a}))_{i \in (N \backslash \bar{K}) \backslash \{j\}}, \bar{T}_j(\bar{a}))$, $\bar{u}_j = U_j\left(\bar{a}, \bar{T}_j(\bar{a})\right)$ and $\bar{u}_i = U_i(a^*, T_i^*(a^*))$ for all $i \in (N \backslash \bar{K}) \backslash \{j\}$. Then $\bar{u}_{N_0 \backslash \bar{K}}$ BC-blocks u^* by $N_0 \backslash \bar{K}$, which contradicts $u^* \in BCcore_\Gamma$. (Case 2): $m(T^*) > m(\bar{T}_j, T_{-j}^*)$. Note that $a^* \in M(T^*)$ implies $T_j^*(a^*) > 0$. There exists $\varepsilon_j > 0$ such that $\hat{T}_j(a^*) = T_j^*(a^*) - \varepsilon_j > 0$ and $\hat{T}_j(a) = 0$ for all $a \in A \setminus \{a^*\}$ so that $a^* \in M(\hat{T}_j, T_{-j}^*)$ and $U_j(a^*, \hat{T}_j(a^*)) > U_j(a^*, T_j^*(a^*))$. Denote $\hat{u}_0 = U_0(a^*, \hat{T}_j(a^*), T_{-j}^*(a^*))$, $\hat{u}_j = U_j(a^*, \hat{T}_j(a^*))$ and $\hat{u}_i = U_i(a^*, T_i^*(a^*))$ for all $i \in N \setminus \{j\}$. Note that $\hat{u}_i \geq u_i^*$ for all $i \in N$ and $\hat{u}_j > u_j^*$. If $\hat{u} \in BCcore_\Gamma$, then $u^* \notin \overline{BCcore_\Gamma}$, which is a contradiction. Hence, $\hat{u} \notin BCcore_\Gamma$ holds. Then for some $S \subseteq N$, there exists $\bar{\bar{u}}_{S_0} \in \mathcal{U}_\Gamma(S_0)$ such that $\bar{\bar{u}}_{S_0}$ BC-blocks $\hat{u}$ and $\bar{\bar{u}}_{S_0} \neq u_{S_0}^*$.[30] Since $\hat{u}$ and u^* are supported by $(a^*, \hat{T}_j, T_{-j}^*)$ and (a^*, T^*), it must be $\bar{\bar{u}}_0 \geq u_0^*$.[31] However, this implies $\bar{\bar{u}}_{S_0}$ BC-blocks u^*, which contradicts $u^* \in BCcore_\Gamma$. $\square$

Before stating Proposition 5 and Proposition 6, it is useful to prove the following two lemmas.

Lemma 1. In every menu auction game Γ, if an allocation u^* is supported by a truthful Nash equilibrium (a^*, T^*) in Γ, then $u_{J_0}^* \in \overline{BCcore_{\Gamma \setminus (T_k^*)_{k \in N \setminus J}}}$ for all non-empty $J \subseteq N$ where $u_{J_0}^* \equiv (u_0^*, (u_i^*)_{i \in J})$.

Proof. First, we show $u_{J_0}^* \in BCcore_{\Gamma \setminus (T_k^*)_{k \in N \setminus J}}$ for all non-empty $J \subseteq N$. Suppose $u_{J_0}^* \notin BCcore_{\Gamma \setminus (T_k^*)_{k \in N \setminus J}}$ for some non-empty $J \subseteq N$. For some non-empty $S \subseteq J$, there exists $\tilde{u}_{S_0} \in \mathcal{U}_{\Gamma \setminus (T_k^*)_{k \in N \setminus J}}(S_0)$ supported by $(\tilde{a}, (\tilde{T}_i)_{i \in S}) \in A \times (\mathcal{T}_i)_{i \in S}$ such that $\tilde{u}_i \geq u_i^*$ for all $i \in S_0$ and $\tilde{u}_j > u_j^*$ with $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$ for some $j \in S$,[32] where $\tilde{u}_0 = U_0(\tilde{a}, (\tilde{T}_i(\tilde{a}))_{i \in S}, (T_k^*(\tilde{a}))_{k \in N \setminus J})$, $u_0^* = U_0(a^*, T^*(a^*))$, and for all $i \in S$, $\tilde{u}_i = U_i(\tilde{a}, \tilde{T}_i(\tilde{a}))$ and $u_i^* = U_i(a^*, T_i^*(a^*))$. There exists $\varepsilon_j > 0$ such that $\hat{T}_j(\tilde{a}) = \tilde{T}_j(\tilde{a}) + \varepsilon_j < \omega_j(\tilde{a})$ and $\hat{T}_j(a) = 0$ for all $a \in A \setminus \{\tilde{a}\}$ so that $U_j((\tilde{a}, \hat{T}_j(\tilde{a})) > U_j(a^*, T_j^*(a^*))$ and $U_0(\tilde{a}, \hat{T}_j(\tilde{a}), T_{-j}^*(\tilde{a})) > U_0(\tilde{a}, \tilde{T}_j(\tilde{a}), T_{-j}^*(\tilde{a}))$. For all $i \in S$, since

