Recursive Ambiguity and Machina's Examples*

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Abstract

Machina (2009, 2012) lists a number of situations where standard models of ambiguity aversion are unable to capture plausible features of ambiguity attitudes. Most of these problems arise in choice over prospects involving three or more outcomes. We show that the recursive non-expected utility model of Segal (1987) is rich enough to accommodate all these situations.

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1 Introduction

Ellsberg (1961) suggested a number of thought experiments that challenge Savage's (1954) subjective expected utility theory and the sure-thing principle. These experiments reveal different attitudes toward risk (objectively given probabilities) and uncertainty or ambiguity (unknown probabilities). Mainly motivated by these examples, several formal models have been proposed to accommodate ambiguity aversion. Choquet expected utility is the standard model used in the literature to explain the Ellsberg paradox (see Schmeidler (1989) and Gilboa (1987)). Other well-known models are, for example, maximin expected utility (Gilboa and Schmeidler (1989)), variational preferences (Maccheroni, Marinacci, and Rustichini (2006)), α-maxmin (Ghirardato, Maccheroni, and Marinacci (2004)), and the smooth model of ambiguity aversion (Klibanoff, Marinacci, and Mukerji (2005)).

Ellsberg experiments involve binary bets (that is, the ambiguous prospects have only two outcomes). Machina (2009) offers examples of ambiguous choice problems which involve three or more outcomes and which cannot be handled by Choquet expected utility. Baillon, L'Haridon, and Placido (2012) show that these examples pose difficulties not only for Choquet expected utility, but also for the other four models mentioned above. In a follow-up paper, Machina (2012) offers more examples of non-binary bets and explains why they pose new difficulties to many of the above models. Evidently, the source of these models' inflexibility is a certain degree of event-separability that is built into each one of them.

Our aim in this work is to show that all of Machina's examples can easily be handled by the recursive non-expected utility model of Segal (1987, 1990). According to this model, the decision maker contemplates possible probabilistic realizations of the given uncertainty, and computes for each of them its subjective value for him. He then views the uncertain prospect as a lottery over these subjective values, using his personal beliefs over the possible realizations. As the decision maker does not maximize expected utility, inseparability between events is an integral part of this model.

Machina's aim was not to investigate behavioral patterns of ambiguity, but to show that several models of ambiguity aversion are unable to capture features of ambiguity attitudes which can be revealed in choice over prospects involving three or more outcomes. Consequentially, we are not proposing a general theory or conditions under which a particular pattern of behavior will be observed. Instead, we provide some simple examples demonstrating that the recursive model is rich enough to accommodate these possible attitudes.

The reminder of the paper is organized as follows: Section 2 reviews the recursive non-expected utility model. Section 3 describes Machina's examples; the examples in Sections 3.1 and 3.2 are taken from Machina (2009), while the rest of the examples are taken from Machina (2012). For each example, we mention which models are inconsistent with it, and show, by means of examples, that it is consistent with recursive non-expected utility.

2 Recursive Non-Expected Utility

Let $[w, b]$ be an interval of monetary prizes, and let $S = \{s_1, \dots, s_n\}$ be a finite state space. Consider a certain random variable $x = (x_1, s_1; \dots; x_n, s_n)$. The decision maker does not know the probabilities of (all) the states $s_1, \dots, s_n$, but he has possible probability measures for them. For simplicity, assume that there are m such possible measures, $P^j = (p_1^j, \dots, p_n^j)$, $j = 1, \dots, m$. The decision maker believes that there is probability q^j that the true measure is P^j. He therefore views the ambiguous prospect as a two-stage lottery, where with probability q^j he'll play the lottery $X^j = (x_1, p_1^j; \dots; x_n, p_n^j)$, $j = 1, \dots, m$.

The decision maker is using a non-expected utility functional V to evaluate single-stage lotteries. Denote by c^j the certainty equivalent of lottery X^j, that is, the number that satisfies

$$V(c^j, 1) = V(x_1, p_1^j; \dots; x_n, p_n^j)$$

The decision maker replaces each of the second-stage lotteries $X^1, \dots, X^m$ with its certainty equivalent using the functional V, thus obtaining the simple lottery $(c^1, q^1; \dots; c^m,\ q^m)$. He then computes the V value of this lottery, $V(c^1, q^1; \dots; c^m, q^m)$, which is his subjective value of the ambiguous random variable x.

