Household Formation and Markets*

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Abstract

We explore whether stable matchings and trade in commodities can coexist. For this purpose, we consider competitive markets for multiple commodities with endogenous formation of one- or two-person households. Within each two-person household, individuals obtain utility from his/her own private consumption, from discrete actions such as job choice, from the partner's observable characteristics such as appearance and hobbies, from some of the partner's consumption vectors, and from the partner's action choices. We investigate competitive market outcomes with an endogenous household structure in which no individual and no man/woman-pair can deviate profitably. We find a set of sufficient conditions under which a stable matching equilibrium exists. We further establish the first welfare theorem for this economy.

Keywords: endogenous household formation, consumption externalities, stable matching equilibrium, efficiency

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1 Introduction

In their seminal contribution, Gale and Shapley (1962) show that stable matching of partners obtains. The main results of the subsequent literature on two-sided matching are surveyed in Roth and Sotomayor (1990). A parallel literature, starting with the seminal paper of Shapley and Shubik (1972), has established the existence of stable outcomes in assignment games. In all of these models, markets for commodities are inactive simply because there exists at most one tradeable commodity.[1]

Gersbach and Haller (2011) show that two tradeable commodities may already endanger stable matchings in finite populations. They present an example with two private commodities and household formation reducible to a two-sided matching problem in which stable matchings and market clearing cannot be achieved simultaneously. The reason is that when households trade actively, different matchings may be associated with different price systems that clear commodity markets if consumption externalities are present or if group externalities are not separable. In such cases, individuals may find it optimal, for instance, to split at the going market prices to reduce negative consumption externalities. However, when the household structure changes and market clearing prices change, individuals may find it optimal to remarry or match again. Incompatibility of stable matching and market clearing does not disappear under replication.

It remains open, however, whether the non-existence problem is a consequence of having a finite population. In this paper, therefore, we explore the compatibility of stable household structures and market clearing for a continuum of individuals. In addition to the standard matching model and to Gersbach and Haller (2011), we also allow for a richer interaction of household members.

More specifically, we consider a market economy in which any two partners of opposite sex can form a household. Within each two-person household, externalities from the partner's commodity consumption in some categories and unpriced actions are allowed. Each individual has two characteristics: observable characteristics, which may include a taste component, and unobservable taste characteristics. Each individual gets utility from its own private consumption and discrete actions (in particular job choice), the partner's observable characteristics (such as appearance and known hobbies), the partner's consumption vector in certain categories, and the action choice of the partner.[2]

[1] A noteworthy exception are Drèze and Greenberg (1980).

[2] Remark 1 discusses in more detail the rationale for an endogenous discrete action

Partners in a household jointly choose consumption bundles and actions. They achieve intra-household Pareto efficiency. Such negotiated outcomes have to be stable, i.e., they have to be immune against (a) deviation of individuals who would fare better on their own and (b) a deviation of an arbitrary pair of individuals who could form a household and could choose a feasible allocation of commodities and actions. Stable matching and market clearing together define a stable matching equilibrium.

Our main result is a set of sufficient conditions under which a stable matching equilibrium exists. We illustrate the advantage of the continuum version over the corresponding finite population model by means of an example based on the motivating counter-example in Gersbach and Haller (2011). We further show a first welfare theorem. We illustrate the theorem by an example and describe how the qualitative properties of the equilibria differ from equilibria that could be obtained with the continuum version of the club model.

The paper is organized as follows. In the next section we elaborate further on the relation to the literature. In section three, we introduce the model and define matching and feasible allocations in the continuum economy. In section four, we define a stable matching equilibrium and illustrate it by an example. In section five, we state our main result. Section six is devoted to welfare analysis. There we show that every stable matching equilibrium is efficient under an appropriate definition of Pareto efficiency of allocations. Section seven provides the example that illustrates the welfare conclusions and the difference from the club literature. Section eight concludes. The proof of the main result is contained in the Appendix.

2 Relation to the Literature

Although general equilibrium models with multi-member households as outlined in Gersbach and Haller (2010, 2011) motivate our current investigation, the present model does not constitute a continuum version of Gersbach and Haller (2011). It is more restrictive in that it is focused on the formation of two-person and singleton households; but it is more general and richer regarding the features of individuals and households as will be discussed below.[3] By focussing on one- or two-person households, we follow the bulk of the literature dealing with households:

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choice.

[3]In general, Gersbach and Haller do not impose any a priori restrictions on household or group size — while individual preferences may prevent large groups from being formed in equilibrium.

- The vast majority of the empirical literature deals with decision making of one- or two-person households or households with children and one or two adults. For details and discussion, see Bourguignon and Chiappori (1992, 1994), Kapteyn and Koreman (1992), Bergstrom (1997) and Apps and Rees (2009).

- Most of the theoretical literature on household formation is devoted to two-sided matching, where the population is divided into males and females and individuals either remain single or form a heterosexual couple. Gale and Shapley (1962), Shapley and Shubik (1972) and Roth and Sotomayor (1990) are the main references mentioned before. Becker (1973) makes the same basic assumption. He derives an assortative matching when all individuals on each side have the same preference ordering over individuals on the other side.

Existence of stable matchings. Gale and Shapley (1962) consider a finite set of males and a finite set of females. Each person has preferences for members of the other sex or remaining single. There are no commodities involved. A stable matching is a partition of the population into heterosexual pairs and singletons so that (i) no matched individual would rather be single and (ii) no two individuals of opposite sex would both prefer being matched with each other to their current status. The authors show, among other things, existence of stable matchings. In the classical assignment game of Shapley and Shubik (1972), there is a single commodity or money. In our context, their model can be interpreted as follows. The population is divided into two groups, say M and W. Individual $i \in M$ has endowment $e_i \geq 0$, utility $u_i = x_i$ when single and consuming $x_i \geq 0$, and utility $u_i = -c_i + x_i$ when matched with some $j \in W$ and consuming x_i. Individual $j \in W$ has endowment $\widetilde{e}_j$, utility $v_j = \widetilde{x}_j$ when single and consuming $\widetilde{x}_j \geq 0$ and enjoys utility $v_j = h_{ij} + \widetilde{x}_j$ when matched with $i \in M$ and consuming $\widetilde{x}_j$. It is assumed that $c_i \geq 0$ for all i and $h_{ij} \geq 0$ for all ij. Shapley and Shubik (1972) show that without budget constraints, there exists a stable matching of the following form: There exist transfers $t_i \geq 0, i \in M$, such that $x_i = e_i$ if $i \in M$ remains single; $\widetilde{x}_j = \widetilde{e}_j$ if $j \in W$ remains single; $x_i = e_i + t_i$ and $\widetilde{x}_j = \widetilde{e}_j - t_i$ in case the match $\{i, j\}$ obtains. This constitutes a stable matching equilibrium in our sense provided the endowments $\widetilde{e}_j$ are sufficiently large, that is $\widetilde{e}_j \geq t_i$ for all matched pairs $\{i, j\}$. Notice that there is no active trade across households. The existence theorem of Alkan and Gale (1990) applies to a variety of cases, including the model of Crawford and Knoer (1981) with a perfectly divisible good. Their model can be recast as a model of household formation with a single good in our sense.

The influential contribution by Hatfield and Milgrom (2005) on matching with contracts synthesizes several previous approaches. It encompasses Gale and Shapley (1962), the discrete version of Crawford and Knoer (1981), Kelso and Crawford (1982) — who introduce a gross substitutes condition, allowing for many-to-one matchings and endogenous salaries — and a number of other models with applications to marriage markets, labor markets, school, college and ROTC branch choice, assignment of medical interns to hospitals, kidney exchange, etc. Their model is discrete, their analysis relies on arguments from lattice theory, and markets for commodities are absent. They show existence of stable allocations.[4]

Non-Existence of a stable matching equilibrium. Stable matchings in the previous models of matching constitute competitive equilibria in the sense of Gersbach and Haller (2011) and stable matching equilibria as defined in the present paper. Our main concern, however, are models with pairwise matching and active commodity markets that can serve as general models of household formation and "marriage markets". With active trade in commodity markets, we and the matching literature face a serious challenge, illustrated by the intriguing features of the aforementioned example in Gersbach and Haller (2011):

1. Given a price system p and the associated affordable utility allocations for each potential single household or couple, a stable matching exists. Let $\mathcal{S}(p)$ denote the set of those stable matchings.

2. For each matching μ, there exists a market clearing price system. Let $\mathfrak{P}(\mu)$ denote the set of such price systems.

3. However, the correspondence $p \mapsto \mathfrak{P}(\mathcal{S}(p))$ does not have a fixed point. This means that despite favorable conditions for the existence of stable matchings and market clearing prices, a stable matching and market clearing cannot be achieved simultaneously. A stable matching equilibrium does not exist. This example shows that active trade across households poses a challenge with regard to existence of stable outcomes not only for Gersbach and Haller, but also for the traditional matching literature.

Existence with a continuum of consumers. We consider a two-sided matching model with a continuum of males, a continuum of females, finitely many male types, finitely many female types, $I + K$ commodities where $I \geq 1$ and $K \geq 1$, and with a constant returns to scale production sector. We set out to identify sufficient conditions for the existence of a

[4]See Aygün and Sönmez (2013) for a qualification.

stable matching equilibrium. Since we are primarily interested in endogenous household formation, we augment the model with several pertinent features of households:

- Discrete actions that affect endowments, consumption sets, and utilities (e.g., an agent's job choice).
- Both observable and unobservable traits (types).
- Intra-household consumption externalities.
- Joint budget constraint for a couple.
- Pareto efficient choices within households.
- Active commodity markets.

The first two features constitute innovations relative to Gersbach and Haller (2011). Insofar, the current model is more general in some respects while being more restrictive in others.

Despite a superficial resemblance between our model and the ones in the literature on two-sided matching, there are no direct methodological connections between them. Two-sided matching theory is concerned with matching markets in a partial equilibrium setting. To find and to characterize equilibria, it primarily resorts to lattice theory, uses Tarski's fixed point theorem and takes an algorithmic approach. In contrast, the current paper deals with a continuum of atomless agents who form households. To establish existence of equilibrium and to prove the first welfare theorem, we extend the standard general equilibrium approach, by developing a global mapping with seven submappings to handle individual choices, matching of continua of agents, and market clearing simultaneously. A suitably constructed global mapping will ultimately allow to apply Kakutani's fixed point theorem.[5] In contrast to discrete matching models, we cannot rely on algorithms that converge to an equilibrium in finitely many steps.

Reexamination of the non-existence example from Gersbach and Haller (2011) in its continuum version proves instructive in several respects: Closer examination reveals the severity of the non-existence problem. It shows that non-existence in the original example does not go away under suitable replication, that it is not merely a small number or integer problem. We find, however, that existence is restored in the continuum model. Moreover, the

[5]Recently, the matching literature has used topological fixed point arguments as well. E.g., Azevedo and Hatfield (2013) apply Brouwer's fixed point theorem and Che, Kim and Kojima (2014) apply the Kakutani-Fan-Glicksberg fixed point theorem.

stable matching equilibrium of the continuum model indicates how to construct approximate equilibria for sufficiently large replicas of the original model.

Clubs. Some of the literature lists households as examples of clubs. Indeed, most club models also deal with an endogenous partition of the population into groups and some allow for the competitive market allocation of multiple private goods as well. See in particular Cole and Prescott (1997), Ellickson (1979), Ellickson, Grodal, Scotchmer, and Zame (1999, 2001), Gilles and Scotchmer (1997, 1998), Wooders (1988, 1989, 1997). However, the members of a typical club do not reach a collective decision regarding their consumption of private goods. Procurement of private goods remains an individual decision and is subject to an individual budget constraint. Moreover, this paper allows for

(i) consumption externalities;

(ii) externalities from observable types;

(iii) no discrimination by their preference type.

In contrast, Ellickson, Grodal, Scotchmer, and Zame (1999) allow for (ii) and (iii), but not (i); Allouch, Conley, and Wooders (2007) allow for unbounded size coalitions, and (ii), but neither (i) nor (iii); Konishi (2010 and 2013) allow for (i) and (iii), but not (ii). We comment further on the assumptions and the strategy of proof after the statement of the theorem. The proof itself if given in the Appendix.

While existence of equilibrium is a crucial issue, qualitative properties of equilibria are equally important. The equilibria obtained in club models and stable matching equilibria tend to be different. Gersbach and Haller (2010) amend the concept of valuation equilibrium à la Gilles and Scotchmer (1997) by two modest requirements that should hold in the presence of externalities: a) A group (club, household) member should not afford an alternative private consumption bundle so that *ceteris paribus* the individual is better off. b) A group member should not be better off going single. They present a two-person example where the stable matching equilibrium is strongly Pareto optimal whereas none of the valuation equilibria is weakly Pareto optimal. In the continuum context, the model of Ellickson et al. (1999) is often taken as a yardstick. In section seven, we develop an example where the stable matching equilibrium is Pareto optimal and the corresponding transfer equilibria are not.

3 The Basic Model

The basic model describes and defines consumer and household characteristics, the production sector, and feasible allocations.

3.1 Individuals and Couples

There is a continuum of individuals with two different kinds of characteristics — observable characteristics ("crowding characteristics" in the terminology of Conley and Wooders (1997)) and unobservable taste characteristics. The set of observable characteristics is partitioned into two finite non-empty sets M and W. The sets M and W denote a list of male types and female types, respectively. An element $m \in M$ ($w \in W$) describes a type m male's (type w female's) observable characteristics — his (her) appearance and observable hobbies etc. — that may be cared for by a partner $w \in W$ ($m \in M$). For expositional purposes, we will assume that each male (female) will be either matched with a female (male) or stay single. For each $m \in M$ ($w \in W$), there is a set Θ of unobservable (taste) characteristics, with generic elements $\theta, \tilde{\theta}$, etc. That is, the set of individual types is denoted by $(M \cup W) \times \Theta$. This specification is for notational simplicity. It is easy to accommodate observable-type-dependent taste sets Θ^m for $m \in M$ and Θ^w for $w \in W$. In such a case, we can just assume that $\Theta = (\cup_{m \in M} \Theta^m) \cup (\cup_{w \in W} \Theta^w)$, postulating that for all $\theta \notin \Theta^m$, the population measure of m-type having taste-type θ is zero and for all $\theta \notin \Theta^w$, the population measure of w-type having taste-type θ is zero. Whenever warranted, we distinguish sets and variables attributed to females by $\tilde{}$.

Asymmetric information in the form of unobservable (taste) characteristics is not crucial for our analysis. The model and the proofs would work perfectly well when all taste characteristics were observable. But our approach is not limited to that case. Hence we choose the more general setting, allowing for asymmetric information, where our approach still applies. This is possible because of two reasons. First, stability in our equilibrium concept can be viewed as a sort of "long run" outcome immune to deviations. In a dynamic setting, outcomes may depend on how information is revealed during the bargaining process — and might be inefficient. Second, our analysis rests on the fact that unobservable characteristics do not affect endowments and consumption sets and individuals do not care for a partner's θ. As a consequence, a stable matching equilibrium remains a stable matching equilibrium when ceteris paribus the θ's become observable — while the converse is not obvious.

We assume that M, W and Θ are all finite sets. We use the distribution approach to describe consumers: Let $N^{(m,\theta)}$ ($N^{(w,\theta)}$) be the population measure of type (m, θ) $((w, \theta))$ with $\sum_{(\ell,\theta)\in(M\cup W)\times\Theta} N^{(\ell,\theta)} = N > 0$. To make the analysis relevant at all, we further assume that both the male and the female population have positive measure: $\sum_{(m,\theta)\in M\times\Theta} N^{(m,\theta)} = N^M > 0$ and $\sum_{(w,\theta)\in W\times\Theta} N^{(w,\theta)} = N^W > 0$.

3.2 Actions

Let A be a finite non-empty action set that is common to all individuals.[6] A typical example for an action is a consumer's job choice. But the consumer's opportunities are not necessarily limited to job choice. E.g., an action can indicate the number of kids to have, or which spouse primarily takes care of kids etc. Each job may have a different wage rate and different time commitment. That is, a person's leisure endowment in particular is dependent on her job choice, not only because wage rates are different[7], but also because the time available for leisure consumption can differ. For example, the minimum working hours or commuting time can vary. Moreover, if action a means that a spouse commits to taking care of kids primarily, his (her) leisure endowment would shrink since his (her) time to spend on the job or leisure needs to be reduced.

3.3 Consumption Sets, Endowments, and Job Choice

Sets $\mathcal{I} = \{1, ..., I\}$ and $\mathcal{K} = \{I + 1, ..., I + K\}$ denote the set of commodities without externalities (as when reading books), and the set of commodities with externalities (such as from smoking) to and from the partner if matched, respectively. These sets are common to all individuals, regardless whether they are matched or single. Each male with observable type $m \in M$ has an endowment bundle dependent on his action, given by a mapping $e^m : A \to \mathbb{R}_+^{I+K}$. Similarly, a female of observable type $w \in W$ has an endowment bundle given by a mapping $\tilde{e}^w : A \to \mathbb{R}_+^{I+K}$. The consumption set of each person can depend on the action taken by the person and, therefore, consumption sets are represented by correspondences $X^m : A \twoheadrightarrow \mathbb{R}_+^{I+K}$ and $\tilde{X}^w : A \twoheadrightarrow \mathbb{R}_+^{I+K}$. $X_{\mathcal{I}}^m(a) \subset \mathbb{R}_+^I$ and $X_{\mathcal{K}}^m(a) \subset \mathbb{R}_+^K$ describe the projections of $X^m(a)$ on the first I commodities and the last K commodities, respectively. The projections $\tilde{X}_{\mathcal{I}}^w(\tilde{a})$ and $\tilde{X}_{\mathcal{K}}^w(\tilde{a})$ are defined similarly.

[6]This is again for notational simplicity. We can obtain the same results allowing action sets to be dependent on observable types: For $m \in M$ and $w \in W$, action sets A^m and A^w are finite. Regarding endogenous discrete action choices, see Remark 1.

[7]The type of labor being different implies that the type of leisure is different: There exists a difference in opportunity costs due to wage rate differentials.

If consumers cannot buy more leisure than their time endowments, the consumption set typically cannot be $\mathbb{R}_+^{I+K}$ itself. For example, suppose that there are J jobs representing different types of labor inputs and associated leisure choices. Then, a type m man's consumption set and endowment are dependent on the chosen action $a \in A$ in the following manner: Let $\{I+1, ..., I+J\} = \mathcal{J} \subset \mathcal{K}$ be the set of different types of labor ($J \leq K$).[8] If consumer m's choice is $a \in A$, then his job choice associated with his action choice $a \in A$ is some $j(a) \in \{I+1, ..., I+J\}$, his endowment given $a \in A$ is

$$e^m(a) = (\underbrace{e_1^m(a), ..., e_I^m(a)}_{i \in \mathcal{I}}; \underbrace{0, ..., 0, e_{j(a)}^m(a), 0, ..., 0}_{j(a) \in \mathcal{J}}; \underbrace{e_{I+J+1}^m(a), ..., e_{I+K}^m(a)}_{k \in \mathcal{K} \setminus \mathcal{J}}),$$

and his consumption set is, assuming that he cannot buy more leisure than his time endowment:

$$X^m(a) \subseteq \mathbb{R}_+^I \times \{0\} \times ... \times \{0\} \times [0, e_{j(a)}^m(a)] \times \{0\} \times ... \times \{0\} \times \mathbb{R}_+^{K-J}.$$

This rich setup allows for many different situations: For example, if by its nature, job $j(a)$ requires at least T hours of working, the $j(a)$-axis of $X^m(a)$ can be written as $[0, e_{j(a)}^m(a) - T]$.

We will assume that $\cup_{a \in A} j(a) = \mathcal{J}$ which means that for every job $j \in \mathcal{J}$, there exists an action that entails that job.

We further assume that (non-externality) commodity 1 in $\mathcal{I}$ is a special good as we shall specify in more detail in Section 4. In particular, we write for all $m \in M$ and all $a \in A$, $X^m(a) = [0, \infty) \times X_{-\{1\}}^m(a)$ where $X_{-\{1\}}^m(a)$ denotes type m's consumption set for all other commodities. For all $w \in W$ and all $\tilde{a} \in A$, we define $\tilde{e}^w(\tilde{a})$ and $\tilde{X}^w(\tilde{a})$ in a similar manner.

Remark 1 (Discrete action choices). We have opted for discrete action spaces for two reasons. First, job choice and its effects on endowment are best expressed by means of a discrete choice. Second, a continuous action space does not simplify the analysis in our context, but rather complicates it, both in notation and analytical details. Lateron, we exploit the finiteness of A and resort to extended household types like (m, θ, a). A finite action space yields a finite extended type set, which would no longer be the case with a continuous action space.

3.4 Preferences

Male type $(m, \theta) \in M \times \Theta$ has the following utility presentation. If he is matched and his partner has (observable) type $w \in W$, then his utility

[8] By positing $\mathcal{J} \subset \mathcal{K}$, we are assuming that leisure consumption by a man/woman affects his/her spouse's utility.

function is
$u^{(m,\theta)} : (\cup_{a\in A} X^m(a) \times \{a\}) \times \{w\} \times (\cup_{\tilde{a}\in A} \tilde{X}^w_{\mathcal{K}}(\tilde{a}) \times \{\tilde{a}\}) \to \mathbb{R}$ such that

$$u^{(m,\theta)} = u^{(m,\theta)}(x_{\mathcal{I}}, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}),$$

where $a \in A$ and $x = (x_{\mathcal{I}}, x_{\mathcal{K}}) \in X^m(a)$ are his own action and consumption vector and $\tilde{a} \in A$ and $\tilde{x}_{\mathcal{K}} \in \tilde{X}^w_{\mathcal{K}}(\tilde{a})$ are his female partner's action and (mutually relevant) consumption vector. Note that a male of type (m, θ) does not care what preference type $\tilde{\theta}$ his partner has. He cares only about her observable type $w \in W$ and her choice $(\tilde{x}_{\mathcal{K}}, \tilde{a})$. If he is single, his utility function is simply $u^{(m,\theta)} : (\cup_{a\in A} X^m(a) \times \{a\}) \times \{\emptyset\} \to \mathbb{R}$ so that

$$u^{(m,\theta)} = u^{(m,\theta)}(x, a; \emptyset),$$

where $a \in A$ and $x = (x_{\mathcal{I}}, x_{\mathcal{K}}) \in X^m(a)$. Note that there is no externality in consumption in this case. The same comment applies to the case of w being single.

Female type $(w, \tilde{\theta}) \in W \times \Theta$ has utility function
$u^{(w,\tilde{\theta})} : (\cup_{\tilde{a}\in A} \tilde{X}^w(\tilde{a}) \times \{\tilde{a}\}) \times \{m\} \times (\cup_{a\in A} X^m_{\mathcal{K}}(a) \times \{a\}) \to \mathbb{R}$ such that

$$u^{(w,\tilde{\theta})} = u^{(w,\tilde{\theta})}(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a)$$

if she is matched with observable male type $m \in M$. She has utility function $u^{(w,\tilde{\theta})} : (\cup_{\tilde{a}\in A} \tilde{X}^w(\tilde{a}) \times \{\tilde{a}\}) \times \{\emptyset\} \to \mathbb{R}$ so that

$$u^{(w,\tilde{\theta})} = u^{(w,\tilde{\theta})}(\tilde{x}, \tilde{a}; \emptyset),$$

if she is single.

3.5 Matching

In order to define an allocation, we impose assumptions on the populations of individuals and the number of couples. Let $\bar{\Gamma}^C = M \times \Theta \times W \times \Theta$ denote the set of couple types, $\bar{\Gamma}^M = M \times \Theta$ denote the set of male types, and $\bar{\Gamma}^W = W \times \Theta$ denote the set of female types. Then $\bar{\Gamma} = \bar{\Gamma}^C \cup \bar{\Gamma}^M \cup \bar{\Gamma}^W$ would be the set of all possible household types. However, it proves more convenient to augment a household type by its members' actions because household members' endowments and consumption sets depend on their action choices and, thus, the set of feasible allocations in each household depends on the actions taken by its members. Augmented or extended household type sets are defined as follows. Let $\Gamma^C = M \times \Theta \times A \times W \times \Theta \times A$, $\Gamma^M = M \times \Theta \times A$, $\Gamma^W = W \times \Theta \times A$, and $\Gamma = \Gamma^C \cup \Gamma^M \cup \Gamma^W$.[9] For example, $\gamma \in \Gamma^C$ assumes the

[9] Working with augmented household types also helps overcome the non-convexity problem associated with discrete action set A.

form $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a})$ which is a couple consisting of a male of observable type m who has unobservable taste type θ and takes action a and a female of observable type w who has unobservable taste type $\tilde{\theta}$ and takes action $\tilde{a}$.