[30] If $\bar{\bar{u}}_{S_0} = u_{S_0}^*$, then $\hat{u}_i \geq \bar{\bar{u}}_i$ for all $i \in S_0$. This implies $\bar{\bar{u}}_{S_0}$ cannot BC-block $\hat{u}$.

[31] Otherwise, we can construct $\hat{u}$ such that it is not BC-blocked by any $\bar{\bar{u}}_{\tilde{S}_0}$ for all $\tilde{S} \subseteq N$.

[32] The auctioneer is not maximizing if there is no $j \in S$ such that $\tilde{u}_j > u_j^*$ with $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$.

T_i^* is a truthful bidding menu, we have $T_i^*(\tilde{a}) \geq \tilde{T}_i(\tilde{a})$ so that $U_0(\tilde{a}, \tilde{T}_j(\tilde{a}), T_{-j}^*(\tilde{a})) \geq U_0(\tilde{a}, (\tilde{T}_i(\tilde{a}))_{i\in S}, (T_k^*(\tilde{a}))_{k\in N\setminus J})$. Therefore, $U_0(\tilde{a}, \hat{T}_j(\tilde{a}), T_{-j}^*(\tilde{a})) > U_0(a^*, T^*(a^*))$ implies $\tilde{a} \in M(\hat{T}_j, T_{-j}^*)$, which contradicts (a^*, T^*) being a Nash equilibrium.

It remains to show $u_{J_0}^* \in \overline{BCcore_{\Gamma\setminus(T_k^*)_{k\in N\setminus J}}}$ for all non-empty $J \subseteq N$. Suppose not. There exists $\tilde{\tilde{u}}_{J_0} \in \overline{BCcore_{\Gamma\setminus(T_k^*)_{k\in N\setminus J}}}$ supported by $(\tilde{\tilde{a}}, (\tilde{\tilde{T}}_i)_{i\in J}) \in A \times (\mathcal{T}_i)_{i\in J}$ such that for all $i \in J$, $\tilde{\tilde{u}}_i \geq u_i^*$ and for some $j' \in J$, $\tilde{\tilde{u}}_{j'} > u_{j'}^*$ and $\tilde{\tilde{T}}_{j'}(\tilde{\tilde{a}}) < \omega_{j'}(\tilde{\tilde{a}})$. By Proposition 4, $\tilde{\tilde{u}}_{J_0}$ can be supported by a TNE $(\tilde{\tilde{a}}, (T_i^{\tilde{\tilde{u}}_i})_{i\in J})$ in $\Gamma\setminus(T_k^*)_{k\in N\setminus J}$ implying $m(T^*) \geq m((T_i^{\tilde{\tilde{u}}_i})_{i\in J}, (T_k^*)_{k\in N\setminus J})$. There are two cases, both of which contradict (a^*, T^*) being a Nash equilibrium. (Case 1): $m(T^*) = m((T_i^{\tilde{\tilde{u}}_i})_{i\in J}, (T_k^*)_{k\in N\setminus J})$. There exists $\varepsilon_{j'} > 0$ such that $\bar{T}_{j'}(\tilde{\tilde{a}}) = T_{j'}^{\tilde{\tilde{u}}_{j'}}(\tilde{\tilde{a}}) + \varepsilon_{j'} < \omega_{j'}(\tilde{\tilde{a}})$ and $\bar{T}_{j'}(a) = 0$ for all $a \in A\setminus\{\tilde{\tilde{a}}\}$ so that $\tilde{\tilde{a}} \in M(\bar{T}_{j'}, T_{-j'}^*)$ and $U_{j'}(\tilde{\tilde{a}}, \bar{T}_{j'}(\tilde{\tilde{a}})) > U_{j'}(a^*, T_{j'}^*(a^*))$. This contradicts (a^*, T^*) being a Nash equilibrium; (Case 2): $m(T^*) > m((T_i^{\tilde{\tilde{u}}_i})_{i\in J}, (T_k^*)_{k\in N\setminus J})$. Then $m(T^*) > m(T_{j''}^{\tilde{\tilde{u}}_{j''}}, T_{-j''}^*)$ for some $j'' \in J$. Note that $a^* \in M(T^*)$ implies $T_{j''}^*(a^*) > 0$. There exists $\varepsilon_{j''} > 0$ such that $\bar{\bar{T}}_{j''}(a^*) = T_{j''}^*(a^*) - \varepsilon_{j''} > 0$ and $\bar{\bar{T}}_{j''}(a) = 0$ for all $a \in A\setminus\{a^*\}$ so that $a^* \in M(\bar{\bar{T}}_{j''}, T_{-j''}^*)$ and $U_{j''}(a^*, \bar{\bar{T}}_{j''}(a^*)) > U_{j''}(a^*, T_{j''}^*(a^*))$ and $\bar{\bar{T}}_{j''}(a^*) < \omega_{j''}(a^*)$. This contradicts (a^*, T^*) being a Nash equilibrium.□