Of course, the decision maker may instead reduce the two-stage lottery into a simple lottery by computing the probabilities of the final outcomes. This is known as the reduction of compound lotteries axiom, and together with the above recursive procedure is known to imply expected utility theory (see Samuelson (1952)). The procedure we use must therefore violate the reduction axiom and expected utility theory. For further analysis, see Segal (1987, 1990).[1]

[1]Halevy (2007) provides evidence in favor of the recursive, non-expected utility model. Approximately 40% of his subjects were classified as having preferences that are consistent with that model.

Identifying ambiguity with a compound lottery which the decision maker fails to reduce, does not depend on the specific functional V used in the evaluation procedure described above. Accordingly, we conduct our analysis using two well known non-expected utility functionals: the rank-dependent utility of Quiggin (1982) and Gul's (1991) model of disappointment aversion, which we now formally describe.

For $x_1 \leqslant \ldots \leqslant x_n$, the rank-dependent value $V(x_1, p_1; \ldots; x_n, p_n)$ is given by

$$u(x_n)f(p_n) + \sum_{i=1}^{n-1} u(x_i)[f(\sum_{j=i}^{n} p_j) - f(\sum_{j=i+1}^{n} p_j)] \tag{1}$$

where $f : [0,1] \to [0,1]$ is strictly increasing and onto, and $u : [w,b] \to \Re$ is increasing. Segal (1987) provided sufficient conditions on the probability-weighting function f, under which a binary non-ambiguous lottery is preferred to an ambiguous (compound) one.

In Gul's (1991) disappointment aversion model, the support of any non-degenerate lottery is divided into two groups, the elating outcomes (which are preferred to the lottery) and the disappointing outcomes (which are worse than the lottery). The decision maker values lotteries by taking their "expected utility," except that disappointing outcomes get a uniformly greater (or smaller) weight that depends on the value of a single parameter β, the coefficient of disappointment aversion. The disappointment aversion value of a lottery X, $V(X; \beta, u)$, is the unique v that solves

$$v = \frac{\sum_{\{x|u(x) \geq v\}} p(x)u\,(x) + (1+\beta)\sum_{\{x|u(x)<v\}} p(x)u\,(x)}{1 + \beta \sum_{\{x|u(x)<v\}} p(x)} \tag{2}$$

where $\beta \in (-1, \infty)$ and $u : [w,b] \to \Re$ is increasing. Note that in any non-degenerate lottery, the highest outcome is elating and the lowest outcome is disappointing.

Under the interpretation that ambiguity aversion amounts to preferring the objective (unambiguous) simple lottery to the (ambiguous) compound one, Artstein-Avidan and Dillenberger (2011) show that a disappointment averse decision maker with $\beta > 0$ displays ambiguity aversion for *any* possible beliefs he might hold about the probability distribution over the states. Furthermore, this result is valid for arbitrary number of outcomes.[2]

[2] This assertion is not specific to Gul's model but applies to any member of the class of preferences characterized in Dillenberger (2010).

3 Addressing Machina's Examples

The examples in Sections 3.1 and 3.2 are taken from Machina (2009). The other examples are taken from Machina (2012).

3.1 The 50:51 Example

An urn contains 101 balls, each carries one of the numbers $1, \ldots, 4$. Of these, 50 are marked either 1 or 2 and 51 are marked either 3 or 4. Let E_i denote the event "a ball marked i is drawn" and consider the following four acts:

	50 balls		51 balls	
Act	E_1	E_2	E_3	E_4
f_1	8,000	8,000	4,000	4,000
f_2	8,000	4,000	8,000	4,000
f_3	12,000	8,000	4,000	0
f_4	12,000	4,000	8,000	0

By the sure-thing principle (Savage (1954)),[3] $f_1 \succeq f_2$ iff $f_3 \succeq f_4$. Machina showed that a property of Choquet expected utility, called tail-separability, similarly implies that f_1 is preferred to f_2 if and only if f_3 is preferred to f_4. Nevertheless, Machina (2009, Sec. II) invokes an Ellsberg-like argument that f_4 could be preferred to f_3 even though f_1 were preferred to f_2, which accordingly violates Choquet expected utility theory. Moreover, Baillon, L'Haridon, and Placido (2012) show that the same holds true for the maximin expected utility and α-maximin. For variational preferences and the smooth model of ambiguity aversion, they show that $f_1 \succeq f_2$ implies $f_3 \succeq f_4$.

