

Transitive Regret over Statistically Independent Lotteries

April 2, 2012

Abstract

Preferences may arise from regret, i.e., from comparisons with alternatives forgone by the decision maker. We show that when the choice set consists of pairwise statistically independent lotteries, transitive regret-based behavior is consistent with betweenness preferences and with a family of preferences that is characterized by a consistency property. Examples of consistent preferences include CARA, CRRA, and anticipated utility.

Keywords: Regret, transitivity, non-expected utility

1 Introduction

One of the most convincing psychological alternatives to expected utility theory is regret theory, which was independently developed by Bell [1] and by Loomes and Sugden [8]. The basic form of the theory applies to choices made between pairs of random variables. While in Savage's [12] model the decision maker evaluates a random variable by weighting its outcomes, regret theory suggests that the decision maker should also take into consideration the alternative outcome from the other random variable. This comparison may cause rejoicing — if the actual outcome is better than the alternative — or regret.

Both Bell and Loomes and Sugden assumed that the evaluation of the regret should be additive. That is, for two random variables $X = (x_1, s_1; \dots; x_n, s_n)$ and $Y = (y_1, s_1; \dots; y_n, s_n)$,

$$X \succeq Y \iff \sum_i p(s_i)\psi(x_i, y_i) \geq 0$$

where $p(s_i)$ is the probability of event s_i and ψ is a regret function. If $\psi(x, y) = u(x) - u(y)$ then this theory reduces to expected utility theory and it is easy to verify that unless this is the case, such preferences cannot be transitive. One may suspect the restrictive additive form to be the source of intransitive cycles, but as is proved in Bikhchandani and Segal [2], intransitivity is built into the regret model: even when one adopts the more general form

$$X \succeq Y \iff V(\psi(x_1, y_1), p(s_1); \dots; \psi(x_n, y_n), p(s_n)) \geq 0, \quad (1)$$

where V is any increasing (with respect to first-order stochastic dominance) functional, transitivity still implies expected utility.

Regret theory can be used to compare statistically independent lotteries (see Loomes and Sugden [8]), where the regret felt upon winning x_i and not y_j is weighted by $p_i q_j$. But consider a gambler who chooses to play the Roulette instead of Craps. While betting on Black in Roulette when the outcome turns out Red, it seems unnatural that he will compare this outcome to each specific roll of the dice in a Craps game he did not play (and probably did not even observe). Rather, it may drive him

to regret the fact that he did not play Craps, and the regret is with respect to the whole alternative distribution. Such feelings are the topic of the present paper.

We discuss a choice problem between two statistically independent lotteries $X = (x_1, p_1; \dots; x_n, p_n)$ and $Y = (y_1, q_1; \dots; y_m, q_m)$. When evaluating the lottery X , the decision maker forecasts his ex-post feelings, and considers his regret or rejoice when he will know that he won x_i but did not play the lottery Y .¹ Formally, we analyze binary relations that are defined by $X \succeq Y$ iff

$$\begin{aligned} V(\psi(x_1, Y), p_1; \dots; \psi(x_n, Y), p_n) &\geq 0 \\ &\geq V(\psi(y_1, X), q_1; \dots; \psi(y_m, X), q_m) \end{aligned} \tag{2}$$

where $\psi(x, Y)$ is the rejoice or regret felt by the decision maker upon learning that he won x in lottery X which he chose to play out of the set $\{X, Y\}$.² We call this property *distribution regret*. The question we ask is this: Under what conditions are distribution-regret relations transitive?

Unlike [2], where it was shown that only expected utility preferences are consistent with both eq. (1) and transitivity, here we identify two families of preferences that satisfy distribution regret, i.e. eq. (2), and transitivity. The first are betweenness preferences, according to which $X \succeq Y$ iff for all $\alpha \in [0, 1]$, $X \succeq \alpha X + (1 - \alpha)Y \succeq Y$ (see Chew [3, 4], Fishburn [6], and Dekel [5]). The other family is new and is characterized by a consistency property which includes, as a special case, constant risk-aversion preferences. We also offer conditions over the regret preferences under which these two families are the only preferences to satisfy distribution regret and transitivity.