Remark 2 (Endogenous action choices). Introducing extended types when proving existence is not the same as making extended types primitives of the model, that is making action choice exogenous. To illustrate this point, let us consider an example of matching where consumption of commodities is absent. There exist one male type m with Lebesgue measure 1 and one female type w of Lebesgue measure 1. The sole unobservable type can be ignored. The binary action space is $A = \{-1, 1\}$. The utility of a single person who takes action a is a. The utility of a member of a couple where one partner takes action a and the other takes action a' is $a + a'$. In equilibrium, all individuals get matched and everybody chooses action $a = 1$. This outcome is also achieved in the unique stable matching when there are two exogenous extended types, $(m, 1)$ and $(w, 1)$, each with Lebesgue measure 1. Now suppose that all four extended types, $(m, 1)$, $(m, -1)$, $(w, 1)$, and $(w, -1)$ are exogenously given, each with Lebesgue measure 1/2. Then the unique stable matching yields couples of the type $(m, 1; w, 1)$ and unmatched types $(m, -1)$ and $(w, -1)$. This is different from the original equilibrium allocation with endogenous action choice. More elaborate examples with commodity markets can be constructed.

A **matching** is a mapping $\mu : \Gamma \to \mathbb{R}_+$ such that $\mu(\gamma)$ is the Lebesgue measure of households of augmented type $\gamma \in \Gamma$.

In our formal derivations, we shall further assume **Measurement Consistency (MC).** This is a technical assumption how population sizes line up in a continuum economy. In our context, it requires that

- the measure $\mu(m, \theta, a; w, \tilde{\theta}, \tilde{a})$ of couples of extended type $(m, \theta, a; w, \tilde{\theta}, \tilde{a})$,
- the measure $\mu^{(m,\theta)}(m, \theta, a; w, \tilde{\theta}, \tilde{a})$ of the sub-population of males of type (m, θ) belonging to couples of extended type $(m, \theta, a; w, \tilde{\theta}, \tilde{a})$ and
- the measure $\mu^{(w,\tilde{\theta})}(m, \theta, a; w, \tilde{\theta}, \tilde{a})$ of the sub-population of females of type $(w, \tilde{\theta})$ belonging to couples of extended type $(m, \theta, a; w, \tilde{\theta}, \tilde{a})$

coincide. MC has been introduced by Kaneko and Wooders (1986), to properly account for resources consumed by finite coalitions in a continuum economy. When MC holds in our context, that is when $\mu(m, \theta, a; w, \tilde{\theta}, \tilde{a}) = \mu^{(m,\theta)}(m, \theta, a; w, \tilde{\theta}, \tilde{a}) = \mu^{(w,\tilde{\theta})}(m, \theta, a; w, \tilde{\theta}, \tilde{a})$, we can take $e^m(a) + \tilde{e}^w(\tilde{a})$ as the endowment of a couple of extended type $(m, \theta, a; w, \tilde{\theta}, \tilde{a})$. Otherwise, it is unclear how such a couple's endowment would be determined. We will assume MC throughout the paper.

Now, we can define feasibility of matchings. A matching μ is **feasible** if we have:

(F1) $\sum_{(w,\tilde{\theta},a,\tilde{a})\in W\times\Theta\times A\times A}\mu(m,\theta,a;w,\tilde{\theta},\tilde{a})+\sum_{a\in A}\mu(m,\theta,a)=N^{(m,\theta)}$ for all $(m,\theta)\in M\times\Theta$.

(F2) $\sum_{(m,\theta,a,\tilde{a})\in M\times\Theta\times A\times A}\mu(w,\tilde{\theta},\tilde{a};m,\theta,a)+\sum_{\tilde{a}\in A}\mu(w,\tilde{\theta},\tilde{a})=N^{(w,\tilde{\theta})}$ for all $(w,\tilde{\theta})\in W\times\Theta$.

3.6 Production

We next introduce production. We assume that the aggregate production technology exhibits constant returns to scale so that there will not be profits in equilibrium. We denote the aggregate production set by $Y\subset\mathbb{R}^{I+K}$.

3.7 Feasible Allocations

Let the household consumption correspondence $\mathcal{X}:\Gamma\twoheadrightarrow\mathbb{R}^{I+K}\cup(\mathbb{R}^{I+K})^2$ be given by $\mathcal{X}(m,\theta,a)=X^m(a)$, $\mathcal{X}(w,\tilde{\theta},\tilde{a})=\tilde{X}^w(\tilde{a})$, and $\mathcal{X}(m,\theta,a,w,\tilde{\theta},\tilde{a})=X^m(a)\times\tilde{X}^w(\tilde{a})$.

A **symmetric household consumption allocation** is given by a selection $\mathbf{x}$ of $\mathcal{X}:\Gamma\twoheadrightarrow\mathbb{R}^{I+K}\cup(\mathbb{R}^{I+K})^2$, that is $\mathbf{x}:\Gamma\to\mathbb{R}^{I+K}\cup(\mathbb{R}^{I+K})^2$ such that $\mathbf{x}(\gamma)\in\mathcal{X}(\gamma)$ for $\gamma\in\Gamma$. Frequently, we shall write x^γ instead of $x(\gamma)$. Let $\mathfrak{X}$ be the set of all selections of $\mathcal{X}$, that is, the set of symmetric household consumption allocations. In a symmetric household consumption allocation $\mathbf{x}\in\mathfrak{X}$,

1. for each $\gamma=(m,\theta,a;w,\tilde{\theta},\tilde{a})\in\Gamma^C$, $(x^\gamma;\tilde{x}^\gamma)\in\mathcal{X}(\gamma)$ is the consumption combination of couples of type γ;

2. for each $\gamma=(m,\theta,a)\in\Gamma^M$, $x^\gamma\in\mathcal{X}(\gamma)$ is the consumption bundle of singles of type γ;

3. for each $\gamma=(w,\tilde{\theta},\tilde{a})\in\Gamma^W$, $\tilde{x}^\gamma\in\mathcal{X}(\gamma)$ is the consumption bundle of singles of type γ.

Three details ought to be noted here. First, taste types θ, $\tilde{\theta}$ do not affect feasibility of household consumption. Second, this definition presumes that households of the same type choose identical consumption, an "equal treatment property" for households.[10] Third, although we list a consumption plan for all possible types of households, we do not require that all types of households must be present in equilibrium.

A **(symmetric) consumption allocation** is a pair $(\mathbf{x}, \mu)$ where $\mathbf{x} \in \mathfrak{X}$ is a symmetric household consumption allocation and μ is a matching. A **feasible allocation** is a triple $(\mathbf{x}, \mu, y)$ such that $(\mathbf{x}, \mu)$ is a consumption allocation and $y \in Y$ is a production vector satisfying

$$\left.\begin{array}{rl} & \sum_{\gamma\in\Gamma^C} \mu(\gamma)\,(x^\gamma + \tilde{x}^\gamma) + \sum_{\gamma\in\Gamma^M} \mu(\gamma)x^\gamma + \sum_{\gamma\in\Gamma^W} \mu(\gamma)\tilde{x}^\gamma \\ \leq & \sum_{\gamma\in\Gamma^C} \mu(\gamma)\,(e^\gamma + \tilde{e}^\gamma) + \sum_{\gamma\in\Gamma^M} \mu(\gamma)e^\gamma + \sum_{\gamma\in\Gamma^W} \mu(\gamma)\tilde{e}^\gamma + y \end{array}\right\} \quad (1)$$

where we set $e^\gamma = e^m(a)$ for $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$ or $\gamma = (m, \theta, a) \in \Gamma^M$ and $\tilde{e}^\gamma = \tilde{e}^w(\tilde{a})$ for $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$ or $\gamma = (w, \tilde{\theta}, \tilde{a}) \in \Gamma^W$.
A production plan $y \in Y$ is feasible if it is part of a feasible allocation $(\mathbf{x}, \mu, y)$. For later use we establish sufficient conditions to ensure that the set of feasible production plans is bounded.

Lemma 1 *Suppose the aggregate production set $Y \subset \mathbb{R}^{I+K}$ is a closed and convex set satisfying (i) $Y + \mathbb{R}^{I+K}_- \subseteq Y$ (free disposal), (ii) $Y \cap \mathbb{R}^{I+K}_+ = \{0\}$ (no free lunch), and (iii) $ty \in Y$ for all $y \in Y$ and all $t > 0$ (constant returns to scale). Then the set of feasible production plans is bounded.*

Proof. There exists $\bar{b} \gg 0$ such that the aggregate endowment

$$\mathfrak{e} = \sum_{\gamma\in\Gamma^C} \mu(\gamma)\,(e^\gamma + \tilde{e}^\gamma) + \sum_{\gamma\in\Gamma^M} \mu(\gamma)e^\gamma + \sum_{\gamma\in\Gamma^W} \mu(\gamma)\tilde{e}^\gamma$$

satisfies $0 \leq \mathfrak{e} \ll \bar{b}$ regardless of the distribution μ of augmented household types and action choices by households. Then the feasibility condition (1) requires $0 \leq$ "lhs of (1)" $\leq \mathfrak{e} + y \ll \bar{b} + y$ (and, consequently, $-\bar{b} \ll y$).

Application of assertion (2) in Debreu (1959, 5.4) to the Robinson Crusoe economy with consumption set $\mathbb{R}^{I+K}_+$, endowment bundle $\bar{b}$, and production set Y with the hypothesized properties yields an upper bound b^o (and a lower bound) for feasible production plans when exact market clearing is imposed. Such an upper bound will also do in our context: In the Robinson Crusoe

[10] Obviously, we can assign different consumption-action combinations to two households of the same type. In this sense, the equilibrium concept proposed below is not the most general one. More general definitions using the "distribution approach" can be found in Mas-Colell (1984) or Zame (2007). Notice, however, that we need Pareto indifference among realized equilibrium outcomes since consumers are free to choose available policies in the market. We will assume convex preferences for our main theorem. Therefore, we essentially lose nothing by choosing the simpler definition.

economy, $x = \bar{b} + y$ for $x \in \mathbb{R}_+^{I+K}$ and $y \in Y$ implies $y \leq b^0$. Hence for $y \in Y$: $0 \leq \mathfrak{e} + y$ implies $0 \ll \bar{b} + y$, therefore $x = \bar{b} + y$ for some $x \in \mathbb{R}_+^{I+K}$ and $y \leq b^0$. □

4 Equilibrium Analysis

In this section we specify first the choices available to households and next how households decide. Then we introduce the equilibrium concept, stable matching equilibrium. Finally, we present an example, the continuum version of Example 3 in Gersbach and Haller (2011).

4.1 Household Decisions

Let $\Delta = \{p = (p_{\mathcal{I}}, p_{\mathcal{K}}) \in \mathbb{R}_+^{I+K} : \sum_{i \in \mathcal{I}} p_i + \sum_{k \in \mathcal{K}} p_k = 1\}$ be the set of price vectors. For $p \in \Delta$, a couple of observable type (m, w) has (for any $\theta, \tilde{\theta} \in \Theta$) the budget set dependent of their actions $(a, \tilde{a}) \in A \times A$,

$$\mathcal{B}^{(m,w;a,\tilde{a})}(p) \equiv \left\{ (x, \tilde{x}) \in X^m(a) \times X^w(\tilde{a}) \;\middle|\; \begin{array}{l} p \cdot (x + \tilde{x}) \\ \leq p \cdot (e^m(a) + \tilde{e}^w(\tilde{a})) \end{array} \right\}. \tag{2}$$

The members of that household determine their consumption-action bundles either by negotiating or independently. If consumption or action externalities within the household are absent, then the members can choose their consumption-action vectors independently, achieving an intra-household efficient allocation. However, as is well known, independent decisions need not lead to intra-household efficiency in the presence of externalities. Hence in general, they jointly decide what actions a and $\tilde{a}$ and what consumption vectors causing externalities — $x_{\mathcal{K}}^m$ and $\tilde{x}_{\mathcal{K}}^w$ — to choose, and divide their residual aggregated income $pe^m(a) + p\tilde{e}^w(\tilde{a}) - px_{\mathcal{K}}^m - p\tilde{x}_{\mathcal{K}}^w$ into residual income shares B and $\tilde{B}$, so that $B + \tilde{B} = pe^m(a) + p\tilde{e}^w(\tilde{a}) - p_{\mathcal{K}} x_{\mathcal{K}}^m - p_{\mathcal{K}} \tilde{x}_{\mathcal{K}}^w$. That is, the members can negotiate over B and $\tilde{B}$ by taking their outside options (by deviating unilaterally or by finding another partner) into account.

Therefore, it is natural to think about a contract over consumption and actions together with a budget share agreement between the partners. Note that both members of a household contribute their endowments to the household joint budget first and then receive budget shares (allowances) for consumption of goods in $\mathcal{I}$. If for example, good $j \in \mathcal{J}$ is a type of leisure (based on the male partner's job choice a), then p_j is the corresponding wage rate (or opportunity cost of leisure). In this case, he has to contribute $p_j e_j^m(a)$ to the household first, and buy back afterwards some of the leisure time x_j^m by using his "allowance" (that is residual income share) B. This is the same as deciding on labor supply $e_j^m(a) - x_j^m$ yielding net allowance $B - p_j x_j^m$. Here, a further issue is that such a contract must be formed without knowing the true preference types of partners, although the contract can be contingent on their observable types and reported taste types.

A single's budget constraint is easier to describe. A type m male has a budget set (for any $\theta \in \Theta$) dependent on his action $a \in A$,

$$\mathcal{B}^{(m,a)}(p) \equiv \{x \in X^m(a) : px \leq pe^m(a)\}. \tag{3}$$

A type (m, θ) male chooses $a \in A$ and $x \in \mathcal{B}^{(m,a)}(p)$ to maximize his utility $u^{(m,\theta)}(x, a; \emptyset)$. Similarly, a type w female has a budget set (for any $\tilde{\theta} \in \Theta$) dependent on her action $\tilde{a} \in A$,

$$\mathcal{B}^{(w,\tilde{a})}(p) \equiv \left\{\tilde{x} \in \tilde{X}^w(\tilde{a}) : p\tilde{x} \leq p\tilde{e}^w(\tilde{a})\right\}. \tag{4}$$

A type $(w, \tilde{\theta})$ female chooses $\tilde{a} \in A$ and $\tilde{x} \in \mathcal{B}^{(w,\tilde{a})}(p)$ to maximize her utility $u^{(w,\tilde{\theta})}(\tilde{x}, \tilde{a}; \emptyset)$.

4.2 Negotiations over Intra-Household Allocation (Consumption and Actions)

We think of an ideal situation of negotiations within a household (between partners). They try to achieve an intra-household Pareto efficient allocation given their reported preference types (and their observable types) that is immune to joint deviations with other partners (with negotiated allocations) and to single deviations as well.[11] Suppose that (observable) types m and w met discussing their potential intra-household allocation. They report their taste types (truthfully or manipulatively) θ and $\tilde{\theta}$ to each other, and jointly choose their actions and consumption vectors that cause externalities to the partners. An intra-household Pareto-efficient allocation given the participation constraints needs to ensure that their utilities from the allocation exceed or equal $U^{(m,\theta)*}$ and $U^{(w,\tilde{\theta})*}$, respectively, where $U^{(m,\theta)*}$ and $U^{(w,\tilde{\theta})*}$ are the outside options for these types. For the purpose of the following analysis of

[11]Notice that intra-household Pareto efficiency or collective rationality does not presume that the partners follow a specific bargaining or decision-making protocol when negotiating. In principle, any Pareto efficient outcome can be agreed upon while we are not specifying a particular procedure by which it is reached. The important feature is that in equilibrium (in a stable matching), none of the partners has an incentive to deviate.

premarital negotiations, it is convenient to consider indirect utility functions:

$$\begin{aligned} & V^{(m,\theta)}(p, B, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) \\ = & \max_{x_{\mathcal{I}} \in X^m_{\mathcal{I}}(a)} u^{(m,\theta)}(x_{\mathcal{I}}, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) \\ & \text{subject to } \sum_{i \in \mathcal{I}} p_i x_i \leq B; \end{aligned}$$

$$\begin{aligned} & V^{(w,\tilde{\theta})}(p, \tilde{B}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a) \\ = & \max_{\tilde{x}_{\mathcal{I}} \in X^w_{\mathcal{I}}(\tilde{a})} u^{(w,\tilde{\theta})}(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a) \\ & \text{subject to } \sum_{i \in \mathcal{I}} p_i \tilde{x}_i \leq \tilde{B}. \end{aligned}$$

These indirect utility functions describe what a member of a household can achieve when an agreement over consumption and action vectors $(x_{\mathcal{K}}, a, \tilde{x}_{\mathcal{K}}, \tilde{a})$ and an expenditure sharing rule (allowances for commodity consumption without externalities) $(B, \tilde{B})$ has been reached. Now, we will consider negotiations between husband and wife — or more generally, male and female partner. Let us denote again the wife's consumption-action vector by tildes ($\tilde{\ }$). Consider a couple consisting of types (m, θ) and $(w, \tilde{\theta})$. A **feasible plan for observable types** m **and** w **under** p is a list $(B, x_{\mathcal{K}}, a; \tilde{B}, \tilde{x}_{\mathcal{K}}, \tilde{a})$ such that

$$B + \sum_{k \in \mathcal{K}} p_k x_k + \tilde{B} + \sum_{k \in \mathcal{K}} p_k \tilde{x}_k \leq p e^m(a) + p \tilde{e}^w(\tilde{a}).$$

Denote the set of feasible plans for observable types m and w under p by $\mathcal{C}[m, w; p]$. The *negotiation problem* between (m, θ) and $(w, \tilde{\theta})$ is to find a feasible plan $(B, x_{\mathcal{K}}, a; \tilde{B}, \tilde{x}_{\mathcal{K}}, \tilde{a}) \in \mathcal{C}[m, w; p]$ that is agreeable to them.

If they report their preference types θ and $\tilde{\theta}$ before negotiation takes place, then the negotiation problem may become the one to find an **intra-household Pareto efficient allocation** for types (m, θ) and $(w, \tilde{\theta})$: a feasible plan $(B, x_{\mathcal{K}}, a; \tilde{B}, \tilde{x}_{\mathcal{K}}, \tilde{a}) \in \mathcal{C}[m, w; p]$ such that there is no other feasible plan $(B', x'_{\mathcal{K}}, a'; \tilde{B}', \tilde{x}'_{\mathcal{K}}, \tilde{a}') \in \mathcal{C}[m, w; p]$ with

(i) $V^{(m,\theta)}(p, B', x'_{\mathcal{K}}, a'; w, \tilde{x}'_{\mathcal{K}}, \tilde{a}') \geq V^{(m,\theta)}(p, B, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a})$,

(ii) $V^{(w,\tilde{\theta})}(p, \tilde{B}', \tilde{x}'_{\mathcal{K}}, \tilde{a}'; m, x'_{\mathcal{K}}, a') \geq V^{(w,\tilde{\theta})}(p, \tilde{B}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a)$,

(iii) $V^{(m,\theta)}(p, B', x'_{\mathcal{K}}, a'; w, \tilde{x}'_{\mathcal{K}}, \tilde{a}') \geq U^{(m,\theta)*}$ and
$V^{(w,\tilde{\theta})}(p, \tilde{B}', \tilde{x}'_{\mathcal{K}}, \tilde{a}'; m, x'_{\mathcal{K}}, a') \geq U^{(w,\tilde{\theta})*}$,

and at least one strict inequality in (i) or (ii).

Given that there are unobservable characteristics, it is perfectly legitimate for types (m, θ) and $(w, \tilde{\theta})$ to report (m, ϑ) and $(w, \tilde{\vartheta})$ before they enter the negotiation stage. Note that even if (m, θ) pretends to be (m, ϑ), his partner $(w, \tilde{\theta})$ (or actually $(w, \tilde{\vartheta})$) does not care what actual taste he has, as long as his observable type is m (his appearance and observable hobbies etc.) and both agree with his consumption and action vectors $(x_{\mathcal{K}}, a)$.

The situation would be different in a model where each couple is committed to a specific bargaining protocol or mechanism, for instance a specific asymmetric Nash bargaining solution as in Gori (2010). Then the partner's taste parameters would matter and misrepresentation could prove advantageous. The outcome might also be different when a couple engaged in noncooperative bargaining under incomplete information.

4.3 Stable Matching Equilibrium

Our concept of equilibrium requires a feasible allocation such that (i) men and women are free to choose a partner and negotiate a budget-feasible intra-household allocation with that partner or, alternatively, stay single; (ii) there is no pair of male and female types who can be better off by deviating from the equilibrium allocation by negotiating their after-deviation intra-household allocation. Our equilibrium concept is described formally as follows.

Definition. A **stable matching equilibrium** is a quadruple $(p, \mathbf{x}, \mu, y)$ where
$p \in \Delta$ is a price system;
$(\mathbf{x}, \mu, y)$ is a feasible allocation
and the following conditions 1-4 hold:

1. Single-Household Efficiency

   1.a: For all $\gamma = (m, \theta, a) \in \Gamma^M$ with $\mu(\gamma) > 0$, (x^γ, a) maximizes $u^{(m,\theta)}(x, a; \emptyset)$ subject to $(x, a) \in \mathcal{B}^{(m,a)}(p)$.

   1.b: For all $\gamma = (w, \tilde{\theta}, \tilde{a}) \in \Gamma^W$ with $\mu(\gamma) > 0$, $(\tilde{x}^\gamma, \tilde{a})$ maximizes $u^{(w,\theta)}(\tilde{x}, \tilde{a}; \emptyset)$ subject to $(\tilde{x}, \tilde{a}) \in \mathcal{B}^{(w,\tilde{a})}(p)$.

2. Stable Matching I (immunity to joint deviations)

2.a: If $\gamma = (\overline{m}, \overline{\theta}, \bar{a}; w, \tilde{\theta}, \tilde{a}), \delta = (m, \theta, a; \widehat{w}, \widehat{\theta}, \hat{a}) \in \Gamma^C$ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, then there is no feasible plan for observable types $\overline{m}$ and $\widehat{w}$ under p, $(B', x'_{\mathcal{K}}, a'; B'', x''_{\mathcal{K}}, a'') \in \mathcal{C}[\overline{m}, \widehat{w}; p]$, such that $V^{(\overline{m}, \overline{\theta})}(p, B', x'_{\mathcal{K}}, a'; \widehat{w}, x''_{\mathcal{K}}, a'') \geq u^{(\overline{m}, \overline{\theta})}(x^{\gamma}_{\mathcal{I}}, x^{\gamma}_{\mathcal{K}}, a^{\gamma}; w, \tilde{x}^{\gamma}_{\mathcal{K}}, \tilde{a}^{\gamma})$ and $V^{(\widehat{w}, \widehat{\theta})}(p, B'', x''_{\mathcal{K}}, a''; \overline{m}, x'_{\mathcal{K}}, a') \geq u^{(\widehat{w}, \widehat{\theta})}(\tilde{x}^{\delta}_{\mathcal{I}}, \tilde{x}^{\delta}_{\mathcal{K}}, \tilde{a}^{\delta}; m, x^{\delta}_{\mathcal{K}}, a^{\delta})$ with at least one strict inequality.

2.b: If $\gamma = (\overline{m}, \overline{\theta}, \bar{a}; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C, \delta = (\widehat{w}, \widehat{\theta}, \hat{a}) \in \Gamma^W$ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, then there is no feasible plan for observable types $\overline{m}$ and $\widehat{w}$ under p, $(B', x'_{\mathcal{K}}, a'; B'', x''_{\mathcal{K}}, a'') \in \mathcal{C}[\overline{m}, \widehat{w}; p]$, such that $V^{(\overline{m}, \overline{\theta})}(p, B', x'_{\mathcal{K}}, a'; \widehat{w}, x''_{\mathcal{K}}, a'') \geq u^{(\overline{m}, \overline{\theta})}(x^{\gamma}_{\mathcal{I}}, x^{\gamma}_{\mathcal{K}}, a^{\gamma}; w, \tilde{x}^{\gamma}_{\mathcal{K}}, \tilde{a}^{\gamma})$ and $V^{(\widehat{w}, \widehat{\theta})}(p, B'', x''_{\mathcal{K}}, a''; \overline{m}, x'_{\mathcal{K}}, a') \geq u^{(\widehat{w}, \widehat{\theta})}(\tilde{x}^{\delta}, \tilde{a}^{\delta}; \emptyset)$ with at least one strict inequality.