Lemma 2. In every menu auction game Γ with $|N| \geq 2$, if an outcome (a^*, T^*) is self-enforcing in Γ, then for all non-empty $S \subsetneq N$, $u_{S_0}^* \in \overline{BCcore_{\Gamma\setminus(T_k^*)_{k\in N\setminus S}}}$ where $u_{S_0}^* \in \mathcal{U}_\Gamma(S_0)$ is supported by $(a^*, (T_i^*)_{i\in S})$ in $\Gamma\setminus(T_k^*)_{k\in N\setminus S}$.

Proof. Consider $|N| = 2$. For all $i \in N$, (a^*, T^*) is self-enforcing in Γ if (a^*, T_i^*) is a CPNE in $\Gamma\setminus T_{-i}^*$. By definition, (a^*, T_i^*) is a CPNE in $\Gamma\setminus T_{-i}^*$ if and only if it is a Nash equilibrium in $\Gamma\setminus T_{-i}^*$. If $(u_0^*, u_i^*) \notin \overline{BCcore_{\Gamma\setminus T_{-i}^*}}$ for some $i \in N$, then

(a^*, T^*) cannot be a Nash equilibrium in Γ. The rest of argument will be completed by induction.

Consider $|N| > 2$. By the induction assumption, $u^*_{S_0} \in \overline{BCcore_{\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash S}}}$ for all $S \subseteq N$ with $|S| < |N| - 1$. Suppose $u^*_{\tilde{S}_0} \notin \overline{BCcore_{\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}}}$ for some $\tilde{S} \subseteq N$ with $|\tilde{S}| = |N| - 1$. There exists $\tilde{u}_{\tilde{S}_0} \in \overline{BCcore_{\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}}}$ supported by $(\tilde{a}, (\tilde{T}_i)_{i \in \tilde{S}}) \in A \times (\mathcal{T}_i)_{i \in \tilde{S}}$ in $\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}$ such that $\tilde{u}_i \geq u^*_i$ for all $i \in \tilde{S}$, and $\tilde{u}_j > u^*_j$ and $\tilde{T}_j\left(\tilde{a}\right) < \omega_j\left(\tilde{a}\right)$ some $j \in \tilde{S}$. By Proposition 4, $(\tilde{a}, (T^{\tilde{u}_i}_i)_{i \in \tilde{S}})$ is a TNE in $\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}$. Then, by Lemma 1, for all non-empty $J \subseteq \tilde{S}$, $\tilde{u}_{J_0} \in \overline{BCcore_{\Gamma \backslash ((T^*_k)_{k \in N \backslash \tilde{S}}, (T^{\tilde{u}_i}_i)_{i \in \tilde{S} \backslash J})}}$. Hence, $(\tilde{a}, (T^{\tilde{u}_i}_i)_{i \in \tilde{S}})$ is self-enforcing in $\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}$ as there is no credible deviation in $\Gamma \backslash ((T^*_k)_{k \in N \backslash \tilde{S}}, (T^{\tilde{u}_i}_i)_{i \in \tilde{S} \backslash \hat{S}})$ for all $\hat{S} \subseteq \tilde{S}$. Since $U_j(\tilde{a}, T^{\tilde{u}_j}_j\left(\tilde{a}\right)) > U_j(a^*, T^*_j\left(a^*\right))$ and $T^{\tilde{u}_j}_j\left(\tilde{a}\right) < \omega_j\left(\tilde{a}\right)$, the outcome (a^*, T^*) cannot be a CPNE in $\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}$. This contradicts (a^*, T^*) being self-enforcing in Γ. Hence, $u^*_{\tilde{S}_0} \in \overline{BCcore_{\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash \tilde{S}}}}$ for all $\tilde{S} \subsetneq N$. By induction, Lemma 2 is proved.□