The paper is organized as follows. The model and a simplification of the regret function ψ that is due to transitivity are described in Section 2. In Section 3 we show that betweenness preferences satisfy distribution regret with a linear regret functional V ; moreover, if eq. (2) holds with a linear V then preferences must be

¹These feelings do not have to agree with his initial preferences over lotteries. That is, at this stage we do not rule out the possibility of preference for the outcome x_i over Y together with anticipated regret if X is chosen and x_i is drawn. But this will be ruled out by transitivity.

²An additive form of such preferences was suggested without a further discussion by Machina [9] and by Starmer [13]. We provide additional results in Section 3 below.

betweenness. Consistent preferences are defined in Section 4 and shown to satisfy distribution regret. We conclude in Section 5.

2 The model and preliminary results

The choice set, denoted by \mathcal{L} , is the set of finite-valued lotteries with outcomes in a set $\mathcal{D} \subseteq \mathfrak{R}$. When comparing a pair of lotteries, the decision maker evaluates each possible outcome of one lottery against the entire probability distribution of the other. Thus, in evaluating $X = (x_1, p_1; \dots; x_n, p_n)$ against $Y = (y_1, q_1; \dots; y_m, q_m)$, the decision maker considers his feelings of regret or rejoicing if he were to obtain outcome x_i after choosing X over the alternative lottery Y . This evaluation is conducted for each outcome x_i of X . Implicit in this formulation is the assumption that X and Y are independent lotteries: the probability distribution of Y is unchanged after the outcome of X becomes known. With this background, we have the following definitions.

Definition 1 *The continuous function $\psi : \mathcal{D} \times \mathcal{L} \rightarrow \mathfrak{R}$ is a regret function if for all x and Y , $\psi(x, Y)$ is strictly increasing in x and strictly decreasing as Y increases in the sense of first-order stochastic dominance.*

If the lottery X yields x then $\psi(x, Y)$ is a measure of the decision maker's *ex post* feelings of regret or rejoicing about the choice of X over Y . This leads to the next definition:

Definition 2 *Let $X, Y \in \mathcal{L}$. The regret lottery evaluating the choice of X over Y is*

$$\Psi(X, Y) = (\psi(x_1, Y), p_1; \dots; \psi(x_n, Y), p_n)$$

Denote the set of regret lotteries by $\mathcal{R} = \{\Psi(X, Y) : X, Y \in \mathcal{L}\}$.

Thus, $\Psi(X, Y)$ is the ex ante regret lottery the decision maker uses in evaluating the choice of X over Y .³

³For brevity we refer to ψ and Ψ as regret functions and regret lotteries respectively, even though they encompass both regret and rejoicing.

Definition 3 A preference relation (that is, a complete and transitive relation) \succeq over \mathcal{L} is distribution-regret based if there is a regret lottery Ψ and a continuous functional $V : \mathfrak{R} \rightarrow \mathfrak{R}$ such that

$$X \succeq Y \quad \text{iff} \quad V(\Psi(X, Y)) \geq 0 \quad \text{iff} \quad 0 \geq V(\Psi(Y, X)). \quad (3)$$

The aim of this paper is to find conditions over a preference relation such that it will satisfy distribution regret. Formally, we ask under what circumstances will a preference relation satisfy eq. (3) above.

Our first observation is that transitivity of a distribution-regret relation leads to an enormous simplification of the regret function. Instead of evaluating the regret of receiving the outcome x out of X when the alternative lottery was Y , one can evaluate regret with respect to the certainty equivalent of Y . In other words, if Y and Y' are equally attractive, then the regret of x with respect to both is the same.

Lemma 1 Let \succeq be a distribution-regret preference relation. Then \succeq admits a two-dimensional regret function $\psi^* : \mathcal{D} \times \mathcal{D} \rightarrow \mathfrak{R}$ and a regret functional V^* such that

$$\begin{aligned} X \succeq Y &\iff V^*(\psi^*(x_1, c_Y), p_1; \dots; \psi^*(x_n, c_Y), p_n) \geq 0 \\ &\iff V^*(\psi^*(y_1, c_X), q_1; \dots; \psi^*(y_m, c_X), q_m) \leq 0 \end{aligned}$$

where c_X and c_Y are the certainty equivalents of X and Y respectively.