2.c: If $\gamma = (\overline{m}, \overline{\theta}, \bar{a}) \in \Gamma^M, \delta = (m, \theta, a; \widehat{w}, \widehat{\theta}, \hat{a}) \in \Gamma^C$ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, then there is no feasible plan for observable types $\overline{m}$ and $\widehat{w}$ under p, $(B', x'_{\mathcal{K}}, a'; B'', x''_{\mathcal{K}}, a'') \in \mathcal{C}[\overline{m}, \widehat{w}; p]$, such that $V^{(\overline{m}, \overline{\theta})}(p, B', x'_{\mathcal{K}}, a'; \widehat{w}, x''_{\mathcal{K}}, a'') \geq u^{(\overline{m}, \overline{\theta})}(x^{\gamma}, a^{\gamma}; \emptyset)$ and $V^{(\widehat{w}, \widehat{\theta})}(p, B'', x''_{\mathcal{K}}, a''; \overline{m}, x'_{\mathcal{K}}, a') \geq u^{(\widehat{w}, \widehat{\theta})}(\tilde{x}^{\delta}_{\mathcal{I}}, \tilde{x}^{\delta}_{\mathcal{K}}, \tilde{a}^{\delta}; m, x^{\delta}_{\mathcal{K}}, a^{\delta})$ with at least one strict inequality.

2.d: If $\gamma = (\overline{m}, \overline{\theta}, \bar{a}) \in \Gamma^M, \delta = (\widehat{w}, \widehat{\theta}, \hat{a}) \in \Gamma^W$ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, then there is no feasible plan for observable types $\overline{m}$ and $\widehat{w}$ under p, $(B', x'_{\mathcal{K}}, a'; B'', x''_{\mathcal{K}}, a'') \in \mathcal{C}[\overline{m}, \widehat{w}; p]$, such that $V^{(\overline{m}, \overline{\theta})}(p, B', x'_{\mathcal{K}}, a'; \widehat{w}, x''_{\mathcal{K}}, a'') \geq u^{(\overline{m}, \overline{\theta})}(x^{\gamma}, a^{\gamma}; \emptyset)$ and $V^{(\widehat{w}, \widehat{\theta})}(p, B'', x''_{\mathcal{K}}, a''; \overline{m}, x'_{\mathcal{K}}, a') \geq u^{(\widehat{w}, \widehat{\theta})}(\tilde{x}^{\delta}, \tilde{a}^{\delta}; \emptyset)$ with at least one strict inequality.

3. Stable Matching II (immunity to single deviations)

   For each $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$ with $\mu(\gamma) > 0$:

$$u^{(m,\theta)}(x^{\gamma}_{\mathcal{I}}, x^{\gamma}_{\mathcal{K}}, a; w, \tilde{x}^{\gamma}_{\mathcal{K}}, \tilde{a}) \geq \sup_{a' \in A} \sup_{x \in \mathcal{B}^{(m,a')}(p)} u^{(m,\theta)}(x, a'; \emptyset);$$

$$u^{(w,\tilde{\theta})}(\tilde{x}^{\gamma}_{\mathcal{I}}, \tilde{x}^{\gamma}_{\mathcal{K}}, \tilde{a}; m, x^{\gamma}_{\mathcal{K}}, a) \geq \sup_{\tilde{a}' \in A} \sup_{\tilde{x} \in \mathcal{B}^{(w,\tilde{a}')}(p)} u^{(w,\tilde{\theta})}(\tilde{x}, \tilde{a}'; \emptyset).$$

4. Profit Maximization

   $py \geq py'$ for all $y' \in Y$.

Remark 3. Conditions 2.a–2.d refer respectively, to the case where a male and female cannot do better by forming a couple if they currently belong to a) two two-person households of extended types γ and δ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, or b) a two-person household of extended type γ and a single household of extended type δ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, or c) a single household of extended type γ and a two-person household of extended type δ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$, or d) two single households of extended types γ and δ with $\mu(\gamma) > 0$ and $\mu(\delta) > 0$.

Remark 4. The reason that supremum in condition 3 is used is that there may not be optimal consumption plans for nonexisting (negligible) household types with constituent characteristics (m, θ) or $(w, \tilde{\theta})$ such that $N^{(m,\theta)} = 0$ or $N^{(w,\tilde{\theta})} = 0$. Note that intra-household allocations for γ with $\mu(\gamma) = 0$ play no role. They are included in the definition only for simplicity of notation.

Next we present an example that illustrates the differences between a continuum model and the corresponding finite population model.

4.4 Examples

We begin with the continuum version of Example 3 in Gersbach and Haller (2011). There are only two observable types, m and w, and one unobservable type θ_0 which we can ignore. We assume $N^m = 2$ and $N^w = 1$. There are two commodities, one without and one with externalities in consumption, and a single action a_0 which we ignore in the sequel. Thus $I = K = 1$ and $\Theta = \{\theta_0\}, A = \{a_0\}$. Endowments are given by $e^m = (0, 1), \tilde{e}^w = (1, 1)$. Preferences are represented by the functions[12]

$$\begin{aligned}
u^m(x_{\mathcal{I}}, x_{\mathcal{K}}; w, \tilde{x}_{\mathcal{K}}) &= \ln x_{\mathcal{K}};\\
u^m(x_{\mathcal{I}}, x_{\mathcal{K}}; \emptyset) &= \ln x_{\mathcal{K}};\\
u^w(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}; m, x_{\mathcal{K}}) &= \rho \ln \tilde{x}_{\mathcal{I}} + (1-\rho) \ln (\max\{0, \tilde{x}_{\mathcal{K}} - k x_{\mathcal{K}}\}) + g;\\
u^w(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}; \emptyset) &= \rho \ln \tilde{x}_{\mathcal{I}} + (1-\rho) \ln \tilde{x}_{\mathcal{K}}
\end{aligned}$$

with parameters $0 < \rho < 1$, $0 < k < 1$, $0 < g$ and the convention $\ln 0 = -\infty$. In the finite version, that is Example 3 of Gersbach and Haller, there are two males and one female, and commodity 2 is the one that causes externalities. For certain parameter values, the finite version does not have an equilibrium nor does any replica of it.

Now let us turn to the continuum version. We are going to show existence of equilibrium for all parameter constellations. If the finite version has an equilibrium, then obviously, the continuum model has one as well. Suppose

[12] This example does not satisfy local nonsatiation in good $\mathcal{I}$ for males, as assumed in our main theorem. But we can modify the utility function to $u^m(x_{\mathcal{I}}, x_{\mathcal{K}}; w, \tilde{x}_{\mathcal{K}}) = \epsilon x_{\mathcal{I}} + \ln x_{\mathcal{K}}$ for $\epsilon > 0$ small enough without modifying the result.

the finite version does not have an equilibrium. Let $\xi = \mu((m,w))$ denote the fraction of females that are matched. Let us consider price systems of the form $p = (p_{\mathcal{I}}, p_{\mathcal{K}}) = (1, p_{\mathcal{K}})$. Note that for convenience, we choose here a different price normalization than in the main model. With this normalization, $p \notin \Delta$ unless $p_{\mathcal{K}} = 0$. In equilibrium, necessarily $x_{\mathcal{I}} = 0, x_{\mathcal{K}} = 1$. Taking this as a constraint, a female's demand for the $\mathcal{K}$-commodity is

$(1-\rho)(1+p_{\mathcal{K}})/p_{\mathcal{K}}$ if single and

$(1-\rho)(1+p_{\mathcal{K}})/p_{\mathcal{K}} + \rho k$ if matched.

Hence aggregate female demand for the $\mathcal{K}$-commodity is

$(1-\rho)(1+p_{\mathcal{K}})/p_{\mathcal{K}} + \rho\xi k$

which has to equal 1 to clear the market. Therefore, the market clearing price is

$$p^*_{\mathcal{K}} = \frac{1-\rho}{\rho(1-\xi k)}.$$

The resulting demands are

$(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}) = (1 + p^*_{\mathcal{K}}\rho\xi k, 1-\rho\xi k) = (1 + (1-\rho)\xi k/(1-\xi k), 1-\rho\xi k)$ for single females and $(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}) = (1 - p^*_{\mathcal{K}}(1-\xi)\rho k, 1 + (1-\xi)\rho k)$ for matched females. Hence the corresponding utilities are continuous functions of $\xi \in [0,1]$. By assumption, females prefer to be matched if $\xi = 0$ and females prefer to be single if $\xi = 1$. By the intermediate value theorem, there exists a $\xi \in (0,1)$ so that females are indifferent between being single and being matched. For such a value of ξ, there exists an equilibrium with $\mu((m,w)) = \xi, \mu(w) = 1-\xi, \mu(m) = 2-\xi$ where both males and females are indifferent between being single and being matched.

Incidentally, $\rho = k = 1/2, g = \ln 2$ yields a unique $\xi = 2(\sqrt{5.5}-2)$ which is an irrational number. Therefore, in that case there does not exist an equilibrium for any replica of the finite model. Moreover, $\xi = 0.69041...$ for this parameter constellation.

There are several ways to construct approximate equilibria for sufficiently large replica economies in the model with a finite number of agents. Suppose, for instance, that 6904 out of 10000 females are actually matched under the above parameter constellation when there are 10000 females and 20000 males. At the corresponding market clearing price $p^*_{\mathcal{K}}$, all females would prefer to be matched. The unmatched ones incur a small utility loss relative to the matched ones. As a second scenario, suppose that $p^*_{\mathcal{K}}$ is the price corresponding to $\xi = 2(\sqrt{5.5}-2)$, at which females are indifferent between being matched and remaining single. Further let 6904 of the 10000 females be matched and get their optimal demand. Of the single females, 3095 also consume their optimal bundle. One of the single females receives the residual endowment bundle which is approximately her optimal consumption. Further approximate equilibria can be devised along these lines.

In a continuum economy, the action of a single agent or a pair of agents (forming a new household or going single) does not affect the price system. In a finite economy the behavioral assumption that an agent takes prices as given is, indeed, an assumption rather than an implication. For instance, in a finite pure exchange economy, actions taken prior to competitive exchange can affect equilibrium prices even if consumers are price takers during the ensuing competitive exchange. Instances are manipulation via withholding of endowments studied by Postlewaite (1979), Safra (1985), Haller (1988) and others or the transfer paradox analyzed by Samuelson (1952, 1954), among others. In a similar vein, in the original finite example, the female may realize that a switch of her marital or matching status alters the equilibrium price.[13] She then might want to manipulate the outcome by choosing the status that would give her higher utility after the optimal consumption choice under the corresponding prices. Of course, the overall outcome need not be a stable matching equilibrium. For instance, consider the above parameter constellation $\rho = k = 1/2, g = \ln 2$. If the household structure with all singletons was fixed, then the female's equilibrium utility would be 0. If a household structure was in place where the female is matched with one of the males, then her subsequent equilibrium utility would be 0 at best, but could be worse — provided that the male partner is guaranteed his outside option value 0. Now suppose that $\rho = k = 1/2$ and that g is slightly less than $\ln 2$. Then the female would fare strictly better when she chose to remain single — unless a male partner was satisfied with less than what he can obtain when single. In fact, all consumers being single constitutes a stable matching equilibrium outcome in case everybody is farsighted.

It is more intriguing that under certain model specifications, stable matching equilibria exist and all stable matching equilibria are prone to manipulation. To see this, let us consider a different example.

Suppose that there are two consumers, one man and one woman. They have the following utility functions (without consumption externalities):[14]

$$\begin{aligned} u^m(x_1, x_2; married) &= x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}; \\ \tilde{u}^w(\tilde{x}_1, \tilde{x}_2; married) &= \tilde{x}_1^{\frac{1}{2}} \tilde{x}_2^{\frac{1}{2}}; \\ u^m(x_1, x_2; single) &= x_1^{\frac{2}{3}} x_2^{\frac{1}{3}} - 0.2; \\ \tilde{u}^w(\tilde{x}_1, \tilde{x}_2; single) &= \tilde{x}_1^{\frac{2}{3}} \tilde{x}_2^{\frac{1}{3}} - 0.1. \end{aligned}$$

[13]We are indebted to the referee for suggesting this possibility.

[14]The example involves separable and non-separable group externalities. In particular, individual preference for a consumption bundle over an other one depends on whether or not they enter into marriage and thus on whether the partner is present or not. Typical examples for such commodities are clothing or cars.

Their endowments are $e^m = (2, 0)$, $\tilde{e}^w = (0, 2)$. Let good 2 be the numéraire: $p_2 = 1$.

If they are married, the market clearing price is $p_1 = 1$ and their efficient consumption choices are of the form $(x, \tilde{x}) = ((x_1, x_2), (2 - x_1, 2 - x_2))$ with $0 \leq x_1 = x_2 \leq 2$, with resulting utilities $u^m = x_1$, $\tilde{u}^w = 2 - x_1$. The currently married man who does not expect a price change caused by a change of his marital status assumes the budget constraint to be $x_1 + x_2 = 2$ when going single. Consequently, he expects optimal consumption $(x_1, x_2) = \left(\frac{4}{3}, \frac{2}{3}\right)$ and the resulting utility $u_*^m = \left(\frac{4}{3}\right)^{\frac{2}{3}} \left(\frac{2}{3}\right)^{\frac{1}{3}} - 0.2 \approx 0.85827$. Similarly, the currently married woman who does not expect a price change when going single, expects the optimal consumption $(\tilde{x}_1, \tilde{x}_2) = \left(\frac{4}{3}, \frac{2}{3}\right)$ and the resulting utility $\tilde{u}_*^w = u_*^m + 0.1 \approx 0.95827$. Hence there are stable matching equilibria where they are married and the equilibrium consumption is of the form $(x, \tilde{x}) = ((x_1, x_2), (2 - x_1, 2 - x_2))$ with $u_*^m \leq x_1 = x_2 \leq 2 - \tilde{u}_*^w$. In fact, these are the only stable matching equilibria.

If they are singles, the market clearing price is $p_1 = 2$, and the resulting utilities are $u_o^m = \left(\frac{4}{3}\right)^{\frac{2}{3}} \left(\frac{4}{3}\right)^{\frac{1}{3}} = \frac{4}{3} - 0.2 \approx 1.13$ and $\tilde{u}_o^w = \left(\frac{2}{3}\right)^{\frac{2}{3}} \left(\frac{2}{3}\right)^{\frac{1}{3}} = \frac{2}{3} - 0.1 \approx 0.567$. It follows that (a) there are no stable matching equilibria where they are single; however, (b) the man prefers to be single if he takes into account the price change caused by a change of his marital status and (c) with farsightedness of both individuals, there exists a set of stable matching equilibria where they are married and the equilibrium consumption is of the form $(x, \tilde{x}) = ((x_1, x_2), (2 - x_1, 2 - x_2))$ with $u_o^m \leq x_1 = x_2 \leq 2 - \tilde{u}_o^w$, which is disjoint from the previous set based on outside option values u_*^m and $\tilde{u}_*^w$.

5 The Main Result

Only in fairly simple cases like the example of subsection 4.4 can one resort to elementary tools like the intermediate value theorem. In general, more advanced techniques are warranted. The main result of our paper encompasses the previous example and is stated below. Recall $X^m(a) = [0, \infty) \times X_{-\{1\}}^m(a)$ where $X_{-\{1\}}^m(a) \subset \mathbb{R}_+^{I+K-1}$ denotes type m's consumption set for all other commodities. Similarly, for all $w \in W$ and all $\tilde{a} \in A$, $\tilde{X}^w(\tilde{a}) = [0, \infty) \times \tilde{X}_{-\{1\}}^w(\tilde{a})$ where $\tilde{X}_{-\{1\}}^w(\tilde{a}) \subset \mathbb{R}_+^{I+K-1}$.

Theorem. There exists a stable matching equilibrium under the following assumptions:

1. For all $m \in M$ and all $a \in A$, $X^m(a) \subset \mathbb{R}_+^{I+K}$ is closed and convex with (i) $X^m(a) = [0, \infty) \times X^m_{\{-1\}}(a)$ and (ii) $e^m_1(a) > 0$.
   For all $w \in W$ and all $\tilde{a} \in A$, $\tilde{X}^w(\tilde{a}) \subset \mathbb{R}_+^{I+K}$ is closed and convex with (i) $\tilde{X}^w(\tilde{a}) = [0, \infty) \times \tilde{X}^w_{\{-1\}}(a)$ and (ii) $\tilde{e}^w_1(\tilde{a}) > 0$.

2. For all $(m, \theta) \in M \times \Theta$, all $w \in W$ and all $a, \tilde{a} \in A$, $u^{(m,\theta)}(x_{\mathcal{I}}, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a})$ is continuous and quasi-concave in $x_{\mathcal{I}}$, $x_{\mathcal{K}}$, and $\tilde{x}_{\mathcal{K}}$, and satisfies local nonsatiation in $\mathcal{I}$; and for all $(w, \tilde{\theta}) \in W \times \Theta$, all $m \in M$ and all $a, \tilde{a} \in A$, $u^{(w,\tilde{\theta})}(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a)$ is continuous and quasi-concave in $\tilde{x}_{\mathcal{I}}$, $\tilde{x}_{\mathcal{K}}$, and $x_{\mathcal{K}}$, and satisfies local nonsatiation in $\mathcal{I}$.

3. For all $(m, \theta) \in M \times \Theta$:
   (a) $u^{(m,\theta)}(0, x_{-1}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) = \underline{u}^{(m,\theta)} \equiv \min_{a' \in A} \min u^{(m,\theta)}(X^m(a'))$ for all $w \in W$, all $a, \tilde{a} \in A$, all $x_{-1} \in X^m_{-\{1\}}(a)$, and all $\tilde{x}_{\mathcal{K}} \in \tilde{X}^w_{\mathcal{K}}(\tilde{a})$;
   (b) There exists $a \in A$ such that $e^m(a) \in X^m(a)$ and $\underline{u}^{(m,\theta)} < u^{(m,\theta)}(e^m(a), a; \emptyset)$.
   For all $(w, \tilde{\theta}) \in W \times \Theta$:
   ($\tilde{\text{a}}$) $u^{(w,\tilde{\theta})}(0, \tilde{x}_{-1}, \tilde{a}; m, x_{\mathcal{K}}, a) = \underline{u}^{(w,\tilde{\theta})} \equiv \min_{\tilde{a}' \in A} \min u^{(w,\tilde{\theta})}(X^w(\tilde{a}'))$ for all $m \in M$, all $a, \tilde{a} \in A$, all $\tilde{x}_{-1} \in X^w_{-\{1\}}(\tilde{a})$, and all $x_{\mathcal{K}} \in X^m_{\mathcal{K}}(a)$;
   ($\tilde{\text{b}}$) There exists $\tilde{a} \in A$ such that $\tilde{e}^w(\tilde{a}) \in \tilde{X}^w(\tilde{a})$ and $\underline{u}^{(w,\tilde{\theta})} < u^{(w,\tilde{\theta})}(\tilde{e}^w(\tilde{a}), \tilde{a}; \emptyset)$.

4. There exists $(\overline{m}, \bar{\theta}) \in M \times \Theta$ with $N^{(\overline{m},\bar{\theta})} > 0$ and the following characteristics:

   (i) For all $a \in A$, $X^{\overline{m}}_{\mathcal{I} \cup \mathcal{K} \setminus \mathcal{J}}(a) = \mathbb{R}_+^{I+K-J}$.

   (ii) For all $a \in A$, $e^{\overline{m}}_i(a) > 0$ for all $i \in \mathcal{I}$, $e^{\overline{m}}_k(a) > 0$ for all $k \in \mathcal{K} \setminus \mathcal{J}$, and $e^{\overline{m}}_{j(a)}(a) > 0$;

   (iii) There exists $\hat{u}^{(\overline{m},\bar{\theta})} : \mathbb{R}_+^{I+K-J} \to \mathbb{R}$ such that
   (a) $u^{(\overline{m},\bar{\theta})}(x, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) = u^{(\overline{m},\bar{\theta})}(x, a; \emptyset) = \hat{u}^{(\overline{m},\bar{\theta})}(x_{\mathcal{I} \cup \mathcal{K} \setminus \mathcal{J}})$ for all $a, \tilde{a} \in A$, $w \in W$, $x \in X^{\overline{m}}(a)$, and $\tilde{x}_{\mathcal{K}} \in \tilde{X}^w_{\mathcal{K}}(\tilde{a})$;
   (b) $\hat{u}^{(\overline{m},\bar{\theta})}$ is strictly quasi-concave; and
   (c) for all $x, x' \in \mathbb{R}_+^{I+K-J}$, there exists $x''_1 > 0$ such that $\hat{u}^{(\overline{m},\bar{\theta})}(x''_1, x_{\mathcal{I} \cup \mathcal{K} \setminus \mathcal{J} - \{1\}}) > \hat{u}^{(\overline{m},\bar{\theta})}(x'_{\mathcal{I} \cup \mathcal{K} \setminus \mathcal{J}})$.

5. The aggregate production set $Y \subset \mathbb{R}^{I+K}$ is a closed and convex set satisfying (i) $Y + \mathbb{R}_-^{I+K} \subseteq Y$ (free disposal), (ii) $Y \cap \mathbb{R}_+^{I+K} = \{0\}$ (no free lunch), and (iii) $ty \in Y$ for all $y \in Y$ and all $t > 0$ (constant returns to scale).

The proof of the Theorem is involved, and is provided in the Appendix. Here we provide a brief road map first. Then we discuss how the assumptions of our theorem are used.

Road Map for the Proof. Ultimately, our proof relies on an application of Kakutani's fixed point theorem to a correspondence φ consisting of seven components $\beta, \nu, \zeta, \tau, \pi, \eta$, and ρ. Several of the components are constructed by suitably combining correspondences and mappings from the literature. Other components are specifically designed for our purposes.

We use the following standard mappings from the literature:

ζ is an excess demand mapping.

τ is the industry supply mapping (correspondence).

π is a variation of the Gale-Nikaido mapping of Debreu (1959, 5.6).

Four particular "mappings" are specifically designed for our purposes. First, the most involved mapping, denoted β, assigns a consumption plan to each extended household that will be optimal at a fixed point. Construction of β includes a Shafer-Sonnenschein mapping (Shafer and Sonnenschein 1975, Greenberg 1979, Ray and Vohra 1997). Second, we design a household choice mapping α which does not directly enter φ. It assigns to each consumer type, say (m, θ), the extended household types which would yield the highest utility $U^{(m,\theta)*}$ to the consumer type. Since preferences, feasibility and affordability depend only on actions, observable types and the true unobservable types, β and α can be constructed in such a way that type (m, θ) cannot gain from pretending to be type (m, θ'), for example. Third, similar to (Konishi 1996), we devise a population mapping ν that distributes the mass (Lebesgue measure) of each consumer type over its best extended household types, that is those distinguished by α. To achieve measurement consistency, we introduce a hypothetical demographic designer (DD) who chooses a distribution over extended household types. DD is a price taker and thus very different from a matchmaker, platform operator or club operator: DD's profit maximizing choices define the correspondence η. Finally, we construct a Gale-Nikaido-type mapping ρ that would guarantee positive profits for DD if he failed to achieve measurement consistency. At the fixed point, measurement inconsistency would imply an infeasible outcome and a contradiction.

After introducing all mappings, we truncate consumption sets and the production set and adapt all the mappings for the truncated economy except for the price mapping π. Then, we conclude that all of the assumptions of Kakutani's fixed point theorem are satisfied for all components of φ, except π. We show the missing part, upper hemi-continuity of π, and then apply

Kakutani's fixed point theorem to φ on the truncated domain.

Points (A)-(G) of the proof argue that a fixed point yields a stable matching equilibrium with free disposal for the truncated economy. Point (H) demonstrates how the truncation can be removed and exact market clearing can be obtained.