We now analyze the four acts $f_1, \ldots, f_4$ using the recursive model, where V is of Gul's (1991) disappointment aversion model (2), with $u(x) = x$ and $\beta = 0.2$.[4] In that case, $V(X)$ is also the certainty equivalent of X.

Suppose that the decision maker believes that 25 balls are marked 1 and 25 balls are marked 2. With respect to the composition of the other 51 balls, he believes that it is equally likely that either all of them are marked 3 or

[3]By the sure-thing principle, if on E, $f_1 = f_2$ and $f_3 = f_4$, and on $\neg E$, $f_1 = f_3$ and $f_2 = f_4$, then $f_1 \succeq f_2$ iff $f_3 \succeq f_4$. In our case, $E = E_1 \cup E_4$ and $\neg E = E_2 \cup E_3$.

[4]Positive β implies that the decision maker is disappointment averse and, as is shown in Gul (1991, Theorem 3), is also risk averse.

all of them are marked 4.[5] In the acts $f_1, \dots, f_4$ we deal with the following simple lotteries (to simplify notation, we divide all outcomes by 1,000).

$\boldsymbol{f_1}$**:** $X^1 = (8, \frac{50}{101}; 4, \frac{51}{101})$

$\boldsymbol{f_2}$**:** $X^2 = (8, \frac{76}{101}; 4, \frac{25}{101})$ and $X^3 = (8, \frac{25}{101}; 4, \frac{76}{101})$

$\boldsymbol{f_3}$**:** $X^4 = (12, \frac{25}{101}; 8, \frac{25}{101}; 4, \frac{51}{101})$ and $X^5 = (12, \frac{25}{101}; 8, \frac{25}{101}; 0, \frac{51}{101})$

$\boldsymbol{f_4}$**:** $X^6 = (12, \frac{25}{101}; 8, \frac{51}{101}; 4, \frac{25}{101})$ and $X^7 = (12, \frac{25}{101}; 4, \frac{25}{101}; 0, \frac{51}{101})$

Using eq. (2), we obtain that $c^1 = 5.798$, $c^2 = 6.867$, and $c^3 = 4.860$. The certainty equivalent of f_2 is thus the certainty equivalent of $(6.867, \frac{1}{2}; 4.860, \frac{1}{2})$, which is $= 5.773$. Hence $f_1 \succ f_2$.

Likewise, $c^4 = 6.698$, $c^5 = 4.496$, and the certainty equivalent of f_3 is the certainty equivalent of $(6.698, \frac{1}{2}; 4.496, \frac{1}{2})$ which is 5.497. On the other hand, $c^6 = 7.811$, $c^7 = 3.597$, and and the certainty equivalent of f_4 is the certainty equivalent of $(7.811, \frac{1}{2}; 3.597, \frac{1}{2})$ which is 5.513. Hence $f_4 \succ f_3$.

It is worth noting that Gul's model is just an example. In fact, since the preferences $f_1 \succ f_2$ and $f_4 \succ f_3$ do not violate first-order stochastic dominance, it is easy to find arbitrary choice of certainty equivalents that will be consistent with these preferences. For example, let $c^1 = 6$, $c^2 = 7$, $c^3 = 5$, $c^4 = 10$, $c^5 = 9$, $c^6 = 11$, and $c^7 = 8$. Next, let $X^8 = (7, \frac{1}{2}; 5, \frac{1}{2})$ with $c^8 = 5.5$, $X^9 = (10, \frac{1}{2}; 9, \frac{1}{2})$ with $c^9 = 9.5$, and $X^{10} = (11, \frac{1}{2}; 8, \frac{1}{2})$ with $c^{10} = 9.6$, and we have $f_1 \succ f_2$ while $f_4 \succ f_3$.