Lemma 1 shows that every allocation supported by a TNE is also in the bidder-optimal BC-core of every component games, and Lemma 2 shows that every allocation supported by a self-enforcing outcome is also in the bidder-optimal BC-core of every component games. Since every CPNE is self-enforcing, Proposition 5 can be shown readily.

Proposition 5. In every menu auction game Γ, every truthful Nash equilibrium is a coalition-proof Nash equilibrium.

Proof. It is trivial when $|N| = 1$. Consider $|N| \geq 2$. Let (a^*, T^*) be a TNE in Γ. We proceed by induction on $S \subseteq N$. By induction assumption, for all non-empty $J \subseteq S$, $(a^*, (T^*_j)_{j \in J})$ is self-enforcing in $\Gamma \backslash \left(T^*_k\right)_{k \in N \backslash J}$. By Lemma 1, for all non-empty

$J \subseteq S$, if $u^*_{J_0} \in \mathcal{U}_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}(J_0)$ is supported by a TNE $(a^*, (T^*_j)_{j \in J})$ in $\Gamma \backslash (T^*_k)_{k \in N \backslash J}$, then $u^*_{J_0} \in \overline{BCcore_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}}$. By Lemma 2, a self-enforcing allocation must be in the bidder-optimal BC-core in every component games. Therefore, for all non-empty $J \subseteq S$, there exists no self-enforcing allocation $\tilde{u}_{J_0} \in \mathcal{U}_{\Gamma \backslash (T^*_k)_{k \in N \backslash J}}(J_0)$ supported by $(\tilde{a}, (\tilde{T}_i)_{i \in J}) \in A \times (\mathcal{T}_i)_{i \in J}$ in $\Gamma \backslash (T^*_k)_{k \in N \backslash J}$ such that $\tilde{u}_i \geq u^*_i$ for all $i \in J$, and $\tilde{u}_{i'} > u^*_{i'}$ and $\tilde{T}_{i'}(\tilde{a}) < \omega_{i'}(\tilde{a})$ for some $i' \in J$. Hence, $(a^*, (T^*_j)_{j \in S})$ is a CPNE in $\Gamma \backslash (T^*_k)_{k \in N \backslash S}$. By induction, (a^*, T^*) is a CPNE in Γ.$\square$

Proposition 6. In every menu auction game Γ, every allocation u^* supported by a coalition-proof Nash equilibrium is in the bidder-optimal BC-core.

Proof. Suppose not. There exists $\tilde{u} \in \overline{BCcore_{\Gamma}}$ supported by $(\tilde{a}, \tilde{T}) \in A \times \mathcal{T}$ such that $\tilde{u}_i \geq u^*_i$ for all $i \in N$, and $\tilde{u}_j > u^*_j$ and $\tilde{T}_j(\tilde{a}) < \omega_j(\tilde{a})$ for some $j \in N$. From Proposition 4 and Proposition 5, $\tilde{u}$ is supported by a CPNE so $\tilde{u}$ is also supported by a self-enforcing outcome. However, the allocation u^* supported by a CPNE implies that there exists no self-enforcing allocation weakly improves all bidders and strictly improves some budget-unconstrained bidders. This is a contradiction.$\square$

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