Remark 5. While our proof relies on the finiteness of M, W, and Θ, the simultaneous presence of hidden information, inter-household trade and intra-household consumption externality makes it impossible to use existing approaches to prove existence. Specifically, Kaneko and Wooders (1986) prove the non-emptiness of the f-core in any finite type characteristic function-form game. We cannot apply their theorem to show the existence of stable marriage equilibrium since (i) inter-household trade is feasible in our model, and (ii) preference type θ is hidden information. Also Ellickson et al. (1999) and Allouch et al. (2009) cannot be used either to prove the Theorem, since we have intra-household consumption externalities.[15] Still, despite the difficulties of directly applying previous results, club theory suggests different ways to prove existence of a stable matching equilibrium. A conceivable alternative approach might be to show core equivalence like Conley and Wooders (2001), e.g., and non-emptiness of the core like for instance Kaneko and Wooders (1986). Under our assumptions, taking that route appears as formidable a task as the route taken here, but could be explored in future research. Showing that core allocations can be decentralized by a price system would, perhaps, be the greatest challenge.

Comments on the Assumptions. Assumptions 1 and 2 are standard. Assumption 1 (ii) requires that every type has a positive endowment with commodity 1. Due to assumption 4, the price of commodity 1 will be positive in equilibrium. The combination of these properties implies that every type will have positive income irrespective of his/her choice of extended household type. Assumption 2 further requires quasi-concavity of an individual's utility function including its spouse's consumption vector (for each possible spouse and each action vector). We need quasi-concave utility because of consumption externalities within a couple. It is assumed in order to find a Pareto optimal intra-household allocation.[16]

Assumption 3 is a variation of a standard assumption when the consump-

[15]See Gersbach and Haller (2010) for the (lack of) equivalence of "club models" and "household models".

[16]This assumption was used in Konishi (2010) in a more restrictive local externality case. See also Konishi (2013). Specifically, we will use the Shafer-Sonnenschein (1975) mapping for the existence of a Pareto-efficient equilibrium. Thus, we need that the intersection of upper contour sets has an open graph (continuity) and is (semi) convex-valued (convex preferences).

tion set is not connected (indivisible commodities: see Mas-Colell (1977), Wooders (1978), and Ellickson (1979) for the spirit of this assumption). This assumption is the simplest way to achieve the closed graph property of the demand correspondence (Mas-Colell 1977).

Assumption 4 requires that there is a type of man (alternatively, a type of woman) who only cares about non-leisure commodity consumption (no concern about his job and his partner), whose endowment is positive for all commodities relative to his job choice, and who strongly prefers commodity 1. The implication of Assumption 4 is that a type $(\bar{m}, \bar{\theta})$ man will choose the highest paid job for this type, has positive wealth at any price system, and would consume an unbounded consumption vector if commodity 1's price went to zero. This boundary behavior contradicts feasibility, by Lemma 1. Thus, assumption 4 together with assumption 1 assures that the price of commodity 1 is positive in equilibrium, which avoids the violation of lower hemi-continuity of budget sets. We are going to use the technique illustrated in the proof of Proposition 17.C.1 in Mas-Colell et al. (1995).

Assumption 5 is standard.[17] Assumption 5 (i), free disposal in production, assures that there is exact market clearing in equilibrium, that is, where aggregate excess demand z (= lhs of (1) $-$ rhs of (1)) satisfies $z = 0$. Without assumption 5 (i), only a "free-disposal equilibrium", that is one with $z \leq 0$, $z \neq 0$ may occur.

To see this, consider a pure exchange example, that is $Y = \{0\}$, with only two observable types, m and w, a single unobservable type θ_0 (which we ignore), $N^m = 1/2$, and $N^w = 1/2$. There are two commodities, one without and one with externalities in consumption, and a single action a_0 (which we ignore). Thus $I = K = 1$ and $\Theta = \{\theta_0\}, A = \{a_0\}$. Endowments are given by $e^m = e^w = (1/2, 1)$. Preferences are represented by the functions

$$
\begin{aligned}
u^m(x_{\mathcal{I}}, x_{\mathcal{K}}; w, \tilde{x}_{\mathcal{K}}) &= -\exp(-(x_{\mathcal{I}} - \tilde{x}_{\mathcal{K}} + 1)); \\
u^m(x_{\mathcal{I}}, x_{\mathcal{K}}; \emptyset) &= -\exp(-x_{\mathcal{I}}); \\
u^w(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}; m, x_{\mathcal{K}}) &= -\exp(-(\tilde{x}_{\mathcal{I}} - x_{\mathcal{K}} + 1)); \\
u^w(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}; \emptyset) &= -\exp(-\tilde{x}_{\mathcal{I}}).
\end{aligned}
$$

Then for each $\chi \in [0, 1]$, $\mu((m, w)) = 1/2$, $p = (1, 0)$, $(x_{\mathcal{I}}, x_{\mathcal{K}}) = (\chi, 0)$ for all males and $(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}) = (1 - \chi, 0)$ for all females constitutes a stable matching equilibrium with $z = (0, -1)$. Indeed, these are all stable matching equilibria.

[17]Any convex (decreasing returns to scale) technology can be described by constant returns to scale technology by introducing managerial inputs as consumers' endowment (see McKenzie, 1959).

6 Welfare Analysis

Let $(\mathbf{x}, \mu, y)$ and $(\mathbf{x}', \mu', y')$ be two feasible allocations. Informally, we would like to say that $(\mathbf{x}', \mu', y')$ improves upon $(\mathbf{x}, \mu, y)$ if every consumer is at least as well off at $(\mathbf{x}', \mu', y')$ as at $(\mathbf{x}, \mu, y)$ and a group of consumers of positive measure is better off at $(\mathbf{x}', \mu', y')$. Since we take the distributional approach, such a statement requires that the two allocations are aligned the right way.

To explain the matter, let us take a short digression. Let us consider a pure exchange economy with two commodities and generic consumption bundles $(x_1, x_2) \in \mathbb{R}^2_+$. There is a continuum of consumers. Each consumer has the utility representation $u_i(x_1, x_2) = x_1 x_2$ and endowment bundle $(1/2, 1/2)$. Consider two feasible allocations $\mathbf{x}$ and $\widehat{\mathbf{x}}$. In $\mathbf{x}$, half of the consumers have consumption $x' = (1/10, 2/3)$ with utility $u_i(x') = 1/15$ and half of the consumers have consumption $x'' = (9/10, 1/3)$ with utility $u_i(x'') = 3/10$. In $\widehat{\mathbf{x}}$, half of the consumers have consumption $\widehat{x}' = (0.2, 0.6)$ with utility $u_i(\widehat{x}') = 0.12$ and half of the consumers have consumption $\widehat{x}'' = (0.8, 0.4)$ with utility $u_i(\widehat{x}'') = 0.32$. Can we say that everybody is better off in $\widehat{\mathbf{x}}$? The answer depends on further distributional details. If in fact half of the consumers get x' under $\mathbf{x}$ and $\widehat{x}'$ under $\widehat{\mathbf{x}}$ and half of the consumers get x'' under $\mathbf{x}$ and $\widehat{x}''$ under $\widehat{\mathbf{x}}$, then everybody is better off in $\widehat{\mathbf{x}}$. But this is not the case if half of the consumers get x' under $\mathbf{x}$ and $\widehat{x}''$ under $\widehat{\mathbf{x}}$ and half of the consumers get x'' under $\mathbf{x}$ and $\widehat{x}'$ under $\widehat{\mathbf{x}}$. Nor is it the case if, for instance, 20% of the consumers get x' under $\mathbf{x}$ and $\widehat{x}'$ under $\widehat{\mathbf{x}}$, 30% get x' under $\mathbf{x}$ and $\widehat{x}''$ under $\widehat{\mathbf{x}}$, 20% get x'' under $\mathbf{x}$ and $\widehat{x}''$ under $\widehat{\mathbf{x}}$, and 30% get x'' under $\mathbf{x}$ and $\widehat{x}'$ under $\widehat{\mathbf{x}}$.

Strictly speaking, then $(\mathbf{x}', \mu', y')$ **potentially improves upon** $(\mathbf{x}, \mu, y)$ if consumers can be aligned in such a way that every consumer is at least as well off at $(\mathbf{x}', \mu', y')$ as at $(\mathbf{x}, \mu, y)$ and a group of consumers of positive measure is better off at $(\mathbf{x}', \mu', y')$.

That means in detail (where at least one of the inequalities $\geq$ in (A) or (B) is strict):

(A) For all $(m, \theta) \in M \times \Theta$:

Let $\Gamma^{(m,\theta)} = \{(m, \theta, a) : a \in A\} \cup \{(m, \theta, a; w, \tilde{\theta}, \tilde{a}) : a \in A, (w, \tilde{\theta}, \tilde{a}) \in \Gamma^W\}$. There exist $\lambda^{(m,\theta)}(\gamma, \delta) \geq 0$ for $(\gamma, \delta) \in \Gamma^{(m,\theta)} \times \Gamma^{(m,\theta)}$ such that:[18]

- $\sum_\delta \lambda^{(m,\theta)}(\gamma, \delta) = \mu'(\gamma)$ for all $\gamma \in \Gamma^{(m,\theta)}$.

[18]Here the notation $x'(\gamma)$ and $\tilde{x}'(\gamma)$ instead of x'^γ and $\tilde{x}'^\gamma$, respectively, proves more transparent.

- $\sum_\gamma \lambda^{(m,\theta)}(\gamma,\delta) = \mu(\delta)$ for all $\delta \in \Gamma^{(m,\theta)}$.
- If $\gamma = (m,\theta,a'), \delta = (m,\theta,a) \in \Gamma^M$ with $\lambda^{(m,\theta)}(\gamma,\delta) > 0$, then $u^{(m,\theta)}(x'(\gamma), a'; \emptyset) \geq u^{(m,\theta)}(x(\delta), a; \emptyset)$.
- If $\gamma = (m,\theta,a') \in \Gamma^M$, $\delta = (m,\theta,a; w,\tilde{\theta},\tilde{a}) \in \Gamma^C$ with $\lambda^{(m,\theta)}(\gamma,\delta) > 0$, then $u^{(m,\theta)}(x'(\gamma), a'; \emptyset) \geq u^{(m,\theta)}(x(\delta), a; w, \tilde{x}_{\mathcal{K}}(\delta), \tilde{a})$.
- If $\gamma = (m,\theta,a'; w',\tilde{\theta}',\tilde{a}'), \delta = (m,\theta,a; w,\tilde{\theta},\tilde{a}) \in \Gamma^C$ with $\lambda^{(m,\theta)}(\gamma,\delta) > 0$, then $u^{(m,\theta)}(x'(\gamma), a'; w', \tilde{x}'_{\mathcal{K}}(\gamma), \tilde{a}') \geq u^{(m,\theta)}(x(\delta), a; w, \tilde{x}_{\mathcal{K}}(\delta), \tilde{a})$.
- If $\gamma = (m,\theta,a'; w',\tilde{\theta}',\tilde{a}') \in \Gamma^C, \delta = (m,\theta,a) \in \Gamma^M$ with $\lambda^{(m,\theta)}(\gamma,\delta) > 0$, then $u^{(m,\theta)}(x'(\gamma), a'; w', \tilde{x}'_{\mathcal{K}}(\gamma), \tilde{a}') \geq u^{(m,\theta)}(x(\delta), a; \emptyset)$.

(B) For all $(w,\tilde{\theta}) \in W \times \Theta$:

Let $\Gamma^{(w,\tilde{\theta})} = \{(w,\tilde{\theta},\tilde{a}) : \tilde{a} \in A\} \cup \{(w,\tilde{\theta},\tilde{a}; m,\theta,a) : \tilde{a} \in A, (m,\theta,a) \in \Gamma^M\}$. There exist $\lambda^{(w,\tilde{\theta})}(\gamma,\delta) \geq 0$ for $(\gamma,\delta) \in \Gamma^{(w,\tilde{\theta})} \times \Gamma^{(w,\tilde{\theta})}$ such that:

- $\sum_\delta \lambda^{(w,\tilde{\theta})}(\gamma,\delta) = \mu'(\gamma)$ for all $\gamma \in \Gamma^{(w,\tilde{\theta})}$.
- $\sum_\gamma \lambda^{(w,\tilde{\theta})}(\gamma,\delta) = \mu(\delta)$ for all $\delta \in \Gamma^{(w,\tilde{\theta})}$.
- If $\gamma = (w,\tilde{\theta},\tilde{a}'), \delta = (w,\tilde{\theta},\tilde{a}) \in \Gamma^W$ with $\lambda^{(w,\tilde{\theta})}(\gamma,\delta) > 0$, then $u^{(w,\tilde{\theta})}(\tilde{x}'(\gamma), \tilde{a}'; \emptyset) \geq u^{(w,\tilde{\theta})}(\tilde{x}(\delta), \tilde{a}; \emptyset)$.
- If $\gamma = (w,\tilde{\theta},\tilde{a}'), \in \Gamma^W$, $\delta = (w,\tilde{\theta},\tilde{a}; m,\theta,a) \in \Gamma^C$ with $\lambda^{(w,\tilde{\theta})}(\gamma,\delta) > 0$, then $u^{(w,\tilde{\theta})}(\tilde{x}'(\gamma), \tilde{a}'; \emptyset) \geq u^{(w,\tilde{\theta})}(\tilde{x}(\delta), \tilde{a}; m, x_{\mathcal{K}}(\delta), a)$.
- If $\gamma = (w,\tilde{\theta},\tilde{a}'; m',\theta',a'), \delta = (w,\tilde{\theta},\tilde{a}; m,\theta,a) \in \Gamma^C$ with $\lambda^{(w,\tilde{\theta})}(\gamma,\delta) > 0$, then $u^{(w,\tilde{\theta})}(\tilde{x}'(\gamma), \tilde{a}'; m', x'_{\mathcal{K}}(\gamma), a') \geq u^{(w,\tilde{\theta})}(\tilde{x}(\delta), \tilde{a}; m, x_{\mathcal{K}}(\delta), a)$.
- If $\gamma = (w,\tilde{\theta},\tilde{a}'; m',\theta',a') \in \Gamma^C, \delta = (w,\tilde{\theta},\tilde{a}) \in \Gamma^W$ with $\lambda^{(w,\tilde{\theta})}(\gamma,\delta) > 0$, then $u^{(w,\tilde{\theta})}(\tilde{x}'(\gamma), \tilde{a}'; m', x'_{\mathcal{K}}(\gamma), a') \geq u^{(w,\tilde{\theta})}(\tilde{x}(\delta), \tilde{a}; \emptyset)$.

Definition. A feasible allocation $(\mathbf{x}, \mu, y)$ is a **Pareto optimal allocation** (or a **Pareto optimum**) if there is no feasible allocation $(\mathbf{x}', \mu', y')$ that potentially improves upon $(\mathbf{x}, \mu, y)$.

Proposition 2 *Suppose that* $(p, \mu, \mathbf{x}, y)$ *is a stable matching equilibrium, consumers are locally non-satiated with respect to consumption of commodities without externalities, and the production set satisfies constant returns to scale. Then* $(\mathbf{x}, \mu, y)$ *is a Pareto optimal allocation.*

Note that the assumptions of the main theorem imply that consumers are locally non-satiated with respect to consumption of commodity 1.

Proof. Suppose that $(p, \mu, \mathbf{x}, y)$ is a stable matching equilibrium, consumers are locally non-satiated with respect to consumption of commodities without externalities, and $(\mathbf{x}, \mu, y)$ is not a Pareto optimal allocation. Let then $(\mathbf{x}', \mu', y')$ be a feasible allocation that potentially improves upon $(\mathbf{x}, \mu, y)$. That is, (A) and (B) hold, with at least one of the inequalities $\geq$ being strict.

Now take for instance two extended male types $\gamma = (m, \theta, a')$ and $\delta = (m, \theta, a)$ with $\lambda^{(m,\theta)}(\gamma, \delta) > 0$. Then $u^{(m,\theta)}(x'(\gamma), a'; \emptyset) \geq u^{(m,\theta)}(x(\delta), a; \emptyset)$. Since $\lambda^{(m,\theta)}(\gamma, \delta) > 0$ — and hence $\mu(\delta) > 0$ — and $(x(\delta), a)$ is an equilibrium choice of a consumer of basic type (m, θ), there are no alternative action a'' and no alternative consumption x'' in the corresponding budget set such that $u^{(m,\theta)}(x'', a''; \emptyset) > u^{(m,\theta)}(x(\delta), a; \emptyset)$. With the assumed local non-satiation, this implies, by the standard argument,

(i) $px'(\gamma) > pe^m(a')$ in case $u^{(m,\theta)}(x'(\gamma), a'; \emptyset) \geq u^{(m,\theta)}(x(\delta), a; \emptyset)$.

(ii) $px'(\gamma) = pe^m(a')$ in case $u^{(m,\theta)}(x'(\gamma), a'; \emptyset) = u^{(m,\theta)}(x(\delta), a; \emptyset)$.

Proceeding in a similar way in all other cases, we obtain $px'(\gamma) \geq pe^m(a')$, $p\tilde{x}'(\gamma) \geq p\tilde{e}^w(\tilde{a}')$, and $p(x'(\gamma) + \tilde{x}'(\gamma)) \geq p(e^m(a') + \tilde{e}^w(\tilde{a}'))$, respectively, for $\gamma = (m, \theta, a') \in \Gamma^M$, $\gamma = (w, \tilde{\theta}, \tilde{a}') \in \Gamma^W$, $\gamma = (m, \theta, a'; w', \tilde{\theta}', \tilde{a}') \in \Gamma^C$, respectively, with $\mu(\gamma) > 0$. And at least one of the derived inequalities is strict.

Because of constant returns to scale, $py = 0$ and $py' \leq 0$. It follows that for the allocation $(\mathbf{x}', \mu', y')$, $p\cdot$(lhs of (1)) exceeds $p\cdot$(rhs of (1)). Since $p \geq 0$, $(\mathbf{x}', \mu', y')$ must therefore violate (1), in contradiction to its presumed feasibility. □

Remark 6. We obtain a first welfare theorem for a model where non-market actions are part of the marriage or matching agreement. Versions of the first welfare theorem in matching models with actions when there are no markets for commodities have been established in Nöldeke and Samuelson (2014). In a two-sided matching model without markets for commodities, they consider actions called "investments" that affect the quality of a match. If like in our setting, investments (actions) and other terms of a match are negotiated simultaneously, they obtain versions of the first and the second welfare theorem. In contrast, when investments are chosen first and matches occur later, then the previous efficient equilibria persist, but inefficient equilibrium outcomes can also emerge.

What is needed in the proof are local non-satiation, exact market clearing, and zero equilibrium profits. Sufficient for the latter are constant returns to scale (assumed in the proposition and in the main theorem) and certain instances of increasing returns to scale. In a pure exchange setting, the first two conditions — local non-satiation and exact market clearing — will do.

With additional notational and expositional effort, one can show that an equilibrium allocation cannot be improved upon by a feasible allocation where individuals or couples of the same type can differ in their consumptive decisions. Also, along the lines of the foregoing proof, a core inclusion result can be obtained instead of the first welfare theorem.

7 Welfare Comparison

In this section, we perform equilibrium and welfare analysis for a specific example. We compare the outcome with the club equilibrium of Ellickson, Grodal, Scotchmer and Zame (1999) for this example. While the stable matching equilibrium is Pareto efficient, the club equilibrium is not. Mutatis mutandis, the comparison applies to the club equilibrium of Allouch, Conley and Wooders (2009) as well.

7.1 Set-up

We consider an economy with a continuum $[0, 1]$ of consumers. There are two observable types M and W. Consumers in $[0, \alpha]$ are males (m) while consumers in $(\alpha, 1]$ are females (w) with $1/2 < \alpha < 1$. There are two commodities, both with externalities in consumption. Each consumer has endowment $e^m = (5, 4)$ (males) and $\tilde{e}^w = (3, 0)$ (females). There is one unobservable type θ_0 and a single action a_0, which we can ignore in the sequel as they would only add to the notational description, but not to matching, behavior and equilibrium allocations.

To ease the presentation, we denote by $(x^z_{m1}, x^z_{m2}) := x_{\mathcal{K}}$ the consumption of the male and by $(x^z_{w1}, x^z_{w2}) := \tilde{x}_{\mathcal{K}}$ the consumption of the female for the first and second commodity, respectively. Let $z \in \{g, a\}$, where g denotes matched consumers and a denotes single consumers.

When both types of consumers are single, we have

$$\begin{aligned} u^a_m := u^m(x_{\mathcal{K}}; \emptyset) &= \tfrac{1}{2}\sqrt{x^a_{m1}} + \tfrac{1}{2}x^a_{m2}; \\ u^a_w := u^w(\tilde{x}_{\mathcal{K}}; \emptyset) &= \tfrac{1}{2}\sqrt{x^a_{w1}}. \end{aligned}$$

If a female and a male are matched, utilities are given by:

$$u_m^g := u^m(x_{\mathcal{K}}; w, \tilde{x}_{\mathcal{K}}) = \tfrac{1}{2}\sqrt{x_{m1}^g(x_{w2}^g + 1)} + \tfrac{1}{2}x_{m2}^g;$$
$$u_w^a := u^w(\tilde{x}_{\mathcal{K}}; m, x_{\mathcal{K}}) = \tfrac{1}{2}\sqrt{x_{w1}^g(x_{m1}^g + 1)}.$$

The example exhibits cross consumption externalities. Suppose for instance that the first commodity is dining and the second one is clothes. The male enjoys nice clothes worn by himself or by his partner when they have dinner together. The female enjoys having dinner together.

Throughout the example we normalize prices by setting the price of the first commodity to 1. The price of the second commodity, denoted by p_2, is determined in equilibrium.

7.2 Full Bargaining Power for Females

We next consider equilibria in which females have all the bargaining power. We will specifically compute equilibrium values for $\alpha = \frac{2}{3}$.