Proof: For $y \in \mathcal{D}$, define $\psi^*(x, y) = \psi(x, \delta_y)$, where δ_y denotes, with a little abuse of notation, the lottery that yields y with probability 1, and define

$$V^*(\psi^*(x_1, y), p_1; \dots; \psi^*(x_n, y), p_n) = V(\psi(x_1, \delta_y), p_1; \dots; \psi(x_n, \delta_y), p_n)$$

Then, by transitivity,

$$\begin{aligned} X \succeq Y &\iff \\ X \succeq \delta_{c_Y} &\iff \\ V(\psi(x_1, \delta_{c_Y}), p_1; \dots; \psi(x_n, \delta_{c_Y}), p_n) \geq 0 &\iff \\ V^*(\psi^*(x_1, c_Y), p_1; \dots; \psi^*(x_n, c_Y), p_n) \geq 0 & \end{aligned}$$

■

The requirement in Lemma 1 that the relation is a preference relation is restrictive, as not all distribution-regret relations are transitive. The following is an example of such a relation.

Example 1 Let

$$\psi(x, Y) = x^3 + x^2 + x + x^2\mathbb{E}[Y] - \mathbb{E}[Y^3] - \mathbb{E}[Y^2] - \mathbb{E}[Y] - x\mathbb{E}[Y^2]$$

and let

$$V(\psi(x_1, Y), p_1; \dots; \psi(x_n, Y), p_n) = \sum_i p_i \psi(x_i, Y)$$

Therefore, by eq. (3), $X \succeq Y$ iff

$$\begin{aligned} & \mathbb{E}[X^3] + \mathbb{E}[X^2] + \mathbb{E}[X] + \mathbb{E}[X^2]\mathbb{E}[Y] - \\ & \mathbb{E}[Y^3] - \mathbb{E}[Y^2] - \mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y^2] \geq 0 \end{aligned}$$

Let

- $X = (0.9, \frac{315}{1050}; 0.5, \frac{331}{1050}; 0.2, \frac{404}{1050})$
- $Y = (0.8, \frac{1}{2}; 0.2, \frac{1}{2})$
- $Z = (0.6, \frac{185}{198}; 0.3, \frac{13}{198})$

and obtain that $X \sim Y$, $Y \sim Z$, but $Z \succ X$. ■

As a consequence of Lemma 1, we use the following definition of distribution regret without loss of generality:

Definition 4 *The preference relation \succeq is distribution-regret based if there exists*

- a. *A continuous function $\psi : \mathcal{D} \times \mathcal{D} \rightarrow \mathfrak{R}$, strictly increasing in the first argument, strictly decreasing in the second argument, and such that for all x, y , $\psi(x, x) = \psi(y, y)$.*
- b. *A continuous functional $V : \mathcal{R} \rightarrow \mathfrak{R}$ such that*

$$X \succeq Y \quad \text{iff} \quad V(\Psi(X, c_Y)) \geq 0 \quad \text{iff} \quad 0 \geq V(\Psi(Y, c_X)),$$

where $\Psi(X, c_Y) = (\psi(x_1, c_Y), p_1; \dots; \psi(x_n, c_Y), p_n)$ is the regret lottery evaluating the choice of X over δ_{c_Y} (hence over Y), $\Psi(Y, c_X)$ is the regret lottery evaluating the choice of Y over δ_{c_X} (hence over X), and \mathcal{R} is the set of such regret lotteries.

For any x and x' , let $X = Y = \delta_x$ and $X' = Y' = \delta_{x'}$. Then $X \sim Y$ and $X' \sim Y'$ imply that $V((\psi(x, x), 1)) = 0$ and $V((\psi(x', x'), 1)) = 0$. Thus, $\psi(x, x) = \psi(x', x')$. Hence, the assumption that for all x, y , $\psi(x, x) = \psi(y, y)$ (Definition 4a) is necessary.

3 Betweenness preferences

The preferences \succeq satisfy betweenness [3, 4, 6, 5] if all indifference curves are linear; that is, if $X \succeq Y$ implies $X \succeq \alpha X + (1 - \alpha)Y \succeq Y$ for all $\alpha \in [0, 1]$. In this section we show that all betweenness preferences satisfy distribution regret.