3.2 The Reflection Example

Consider the following acts.

	50 balls		50 balls	
Act	E_1	E_2	E_3	E_4
f_5	4,000	8,000	4,000	0
f_6	4,000	4,000	8,000	0
f_7	0	8,000	4,000	4,000
f_8	0	4,000	8,000	4,000

[5]This particular choice is not crucial for our result. That is, the argument could be made with many other possible compositions of the urn. The recursive non-expected utility model does not pin down the beliefs of the decision maker. Our aim is thus to make our point using simple and plausible possible beliefs.

The two acts f_5 and f_8 reflect each other and the decision maker should therefore be indifferent between them. Likewise, f_6 should be indifferent to f_7. As by the Choquet expected utility model $f_5 \succeq f_6$ iff $f_7 \succeq f_8$, it follows that $f_5 \sim f_6$ (and $f_7 \sim f_8$). Yet, as is argued by Machina (2009, Sec. III), ambiguity attitudes may well suggest strict preference within each pair.

Let $\alpha, \beta, \gamma, \delta$ be a list of possible numbers of balls of the four types in the urn, where $\alpha + \beta = \gamma + \delta = 50$. Denote by $q(\alpha, \beta, \gamma, \delta)$ the probability the decision maker attaches to the event "the composition of the urn is $\alpha, \beta, \gamma, \delta$." We say that such beliefs are symmetric if

$$q(\alpha, \beta, \gamma, \delta) = q(\beta, \alpha, \delta, \gamma) = q(\gamma, \delta, \alpha, \beta) = q(\delta, \gamma, \beta, \alpha).$$

If beliefs are symmetric, then the recursive model implies $f_5 \sim f_8$ and $f_6 \sim f_7$, yet it does not require $f_5 \sim f_6$. For example, let $q(10, 40, 25, 25) = \frac{1}{4}$. As before, divide all outcomes by 1,000 and obtain that acts f_5 and f_8 become equal-probability lotteries over the certainty equivalents of $X^{11} = (8, \frac{8}{20}; 4, \frac{7}{20}; 0, \frac{5}{20})$, $X^{12} = (8, \frac{2}{20}; 4, \frac{13}{20}; 0, \frac{5}{20})$, $X^{13} = (8, \frac{5}{20}; 4, \frac{13}{20};\ 0, \frac{2}{20})$, and $X^{14} = (8, \frac{5}{20}; 4, \frac{7}{20}; 0, \frac{8}{20})$. Acts f_6 and f_7 yield the lottery $X^{15} = (8, \frac{1}{4}; 4, \frac{1}{2}; 0, \frac{1}{4})$ with probability $\frac{1}{2}$, and with probability $\frac{1}{4}$ each of the lotteries $X^{16} = (8, \frac{4}{10}; 4, \frac{5}{10}; 0, \frac{1}{10})$ and $X^{17} = (8, \frac{1}{10}; 4, \frac{5}{10}; 0, \frac{4}{10})$. Using Gul's functional (2), we obtain that $c^{11} = 4.357$, $c^{12} = 3.238$, $c^{13} = 4.452$, $c^{14} = 3.148$, and $V(f_5) = V(c^{11}, \frac{1}{4}; \ldots; c^{14}, \frac{1}{4}) = 3.744$. On the other hand, $c^{15} = 3.809$, $c^{16} = 5.0$, $c^{17} = 2.593$, and $V(f_6) = V(c^{15}, \frac{1}{2}; c^{16}, \frac{1}{4}; c^{17}, \frac{1}{4}) = 3.745$, hence $f_6 \succ f_5$.

3.3 The Slightly Bent Coin Problem

A coin is flipped and a ball is drawn out of an urn. You know that the coin is slightly bent (but you don't know which side is more likely or the respective probabilities) and that the urn contains two balls, each is either white or black. Which of the following bets do you prefer?