Let us assume that females maximize their utility subject to the budget constraint of the match and males obtain the same utility as if they were going single. The associated Lagrangian is

$$\begin{aligned} L(x_{m1}^g, x_{m2}^g, x_{w1}^g, x_{w2}^g, \mu_1, \mu_2) = & \frac{1}{2}\sqrt{x_{w1}^g(x_{m1}^g + 1)} \\ & + \mu_1 \cdot (8 + 4p_2 - x_{m1}^g - x_{w1}^g - p_2 \cdot (x_{m2}^g + x_{w2}^g)) \\ & + \mu_2 \cdot \Big(\frac{1}{2}\sqrt{x_{m1}^g(x_{w2}^g + 1)} + \frac{1}{2}x_{m2}^g \\ & - \frac{1}{2} \cdot \frac{p_2}{2} - \frac{1}{2}(\frac{5}{p_2} + 4 - \frac{1}{4}p_2)\Big) \end{aligned}$$

which leads to the following system of equations:

$$\frac{\partial L}{\partial x_{m1}^g} = \frac{1}{4}\sqrt{\frac{x_{w1}^g}{x_{m1}^g + 1}} - \mu_1 + \frac{1}{4}\mu_2\sqrt{\frac{x_{w2}^g + 1}{x_{m1}^g}} = 0, \tag{5}$$

$$\frac{\partial L}{\partial x_{m2}^g} = -\mu_1 p_2 + \frac{1}{2}\mu_2 = 0, \tag{6}$$

$$\frac{\partial L}{\partial x_{w1}^g} = \frac{1}{4}\sqrt{\frac{x_{m1}^g + 1}{x_{w1}^g}} - \mu_1 = 0, \tag{7}$$

$$\frac{\partial L}{\partial x_{w2}^g} = -\mu_1 p_2 + \frac{1}{4}\mu_2\sqrt{\frac{x_{m1}^g}{x_{w2}^g + 1}} = 0, \tag{8}$$

$$\frac{\partial L}{\partial \mu_1} = 8 + 4p_2 - x_{m1}^g - x_{w1}^g - p_2 \cdot (x_{m2}^g + x_{w2}^g) = 0, \tag{9}$$

$$\frac{\partial L}{\partial \mu_2} = \frac{1}{2}\sqrt{x_{m1}^g(x_{w2}^g + 1)} + \frac{1}{2}x_{m2}^g - \frac{p_2}{8} - \frac{5}{2p_2} - 2 = 0. \tag{10}$$

This system of equations can be solved as follows:

$$\mu_1 = \frac{1}{4}\sqrt{\frac{x_{m1}^g + 1}{x_{w1}^g}} \qquad \text{from } (7) \tag{11}$$

$$x_{w2}^g = \frac{1}{4}x_{m1}^g - 1 \qquad \text{from } (6, 8) \tag{12}$$

$$\mu_2 = \frac{p_2}{2}\sqrt{\frac{x_{m1}^g + 1}{x_{w1}^g}} \qquad \text{from } (6, 11) \tag{13}$$

$$x_{w1}^g = (1 - \frac{1}{4}p_2)(x_{m1}^g + 1) \qquad \text{from } (5, 11, 12, 13) \tag{14}$$

$$x_{m2}^g = \frac{21}{4} + \frac{7}{p_2} - \frac{2}{p_2}x_{m1}^g \qquad \text{from } (9, 12, 14) \tag{15}$$

$$x_{m1}^g = \frac{p_2^2 - 5p_2 - 8}{2p_2 - 8} \qquad \text{from } (10, 12, 15) \tag{16}$$

Demand

Hence, demand functions are given by

$$D^g_m(p_2) = (x^g_{m1}, x^g_{m2}) = \Big(\frac{p_2^2 - 5p_2 - 8}{2p_2 - 8}\,,\,\frac{21}{4} + \frac{7}{p_2} - \frac{p_2^2 - 5p_2 - 8}{p_2^2 - 4p_2}\Big),$$

$$D^g_w(p_2) = (x^g_{w1}, x^g_{w2}) = \Big(\frac{1}{8}(16 + 3p_2 - p_2^2)\,,\,\frac{p_2^2 - 13p_2 + 24}{8p_2 - 32}\Big).$$

Note that demand is not well defined for $p_2 = 4$.
When males are singles, their demand is given by

$$D^a_m(p_2) = \Big(\frac{p_2^2}{4}, \frac{5}{p_2} + 4 - \frac{1}{4}p_2\Big) =: (x^a_{m1}, x^a_{m2}).$$

Equilibrium Prices

The market clearing condition for good 1 when $\alpha = \frac{2}{3}$ is given as:

$$\frac{1}{3}x^g_{m1} + \frac{1}{3}x^a_{m1} + \frac{1}{3}x^g_{w1} = \frac{13}{3}$$

Thus, the prices $p_2 = 3.19$ and $p_2 = 7.38$ qualify as equilibrium candidates. But $p_2 = 7.38$ can be ruled out, since it would yield $x^g_{w1} < 0$.

Equilibrium Allocation

We obtain the following equilibrium allocation:

p_2	x^g_{m1}	x^g_{m2}	x^g_{w1}	x^g_{w2}	x^a_{m1}	x^a_{m2}	$u^g_m = u^a_m$	u^g_w
3.19	8.53	2.10	1.92	1.13	2.55	4.77	3.18	2.14

Table 1: Consumption and utilities when matching occurs under the assumption that females have full bargaining power

Moreover, we obtain

x^a_{w1}	x^a_{w2}	u^a_w
3	0	0.87

Table 2: Consumption and utility of females if they were not matched

Equilibrium Checks

As can be seen in Table 1 and Table 2, we have $u_w^g > u_w^a$ and $u_m^g = u_m^a$ in all matches. Hence, the exit condition is fulfilled: Single males might have an incentive to form a new match with females by offering them a higher utility than in the existing group. But since matches are constructed by setting males to their reservation utility level, there is no possibility for single males to attract females without suffering utility losses beyond the utility level they can achieve as singles.

We conclude that the constructed equilibrium is indeed a stable matching equilibrium as defined in this paper.

7.3 Remarks

It is straightforward to verify that the ensuing allocation in case $\alpha = \frac{2}{3}$ is Pareto optimal. It is also a direct consequence of our welfare theorem.

We note that the constructed equilibrium cannot occur in continuum versions of the club theory. Moreover, it turns out that it is impossible for individuals to realize gains in a marriage in a club setting. Thus, matching in a club setting becomes irrelevant.

To illustrate, suppose that matches are formed by consistent group formation in Ellickson, Grodal, Scotchmen and Zame (1999). Each potential match is associated with membership prices that add up to zero. All individuals choose memberships in matches and private commodities, given budget-balancing membership prices for the former and a price system for the latter. In equilibrium, no agent wants to deviate from his/her private consumption plan and from his/her group membership plan.

We immediately observe that in any equilibrium $x_{w2}^g = 0$ will be chosen by the female in a match as the choice of x_{w2}^g is made by her individually, and she derives no utility from x_{w2}^g.

As $x_{w2}^g = 0$ in any match and $x_{w2}^a = 0$ for females going single, we next observe that all females choose $x_{w1}^g = x_{w1}^a = 3$, independently of whether they are singles or matched. Moreover, as $x_{w2}^g = x_{w2}^a = 0$, males will demand the same commodity bundle, independently of whether they are matched or not. Hence, in any equilibrium we will have $(x_{w1}^g, x_{w2}^g) = (x_{w1}^a, x_{w2}^a) = (3, 0)$ and $(x_{m1}^g, x_{m2}^g) = (x_{m1}^a, x_{m2}^a) = (5, 4)$, independently of whether individuals are singles or matched.

Moreover, membership prices will be zero for males and females in all matches. However, the equilibrium is not Pareto optimal. In each match, both males and females could achieve higher utility by choosing a positive amount of x_{w2}^g and reducing x_{m2}^g by an equivalent amount (and possibly further changes of x_{m1}^g and x_{w1}^g and trading in commodity markets).

A similar analysis can be performed for $\alpha \in (0, \frac{1}{2}]$. In case $0 < \alpha < \frac{1}{2}$, the analysis is simplified when males enjoy all the bargaining power. Examination of the corresponding club equilibrium proves slightly more cumbersome because it involves positive transfers from the woman to the man in a group. Still, the stable matching equilibria are Pareto optimal whereas the transfer equilibria are not.

8 Conclusion

In this paper, we define an equilibrium for a market economy in which any two partners of opposite sex can form a household. Within each two-person household, externalities from the partner's commodity consumption in some categories and non-priced actions (such as job choice) are allowed. The main result is a set of sufficient conditions under which a stable matching equilibrium exists. We illustrate the advantage of the continuum version over the corresponding finite population model by means of an example based on the motivating counter-example in Gersbach and Haller (2011). We also prove efficiency of every stable matching equilibrium allocation under an appropriate definition of efficiency.

In our model, consumers can choose actions which encompass job choice, an attractive feature absent from prior models of pairwise matching and previous general equilibrium models with endogenous household formation. Typically then, one would not expect a consumer to be endowed with labor of various skills, only with the kind of labor compatible with her action and job choice. But then, a consumer's endowment bundle may lie on the boundary of her consumption set. This in turn tends to cause discontinuity of the budget correspondence and lack of upper hemi-continuity of the demand correspondence, a challenge in proving existence of equilibrium. There are various ways to overcome that obstacle. Assumption 4 of our existence theorem postulates a particular type of consumers who guarantees a boundary condition with respect to the first commodity; cf. Lemma 3. Combined with the further assumptions, this circumvents the upper hemi-continuity problem of demand correspondences. Instead, we could postulate a Cobb-Douglas type of agents of positive measure with strictly positive endowment bundle, who does not impose any externalities and whose utility depends only on own consumption. Such a special consumer type yields the standard boundary condition. As a third alternative, one could follow Mirrlees (1971) and postulate that all kinds of labor are perfect substitutes. Then all kinds of leisure or labor could be measured in efficiency units in terms of some normalized labor input and we could proceed as if normalized labor was the only type of labor input and each consumer was endowed with it.

Our definition of stable matching equilibrium allows for complete and efficient contracts between two partners — binding agreements over their actions and consumption vectors that generate externalities to each other and over residual budget shares to be spent on consumption without externalities. This is an idealized concept that offers the best chance for the first welfare theorem to hold, and indeed, it has been proven in addition to the existence result. The complete contract assumption is certainly very strong. As an alternative, the club literature only requires club members to pay fees in return for local public goods, club goods and projects, and possibly the company of other club members. A consumer is free to spend the remaining budget on private consumption. However, by doing so, the consumers no longer internalize consumption externalities within groups (couples, households, clubs, jurisdictions). The resulting group decisions would no longer be efficient and, as a rule, equilibrium allocations would cease to be Pareto optimal, if intra-group consumption externalities exist. We refer to Gersbach and Haller (2010) for a comparison of the "household model" and the "club model". It could be fruitful to define a stable matching equilibrium concept with incomplete contracting of action and consumption choices and to investigate how the degree of (in)completeness affects equilibrium outcomes. This task is left for future research.

Appendix

We will construct a fixed point mapping $\varphi : \Phi \twoheadrightarrow \Phi$ and its domain Φ in eleven steps. In these steps, we are going to define a number of mappings (functions and correspondences) that constitute φ and whose domains (and ranges) are components of Φ. After truncating some of the domains (and ranges), Φ and φ satisfy the assumptions of Kakutani's fixed point theorem. In a last step, it can be shown that a subsequence of fixed point prices and allocations converges to a stable matching equilibrium when the truncation is gradually removed.

First, in order to prove the theorem, we will introduce a hypothetical demographic designer (DD) and hypothetical gender-dependent membership prices for each household type. Given marriage contracts consisting of a list of household type $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, consumption plans of goods with externalities $(x_{\mathcal{K}}^{\gamma}, \tilde{x}_{\mathcal{K}}^{\gamma})$, and membership fees for men and women, DD proposes a measure h^{γ} for each couple type $\gamma \in \Gamma$. In that sense, she may be called demographic designer. The more common and fancier term match-maker might be misleading in our context. DD charges the corresponding membership fees to the respective men and women, but acts as a price-taker. DD can always choose inaction $\emptyset$: i.e., propose measure zero for any $\gamma \in \Gamma$. Denote male and female membership prices of households of type $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$ together with consumption plans of goods with externalities $(x_{\mathcal{K}}^{\gamma}, \tilde{x}_{\mathcal{K}}^{\gamma})$ by r^m and r^w, respectively. That is, a type (m, θ') man (it is not necessary to have $\theta' = \theta$) can purchase the right to be partnered with a woman of observable type w, consuming $(a, \tilde{a}; x_{\mathcal{K}}^{\gamma}, \tilde{x}_{\mathcal{K}}^{\gamma})$, by paying $r^{m\gamma}$.

His allowance for non-externality commodities in household γ is $pe^m(a) - r^{m\gamma}$, which will be B (and $\tilde{B} = p\tilde{e}^w(\tilde{a}) - r^{w\gamma}$) in a fixed point. (DD receives zero profit.) DD collects $r^{m\gamma} + r^{w\gamma}$ and finances $p_{\mathcal{K}} \cdot (x_{\mathcal{K}}^{\gamma} + \tilde{x}_{\mathcal{K}}^{\gamma})$: I.e., her profit, if any, is $r^{m\gamma} + r^{w\gamma} - p_{\mathcal{K}} \cdot (x_{\mathcal{K}}^{\gamma} + \tilde{x}_{\mathcal{K}}^{\gamma})$. We assume that DD behaves as a price-taker. A membership price vector is written $r = (r^{m\gamma}, r^{w\gamma})_{\gamma \in \Gamma^C}$. Let the supply of households of type γ (with consumption plan of externality-commodities) be $h^{\gamma} \geq 0$, and let the vector of household supply be $h = (h^{\gamma})_{\gamma \in \Gamma^C}$. Then DD's total profit is $\sum_{\gamma \in \Gamma^C} \{r^{m\gamma} + r^{w\gamma} - p_{\mathcal{K}} \cdot (x_{\mathcal{K}}^{\gamma} + \tilde{x}_{\mathcal{K}}^{\gamma})\} h^{\gamma}$. For $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, $B^{\gamma} = pe^m(a) - r^{m\gamma}$ and $\tilde{B}^{\gamma} = p\tilde{e}^w(\tilde{a}) - r^{w\gamma}$ holds if $r^{m\gamma} + r^{w\gamma} - p_{\mathcal{K}} \cdot (x_{\mathcal{K}}^{\gamma} + \tilde{x}_{\mathcal{K}}^{\gamma}) = 0$, since the household budget constraint is written as:

$$B^{\gamma} + \sum_{k \in \mathcal{K}} p_k x_k^{\gamma} + \tilde{B}^{\gamma} + \sum_{k \in \mathcal{K}} p_k \tilde{x}_k^{\gamma} = pe^m(a) + p\tilde{e}^w(\tilde{a}). \tag{17}$$

To be precise, (17) and $r^{m\gamma} + r^{w\gamma} - p_{\mathcal{K}} \cdot (x_{\mathcal{K}}^{\gamma} + \tilde{x}_{\mathcal{K}}^{\gamma}) = 0$ imply $B^{\gamma} + \tilde{B}^{\gamma} = pe^m(a) + p\tilde{e}^w(\tilde{a}) - (r^{m\gamma} + r^{w\gamma})$. The identities $B^{\gamma} = pe^m(a) - r^{m\gamma}$ and

$\tilde{B}^\gamma = p\tilde{e}^w(\tilde{a}) - r^{w\gamma}$ result from the constructions in the fourth and fifth step.

Second, note that production technologies are assumed to exhibit constant returns to scale, and there will be zero profit in equilibrium in our model. However, when we prove existence of equilibrium, we need to specify how profits are distributed in off-equilibrium states. For simplicity, assume that there is only one firm (or many identical firms). Let ψ^m (and ψ^w) be the share of the firm that type m men (type w women) own: That is, $\sum_{m\in M} \psi^m N^m + \sum_{w\in W} \psi^w N^w = 1$. Let $\omega^m(p; y, h)$ $= \psi^m \left[p \cdot y + \sum_{\gamma\in\Gamma^C} \{r^{m\gamma} + r^{w\gamma} - p_{\mathcal{K}} \cdot (x^\gamma_{\mathcal{K}} + \tilde{x}^\gamma_{\mathcal{K}})\} h^\gamma\right]$ and $\omega^w(p; y, h) = \psi^w \left[p \cdot y + \sum_{\gamma\in\Gamma^C} \{r^{m\gamma} + r^{w\gamma} - p_{\mathcal{K}} \cdot (x^\gamma_{\mathcal{K}} + \tilde{x}^\gamma_{\mathcal{K}})\} h^\gamma\right]$ denote the profit accruing to a man with share ψ^m and a woman with share ψ^w, respectively, where $y \in Y$ is a production vector. When we work on the proof of equilibrium existence, we will modify budget constraints in the following manner:

$$\mathcal{B}^{(m,w;a,\tilde{a})}(p; y, h) \equiv \left\{ (x, \tilde{x}) \in X^m(a) \times X^w(\tilde{a}) \;\middle|\; \begin{array}{l} p(x + \tilde{x}) \le p(e^m(a) + \tilde{e}^w(\tilde{a})) \\ +\omega^m(p; y, h) + \omega^w(p; y, h) \end{array} \right\},$$

$$\mathcal{B}^{(m,a)}(p; y, h) \equiv \{x \in X^m(a) : px \le pe^m(a) + \omega^m(p; y, h)\},$$

and

$$\mathcal{B}^{(w,\tilde{a})}(p; y, h) \equiv \left\{\tilde{x} \in \tilde{X}^w(\tilde{a}) : p\tilde{x} \le p\tilde{e}^w(\tilde{a}) + \omega^w(p; y, h)\right\}.$$

Clearly, when $p \cdot y = 0$ and $r^{m\gamma} + r^{w\gamma} - p(x^\gamma_{\mathcal{K}} + \tilde{x}^\gamma_{\mathcal{K}}) = 0$ for all $\gamma \in \Gamma^C$, then $\omega^m(p; y, h) = \omega^w(p; y, h) = 0$ holds, and we go back to our original budget constraints. *Throughout the appendix, we use the above definitions of budget constraints in order to prove the existence of equilibrium.* However, we are going to show that the conditions $\omega^m(p; y, h) = \omega^w(p; y, h) = 0$ are self-confirming: If they are assumed, then $p\cdot y = 0$ and $r^{m\gamma}+r^{w\gamma}-p(x^\gamma_{\mathcal{K}}+\tilde{x}^\gamma_{\mathcal{K}}) = 0$ for all $\gamma \in \Gamma^C$ obtain at a fixed point.

Third, let $\Delta \equiv \{p \in \mathbb{R}^{I+K}_+ : \sum_{\ell\in\mathcal{I}\cup\mathcal{K}} p_\ell = 1\}$, which is a price simplex. We will treat commodity 1 differently in order to assure $p_1 > 0$ in equilibrium, and we let $\mathring{\Delta}_1 \equiv \{p \in \Delta : p_1 > 0\}$ and $\partial\Delta_1 \equiv \{p \in \Delta : p_1 = 0\}$. We will show that any fixed point price vector of our fixed point mapping is in $\mathring{\Delta}_1$ (see below). Let $\bar{e} = \max_{(j,a)\in(M\cup W)\times A}\{\max_{\ell\in\mathcal{I}\cup\mathcal{K}} e^j_\ell(a)\}$ and $R = [-2\bar{e}, 2\bar{e}]$, where $\bar{e}$ denotes the highest possible income a consumer can obtain among all $p \in \Delta$. We allow for negative membership prices, since a couple may cross-subsidize each other. (At least one partner needs to pay a positive price, but the other may get a transfer from him/her.) The membership price set is denoted by $R^{2\Gamma^C}$, with representative elements $r \in R^{2\Gamma^C}$.

Fourth, for all $\gamma \in \Gamma^C$, we construct a mapping $\beta_{\mathcal{K}}^{\gamma}$ that assigns a Pareto-efficient allocation of goods with externalities (those in $\mathcal{K}$) to γ. This procedure needs some preparation. For each $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, let $F_\gamma : \Delta \times R^2 \times Y \times H \twoheadrightarrow X_{\mathcal{K}}^m(a) \times \tilde{X}_{\mathcal{K}}^w(\tilde{a})$ be a correspondence that describes feasible consumption plans of commodities with externalities such that

$$\begin{aligned} & F_\gamma(p, r^{m\gamma}, r^{w\gamma}, y, h) \\ \equiv \; & \left\{ (x_{\mathcal{K}}, \tilde{x}_{\mathcal{K}}) \in X_{\mathcal{K}}^m(a) \times \tilde{X}_{\mathcal{K}}^w(\tilde{a}) \,\middle|\, \begin{array}{l} p_{\mathcal{K}}\,(x_{\mathcal{K}} + \tilde{x}_{\mathcal{K}}) \leq pe^m(a) + p\tilde{e}^w(\tilde{a}) \\ -r^{m\gamma} - r^{w\gamma} + \omega^m(p; y, h) + \omega^w(p; y, h) \end{array} \right\} \end{aligned}$$

if $pe^m(a) + p\tilde{e}^w(\tilde{a}) - r^{m\gamma} - r^{w\gamma} + \omega^m(p; y, h) + \omega^w(p; y, h) \geq 0$ and $F_\gamma(p, r^{m\gamma}, r^{w\gamma}, y, h) = \{0\} \in X_{\mathcal{K}}^m(a) \times \tilde{X}_{\mathcal{K}}^w(\tilde{a})$, otherwise. I.e., in order to obtain a correspondence, we assume that 0 is feasible for household γ even if it is actually infeasible.

By choosing feasible $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, type (m, θ') can obtain ($\theta' = \theta$ is not required):

$$\begin{aligned} & V^{(m,\theta')}(p, pe^m - r^{m\gamma} + \omega^m(p; y, h), x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) \\ = & \left\{ \begin{array}{ll} \left. \begin{array}{c} \max\limits_{x_{\mathcal{I}} \in X_{\mathcal{I}}^m(a)} u^{(m,\theta')}(x_{\mathcal{I}}, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) \\ \text{s.t.} \sum_{i \in \mathcal{I}} p_i x_i \leq pe^m - r^{m\gamma} + \omega^m(p; y, h) \end{array} \right\} & \text{if } pe^m - r^{m\gamma} + \omega^m(p; y, h) \geq 0; \\ \underline{u}^{(m,\theta')} & \text{otherwise;} \end{array} \right. \end{aligned}$$

with $\underline{u}^{(m,\theta')}$ defined in assumption 3(a) of the Theorem.

Similarly, by choosing $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, a type $(w, \tilde{\theta}')$ woman can obtain

$$\begin{aligned} & V^{(w,\tilde{\theta}')}(p, p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h), \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a) \\ = & \left\{ \begin{array}{ll} \left. \begin{array}{c} \max\limits_{\tilde{x}_{\mathcal{I}} \in X_{\mathcal{I}}^w(\tilde{a})} u^{(w,\tilde{\theta}')}(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a) \\ \text{s.t.} \sum_{i \in \mathcal{I}} p_i \tilde{x}_i \leq p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h) \end{array} \right\} & \text{if } p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h) \geq 0; \\ \underline{u}^{(w,\tilde{\theta}')} & \text{otherwise;} \end{array} \right. \end{aligned}$$

with $\underline{u}^{(w,\tilde{\theta}')}$ defined in assumption 3($\tilde{a}$) of the Theorem.

By assumption 3 of the Theorem, at a fixed point no consumer chooses a household that gives him or her a nonpositive income.

We will define a Shafer-Sonnenschein utility function based on intra-household Pareto-efficiency assuming that the types of man and woman are (m, θ) and $(w, \tilde{\theta})$, respectively, for each price vector $p \in \Delta$ and $(r^{m\gamma}, r^{w\gamma}) \in$

R^2. Let $\mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}; y, h] : X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a}) \twoheadrightarrow X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a})$ be defined by $\mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}; y, h](x_\mathcal{K}, \tilde{x}_\mathcal{K}) \equiv$

$$\left\{ (x'_\mathcal{K}, \tilde{x}'_\mathcal{K}) \in X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a}) \;\middle|\; \begin{array}{l} V^{(m,\theta')}(p, pe^m - r^{m\gamma} + \omega^m(p; y, h), x'_\mathcal{K}, a; w, \tilde{x}'_\mathcal{K}, \tilde{a}) \\ > V^{(m,\theta')}(p, pe^m - r^{m\gamma} + \omega^m(p; y, h), x_\mathcal{K}, a; w, \tilde{x}_\mathcal{K}, \tilde{a}), \\ V^{(w,\tilde{\theta})}(p, p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h), \tilde{x}'_\mathcal{K}, \tilde{a}; m, x'_\mathcal{K}, a) \\ > V^{(w,\tilde{\theta})}(p, p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h), \tilde{x}_\mathcal{K}, \tilde{a}; m, x_\mathcal{K}, a) \end{array} \right\}.$$

By continuity of utility functions, the correspondence $\mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}; y, h] : X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a}) \twoheadrightarrow X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a})$ has an open graph. For each $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, the Shafer-Sonnenschein utility function $U^\gamma_{SS}[p, r^{m\gamma}, r^{w\gamma}; y, h] : (X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a}))^2 \to \mathbb{R}_+$ is defined such that

$$\begin{aligned} & U^\gamma_{SS}[p, r^{m\gamma}, r^{w\gamma}; y, h](x'_\mathcal{K}, \tilde{x}'_\mathcal{K}; x_\mathcal{K}, \tilde{x}_\mathcal{K}) \\ = & \min_{(x''_\mathcal{K}, \tilde{x}''_\mathcal{K}) \notin \mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}](x_\mathcal{K}, \tilde{x}_\mathcal{K})} \left\| (x'_\mathcal{K}, \tilde{x}'_\mathcal{K}) - (x''_\mathcal{K}, \tilde{x}''_\mathcal{K}) \right\|. \end{aligned}$$

Note that if $(x'''_\mathcal{K}, \tilde{x}'''_\mathcal{K}) \notin \mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}; y, h](x_\mathcal{K}, \tilde{x}_\mathcal{K})$, then $U^\gamma_{SS}[p, r^{m\gamma}, r^{w\gamma}; y, h](x'''_\mathcal{K}, \tilde{x}'''_\mathcal{K}; x_\mathcal{K}, \tilde{x}_\mathcal{K}) = 0$. Also, $U^\gamma_{SS}[p, r^{m\gamma}, r^{w\gamma}; y, h]$ is continuous after compactifying consumption sets ($\mathcal{P}^\gamma$ has an open graph) and quasi-concave (an intersection of convex upper contour sets is convex). For each $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, let $\beta^\gamma_\mathcal{K} : \Delta \times R^2 \times X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a}) \times Y \times H \twoheadrightarrow X^m_\mathcal{K}(a) \times \tilde{X}^w_\mathcal{K}(\tilde{a})$ be such that

$$\beta^\gamma_\mathcal{K}(p, r^{m\gamma}, r^{w\gamma}, x_\mathcal{K}, \tilde{x}_\mathcal{K}; y, h) = \arg \max_{(x'_\mathcal{K}, \tilde{x}'_\mathcal{K}) \in F_\gamma(p, r^{m\gamma}, r^{w\gamma}, y, h)} U^\gamma_{SS}(x'_\mathcal{K}, \tilde{x}'_\mathcal{K}; x_\mathcal{K}, \tilde{x}_\mathcal{K}).$$

Notice that eventually in the fixed point, $(x^\gamma_\mathcal{K}, \tilde{x}^\gamma_\mathcal{K}) \in \beta^\gamma_\mathcal{K}(p, r^{m\gamma}, r^{w\gamma}, x_\mathcal{K}, \tilde{x}_\mathcal{K}; y, h)$. Then $(x^\gamma_\mathcal{K}, \tilde{x}^\gamma_\mathcal{K}) \in F_\gamma(p, r^{m\gamma}, r^{w\gamma}, y, h)$ and $F_\gamma(p, r^{m\gamma}, r^{w\gamma}, y, h) \cap \mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}; y, h](x^\gamma_\mathcal{K}, \tilde{x}^\gamma_\mathcal{K}) = \emptyset$.