Proposition 1 *Betweenness preferences satisfy distribution regret.*

Proof: For $z \in \mathcal{D}$, let u_z be a vNM utility function that determines the indifference curve through δ_z . Obviously, $X \succeq Y$ iff X is above the indifference curve through Y , which is also the indifference curve through δ_{c_Y} . That is, $X \succeq Y$ iff $E[u_{c_Y}(X)] \geq u_{c_Y}(c_Y)$. Define $\psi(x, c_Y) = u_{c_Y}(x) - u_{c_Y}(c_Y)$ and $V(R) = E[R]$ to obtain

$$\begin{aligned} X \succeq Y &\iff \\ E[u_{c_Y}(X)] &\geq u_{c_Y}(c_Y) \iff \\ E[\Psi(X, c_Y)] &\geq 0 \iff \\ V(\Psi(X, c_Y)) &\geq 0 \end{aligned}$$

■

The proof of Proposition 1 also shows that betweenness preferences admit a linear distribution-regret functional V . In fact, betweenness preferences are unique in this respect.

Proposition 2 *The following two conditions are equivalent:*

- a. *The preference relation \succeq is betweenness.*
- b. *The preference relation \succeq satisfies distribution regret with a linear functional V of the form $V(\Psi(X, c_Y)) = \sum_i p_i \psi(x_i, c_Y)$.*

Proof: By Proposition 1 and its proof, (a) \implies (b). To show that (b) \implies (a), suppose that $V(\Psi(X, Y)) = \sum_i p_i \psi(x_i, c_Y)$. Then

$$X \sim \delta_{c_Y} \implies V(\Psi(X, c_Y)) = \sum_{i=1}^n p_i \psi(x_i, c_Y) = 0$$

$$Y \sim \delta_{c_Y} \implies V(\Psi(Y, c_Y)) = \sum_{i=1}^m q_i \psi(y_i, c_Y) = 0$$

Hence

$$V(\Psi(\frac{1}{2}X + \frac{1}{2}Y, c_Y)) = \sum_{i=1}^n \frac{p_i}{2} \psi(x_i, c_Y) + \sum_{i=1}^m \frac{q_i}{2} \psi(y_i, c_Y) = 0$$

Therefore $\frac{1}{2}X + \frac{1}{2}Y \sim \delta_{c_Y} \sim Y \sim X$, hence the claim. ■

In particular, transitivity together with an additive regret functional of the form in eq. (2) does not imply expected utility when the two lotteries are statistically independent.⁴

4 Consistency

We offer another set of transitive distribution preferences. Unlike betweenness preferences where there is little connection between the shape of different indifference curves, the preferences we discuss in this section exhibit a very high degree of interdependency among indifference curves. We call this property consistency. To illustrate this concept, consider the set of CARA (constant absolute risk aversion)

⁴See also Machina [9], footnote 20.

preferences: $X = (x_1, p_1; \dots; x_n, p_n) \succeq Y = (y_1, q_1; \dots; y_m, q_m)$ iff for every suitable λ ,⁵

$$X + \lambda = (x_1 + \lambda, p_1; \dots; x_n + \lambda, p_n) \succeq Y + \lambda = (y_1 + \lambda, q_1; \dots; y_m + \lambda, q_m).$$

Once we know the shape of one indifference curve, (segments of) all other indifference curves are determined by it.

Definition 5 *The preference relation \succeq over \mathcal{L} is consistent if there is a continuous function $f(x, \lambda)$ such that*

- a. *f is strictly increasing in both x and λ*
- b. *For every x, y in the interior of \mathcal{D} there is λ such that $f(x, \lambda) = y$*
- c. *For all $x, y \in \mathcal{D}$ and λ , $f(x, \lambda) > x$ if and only if $f(y, \lambda) > y$*
- d. *For all $X, Y \in \mathcal{L}$ and for every (relevant) λ ,*

$$X \succeq Y \quad \text{iff} \quad f(X, \lambda) \succeq f(Y, \lambda)$$

where $f(Z, \lambda) := (f(z_1, \lambda), r_1; \dots; f(z_\ell, \lambda), r_\ell)$ for any $Z = (z_1, r_1; \dots; z_\ell, r_\ell)$.

The following are examples of consistent preferences.