I	black	white
heads	8,000	0
tails	-8,000	0

II	black	white
heads	0	0
tails	-8,000	8,000

According to Machina (2012, Sec. IV), it is plausible that an ambiguity averse decision maker will prefer Bet *I* to *II*. The reason is that if the coin is only slightly biased, then betting on the coin flip (as in Bet *I*) is less ambiguous than betting on the color of the ball (as in Bet *II*). Yet he shows that a Choquet expected utility maximizer must be indifferent between the two bets.

Consider first the urn with the two balls. As there is no reason to believe any bias in favor of white or black, we assume that the decision maker believes that the probability of each of the two events "there are two black balls" and "there are two white balls" is q, and the probability of the event "there is one black and one white ball" is $1-2q$.

The analysis of the coin is slightly more involved, as the decision maker does not know the direction in which it is biased (heads or tails), nor does he know the magnitude of the bias (that is, the probabilities $p : 1-p$ of the two sides). For simplicity we assume that the bias of the coin is equally likely to be either ε or $-\varepsilon$. We thus obtain six possible probability distributions over the four possible events.

case #	Pr(h), # of b	prob.	hb	hw	tb	tw
1	$\frac{1}{2}+\varepsilon$, $\#b=2$	$\frac{q}{2}$	$\frac{1}{2}+\varepsilon$	0	$\frac{1}{2}-\varepsilon$	0
2	$\frac{1}{2}-\varepsilon$, $\#b=2$	$\frac{q}{2}$	$\frac{1}{2}-\varepsilon$	0	$\frac{1}{2}+\varepsilon$	0
3	$\frac{1}{2}+\varepsilon$, $\#b=1$	$\frac{1}{2}-q$	$\frac{1}{4}+\frac{\varepsilon}{2}$	$\frac{1}{4}+\frac{\varepsilon}{2}$	$\frac{1}{4}-\frac{\varepsilon}{2}$	$\frac{1}{4}-\frac{\varepsilon}{2}$
4	$\frac{1}{2}-\varepsilon$, $\#b=1$	$\frac{1}{2}-q$	$\frac{1}{4}-\frac{\varepsilon}{2}$	$\frac{1}{4}-\frac{\varepsilon}{2}$	$\frac{1}{4}+\frac{\varepsilon}{2}$	$\frac{1}{4}+\frac{\varepsilon}{2}$
5	$\frac{1}{2}+\varepsilon$, $\#b=0$	$\frac{q}{2}$	0	$\frac{1}{2}+\varepsilon$	0	$\frac{1}{2}-\varepsilon$
6	$\frac{1}{2}-\varepsilon$, $\#b=0$	$\frac{q}{2}$	0	$\frac{1}{2}-\varepsilon$	0	$\frac{1}{2}+\varepsilon$

The payoffs of the two gambles are given by I = (8,000,hb; 0,hw; -8,000,tb; 0,tw) and II = (0,hb; 0,hw; -8,000,tb; 8,000,tw). Using the rank dependent functional (1) with $u(x)=x$ and $f(p)=0.5(3p^2-p^3)$,[6] $\varepsilon=0.05$, and $q=0.25$ we obtain that the certainty equivalents of the six lotteries are:

Lottery	1	2	3	4	5	6
Probability	$q/2$	$q/2$	$.5-q$	$.5-q$	$q/2$	$q/2$
CE in I	-2,071	-3,869	-1,830	-2,654	0	0
CE in II	-5,035	-5,500	-2,092	-2,392	2,066	2,965

[6]This function leads to standard ambiguity aversion even though, since the elasticity of f is decreasing, it does not satisfy Segal's (1987) sufficient conditions.

The V-values (and since $u(x) = x$, the certainty equivalents) of the two options are $V(I) = -2{,}377$ and $V(II) = -2{,}978$, and $I \succ II$, which is consistent with Machina's prediction. On the other hand, setting $\varepsilon = 0.25$ and $q = 0.05$ (that is, the coin is seriously biased but the decision maker believes that the two balls are most likely of different color) we obtain that $V(I) = -2{,}805$ and $V(II) = -2{,}438$, and $II \succ I$.