Fifth, we assign an optimal consumption plan β^γ for non-externality commodities to each type γ.

■ *β^γ for singles:* For $\gamma = (m, \theta, a) \in \Gamma^M$, let $\beta^{(m,\theta,a)} : \Delta \times Y \times H \twoheadrightarrow X^m(a)$ be such that $\beta^{(m,\theta,a)}(p; y, h) \equiv \{x \in \mathcal{B}^{(m,a)}(p; y, h) : u^{(m,\theta)}(x, a; \emptyset) \geq u^{(m,\theta)}(x', a; \emptyset)$ for all $x' \in \mathcal{B}^{(m,a)}(p; y, h)\}$. For $\gamma = (w, \tilde{\theta}, \tilde{a}) \in \Gamma^W$, let $\beta^{(w,\tilde{\theta},\tilde{a})} : \Delta \times Y \times H \twoheadrightarrow \tilde{X}^w(\tilde{a})$ be such that $\beta^{(w,\tilde{\theta},\tilde{a})}(p; y, h) \equiv \{\tilde{x} \in \mathcal{B}^{(w,\tilde{a})}(p; y, h) : u^{(w,\tilde{\theta})}(\tilde{x}, \tilde{a}; \emptyset) \geq u^{(w,\tilde{\theta})}(\tilde{x}', \tilde{a}; \emptyset)$ for all $\tilde{x}' \in \mathcal{B}^{(w,\tilde{a})}(p; y, h)\}$. For each price vector p, production plan y, household supply vector h, and corresponding income, these mappings simply assign the optimal consumption vectors to every single household. With continuous and quasi-concave utility functions, singles' β-correspondence is nonempty-valued, upper hemi-continuous (after the minor modification on $\partial\Delta_1$ made below) and convex-valued (when consumption sets are compactified below by means of suitable truncations).

■ β^γ *for couples:* In order to describe consumers' choices in our fixed point mapping, we need to extend the definition of consumption allocations to $\gamma \in \Gamma^C$. In that case, consumption plans assume the form $\beta^\gamma = (\beta^\gamma_{\mathcal{K}}, \beta^\gamma_{\mathcal{I}})$ where $\beta^\gamma_{\mathcal{K}}$ has already been defined in the fourth step. It remains to construct $\beta^\gamma_{\mathcal{I}}$. We will allow each preference type consumer optimize his/her non-externality commodity consumption plan: i.e., if a type (m, θ') is matched with $(w, \tilde{\theta}')$ at $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a})$ with $(x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}})$ and $(r^{m\gamma}, r^{w\gamma})$, then he is not necessarily to choose $x^\gamma_{\mathcal{I}}$ that is prepared for type (m, θ) since his preference is different from type (m, θ): He should be able to choose any $x_{\mathcal{I}}$ as long as $\sum_{i \in \mathcal{I}} p_i x_i \leq pe^m - r^{m\gamma} + \omega^m(p; y, h)$ is satisfied, because as long as a man's appearance type is the same, his actual preference does not matter for his partner. For each $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$ together with $(x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}})$ and $(r^{m\gamma}, r^{w\gamma})$, and for any $\theta', \tilde{\theta}' \in \Theta$, let

$$
\begin{aligned}
&\mathfrak{b}^{(m,\theta')\gamma}(p, r^{m\gamma}, x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}}; y, h) \\
= &\begin{cases} \left\{ \begin{array}{c} \underset{x_{\mathcal{I}} \in X^m_{\mathcal{I}}(a)}{\arg\max}\, u^{(m,\theta')}(x_{\mathcal{I}}, x_{\mathcal{K}}, a; w, \tilde{x}_{\mathcal{K}}, \tilde{a}) \\ \text{s.t. } \sum_{i \in \mathcal{I}} p_i x_i \leq pe^m - r^{m\gamma} + \omega^m(p; y, h) \end{array} \right\} & \text{if } pe^m - r^{m\gamma} + \omega^m(p; y, h) \geq 0, \\ \{x_{\mathcal{I}} \in X^m_{\mathcal{I}}(a) : x_1 = 0\} & \text{otherwise;} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{b}^{(w,\tilde{\theta}')\gamma}(p, r^{w\gamma}, x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}}; y, h) \\
= &\begin{cases} \left\{ \begin{array}{c} \underset{\tilde{x}_{\mathcal{I}} \in X^w_{\mathcal{I}}(\tilde{a})}{\arg\max}\, u^{(w,\tilde{\theta}')}(\tilde{x}_{\mathcal{I}}, \tilde{x}_{\mathcal{K}}, \tilde{a}; m, x_{\mathcal{K}}, a) \\ \text{s.t. } \sum_{i \in \mathcal{I}} p_i \tilde{x}_i \leq p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h) \end{array} \right\} & \text{if } p\tilde{e}^w - r^{w\gamma} + \omega^w(p; y, h) \geq 0, \\ \left\{\tilde{x}_{\mathcal{I}} \in \tilde{X}^m_{\mathcal{I}}(a) : \tilde{x}_1 = 0\right\} & \text{otherwise.} \end{cases}
\end{aligned}
$$

That is, we assign an optimal no-externality commodity consumption plan for each preference type to describe each preference type's optimal household choice (and excess demand correspondence). For each $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, let $\beta^\gamma_{\mathcal{I}} : \Delta \times R^2 \times X^m_{\mathcal{K}} \times \tilde{X}^w_{\mathcal{K}} \times Y \times H \twoheadrightarrow (X^m_{\mathcal{I}})^\Theta \times (\tilde{X}^w_{\mathcal{I}})^\Theta$ be defined by

$$
\begin{aligned}
&\beta^\gamma_{\mathcal{I}}(p, r^{m\gamma}, r^{w\gamma}, x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}}; y, h) \\
\equiv\ &\Pi_{\theta' \in \Theta} \mathfrak{b}^{(m,\theta')\gamma}(p, r^{m\gamma}, x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}}; y, h) \times \Pi_{\tilde{\theta}' \in \Theta} \mathfrak{b}^{(w,\tilde{\theta}')\gamma}(p, r^{w\gamma}, x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}}; y, h),
\end{aligned}
$$

and let $\beta^\gamma : \Delta \times R^2 \times X^m_{\mathcal{K}} \times \tilde{X}^w_{\mathcal{K}} \times Y \times H \twoheadrightarrow (X^m)^\Theta \times (\tilde{X}^w)^\Theta$ be defined as a Cartesian product of two mappings:

$$\beta^\gamma = (\beta^\gamma_{\mathcal{K}}, \beta^\gamma_{\mathcal{I}}).$$

Abusing notation, let us extend the consumption correspondence $\mathcal{X} : \Gamma^M \cup \Gamma^W \cup \Gamma^C \twoheadrightarrow \mathbb{R}^{I+K} \cup \mathbb{R}^{2K} \times \mathbb{R}^{2I}$ to $\hat{\mathcal{X}} : \Gamma^M \cup \Gamma^W \cup \Gamma^C \twoheadrightarrow \mathbb{R}^{I+K} \cup \mathbb{R}^{2K} \times (\mathbb{R}^I)^{2\Theta}$ by setting $\hat{\mathcal{X}}(m,\theta,a) = \mathcal{X}(m,\theta,a)$ for $\gamma = (m,\theta,a) \in \Gamma^M$, $\hat{\mathcal{X}}(w,\tilde{\theta},\tilde{a}) = \mathcal{X}(w,\tilde{\theta},\tilde{a})$ for $\gamma = (w,\tilde{\theta},\tilde{a}) \in \Gamma^W$, and $\hat{\mathcal{X}}(m,\theta,a,w,\tilde{\theta},\tilde{a}) = (X^m_{\mathcal{K}}(a) \times \tilde{X}^w_{\mathcal{K}}(\tilde{a})) \times (X^m_{\mathcal{I}}(a))^\Theta \times (\tilde{X}^w_{\mathcal{I}}(\tilde{a}))^\Theta$ for $\gamma = (m,\theta,a,w,\tilde{\theta},\tilde{a}) \in \Gamma^C$. Note that $\mathcal{X}(m,\theta,a,w,\tilde{\theta},\tilde{a}) = (X^m_{\mathcal{K}}(a) \times \tilde{X}^w_{\mathcal{K}}(\tilde{a})) \times (X^m_{\mathcal{I}}(a)) \times (\tilde{X}^w_{\mathcal{I}}(\tilde{a}))$ for $\gamma = (m,\theta,a,w,\tilde{\theta},\tilde{a}) \in \Gamma^C$, since we only consider the case of $\theta' = \theta$. Let $\hat{\mathbf{x}}$ denote a selection of $\hat{\mathcal{X}}$ — which is an (extended) consumption plan — and let $\hat{\mathfrak{X}}$ be the collection of all $\hat{\mathbf{x}}$. For $\gamma = (m,\theta,a,w,\tilde{\theta},\tilde{a}) \in \Gamma^C$, we write $\hat{\mathbf{x}}(\gamma) = \hat{x}^\gamma = (\hat{x}^{m,\gamma}_{\mathcal{K}}, \hat{x}^{w,\gamma}_{\mathcal{K}}; (\hat{x}^{(m,\theta'),\gamma}_{\mathcal{I}})_{\theta' \in \Theta}; (\hat{x}^{(w,\tilde{\theta}'),\gamma}_{\mathcal{I}})_{\tilde{\theta}' \in \Theta}) \in (X^m_{\mathcal{K}}(a) \times \tilde{X}^w_{\mathcal{K}}(\tilde{a})) \times (X^m_{\mathcal{I}}(a))^\Theta \times (\tilde{X}^w_{\mathcal{I}}(\tilde{a}))^\Theta$, where $\hat{x}^{m,\gamma}_{\mathcal{K}}$ denotes m's consumption vector of $\mathcal{K}$ commodities when m chooses household type $\gamma = (m,\theta,a,w,\tilde{\theta},\tilde{a}) \in \Gamma^C$, and $\hat{x}^{(m,\theta'),\gamma}_{\mathcal{I}}$ denotes type (m,θ')'s consumption vector of $\mathcal{I}$ commodities when he pretends to be type (m,θ) choosing household type γ. (Similarly, $\hat{x}^{w,\gamma}_{\mathcal{K}}$ and $\hat{x}^{(w,\tilde{\theta}'),\gamma}_{\mathcal{I}}$ denote female member's consumption vectors in household type $\gamma \in \Gamma^C$.) For $\gamma = (m,\theta,a) \in \Gamma^M$, we write $\hat{\mathbf{x}}(\gamma) = \hat{x}^\gamma = (\hat{x}^\gamma_{\mathcal{K}}; \hat{x}^\gamma_{\mathcal{I}}) \in X^m_{\mathcal{K}}(a) \times X^m_{\mathcal{I}}(a)$, and for $\gamma = (w,\tilde{\theta},\tilde{a}) \in \Gamma^W$, we write $\hat{\mathbf{x}}(\gamma) = \hat{x}^\gamma = (\hat{x}^\gamma_{\mathcal{K}}; \hat{x}^\gamma_{\mathcal{I}}) \in \tilde{X}^w_{\mathcal{K}}(\tilde{a}) \times \tilde{X}^w_{\mathcal{I}}(\tilde{a})$. *With slight abuse of notation, we will use an (extended) consumption vector $(\hat{x}^\gamma)_{\gamma \in \Gamma}$ and an (extended) commodity consumption plan $\hat{\mathbf{x}} : \Gamma^M \cup \Gamma^W \cup \Gamma^C \twoheadrightarrow \mathbb{R}^{I+K} \cup \mathbb{R}^{2K} \times (\mathbb{R}^I)^{2\Theta}$ interchangeably.*

■ *From β^γ, $\gamma \in \Gamma$, to β:* The mappings $\beta^\gamma, \gamma \in \Gamma$, compose a mapping $\beta : \Delta \times R^{2\Gamma^C} \times \hat{\mathfrak{X}}^{\Gamma^C}_{\mathcal{K}} \times Y \times H \twoheadrightarrow \hat{\mathfrak{X}}$ as follows: $\beta(p, r, \hat{\mathbf{x}}^{\Gamma^C}_{\mathcal{K}}; y, h) \equiv \Pi_{\gamma \in \Gamma^M} \beta^\gamma(p; y, h) \times \Pi_{\gamma \in \Gamma^W} \beta^\gamma(p; y, h) \times \Pi_{\gamma \in \Gamma^C} \beta^\gamma_{\mathcal{K}}(p, x^\gamma, \tilde{x}^\gamma; y, h) \times \Pi_{\gamma \in \Gamma^C} \beta^\gamma_{\mathcal{I}}(p, x^\gamma, \tilde{x}^\gamma; y, h)$. This mapping β determines the consumption allocation. Note that for all $\gamma = (m,\theta,a,w,\tilde{\theta},\tilde{a}) \in \Gamma^C$, $(\hat{x}^{m,\gamma}_{\mathcal{K}}, \hat{x}^{w,\gamma}_{\mathcal{K}}, \hat{x}^{(m,\theta),\gamma}_{\mathcal{I}}, \hat{x}^{(w,\tilde{\theta}),\gamma}_{\mathcal{I}})$ is a Pareto-optimal allocation for γ. ($\hat{x}^{(m,\theta),\gamma}_{\mathcal{I}}$ is the allocation that realizes if type (m,θ) chooses $\gamma = (m,\theta,a,w,\tilde{\theta},\tilde{a})$). An (extended) commodity consumption plan $\hat{\mathbf{x}}$ is optimal if and only if $\hat{\mathbf{x}} \in \beta(p, r, \hat{\mathbf{x}}^{\Gamma^C}_{\mathcal{K}}; y, h)$.

Sixth, we construct each type's household choice problem. To begin with, we introduce for $m \in M$ the notation $\Gamma^{C|m} \equiv \{\gamma \in \Gamma^C : \gamma = (m,\theta,a;w,\tilde{\theta},\tilde{a})$ for some $(\theta,a;w,\tilde{\theta},\tilde{a}) \in \Theta \times A \times W \times \Theta \times A\} = \{m\} \times \Theta \times A \times W \times \Theta \times A$, the set of elements of Γ^C with M-component m. Similarly, we define $\Gamma^{C|w}$ for $w \in W$, $\Gamma^{M|(m,\theta)}$ for $(m,\theta) \in M \times \Theta$, and $\Gamma^{W|(w,\tilde{\theta})}$ for $(w,\tilde{\theta}) \in W \times \Theta$.

Now male type (m,θ) can choose from $\Gamma^{C|m}$, or being single doing his best by himself. If type (m,θ) chooses to be in a couple of extended type $\gamma = (m^\gamma, \theta^\gamma, a^\gamma; w^\gamma, \tilde{\theta}^\gamma, \tilde{a}^\gamma) \in \Gamma^{C|m}$ (thus, $m^\gamma = m$ while $\theta = \theta^\gamma$ is not necessary), he can achieve $u^{(m,\theta)}(x^{(m,\theta)\gamma}_{\mathcal{I}}, x^\gamma_{\mathcal{K}}, a^\gamma; w^\gamma, \tilde{x}^\gamma_{\mathcal{K}}, \tilde{a}^\gamma)$ when the agreement assigned to γ is $(r^{m\gamma}, x^{\gamma}_{\mathcal{K}}, a^{\gamma}; r^{w\gamma}, \tilde{x}^{\gamma}_{\mathcal{K}}, \tilde{a}^{\gamma})$.

Given $\hat{\mathbf{x}} \in \hat{\mathfrak{X}}$, let

$$\begin{aligned} & U^{(m,\theta)*}(\hat{\mathbf{x}}) \\ \equiv\ & \max\left\{ \max_{\gamma \in \Gamma^{C|m}} u^{(m,\theta)}(\hat{\mathbf{x}}^{(m,\theta),\gamma}_{\mathcal{I}}, \hat{\mathbf{x}}^{\gamma}_{\mathcal{K}}, a^{\gamma}, \tilde{a}^{\gamma}), \max_{\gamma \in \Gamma^{M|(m,\theta)}} u^{(m,\theta)}(\hat{\mathbf{x}}^{\gamma}, a^{\gamma}; \emptyset) \right\}. \end{aligned}$$

$U^{(m,\theta)*}(\hat{\mathbf{x}})$ is the maximal utility male type (m, θ) can achieve under any of the possible choices $\gamma \in \Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}$. Next, we can define type (m, θ)'s household choice correspondence $\alpha^{(m,\theta)}$, by assigning to (m, θ) the extended household types $\gamma \in \Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}$ at which $U^{(m,\theta)*}(\hat{\mathbf{x}})$ is attained: Let $\alpha^{(m,\theta)} : \hat{\mathfrak{X}} \twoheadrightarrow \Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}$ be type (m, θ)'s household choice correspondence given by

$$\begin{aligned} & \alpha^{(m,\theta)}(\hat{\mathbf{x}}) \\ \equiv\ & \left\{ \gamma \in \Gamma^{C|m} : u^{(m,\theta)}(\hat{\mathbf{x}}^{(m,\theta)\gamma}_{\mathcal{I}}, \hat{\mathbf{x}}^{m,\gamma}_{\mathcal{K}}, a^{\gamma}; w^{\gamma}, \hat{\mathbf{x}}^{w,\gamma}_{\mathcal{K}}, \tilde{a}^{\gamma}) = U^{(m,\theta)*}(\hat{\mathbf{x}}_{\Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}}) \right\} \\ & \cup \left\{ \gamma \in \Gamma^{M|(m,\theta)} : u^{(m,\theta)}(\hat{\mathbf{x}}^{\gamma}, a^{\gamma}; \emptyset) = U^{(m,\theta)*}(\hat{\mathbf{x}}_{\Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}}) \right\}. \end{aligned}$$

For type $(w, \tilde{\theta}) \in W \times \Theta$, we can define $\alpha^{(w,\tilde{\theta})} : \hat{\mathfrak{X}} \twoheadrightarrow \Gamma^{C|w} \cup \Gamma^{W|(w,\tilde{\theta})}$ similarly.

The α-mappings are used to define our population mapping ν below. For type $(m, \theta) \in M \times \Theta$, let

$$\mathcal{N}^{(m,\theta)} \equiv \{\mathbf{n}^{(m,\theta)} \in \mathbb{R}_{+}^{\Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}} : \textstyle\sum_{\gamma \in \Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}} n^{(m,\theta)}_{\gamma} = N^{(m,\theta)}\}$$

be the set of population allocations of type (m, θ). Let $\nu^{(m,\theta)} : \hat{\mathfrak{X}} \twoheadrightarrow \mathcal{N}^{(m,\theta)}$ be such that $\nu^{(m,\theta)}(\hat{\mathbf{x}}) = \{\mathbf{n}^{(m,\theta)} \in \mathcal{N}^{(m,\theta)} : n^{(m,\theta)}_{\gamma} > 0 \implies \gamma \in \alpha^{(m,\theta)}(\hat{\mathbf{x}})\}$. $\nu^{(m,\theta)}(\cdot)$ ensures that the set of consumers of male type (m, θ) who choose extended household type $\gamma \in \Gamma^{C|m} \cup \Gamma^{M|(m,\theta)}$ has positive Lebesgue measure only if $\gamma \in \alpha^{(m,\theta)}(\cdot)$, that is only if the choice of γ yields the maximum utility $U^{(m,\theta)*}(\hat{\mathbf{x}})$. Observe that $\alpha^{(m,\theta)}$ has closed graph. Therefore, $\{\gamma \in \Gamma^{C|m} \cup \Gamma^{M|(m,\theta)} | \gamma \notin \alpha^{(m,\theta)}(\hat{\mathbf{x}})\}$ is locally constant and, hence, $\nu^{(m,\theta)}$ is upper hemi-continuous. We can define $\alpha^{(w,\tilde{\theta})} : \hat{\mathfrak{X}} \twoheadrightarrow \Gamma^{C|w} \cup \Gamma^{W|(w,\tilde{\theta})}$ and $\nu^{(w,\tilde{\theta})} : \hat{\mathfrak{X}} \twoheadrightarrow \mathcal{N}^{(w,\tilde{\theta})}$ similarly. Let $\nu : \hat{\mathfrak{X}} \twoheadrightarrow \Pi_{(m,\theta) \in M \times \Theta} \mathcal{N}^{(m,\theta)} \times \Pi_{(w,\tilde{\theta}) \in W \times \Theta} \mathcal{N}^{(w,\tilde{\theta})}$ be the product of $\nu^{(m,\theta)}$'s and $\nu^{(w,\tilde{\theta})}$'s. This is our population mapping. A representative element of $\nu(\hat{\mathbf{x}})$ is denoted by $\mathbf{n} \in \nu(\hat{\mathbf{x}})$.

Seventh, we introduce a supply mapping. A supply mapping $\tau : \Delta \twoheadrightarrow Y$ is such that $\tau(p) = \arg\max_{y \in Y} py$. A representative element of $\tau(p)$ is

$y \in \tau(p)$.

Eighth, we construct an excess demand mapping. An excess demand mapping $\zeta : \hat{\mathfrak{X}} \times \Pi_{(m,\theta)\in M\times\Theta}\mathcal{N}^{(m,\theta)} \times \Pi_{(w,\tilde{\theta})\in W\times\Theta}\mathcal{N}^{(w,\tilde{\theta})} \times Y \to \mathbb{R}^{\mathcal{I}\cup\mathcal{K}}$ satisfies

$$\begin{aligned}
&\zeta_i(\hat{\mathbf{x}}, \mathbf{n}, y) \\
&= \sum_{\substack{(\theta',\tilde{\theta}',\gamma) \\ \in\Theta\times\Theta\times\Gamma^C}} \sum_{\gamma\in\Gamma^C} \left\{ \sum_{\theta'\in\Theta} (\hat{\mathbf{x}}_i^{(m,\theta'),\gamma} - e_i^\gamma(a^\gamma)) n^{(m,\theta'),\gamma} + \sum_{\tilde{\theta}'\in\Theta} (\hat{\mathbf{x}}_i^{(w,\tilde{\theta}'),\gamma} - \tilde{e}_i^\gamma(\tilde{a}^\gamma)) n^{(w,\tilde{\theta}'),\gamma} \right\} \\
&\qquad + \sum_{\gamma\in\Gamma^M} (\hat{\mathbf{x}}_i^\gamma - e_i^\gamma(a)) n^\gamma + \sum_{\gamma\in\Gamma^W} (\hat{\mathbf{x}}_i^\gamma - \tilde{e}_i^\gamma(\tilde{a}^\gamma)) n^\gamma - y_i \quad \text{for all } i \in \mathcal{I},
\end{aligned}$$

and

$$\begin{aligned}
&\zeta_k(\hat{\mathbf{x}}, \mathbf{n}, y) \\
&= \sum_{\substack{(\theta',\tilde{\theta}',\gamma) \\ \in\Theta\times\Theta\times\Gamma^C}} \sum_{\gamma\in\Gamma^C} \left\{ (\hat{\mathbf{x}}_k^{m\gamma} - e_k^\gamma(a^\gamma)) \sum_{\theta'\in\Theta} n^{(m,\theta'),\gamma} + (\hat{\mathbf{x}}_k^{w\gamma} - \tilde{e}_k^\gamma(\tilde{a}^\gamma)) \sum_{\tilde{\theta}'\in\Theta} n^{(w,\tilde{\theta}'),\gamma} \right\} \\
&\qquad + \sum_{\gamma\in\Gamma^M} (\hat{\mathbf{x}}_k^\gamma - e_k^\gamma(a)) n^\gamma + \sum_{\gamma\in\Gamma^W} (\hat{\mathbf{x}}_k^\gamma - \tilde{e}_k^\gamma(\tilde{a}^\gamma)) n^\gamma - y_k \quad \text{for all } k \in \mathcal{K}.
\end{aligned}$$

Again, τ and ζ will be well defined after truncation later on. A representative element of $\zeta(\hat{\mathbf{x}}, \mathbf{n}, y)$ is denoted by $z \in \zeta(\hat{\mathbf{x}}, \mathbf{n}, y)$.