Example 2

- a. If \succeq is CARA, let $f(x, \lambda) = x + \lambda$ and if \succeq is CRRA (constant relative risk aversion), let $f(x, \lambda) = \lambda x$ and obtain that both types of preferences are consistent.⁶
- b. Let \mathcal{L} be the set of lotteries with positive outcomes, and for $\lambda > 1$, let $f(x, \lambda) = (x + 1)^\lambda - 1$. Define the indifference curve through δ_1 to be

$$\left\{ X : (\mathbb{E}[X])^2 + \mathbb{E}[\sqrt{X}] \right\} = 2$$

⁵See Safra and Segal [11] for a discussion of constant risk aversion in general non-expected utility theory.

⁶Because the choice set \mathcal{L} for CRRA preferences consists of lotteries over non-negative numbers, i.e. $\mathcal{D} = \mathfrak{R}_+$, Definition 5c is satisfied.

Define λ_X implicitly by

$$(\mathbb{E}[f(X, \lambda_X)])^2 + \mathbb{E}[\sqrt{f(X, \lambda_X)}] = 2$$

The existence of λ_X follows by continuity. The preferences that are represented by $V(X) = 1/\lambda_X$ are consistent.

- c. Let \succeq be rank dependent⁷ with the utility function u . Let $f(x, \lambda) = u^{-1}(u(x) + \lambda)$ and let $X = (x_1, p_1; \dots; x_n, p_n)$ with $x_1 \leq \dots \leq x_n$. We obtain for the rank dependent functional

$$U(f(X, \lambda)) := U(f(x_1, \lambda), p_1; \dots; f(x_n, \lambda), p_n) = U(X) + \lambda$$

and \succeq is consistent. ■

A preference relation \succeq over lotteries in \mathcal{L} is *consistent on a complete domain* if the domain \mathcal{D} of outcomes of lotteries in \mathcal{L} satisfies the following condition: if for some $x, y \in \mathcal{D}$ and λ , $f(x, \lambda) = y$ then for any $z \in \mathcal{D}$, $f(z, \lambda) \in \mathcal{D}$.

The following is the main result of this section.

Proposition 3 *If the preference relation \succeq is consistent on a complete domain then it satisfies distribution regret.*

Proof: For every $Z \in \mathcal{L}$, define $\lambda(Z)$ to be the number λ such that $f(c_Z, \lambda) = 0$.⁸ The existence of $\lambda(Z)$ is implied by Definition 5. Let U be the representation of \succeq satisfying $U(\delta_x) = x$ for all x . Define $\psi(x, c_Y) = f(x, \lambda(Y))$.⁹ That is, $\psi(x, c_Y)$ is the number into which x is transformed via f by applying $\lambda(Y)$ to it, where $\lambda(Y)$ is the number that transforms the certainty equivalent of Y to 0. For example, if \succeq is

⁷According to the rank dependent theory (Quiggin [10]), for $X = (x_1, p_1; \dots; x_n, p_n)$ with $x_1 \leq \dots \leq x_n$, the preferences \succeq can be represented by

$$U(X) = u(x_1)g(p_1) + \sum_{i=2}^n u(x_i)[g(\sum_{j=1}^i p_j) - g(\sum_{j=1}^{i-1} p_j)]$$

⁸We assume that $0 \in \mathcal{D}$. If this is not the case, then for some $d \in \mathcal{D}$, let $f(c_Z, \lambda(Z)) = d$ and use the normalization $\psi(x, x) = d$ in Definition 4.

⁹Observe that $\psi(x, x) = \psi(x, c_{\delta_x}) = f(x, \lambda(\delta_x)) = 0$.

CARA, then $\lambda(Z) = -c_Z$ and $\psi(x, y) = x - y$. By the completeness of the domain, for any $x \in \mathcal{D}$ we have $f(x, \lambda(Y)) \in \mathcal{D}$. Let

$$\begin{aligned} f(X, \lambda(Y)) &= (f(x_1, \lambda(Y)), p_1; \dots; f(x_n, \lambda(Y)), p_n) \\ &= (\psi(x_1, c_Y), p_1; \dots; \psi(x_n, c_Y), p_n) \\ &= \Psi(X, c_Y) \end{aligned} \tag{4}$$