3.4 The Upper/Lower Tail Problem

Let C denote your certainty equivalent of the lottery $(100, \frac{1}{2}; 0, \frac{1}{2})$. Urn I and urn II contain each one red ball and two other balls, each of them is either white or black. One ball is drawn from an urn of your choice, and the payoffs are given in the following table. Do you prefer to play urn I or II?

	red	black	white
urn I	100	0	C
urn II	0	C	100

Machina shows that none of the models mentioned in the introduction allow the decision maker to have strict preferences between these two bets, that is, they all impose indifference.

Using the analysis of section 3.3 above, the decision maker believes that the probability of two black balls is q, the probability of two white balls is q, and the probability of one black and one white ball is $1 - 2q$. The two urns are thus transformed into two stage lotteries, given by

# of black balls	2	1	0
Probability	q	$1-2q$	q
Urn I	$(0, \frac{2}{3}; 100, \frac{1}{3})$	$(0, \frac{1}{3}; C, \frac{1}{3}; 100, \frac{1}{3})$	$(C, \frac{2}{3}; 100, \frac{1}{3})$
Urn II	$(0, \frac{1}{3}; C, \frac{2}{3})$	$(0, \frac{1}{3}; C, \frac{1}{3}; 100, \frac{1}{3})$	$(0, \frac{1}{3}; 100, \frac{2}{3})$

Using again the rank-dependent functional V with $u(x) = x$ and $f(p) = 0.5(3p^2 - p^3)$ we obtain that $C = 31.25$ and the certainty equivalents of the second stage lotteries are given by $c(0, \frac{2}{3}; 100, \frac{1}{3}) = 14.814$, $c(0, \frac{1}{3}; C, \frac{1}{3}; 100, \frac{1}{3}) = 26.388$, $c(C, \frac{2}{3}; 100, \frac{1}{3}) = 41.435$, $c(0, \frac{1}{3}; C, \frac{2}{3}) = 16.203$, and $c(0, \frac{1}{3}; 100, \frac{2}{3}) = 51.851$. Urn I is thus reduced into the lottery $(41.435, q; 26.388, 1-2q; 14.814, q)$. Urn II is reduced into $(51.851, q; 26.388, 1-2q; 16.203, q)$ which dominates urn I for every $q \in [0, \frac{1}{2})$. Machina predicted indeed that an ambiguity averse decision maker should prefer urn II to urn I.[7]

4 Concluding Remarks

Machina's (2009, 2012) examples are in line with a well-established tradition of "puzzles" in decision theory: A theory implies a specific relationship between a pair of choices, even though thought or actual experiments systematically violate this relationship. Such are, for example, the Allais paradox and the common ratio effect. In a similar way, Machina challenges the links between different decision situations implied by several models.[8]

The aim of the current paper is not to determine the "correct" choices in these decision problems. Rather, we show that those aspects of ambiguity aversion which can emerge in Machina's three-outcome examples, and which cannot be handled by the other major models, can easily be accommodated by the two-stage recursive ambiguity model of Segal (1987). In other words, the recursive utility analysis of ambiguity, while consistent with the standard intuition of ambiguity aversion with respect to Ellsberg (1961) problems, is rich enough *not* to impose connections within Machina's pairs. Machina (2009) pointed out that "the phenomenon of ambiguity aversion is intrinsically one of non separable preferences across mutually exclusive events." In the recursive model there is no separability between the different events because the underlying preference relation of lotteries in non expected utility (hence non separable). In our analysis, we use very small sets of possible beliefs about the true probabilities of the states. Obviously, the more complicated and the richer are the beliefs, the easier it is to provide examples that support Machina's intuition. That our examples are so simple, therefore, supports our claim for the recursive model.

[7]The dominance, or even the preferences of urn II over I is not a global property. Using the (risk averse) functional $V(X) = \mathrm{E}[X] \cdot \mathrm{E}[\sqrt{X}]$ implies that urn I is superior to urn II.

[8]The Allais paradox and the common ratio effect become part of Machina's criticism of the different models discussed by him. They all converge to expected utility on probabilistic lotteries, and as sufficiently rich environment contain "almost" probabilistic events (see Machina (2004)), these models are challenged by all the probabilistic violations of expected utility theory. It is worth noting that the recursive model converges on probabilistic lotteries to the non-expected utility functional V and is therefore not exposed to such criticism.

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