Ninth, a price mapping is a variation of the Gale-Nikaido mapping (Debreu 1959, 5.6) with a modification inspired by Mas-Colell et al. (1995, Proposition 17.C.1). A price mapping $\pi : \mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times \Delta \twoheadrightarrow \Delta$ is such that $\pi(z, p) = \arg\max_{q\in\Delta} qz$ if $p \in \mathring{\Delta}_1$, and $\pi(z, p) = \{q \in \Delta |\ qp = 0\}$ if $p \in \partial\Delta_1$. Clearly, if there is a fixed point price vector $p \in \pi(z, p)$, $p \in \mathring{\Delta}_1$ must hold. (Otherwise, $\|p\|^2 = 0$ which is a contradiction to $p \in \Delta$.)

Tenth, we consider DD's aggregate supply mapping for households with couples $\gamma \in \Gamma^C$ with externality commodity consumption plans $(x_\mathcal{K}^\gamma, \tilde{x}_\mathcal{K}^\gamma) = (\hat{\mathbf{x}}_\mathcal{K}^{m\gamma}, \hat{\mathbf{x}}_\mathcal{K}^{w\gamma})$. An externality commodity consumption plan $(x_\mathcal{K}^\gamma, \tilde{x}_\mathcal{K}^\gamma)$ in type $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a})$ household is regarded as a local public good for exactly two residents who have appearance types m and w. The provision cost is $p(\hat{\mathbf{x}}_\mathcal{K}^{m\gamma}, \hat{\mathbf{x}}_\mathcal{K}^{w\gamma}) = p_\mathcal{K} \cdot (x_\mathcal{K}^\gamma + \tilde{x}_\mathcal{K}^\gamma)$ and DD's revenue is $r^{m\gamma} + r^{w\gamma}$. Let
$H = \{h \in \mathbb{R}_+^{\Gamma^C\cup\{\emptyset\}} : \sum_{\gamma\in\Gamma^C\cup\{\emptyset\}} h^\gamma = N\}$, which will be the set of household

supply allocations. Since total population including both men and women is N, households will be certainly oversupplied. The choice $\emptyset$ assures nonnegative profit of DD. DD's supply correspondence $\eta : R^{2\Gamma^C} \times \Delta \times \hat{\mathfrak{X}}_{\mathcal{K}}^{\Gamma^C} \twoheadrightarrow H$ is such that $\eta(r^m, r^w, p; \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma}) = \arg\max_{h\in H} \sum_{\gamma\in\Gamma^C\cup\{\emptyset\}} \{(r^{m\gamma} + r^{w\gamma} - p(\hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma}))\, h^\gamma\}$, where $r^{m\emptyset} + r^{w\emptyset} = 0$ and $p_{\mathcal{K}} \cdot \left(x_{\mathcal{K}}^{\emptyset} + \tilde{x}_{\mathcal{K}}^{\emptyset}\right) = 0$.

Eleventh, we construct a variation of the Gale-Nikaido price mapping for DD as well. Let $\rho : \Pi_{(m,\theta)\in M\times\Theta}\mathcal{N}^{(m,\theta)} \times \Pi_{(w,\tilde{\theta})\in W\times\Theta}\mathcal{N}^{(w,\tilde{\theta})} \times H \twoheadrightarrow R^{2\Gamma^C}$ be such that

$$\begin{aligned} &\rho(\mathbf{n}, h) \\ = \;& \arg\max_{r\in R^{2\Gamma^C}} \sum_{\gamma\in\Gamma^C} \left[(r^{m\gamma} + r^{w\gamma}) \left\{ \left(\sum_{\theta\in\Theta} n^{(m,\theta)\gamma} - h^\gamma \right)^2 + \left(\sum_{\tilde{\theta}\in\Theta} n^{(w,\tilde{\theta})\gamma} - h^\gamma \right)^2 \right\} \right]. \end{aligned}$$

In order to have a fixed point, $\sum_{\theta\in\Theta} n^{(m,\theta)\gamma} = \sum_{\tilde{\theta}\in\Theta} n^{(w,\tilde{\theta})\gamma} = h^\gamma$ must hold.

Our fixed point mapping is $\varphi : \Phi \twoheadrightarrow \Phi$ where

$$\begin{aligned} \Phi \;=\; & \Delta \times \mathfrak{X} \times R^{2\Gamma^C} \times \prod_{(m,\theta)\in M\times\Theta} \mathcal{N}^{(m,\theta)} \times \prod_{(w,\tilde{\theta})\in W\times\Theta} \mathcal{N}^{(w,\tilde{\theta})} \\ & \times Y \times \mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times H \end{aligned}$$

and φ is composed of

$$\begin{aligned} &\beta : \Delta \times R^{2\Gamma^C} \times \hat{\mathfrak{X}}_{\mathcal{K}}^{\Gamma^C} \times Y \times H \twoheadrightarrow \hat{\mathfrak{X}}, \\ &\nu : \hat{\mathfrak{X}} \twoheadrightarrow \Pi_{(m,\theta)\in M\times\Theta}\mathcal{N}^{(m,\theta)} \times \Pi_{(w,\tilde{\theta})\in W\times\Theta}\mathcal{N}^{(w,\tilde{\theta})}, \\ &\zeta : \hat{\mathfrak{X}} \times \Pi_{(m,\theta)\in M\times\Theta}\mathcal{N}^{(m,\theta)} \times \Pi_{(w,\tilde{\theta})\in W\times\Theta}\mathcal{N}^{(w,\tilde{\theta})} \times Y \rightarrow \mathbb{R}^{\mathcal{I}\cup\mathcal{K}}, \\ &\tau : \Delta \twoheadrightarrow Y, \\ &\pi : \mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times \Delta \twoheadrightarrow \Delta, \\ &\eta : \Delta \times R^{2\Gamma^C} \times \hat{\mathfrak{X}}_{\mathcal{K}}^{\Gamma^C} \twoheadrightarrow H, \text{ and} \\ &\rho : \Pi_{(m,\theta)\in M\times\Theta}\mathcal{N}^{(m,\theta)} \times \Pi_{(w,\tilde{\theta})\in W\times\Theta}\mathcal{N}^{(w,\tilde{\theta})} \times H \twoheadrightarrow R^{2\Gamma^C}, \end{aligned}$$

To be precise,

$$\begin{aligned} &\varphi(p, r, \hat{\mathbf{x}}, \mathbf{n}, z, y, h) \\ = \;& (\pi(z,p), \rho(\mathbf{n}, h), \beta(p, r, \hat{\mathbf{x}}_{\Gamma^C}; y, h), \nu(\hat{\mathbf{x}}), \zeta(\hat{\mathbf{x}}, \mathbf{n}, y), \tau(p), \eta(p, r, \hat{\mathbf{x}})). \end{aligned}$$

We are going to truncate sets in the domain of φ to obtain compactness and to apply Kakutani's fixed point theorem. Note that the aggregate endowment is bounded above, consumption sets are bounded below, and the aggregate production set is convex having no intersection with $\mathbb{R}_+^{I+K}\backslash\{0\}$. By the standard argument, we arrived at the conclusion of Lemma 1 that the set of production vectors $y \in Y$ in all feasible allocations is bounded. Let the feasible production set be Y^f, and $\hat{Y} = \{y \in Y : \overline{y} \geq y \geq \underline{y}\}$ where $\overline{y}, \underline{y} \in \mathbb{R}^{I+K}$ with $\overline{y} \gg 0 \gg \underline{y}$ and $\overline{y} \gg Y^f \gg \underline{y}$. Clearly, $\hat{Y}$ contains Y^f in its interior relative to Y, and $\hat{Y}$ is compact and convex. Similarly, feasible aggregate consumption $\sum_{\gamma\in\Gamma^C} \mu(\gamma)\,(x^\gamma + \tilde{x}^\gamma) + \sum_{\gamma\in\Gamma^M} \mu(\gamma)x^\gamma + \sum_{\gamma\in\Gamma^W} \mu(\gamma)\tilde{x}^\gamma$ is bounded below, since consumption sets are bounded below. By (1), feasible aggregate consumption is bounded above by $\overline{b}+\overline{y}$ where $\overline{b}$ is an upper bound for the aggregate endowment. As a consequence, the set of all feasible excess demand is also bounded.

Let $Z \subset \mathbb{R}^{\mathcal{I}\cup\mathcal{K}}$ be a compact and convex set that contains all feasible excess demand vectors in its interior. In order to allow a small number of consumers consuming a large amount of private goods in an atomless economy, we follow the technique by Aumann (1966). For all $m \in M$ and all $a \in A$, let $X^{ms}(a) \equiv \{x \in X^m(a) : x \leq (s, ..., s)\}$. For each natural number s, $X^{ms}(a)$ is a compact set. Similarly, for all $w \in W$ and all $\tilde{a} \in A$, let $\tilde{X}^{ws}(\tilde{a}) \equiv \{x \in \tilde{X}^w(\tilde{a}) : x \leq (s, ..., s)\}$. We will consider an s-truncated economy with $\hat{Y}$, $(X^{ms}(a))_{m\in M, a\in A}$, and $(\tilde{X}^{ws}(\tilde{a}))_{w\in W,\tilde{a}\in A}$ for a natural number s. Thus, the space $\hat{\mathfrak{X}}$ can also be truncated as $\hat{\mathfrak{X}}^s$ accordingly. We will consider an equilibrium of the s-truncated economy, and take the limit of an equilibrium sequence for $s \to \infty$.

We need to slightly modify all the mappings for the truncated economy except for the price mapping π. The main problem is well known: If for some price p, a consumer's wealth allows for only consumption vectors on the boundary of her consumption set, then her budget correspondence may fail to be (lower hemi-) continuous, and her demand correspondence may fail to be upper hemi-continuous (a violation of Berge's maximum theorem). Note that under assumption 4 (iii-b), if $p_1 > 0$ is assured, no such problem exists: each consumer's wealth is positive. Thus, if the domain is confined to $\mathring{\Delta}_1$, the demand correspondence β is upper hemi-continuous. For other mappings involving prices, we consider two cases: (i) $p \in \mathring{\Delta}_1$ and (ii) $p \in \partial\Delta_1$. Case (ii) is the only at issue, but we simply map all $p \in \partial\Delta_1$ to the entire (compactified) range:

$$\begin{aligned}\beta(p, r, \hat{\mathbf{x}}_{\Gamma^C}; y, h) &= \hat{\mathfrak{X}}^s \\ \tau(p) &= \hat{Y} \\ \eta(p, r, \mathbf{x}) &= H\end{aligned}$$

Clearly, if β, τ, and η are nonempty-valued and upper hemi-continuous in $\mathring{\Delta}_1$, they are also nonempty-valued and upper hemi-continuous in Δ. As we have explained, a fixed point price vector of the mapping φ must lie in $\mathring{\Delta}_1$ and, thus, the above modifications of the mappings do not affect the fixed points.

Hence, the only remaining task we have in order to apply Kakutani's theorem is to show that the mapping π is upper hemi-continuous. We first prove the following lemma:

Lemma 3 *Let $(\overline{m},\bar{\theta})$ be the type described in assumption 4 of the Theorem. Let $p^n, n \in \mathbb{N}$, be a sequence in Δ with $p^n \to p \in \partial\Delta_1$, and for all $n = 1, 2, ...$, let $x^n \in \mathbb{R}_+^{\mathcal{I}\cup\mathcal{K}\backslash\mathcal{J}}$ be*

$$x^n = \arg\max \hat{u}^{(\overline{m},\bar{\theta})}(x) \text{ subject to } \sum_{\ell\in\mathcal{I}\cup\mathcal{K}} p_\ell^n x_\ell \leq B^n \text{ and } x \in X^{\overline{m}} = \mathbb{R}_+^{\mathcal{I}\cup\mathcal{K}\backslash\mathcal{J}}$$

where $B^n = \max_{a\in A} p^n e^{(\overline{m},\bar{\theta})}(a)$.[19] *Then,* $\max\{x_1^n, ..., x_{I+K-J}^n\} \to \infty$ *as $n \to \infty$.*
Moreover, for all s and all n, let

$$x^{ns} = \arg\max \hat{u}^{(\overline{m},\bar{\theta})}(x) \text{ subject to } \sum_{\ell\in\mathcal{I}\cup\mathcal{K}\backslash\mathcal{J}} p_\ell^n x_\ell \leq B^n \text{ and } x \in X^{\overline{m}s}.$$

If $x_\ell^n > s$ holds for some $\ell \in \mathcal{I}\cup\mathcal{K}\backslash\mathcal{J}$, then $x_l^{ns} = s$ for some $l \in \mathcal{I}\cup\mathcal{K}\backslash\mathcal{J}$.

Proof. Note that this type of consumer only cares about his own consumption of non-leisure goods (assumption 4 (iii)), so he always tries to take an action (a job) that maximizes his wealth for each p:

$$a \in \arg\max_{a\in A} \sum_{\ell\in\mathcal{I}\cup\mathcal{K}} p_\ell e_\ell^{(\overline{m},\bar{\theta})}(a)$$

For all $p \in \Delta$, at least one commodity has a positive price. Recalling $\cup_{a\in A} j(a) = \mathcal{J}$, assumption 4 (ii) assures that he has positive wealth for all $p \in \Delta$, even at $p \in \partial\Delta_1$. Since he can be single, he can have his maximum wealth at his disposal: thus, he spends it for non-leisure good consumption. First we consider the unbounded consumption set $X^{\overline{m}}$. Suppose that $\max\{x_1^n, ..., x_{I+K-J}^n\} \to \infty$ does not hold. Then, $\{x^n\}_{n=1}^\infty$ is contained in a compact set. Hence there exists a convergent subsequence, and we can let x be the limit point. Clearly, $\max\{x_1, ..., x_{I+K-J}\} < \infty$. However, by assumption 4 (iii-b), for large enough n, he can achieve higher utility than x by consuming unboundedly higher x_1''. This is a contradiction.

[19] Note that the optimal choice is unique since $\hat{u}^{(\overline{m},\bar{\theta})}$ is assumed to be strictly quasi-concave.

Second, consider the case where the consumption set is truncated by s. Let x^{ns} be the utility maximizing consumption vector within the truncated budget set. Suppose n and $\ell \in \mathcal{I} \cup \mathcal{K} \backslash \mathcal{J}$ are such that $x_\ell^n > s$. We will show that $x_{\ell'}^{ns} = s$ for some $\ell' \in \mathcal{I} \cup \mathcal{K} \backslash \mathcal{J}$. Suppose not, that is, $x_{\ell'}^{ns} < s$ for all $\ell' \in \mathcal{I} \cup \mathcal{K} \backslash \mathcal{J}$. By definition, $x^{ns} \neq x^n$ and $\hat{u}^{(\overline{m},\bar{\theta})}(x^n) \geq \hat{u}^{(\overline{m},\bar{\theta})}(x^{ns})$. Since the utility function is strictly quasi-concave in $X^{\overline{m}}_{\mathcal{I}\cup\mathcal{K}\backslash\mathcal{J}}$, we have $\hat{u}^{(\overline{m},\bar{\theta})}(tx^n + (1-t)x^{ns}) > \hat{u}^{(\overline{m},\bar{\theta})}(x^{ns})$ for all $0 < t < 1$. Since $x_\ell^n > s$ and $x_\ell^{ns} < s$, there is a better consumption vector in the truncated budget set. A contradiction. $\square$

This lemma implies that if $s > \max_{\ell \in \mathcal{I}\cup\mathcal{K}} \frac{\bar{b}_\ell + \overline{y}_\ell}{N^{(\overline{m},\bar{\theta})}}$, type $(\overline{m}, \bar{\theta})$ consumers alone consume more than $\bar{b}_\ell + \overline{y}_\ell$ if $x_\ell^n \to \infty$ and $p^n \to p \in \partial\Delta_1$. Now, we are ready to prove that π is upper hemi-continuous in $\mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times \Delta$ following Mas-Colell et al. (1995, Proposition 17.C.1). As we mentioned before, $\pi(z,p)$ is upper hemi-continuous in $(z,p) \in \mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times \mathring{\Delta}_1$, by Berge's maximum theorem. Thus suppose $(z,p) \in \mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times \partial\Delta_1$, $(z^n, p^n) \to (z,p)$, $q^n \in \pi(z^n, p^n)$. Since π is a compact-valued correspondence from the metric space $\mathbb{R}^{\mathcal{I}\cup\mathcal{K}} \times \Delta$ into the metric space Δ, it suffices to show that q^n has a convergent subsequence whose limit belongs to $\pi(z,p) = \{q \in \Delta | \, qp = 0\}$. Compactness of Δ implies that q^n has a convergent subsequence. Without restriction, we may assume $q^n \to q$. It remains to be shown that $q \in \pi(z,p)$, that is, $qp = 0$.

Take any $\ell \in \mathcal{I} \cup \mathcal{K}$ with $p_\ell > 0$. We shall argue that $q_\ell^n = 0$ for n large enough and thus $q_\ell = 0$ and $q \in \pi(z,p)$. Because $p_\ell > 0$, there is $\epsilon > 0$ such that $p_\ell^n > \epsilon$ for n large enough. If $p^n \in \partial\Delta_1$, then $q_\ell^n = 0$ holds by the definition of π. If $p^n \in \mathring{\Delta}_1$, then by Lemma 3, we have

$$z_\ell^n < \max\{z_1^n, ..., z_{I+K-J}^n\}$$

for n sufficiently large. Thus, $q_\ell^n = 0$ must hold again. This completes the proof of upper hemi-continuity of π.

Next, we apply Kakutani's theorem. For given s, the domain (range) of φ is nonempty, compact and convex. We know that φ is nonempty-valued, upper hemi-continuous, and convex-valued. Thus, by the Kakutani fixed point theorem, for each s-truncated economy, φ has a fixed point $(p^s, r^s, \hat{\mathbf{x}}^s, \mathbf{n}^s, y^s, h^s, z^s)$, where

$p^s \in \pi(z^s, p^s)$,
$r^s \in \rho(\mathbf{n}^s, h^s)$,
$\hat{\mathbf{x}}^s \in \beta(p^s, r^s, \hat{\mathbf{x}}^s_{\Gamma^C}; y^s, h^s)$,
$\mathbf{n}^s \in \nu(\hat{\mathbf{x}}^s)$,
$z^s = \zeta(\hat{\mathbf{x}}^s, \mathbf{n}^s, y^s)$,
$y^s \in \tau(p^s)$, and
$h^s \in \eta(p^s, r^s, \hat{\mathbf{x}}^s)$.

It remains to be shown that the fixed point yields a stable matching equilibrium. In (A)—(F) below, we shall proceed under the assumption that for all $(m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, all $\theta' \neq \theta$, and all $\tilde{\theta}' \neq \tilde{\theta}$, $\mathbf{x}^{(m,\theta')\gamma s} = \mathbf{x}^{(w,\tilde{\theta}')\gamma s} = 0$ and $\mathbf{n}^{(m,\theta')\gamma s} = \mathbf{n}^{(w,\tilde{\theta}')\gamma s} = 0$ holds. In (G), we are going to justify that assumption.

(A) Within the truncated domain, it is clear from the definition of the mapping β that $(x^{(m,\theta),\gamma s}_{\mathcal{I}}, \tilde{x}^{(w,\tilde{\theta}),\gamma s}_{\mathcal{I}}, x^{\gamma s}_{\mathcal{K}}, \tilde{x}^{\gamma s}_{\mathcal{K}})$ is intra-household Pareto-efficient for all $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$ (couples) and $x^{\gamma s}$ and $\tilde{x}^{\tilde{\gamma} s}$ are the best choices for all $\gamma \in \Gamma^M$ and $\tilde{\gamma} \in \Gamma^W$, respectively. Regarding the first claim, we observed at the end of the fourth step that $F_\gamma(p, r^{m\gamma}, r^{w\gamma}, y, h) \cap \mathcal{P}^\gamma[p, r^{m\gamma}, r^{w\gamma}; y, h](x^\gamma_{\mathcal{K}}, \tilde{x}^\gamma_{\mathcal{K}}) = \emptyset$ has to hold. In turn, this means that the indirect utilities $V^{(m,\theta')}(\cdot)$ and $V^{(w,\tilde{\theta}')}(\cdot)$ cannot be improved upon. At the fixed point, $x^{(m,\theta),\gamma s}_{\mathcal{I}}$ and $\tilde{x}^{(w,\tilde{\theta}),\gamma s}_{\mathcal{I}}$ are non-externality consumption bundles that yield those indirect utilities. The observation of footnote 20 below will complete the argument.

Now, we will show that any pair $(\widehat{m}, \widehat{\theta}; \hat{w}, \hat{\theta})$ cannot find an improving deviation from $\mathbf{x}^s \in \mathfrak{X}$ and its action choice. Suppose that there is a feasible intra-household allocation $(x', a'; \tilde{x}', \tilde{a}')$ under p^s such that both $(\widehat{m}, \widehat{\theta})$ and $(\hat{w}, \hat{\theta})$ improve. From mapping ν, all types of agents in $(M \cup W) \times \Theta$ choose their favorite household type from all possible household types. Let $U^{(\widehat{m},\widehat{\theta})*}$ and $U^{(\hat{w},\hat{\theta})*}$ be the payoffs from the choices of $(\widehat{m}, \widehat{\theta})$ and $(\hat{w}, \hat{\theta})$, respectively. Let us consider a type of household $\gamma' = (\widehat{m}, \widehat{\theta}, a'; \hat{w}, \hat{\theta}, \tilde{a}')$. Note that $u^{(\widehat{m},\widehat{\theta})}(x^{\gamma'}, a'; \hat{w}, \tilde{x}^{\gamma'}_{\mathcal{K}}, \tilde{a}') \leq U^{(\widehat{m},\widehat{\theta})*}$ and $u^{(\hat{w},\hat{\theta})}(\tilde{x}^{\gamma'}, \tilde{a}'; \widehat{m}, x^{\gamma'}_{\mathcal{K}}, a') \leq U^{(\hat{w},\hat{\theta})*}$ by construction of ν (or α). This implies $u^{(\widehat{m},\widehat{\theta})}(x^{\gamma'}, a'; \hat{w}, \tilde{x}^{\gamma'}_{\mathcal{K}}, \tilde{a}') < u^{(\widehat{m},\widehat{\theta})}(x', a'; \hat{w}, \tilde{x}'_{\mathcal{K}}, \tilde{a}')$ and $u^{(\hat{w},\hat{\theta})}(\tilde{x}^{\gamma'}, \tilde{a}'; \widehat{m}, x^{\gamma'}_{\mathcal{K}}, a') < u^{(\hat{w},\hat{\theta})}(\tilde{x}', \tilde{a}'; \widehat{m}, x'_{\mathcal{K}}, a')$.[20]

[20]By assumptions 1 and 3 (positive endowment for commodity 1, and essentiality of commodity 1), commodity 1 is consumed in positive amounts in equilibrium. Hence, if an intra-household allocation is strictly improving for one side while the other side is indifferent, then there is another intra-household allocation that improves both sides strictly.

Since $(x^{\gamma'}, \tilde{x}^{\gamma'})$ is an intra-household Pareto-efficient allocation, this is a contradiction. Thus, the fixed point matching is stable.