By consistency

$$\begin{aligned} X \succeq Y &\sim \delta_{c_Y} \iff \\ f(X, \lambda(Y)) \succeq f(Y, \lambda(Y)) &\sim \delta_{f(c_Y, \lambda(Y))} = \delta_0 \iff \\ U(f(X, \lambda(Y))) \geq U(f(Y, \lambda(Y))) &= U(\delta_0) = 0 \end{aligned}$$

Let

$$V(\Psi(X, c_Y)) = U(f(X, \lambda(Y))) = U(\Psi(X, c_Y))$$

(The second equation sign follows by eq. (4)). Therefore

$$\begin{aligned} X \succeq Y &\iff \\ U(\Psi(X, c_Y)) \geq 0 &\iff \\ V(\Psi(X, c_Y)) \geq 0 \end{aligned}$$

Hence \succeq satisfies distribution regret. ■

Not all preferences are consistent. We present next an example that does not satisfy distribution regret and therefore is not consistent.

Example 3 Let \succeq have a linear indifference curve \mathcal{I} such that the preference relation above it is strictly quasi-concave.¹⁰ Let $X, Y, Z \in \mathcal{I}$. Under the supposition that \succeq satisfies distribution regret, we have $V(\Psi(X, c_Z)) = V(\Psi(Y, c_Z)) = 0$, and by the linearity of \mathcal{I} , for all $\alpha \in (0, 1)$, $V(\Psi(\alpha X + (1 - \alpha)Y, c_Z)) = 0$.

Let Z' be a lottery on a non-linear indifference curve above \mathcal{I} and let X' and Y' be such that

$$\Psi(X', c_{Z'}) = \Psi(X, c_Z) \quad \text{and} \quad \Psi(Y', c_{Z'}) = \Psi(Y, c_Z) \tag{5}$$

¹⁰Preferences are strictly quasi-concave iff $X \succeq Y$ implies $\alpha X + (1 - \alpha)Y \succ Y$ for all $\alpha \in (0, 1)$.

By continuity of ψ there exist such X' , Y' , Z' close to \mathcal{I} . Distribution regret implies that $X' \sim Z'$ and $Y' \sim Z'$. It follows from (5) that

$$\Psi(\tfrac{1}{2}X' + \tfrac{1}{2}Y', c_{Z'}) = \Psi(\tfrac{1}{2}X + \tfrac{1}{2}Y, c_Z)$$

Hence $V(\Psi(\tfrac{1}{2}X' + \tfrac{1}{2}Y', c_{Z'})) = 0$ and yet by the assumption that all indifference curves above \mathcal{I} are strictly quasi-concave it follows that $\tfrac{1}{2}X' + \tfrac{1}{2}Y' \approx Z'$, a contradiction. ■

The converse of Proposition 3 is not true. As is demonstrated by the next example, there are distribution-regret preferences that are not consistent.

Example 4 Define the betweenness preferences \succeq by $X \sim \delta_\alpha$ iff $E[u_\alpha(X)] = u_\alpha(\alpha)$, where

$$u_\alpha(x) = \begin{cases} x & \alpha \leq 0 \\ g_\alpha(x) & \alpha > 0 \end{cases} \quad (6)$$

and where for $\alpha > 0$,

$$g_\alpha(x) = \begin{cases} x & x \leq 0 \\ (1 + \alpha)x & x > 0 \end{cases} \quad (7)$$

Let

$$\mu_X^+ = \sum_{i: x_i > 0} p_i x_i$$

and obtain that \succeq can be represented by a function V , given by $X \sim \delta_{E[X]}$ for X such that $E[X] \leq 0$, and for X with $E[X] > 0$, $X \sim \delta_\alpha$ where α solves

$$\begin{aligned} E[X] + \alpha \mu_X^+ &= (1 + \alpha)\alpha \implies \\ \alpha &= \frac{-(1 - \mu_X^+) + \sqrt{(1 - \mu_X^+)^2 + 4E[X]}}{2} \end{aligned} \quad (8)$$

These preferences satisfy betweenness, therefore by Proposition 1, they satisfy distribution regret.

Observe that

$$E[X] > 0 \quad \text{and} \quad \Pr(X < 0) > 0 \implies X \succ \delta_{E[X]} \quad (9)$$

This follows because $\mu_X^\dagger > E[X] > 0$ and if $X \sim \delta_\alpha$, then by eq. (8) we have $\alpha > E[X]$.