(B) The mapping τ assures $0 \leq p^s y^s$, since $0 \in \hat{Y}$.

(C) The mapping η assures that DD makes zero profit from each household type, since otherwise, the resulting allocation is infeasible. To see this, first note that DD can achieve a nonnegative profit by setting $h^{\emptyset} = N$. She can make a positive profit if $r^{m\gamma} + r^{w\gamma} - p(\hat{\mathbf{x}}^{m\gamma}_{\mathcal{K}}, \hat{\mathbf{x}}^{w\gamma}_{\mathcal{K}}) > 0$ for some $\gamma \in \Gamma^C$. Suppose that $\Gamma^C_+ = \{\gamma \in \Gamma^C | \, r^{m\gamma} + r^{w\gamma} - p(\hat{\mathbf{x}}^{m\gamma}_{\mathcal{K}}, \hat{\mathbf{x}}^{w\gamma}_{\mathcal{K}}) > 0\} \neq \emptyset$. Let $\Gamma^C_{++} = \{\gamma \in \Gamma^C_+ | \, h^\gamma > 0\}$. Then $\sum_{\gamma \in \Gamma^C_{++}} h^\gamma = N$ whereas $\sum_{\gamma \in \Gamma^C_{++}} \sum_{\theta \in \Theta} n^{(m,\theta)\gamma} = \sum_{\theta \in \Theta} \sum_{\gamma \in \Gamma^C_{++}} n^{(m,\theta)\gamma} \leq \sum_{\theta \in \Theta} N^{(m,\theta)} = N^M < N$, a contradiction. Hence to the contrary, DD makes zero profit.

(D) The mappings η and ρ assure that the household assignment is measurement consistent. That is, $\sum_{\theta \in \Theta} n^{(m,\theta)\gamma} = \sum_{\tilde{\theta} \in \Theta} n^{(w,\tilde{\theta})\gamma} = h^\gamma$ for all $\gamma \in \Gamma^C$. Suppose that $\sum_{\theta \in \Theta} n^{(m,\theta)\gamma} \neq h^\gamma$ or $\sum_{\tilde{\theta} \in \Theta} n^{(w,\tilde{\theta})\gamma} \neq h^\gamma$ for some $\gamma \in \Gamma^C$. Then according to ρ, $r^{m\gamma} + r^{w\gamma} = 4\bar{e}$. This yields the "infeasible" case $F_\gamma = \{0\}$, provided that the couple's budget is not augmented by a positive term $\omega^m(p; y, h) + \omega^w(p; y, h)$. If we assume that $\omega^m(p; y, h) + \omega^w(p; y, h) = 0$, then household type γ exhibits infeasibility, indeed. But then $r^{m\gamma} + r^{w\gamma} - p(\hat{\mathbf{x}}^{m\gamma}_{\mathcal{K}}, \hat{\mathbf{x}}^{w\gamma}_{\mathcal{K}}) > 0$ which has been shown to be impossible, by the argument in (C). Hence to the contrary, $\sum_{\theta \in \Theta} n^{(m,\theta)\gamma} = \sum_{\tilde{\theta} \in \Theta} n^{(w,\tilde{\theta})\gamma} = h^\gamma$ has to hold for all $\gamma \in \Gamma^C$.

In (C), we have already shown that DD's profits are zero. At the end of the proof, we will show that $py = 0$. The same argument holds for $p^s y^s = 0$ as well. Hence the conditions $\omega^m(p; y, h) = \omega^w(p; y, h) = 0$ are self-confirming: If they are assumed, then $p \cdot y = 0$ and $r^{m\gamma} + r^{w\gamma} - p(x^\gamma_{\mathcal{K}} + \tilde{x}^\gamma_{\mathcal{K}}) = 0$ for all $\gamma \in \Gamma^C$ obtain at a fixed point.

(E) Each household satisfies its budget constraint. Therefore, $p^s z^s \leq 0$.

(F) Next we show $z^s \leq 0$. We apply the argument of the Gale-Nikaido lemma (Debreu 1959, 5.6). To begin with, note that every fixed point satisfies $p^s \in \mathring{\Delta}_1$. Further note that, consequently, the mapping π yields $pz^s \leq p^s z^s$ for all $p \in \Delta$. Suppose for instance that $z^s_1 > 0$. Then, we have $e^1 z^s \leq p^s z^s \leq 0$, while $e^1 z^s = z^s_1 > 0$, where $e^1 = (1, 0, ..., 0) \in \Delta$. This is a contradiction. Thus, $z^s \leq 0$ holds for all s. With the above arguments, we have demonstrated that the fixed point is a stable matching equilibrium for all s (with $z^s \leq 0$).

(G) Concerning the assumption made in (A)—(F), note that if there is a fixed point allocation $(p^s, R^s, \hat{\mathbf{x}}^s, \hat{\mathbf{n}}^s, y^s, h^s, z^s)$, then there is another fixed point allocation $(p^s, R^s, \check{\mathbf{x}}^s, \check{\mathbf{n}}^s, y^s, h^s, z^s)$ such that for all $(m, \theta, a; w, \tilde{\theta}, \tilde{a}) \in \Gamma^C$, all $\theta' \neq \theta$, and all $\tilde{\theta}' \neq \tilde{\theta}$, $\check{\mathbf{x}}^{(m,\theta')\gamma s} = \check{\mathbf{x}}^{(w,\tilde{\theta}')\gamma s} = 0$ and $\check{\mathbf{n}}^{(m,\theta')\gamma s} =$

$\check{\mathbf{n}}^{(w,\tilde{\theta}')\gamma s} = 0$ holds. That is, the latter fixed point is consistent with a truncated stable matching equilibrium $(p^s, \mu^s, \mathbf{x}^s, y^s)$. Since $\sum_{\theta'' \in \Theta} \check{\mathbf{n}}^{(m,\theta'')\gamma s} = \sum_{\tilde{\theta}'' \in \Theta} \check{\mathbf{n}}^{(m,\tilde{\theta}'')\gamma s}$ must hold for all $\gamma \in \Gamma^C$ in a fixed point (otherwise, either $r^{m\gamma s}$ or $r^{w\gamma s}$ will be infeasible by mapping ρ), we can match up each man and each woman. Suppose that $\check{\mathbf{n}}^{(m,\theta')\gamma s} > 0$ and $\check{\mathbf{n}}^{(w,\tilde{\theta}')\gamma s} > 0$ with $\theta' \neq \theta$ and $\tilde{\theta}' \neq \tilde{\theta}$. (Other cases can be treated in the same manner.) By the construction of the α mapping, this implies

$$u^{(m,\theta')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma}, a, \tilde{a}, w) \geq u^{(m,\theta')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma'}, a, \tilde{a}, w),$$

$$u^{(w,\tilde{\theta}')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma}, a, \tilde{a}, m) \geq u^{(w,\tilde{\theta}')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma'}, a, \tilde{a}, m),$$

where $\gamma' = (m, \theta', a; w, \tilde{\theta}', \tilde{a})$. Since the mapping β assigns an intra-household Pareto efficient allocation for true types, $(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma'}, \hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma'}, a, \tilde{a}, w)$ is intra-household Pareto efficient for (m, θ') and $(w, \tilde{\theta}')$. This implies that

$$u^{(m,\theta')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma}, a, \tilde{a}, w) = u^{(m,\theta')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma'}, a, \tilde{a}, w),$$

$$u^{(w,\tilde{\theta}')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma}, a, \tilde{a}, m) = u^{(w,\tilde{\theta}')}(\hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma'}, a, \tilde{a}, m),$$

must hold because choosing $(\hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma})$ at γ' is certainly joint-budget feasible under the price vector p^s. Since each consumer has a quasi-concave utility function, any convex combination of $(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma}, \hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma})$ and $(\hat{\mathbf{x}}_{\mathcal{I}}^{(m,\theta'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{I}}^{(w,\tilde{\theta}'),\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{m\gamma'}, \hat{\mathbf{x}}_{\mathcal{K}}^{w\gamma'})$ achieves the same utility levels for both of (m, θ') and $(w, \tilde{\theta}')$ with $(a, \tilde{a}, m, w)$. Such a convex combination allocation is always budget feasible under p^s. So, these pairs can be moved to *truthful* household type $\gamma' = (m, \theta', a; w, \tilde{\theta}', \tilde{a})$ with the convex combination consumption plan by weighting population. Repeating this procedure until every type of consumer chooses households truthfully, we can create a truthful fixed point $(p^s, R^s, \check{\mathbf{x}}^s, \check{\mathbf{n}}^s, y^s, h^s, z^s)$. By omitting non-truthful arguments (assuming $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a})$, $\check{\mathbf{n}}^{(m,\theta')\gamma s} = 0$ for all $\theta' \neq \theta$, and $\check{\mathbf{n}}^{(w,\tilde{\theta}')\gamma s} = 0$ for all $\tilde{\theta}' \neq \tilde{\theta}$), we can create a (truncated) stable marriage equilibrium by letting $\mu(\gamma) = \check{\mathbf{n}}^{(m,\theta)\gamma s} = \check{\mathbf{n}}^{(w,\tilde{\theta})\gamma s}$ for all $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a})$.

(H) Finally, we enlarge s to infinity. Define an aggregate demand vector for the s-truncated economy given consumption allocation for the s-truncated economy $(\mu^s, (x^s(\gamma), \tilde{x}^s(\gamma))_{\gamma \in \Gamma^C}, (x^s(\gamma))_{\gamma \in \Gamma^M}, (\tilde{x}^s(\gamma))_{\gamma \in \Gamma^W})$. An aggregate demand vector $\underline{\mathbf{x}}^s = ((\underline{x}^s(\gamma), \underline{\tilde{x}}^s(\gamma))_{\gamma \in \Gamma^C}, (\underline{x}^s(\gamma))_{\gamma \in \Gamma^M}, (\underline{\tilde{x}}^s(\gamma))_{\gamma \in \Gamma^W})$ is such that $(\underline{x}^s(\gamma), \underline{\tilde{x}}^s(\gamma)) = (\mu^s(\gamma) x^s(\gamma), \mu^s(\gamma) \tilde{x}^s(\gamma))$ for $\gamma \in \Gamma^C$, $\underline{x}^s(\gamma) = \mu^s(\gamma) x^s(\gamma)$ for $\gamma \in \Gamma^M$, $\underline{\tilde{x}}^s(\gamma) = \mu^s(\gamma) \tilde{x}^s(\gamma)$ for $\gamma \in \Gamma^W$. Since the set of aggregate feasible consumption vectors is compact and $\mu^s \in \mathbb{R}_+^\Gamma$ such that (i) $\sum_{\gamma \in \Gamma : \gamma_{M \times \Theta} = (m,\theta)} \mu^s(\gamma) = N^{(m,\theta)}$ for all for all $\gamma = (m, \theta, a; w, \tilde{\theta}, \tilde{a})$, $(\bar{\mathcal{N}}^{\Gamma^C} \times \bar{\mathcal{N}}^{\Gamma^M} \times \bar{\mathcal{N}}^{\Gamma^W}$, the equilibrium sequence $\{\mu^s, \underline{\mathbf{x}}^s\}_{s=1}^\infty$ has a convergent subsequence. Also, the sequence $\{y^s\}_{s=1}^{\infty}$ has a convergent subsequence, since the set of feasible production plans is compact; and the sequence $\{p^s\}_{s=1}^{\infty}$ has a convergent subsequence, since Δ is compact. Without restriction of generality, we assume that the original sequences converge and $\{\mu^s, \underline{\mathbf{x}}^s, y^s, p^s\} \to \{\mu_\Gamma, \underline{\mathbf{x}}, y, p\}$. For γ with $\mu_\Gamma(\gamma) > 0$, we set $x(\gamma) = \underline{x}(\gamma)/\mu_\Gamma(\gamma)$ with $x^s(\gamma) \to x(\gamma)$ and $\tilde{x}(\gamma) = \underline{\tilde{x}}(\gamma)/\mu(\gamma)$ with $\tilde{x}^s(\gamma) \to \tilde{x}(\gamma)$. For γ with $\mu(\gamma) = 0$, there need not exist accumulation points of $\{x^s(\gamma)\}_{s=1}^{\infty}$ and $\{\tilde{x}^s(\gamma)\}_{s=1}^{\infty}$. However, in the limit, equilibrium conditions 2 and 3 are met. For example, consider the possibility that condition 2.a is not met in the limit. Then, there is a feasible plan for observable types $\overline{m}$ and $\widehat{w}$ under $p \in \Delta$, $(B', x'_{\mathcal{K}}, a'; B'', x''_{\mathcal{K}}, a'') \in \mathcal{C}[\overline{m}, \widehat{w}; p]$, such that $V^{(\overline{m},\overline{\theta})}(p, B'; x'_{\mathcal{K}}, a'; \hat{w}, x''_{\mathcal{K}}, a'') > u^{(\overline{m},\overline{\theta})}(x^\gamma_{\mathcal{I}}, x^\gamma_{\mathcal{K}}, a^\gamma; w, \tilde{x}^\gamma_{\mathcal{K}}, \tilde{a}^\gamma)$ and $V^{(\widehat{w},\widehat{\theta})}(p, B''; x''_{\mathcal{K}}, a''; \overline{m}, x'_{\mathcal{K}}, a') > u^{(\widehat{w},\widehat{\theta})}(\tilde{x}^\delta_{\mathcal{I}}, \tilde{x}^\delta_{\mathcal{K}}, \tilde{a}^\delta; m, x^\delta_{\mathcal{K}}, a^\delta)$. (By assumptions 1 and 3, both parties can be better off, if one party is strictly better off.) Then, by continuity of utility functions and Berge's maximum theorem, the same conditions must hold for $(\dot{B}, \dot{x}_{\mathcal{K}}, a'; \ddot{B}, \ddot{x}_{\mathcal{K}}, a'')$ with $(\dot{B}, \dot{x}_{\mathcal{K}}; \ddot{B}, \ddot{x}_{\mathcal{K}})$ sufficiently close to $(B', x'_{\mathcal{K}}; B'', x''_{\mathcal{K}})$. There exists s_0 such that for $s \geq s_0$, $\mu^s(\gamma) > 0$ and there exists $(\dot{B}, \dot{x}_{\mathcal{K}}, a'; \ddot{B}, \ddot{x}_{\mathcal{K}}, a'') \in \mathcal{C}[\overline{m}, \widehat{w}; p^s]$ with $(\dot{B}, \dot{x}_{\mathcal{K}}; \ddot{B}, \ddot{x}_{\mathcal{K}})$ sufficiently close to $(B', x'_{\mathcal{K}}; B'', x''_{\mathcal{K}})$. This is a contradiction to equilibrium condition 2.a in the s-truncated economy.

$z^s \leq 0$ yields $z \leq 0$ in the limit. Since $p^s y^s = \max p^s \hat{Y}$ for all s, $py = \max p\hat{Y}$. Further $y \in Y^f$ and $py \geq 0$, Y^f is contained in the relative interior of $\hat{Y}$, $\overline{y} \gg 0 \gg \underline{y}$, and Y satisfies constant returns to scale. Therefore, $py = \max pY$ and $py = 0$ must hold. With the free disposal assumption, $y^* = y + z \in Y$. With y^* instead of y, aggregate excess demand becomes $z^* = z - z = 0$. Because of non-satiation and absence of externalities in commodity 1, all households of type γ with $\mu(\gamma) > 0$ exhaust their budget. Therefore, Walras law holds for aggregate excess demand of households. Hence, $py^* = 0$, that is, y^* is a profit maximizing production plan. Thus, $(p, \mu, \mathbf{x}, y^*)$ is a stable matching equilibrium. □

References

[1] Allouch, N., J.P. Conley, and M. Wooders, 2009, Anonymous Price Taking Equilibrium in Tiebout Economies with a Continuum of Agents: Existence and Characterization, *Journal of Mathematical Economics* 45, 492-510.

[2] Alkan, A. and D. Gale, 1990, The Core of the Matching Game, *Games and Economic Behavior* 2, 203-212.

[3] Apps, P. and R. Rees, 2009, *Public Economics and the Household*, Cambridge, UK, Cambridge University Press.

[4] Aumann, R.J., 1966, Existence of Competitive Equilibria in Markets with a Continuum of Traders, *Econometrica* 34, 1-17.

[5] Azevedo, E.M. and J.W. Hatfield, 2013, Complementarity and Multidimensional Heterogeneity in Large Matching Markets, mimeo.

[6] Aygün, O. and T. Sönmez, 2013, Matching with Contracts: Comment, *American Economic Review* 103, 2050-2051.

[7] Becker, G. S., 1973, A Theory of Marriage, Part I, *Journal of Political Economy* 81, 813-846. Reproduced as Chapter 11, in R. Febrero, P.S. Schwartz (Eds.), 1995, *The Essence of Becker*, Stanford, CA, Hoover Institution Press.

[8] Bergstrom, T. C., 1997, A Survey of Theories of the Family, in Rosenzweig, M. and O. Stark, editors, *Handbook of Population and Family Economics*, Ch. 2, pp. 21-79, Amsterdam, Elsevier.

[9] Bourguignon, F. and P.–A. Chiappori, 1992, Collective Models of Household Behavior, *European Economic Review* 36, 355-364.

[10] Bourguignon, F. and P.–A. Chiappori, 1994, The Collective Approach to Household Behavior, in Blundell, R., Preston, I., and I. Walker, editors, *The Measurement of Household Welfare*, pp. 70-85, Cambridge, UK, Cambridge University Press.

[11] Che, Y.-K., Kim, J. and F. Kojima, 2014, Stable Matching in Large Economies, mimeo.

[12] Cole, H. L. and E. C. Prescott, 1997, Valuation Equilibrium with Clubs, *Journal of Economic Theory* 74, 19=39.

[13] Conley, J. P. and M. H. Wooders, 1997, Equivalence of the Core and Competitive Equilibrium in a Tiebout Economy with Crowding Types, *Journal of Urban Economics* 41, 421-440.

[14] Crawford, V. P. and E. M. Knoer, 1981, Job Matching with Heterogeneous Firms and Workers, *Econometrica* 49, 437-450.

[15] Debreu, G., 1959, *Theory of Value*, Wiley, New York.

[16] Drèze, J.H. and J. Greenberg, 1980, Hedonic Coalitions: Optimality and Stability, *Econometrica* 48, 987-1003.

[17] Ellickson, B., 1979, Competitive Equilibrium with Local Public Goods, *Journal of Economic Theory* 21, 46-61.

[18] Ellickson, B., B. Grodal, S. Scotchmer, and W.R. Zame, 1999, Clubs and the Market, *Econometrica* 67, 1185-1217.

[19] Ellickson, B., B. Grodal, S. Scotchmer, and W.R. Zame, 2001, Clubs and the Market: Large Finite Economies *Journal of Economic Theory* 101, 40-77.

[20] Gale, L. and L. Shapley, 1962, College Admissions and the Stability of Marriage, *American Mathematical Monthly* 69(1), 9-15.

[21] Gersbach, H., and H. Haller, 2010, Club Theory and Household Formation, *Journal of Mathematical Economics* 46, 715-724.

[22] Gersbach, H., and H. Haller, 2011, Groups, Collective Decisions and Markets, *Journal of Economic Theory* 146, 275-299.

[23] Gilles, R. P. and S. Scotchmer, 1997, Decentralization in Replicated Club Economies with Multiple Private Goods, *Journal of Economic Theory* 72, 363-387.

[24] Gilles, R. P. and S. Scotchmer, 1998, Decentralization in Club Economies: How Multiple Private Goods Matter, in D. Pines, E. Sadka, and I. Zilcha, editors, *Topics in Public Economics: Theoretical and Applied Analysis*, Ch. 5, pp. 121-138, Cambridge, UK, New York and Melbourne, Cambridge University Press.

[25] Gori, M., 2010, Endogenous Household Formation and Inefficiency in a General Equilibrium Model, Working Paper, Dipartimento di Matematica per le Decisioni, Università degli Studi di Firenze. Available at http://www.dmd.unifi.it/RePEc/flo/WorkingPapers/2010n/DiMaDWP2010-09.pdf

[26] Greenberg, J., 1979, Existence and Optimality of Equilibrium in Labour Managed Economies, *Review of Economic Studies* 46, 419-433.

[27] Haller, H., 1988, Manipulation of Endowments in Replica Economies: an Example, *European Economic Review* 32, 1375-1383.

[28] Hatfield, J. W. and P. R. Milgrom, 2005, Matching with Contracts, *American Economic Review* 95, 913-935.

[29] Kaneko, M., and M.H. Wooders, 1986, The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results, *Mathematical Social Sciences* 2, 105-137.

[30] Kapteyn, A. and P. Kooreman, 1992, Household Labor Supply: What Kind of Data Can Tell Us How Many Decision Makers There Are? *European Economic Review* 36, 365-371.

[31] Kelso, A. S. and V. P. Crawford, 1982, Job Matchings, Coalition Formation, and Gross Substitutes, *Econometrica* 50, 1483=1504.

[32] Konishi, H., 2010, Efficient Mixed Clubs: Nonlinear-Pricing Equilibria with Entrepreneurial Managers (JEA Nakahara Lecture, 2009) *Japanese Economic Review* 61, 35-63.

[33] Konishi, H., 2013, Entrepreneurial Land Developers: Joint Production, Local Externalities, and Mixed Housing Developments, *Journal of Urban Economics* 75, 68-79.

[34] Mas-Colell, A., 1977, Indivisible Commodities and General Equilibrium Theory, *Journal of Economic Theory* 16, 443-456.

[35] Mas-Colell, A., 1984, On Schmeidler's Theorem, *Journal of Mathematical Economics* 13, 201-206.

[36] Mas-Colell, A., M.D. Whinston, and J.R. Green, 1995, *Microeconomic Theory*, New York, Oxford University Press.

[37] McKenzie, W.L., 1959, On the Existence of General Equilibrium for a Competitive Economy, *Econometrica* 27, 54-71.

[38] Mirrlees, J.A., 1971, An Exploration in the Theory of Optimum Income Taxation, *The Review of Economic Studies* 38, 175-208.

[39] Nöldeke, G. and L. Samuelson, 2014, Investment and Competitive Matching, Working Paper, University of Basel, forthcoming in *Econometrica.*

[40] Postlewaite, A., 1979, Manipulation via Endowments, *Review of Economic Studies* 46, 255-262.

[41] Ray, D. and R. Vohra, 1997, Equilibrium Binding Agreements, *Journal of Economic Theory* 73, 30-78.

[42] Roth, A.E. and M.A.O. Sotomayor, 1990, *Two-sided Matching: A Study in Game-theoretic Modeling and Analysis*, Cambridge, Cambridge University Press.

[43] Safra, Z., 1985, Existence of Equilibrium for Walrasian Endowment Games, *Journal of Economic Theory* 37, 366-378.

[44] Samuelson, P.A., 1952, The Transfer Problem and Transport Costs: the Terms of Trade when Impediments are Absent, *The Economic Journal* 62, 278-304.

[45] Samuelson, P.A., 1954, The Transfer Problem and Transport Costs, II: Analysis of Effects of Trade Impediments, *The Economic Journal* 64, 264-289.

[46] Shafer, W., and H. Sonnenschein, 1975, Equilibrium in Abstract Economies without Ordered Preferences" *Journal of Mathematical Economics* 2, 345-348.

[47] Shapley, L.S. and M. Shubik, 1972, The Assignment Game I: The Core, *International Journal of Game Theory* 1, 111-130.

[48] Wooders, M.H., 1978, Equilibria, the Core, and Jurisdiction Structure in Economies with A Local Public Good, *Journal of Economic Theory* 18, 328-348.

[49] Wooders, M. H., 1988, Stability of Jurisdiction Structures in Economies with Local Public Goods, *Mathematical Social Sciences* 15, 29-49.

[50] Wooders, M. H., 1989, A Tiebout Theorem, *Mathematical Social Sciences* 18, 33-55.

[51] Wooders, M. H., 1997, Equivalence of Lindahl Equilibrium with Participation Prices and the Core, *Economic Theory* 9, 115-127.

[52] Zame, W.R., 2007, Incentives, Contracts, and Markets: a General Equilibrium Theory of Firms, *Econometrica* 75, 1453-1500.