Suppose that the preference relation \succeq is consistent. Let $-1 < s, t < 0$, and λ_0 be such that $f(-1, \lambda_0) = s$ and $f(t, \lambda_0) = 0$. For every $z \geq t$ we have

$$E \left[\left(z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z} \right) \right] = t, \quad \text{hence} \quad \left(z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z} \right) \sim (t, 1)$$

Consistency implies that

$$\left(f(z, \lambda_0), \frac{1+t}{1+z}; s, \frac{z-t}{1+z} \right) \sim (0, 1) \tag{10}$$

As the certainty equivalent in eq. (10) is not greater than 0, it follows from the definition of \succeq that

$$\frac{1+t}{1+z} f(z, \lambda_0) + \frac{s(z-t)}{1+z} = 0 \implies f(z, \lambda_0) = \frac{s(t-z)}{1+t}$$

From eq. (6) we have $(-1, \frac{1}{2}; 1, \frac{1}{2}) \sim \delta_0$, hence by consistency and monotonicity

$$\left(s, \frac{1}{2}; f(1, \lambda_0), \frac{1}{2} \right) \sim \delta_{f(0, \lambda_0)} \tag{11}$$

The expected value of the last lottery is

$$\frac{s}{2} + \frac{s t - 1}{2(1+t)} = \frac{st}{1+t} = f(0, \lambda_0) > 0$$

which together with eq. (11) contradicts eq. (9). ■

We have seen so far that betweenness and consistent preferences satisfy distribution regret (Propositions 1 and 3). There are betweenness preferences that are not consistent (Example 4), there are consistent preferences that are not betweenness (e.g., rank dependent, see Example 2c), and there are betweenness preferences that are consistent (e.g., expected utility or weighted utility, see Example 5 below). The next question is whether there are distribution-regret preferences that are neither consistent nor betweenness. We do not know the answer to this question for the general case, but suppose that in addition to distribution regret, the regret function has the following property:

Definition 6 *The regret function ψ is commutative if for all x, x', y, y' ,*

$$\psi(x, x') = \psi(y, y') \implies \psi(x, y) = \psi(x', y') \tag{12}$$

It turns out that if the regret function is commutative, then the converse of Proposition 3 holds.

Proposition 4 *If the preference relation \succeq satisfies distribution regret with a commutative regret function ψ , then it is consistent.*

Proof: Let $d \in \mathfrak{R}$ be such that $\psi(x, x) = d$ for all $x \in \mathcal{D}$. (Definition 4a implies that d exists.) Define $f(x, \lambda) = y$ where $\psi(x, y) = d - \lambda$. That is, for all x and λ ,

$$\psi(x, f(x, \lambda)) = d - \lambda \quad (13)$$

Let $X \succeq Y$, hence by distribution regret, $V(\Psi(X, c_Y)) \geq 0$. As by eq. (13)

$$\psi(c_X, f(c_X, \lambda)) = \psi(c_Y, f(c_Y, \lambda)) = d - \lambda$$

we have from eq. (12) that $\psi(c_X, c_Y) = \psi(f(c_X, \lambda), f(c_Y, \lambda))$, hence

$$\Psi(\delta_{c_X}, c_Y) = \Psi(\delta_{f(c_X, \lambda)}, f(c_Y, \lambda))$$

Thus, $\delta_{c_X} \succeq \delta_{c_Y}$ and distribution regret together imply that $\delta_{f(c_X, \lambda)} \succeq \delta_{f(c_Y, \lambda)}$.

Next, another application of eq. (13) and then eq. (12) implies that

$$\psi(x_i, c_X) = \psi(f(x_i, \lambda), f(c_X, \lambda))$$

Therefore, $\Psi(X, c_X) = \Psi(f(X, \lambda), f(c_X, \lambda))$. Hence,

$$V(\Psi(X, c_X)) = V(\Psi(f(X, \lambda), f(c_X, \lambda))) = 0,$$

and $f(c_X, \lambda)$ is the certainty equivalent of $f(X, \lambda)$. Similarly, $f(c_Y, \lambda)$ is the certainty equivalent of $f(Y, \lambda)$. Hence, consistency. ■

We saw in Example 2 that CARA, CRRA, and rank dependent preferences are all consistent (and hence satisfy distribution regret). We show next that they are commutative. For CARA, let $\psi(x, y) = f(x, \lambda(\delta_y)) = x - y$; for CRRA, let $\psi(x, y) = f(x, \lambda(\delta_y)) = x/y$; and for rank dependent preferences with the utility function u , let $\psi(x, y) = f(x, \lambda(\delta_y)) = u^{-1}(u(x) - u(y))$.

Example 4, which satisfies betweenness, is not consistent. The next example shows that betweenness and consistency do not imply expected utility. It also satisfies

eq. (12). Hence, consistency, commutative ψ , and (non-expected utility) betweenness are compatible.

Example 5 Weighted utility (Chew [3]): Let $X \succeq Y$ iff $\frac{\sum_i \nu(x_i)p_i}{\sum_i \tau(x_i)p_i} > \frac{\sum_j \nu(y_j)q_j}{\sum_j \tau(y_j)q_j}$. Then

$$\begin{aligned} X \succeq Y &\iff \frac{\sum_i \nu(x_i)p_i}{\sum_i \tau(x_i)p_i} \geq \frac{\nu(c_Y)}{\tau(c_Y)} \\ &\iff \sum_i \left[\frac{\nu(x_i)}{\tau(x_i)} - \frac{\nu(c_Y)}{\tau(c_Y)} \right] p_i \geq 0. \end{aligned}$$

Let $\psi(x, x') = \frac{\nu(x)}{\tau(x)} - \frac{\nu(x')}{\tau(x')}$. It is readily verified that ψ is commutative. ■

5 Concluding remarks

We established that betweenness and consistent preferences are two families of transitive preferences that satisfy distribution regret. Moreover, if the regret function ψ is commutative then transitivity and distribution regret implies consistency, and betweenness and consistent preferences become the *only* two families of transitive preferences that satisfy distribution regret. That weighted utility, which is a special case of betweenness preferences, satisfies distribution regret was known from Machina [9] and Starmer [13].¹¹ Our first contribution is to show that *all* betweenness preferences satisfy distribution regret. The second family of preferences we discuss, that of consistent preferences, is a new class that may be of independent interest.

In Bikhchandani and Segal [2] it was shown that if the alternatives X and Y are fully correlated, then transitive regret implies expected utility. In this paper we find that at the other extreme, where X and Y are statistically independent, a large class of non-expected utility models are compatible with transitive regret. The intermediate case of not perfectly correlated X and Y is the subject of future research.

¹¹These authors used the additive regret function in eq. (1) rather than the distribution regret function in eq. (2).

References

- [1] Bell, D., 1982. “Regret in decision making,” *Operations Research*, 30:961–981.
- [2] Bikhchandani, S. and U. Segal, 2010. “Transitive regret,” *Theoretical Economics*, 6:95–108.
- [3] Chew, S.H., 1983. “A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox,” *Econometrica* 51:1065–1092.
- [4] Chew, S.H., 1989. “Axiomatic utility theories with the betweenness property,” *Annals of Operations Research* 19:273–298.
- [5] Dekel, E., 1986. “An axiomatic characterization of preferences under uncertainty,” *Journal of Economic Theory*, 40:304–318.
- [6] Fishburn, P.C., 1983. “Transitive measurable utility,” *Journal of Economic Theory* 31:293–317.
- [7] Fishburn, P.C., 1989. “Non-transitive measurable utility for decision under uncertainty,” *Journal of Mathematical Economics*, 18:187–207.
- [8] Loomes, G. and R. Sugden, 1982. “Regret theory: An alternative theory of rational choice under uncertainty,” *Economic Journal*, 92:805–824.
- [9] Machina, M.J., 1987. “Choice under uncertainty: Problems solved and unsolved,” *Journal of Economic Perspectives* 1:121–154.
- [10] Quiggin, J., 1982. “A theory of anticipated utility,” *Journal of Economic Behavior and Organization*, 3:323–343.
- [11] Safra, Z. and U. Segal, 1998. “Constant risk aversion,” *Journal of Economic Theory*, 83:19–42.
- [12] Savage, L.J., 1954. *The foundations of statistics*, New York: Wiley.

- [13] Starmer, C., 2000. "Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk," *Journal of Economic Literature*, 38:332–382.