Consumer Referrals

Maria Arbatskaya and Hideo Konishi

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Abstract

In many industries, firms reward their customers for making referrals. We analyze the optimal policy mix of price, advertising intensity, and referral fee for monopoly when buyers choose to what extent to refer other consumers to the firm. We find that the firm uses its referral fee, but not its price or advertising level, to manage referrals. When consumers hold correct expectations about the true quality of the product, the firm charges the standard monopoly price. The firm always advertises less when it uses referrals. We extend the analysis to the case where consumer referrals can be targeted.

Keywords: consumer referral policy, word of mouth, referral reward program, targeted advertising, product awareness.

JEL numbers: C7, D4, D8, L1.

Maria Arbatskaya, Department of Economics, Emory University, Atlanta, GA 30322-2240. Phone: (404) 727 2770. Fax: (404) 727 4639. Email: marbats@emory.edu.

Hideo Konishi, Department of Economics, Boston College, 140 Commonwealth Avenue, Chestnut Hill, MA 02467. Tel: (617)-552-1209. Fax: (617)-552-2308. Email: hideo.konishi@bc.edu.

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Firms often pay existing customers for referring potential customers to the firms’ products or services. For example, DIRECTV’s Referral Offer promises a $100 credit to any customer for referring a friend who signs up for the company’s service. Referral policies are adopted in a variety of industries, including banking, health care, web design services, home remodeling, housing, vacation packages, home alarm systems, and high-speed Internet connection. They are used in the recruitment of nurses and technicians, as well as in selling cars, houses, and tickets to sporting events. Private schools, doctors, and daycare centers give out referral bonuses as well.\footnote{A casual observation of referral policies suggests that referral rewards are usually paid out to existing customers for referring new customers who buy the product. Referral payments are typically made in the form of cash, deposit, gift certificate, bonus points, free product or service, or entry into a lottery.} Such referral programs are often seen as “Win/Win/Win” because existing customers, potential customers, and firms all benefit.

This is not surprising, given consumer referrals raise consumer awareness about the product. They can also reduce consumer uncertainty about the product’s quality or fit. For experience goods, recommendations of other people can be more informative than direct advertising because they are more trustworthy. Potential buyers trust their friends and personal acquaintances to provide honest opinions about the product.\footnote{According to Nielsen’s 2012 Global Trust in Advertising Survey of 28,000 Internet respondents from 56 countries, consumers around the world continue to find recommendations from personal acquaintances by far the most credible: 92 percent of respondents trust (“completely” or “somewhat”) recommendations from people they know and 90 percent find these recommendations (“highly” or “somewhat”) relevant. In comparison, ads are found trustworthy or relevant by 30-50 percent of respondents, depending on the media.} Oftentimes, consumers also have superior information about other consumers’ preferences. For example, when consumers belong to a social group or network, the members of that group tend to have similar tastes or simply know more about each other. Consumer referrals can then be better tar-
eted than advertising, which explains why firms often rely on consumer referrals to spread information about the existence of products and their vertical and horizontal characteristics.

This paper explores the features of a firm’s optimal referral policy. Designing an optimal referral policy is complicated by the interactions among a firm’s pricing, advertising, and referral policies. A critical part of our analysis is that we endogenize consumers’ decisions about how engaged they wish to be in making referrals. To our knowledge, no other study has taken such a comprehensive approach to developing an analytical model of consumer referrals.

In our base model, we introduce consumer referrals into an experience good market served by a monopoly. Some consumers can become informed about the existence of the product and its price directly from the firm’s advertisements. These "informed" consumers decide whether or not to purchase the product based on the expected product quality. Consumers who purchase the product recognize its true quality and decide to what extent to refer other consumers, sharing information about the true product quality. The firm’s referral policy provides a monetary reward (referral fee) for each successful referral. Consumers can make multiple referrals at a constant marginal cost. Since referrals are sent independently and at random, in equilibrium, there is congestion in referral messages. The firm can manage referral incentives in our model by changing its policy mix (price, advertising intensity, and referral fee).

In this framework, our main question is whether the firm would set a higher or lower price in the presence of consumer referrals. On the one hand, the referral fee adds to the marginal cost of selling the product, which prompts the firm to raise its price. On the other hand, a higher price reduces the purchase probability, diminishing referral incentives. It is
therefore not clear in which direction the optimal price would move. We also answer the following questions: when would a firm use consumer referrals, would it engage in more or less advertising under referrals, and what are the overall welfare effects of referral policies?

We first characterize the consumer referral equilibrium for any finite number of referring consumers and any policy mix chosen by the firm (Proposition 1). By considering a large population of consumers, we then analyze the firm’s optimal policy and, in particular, its pricing strategy. We find that the profit-maximizing price is the monopoly price for the mixture of consumers it faces, as long as the referral fee is optimally chosen (Proposition 2). In particular, when consumers hold correct expectations about the true quality of the product, the firm charges the standard monopoly price (Corollary 1). The firm uses its referral fee to manage the referral activity and its price to maximize the profitability of sales to consumers aware of its product. We show that when the referral fee is set below the optimal level or the marginal referral cost is increasing, the firm sets its price below the monopoly level. We also show that the firm always advertises less when it uses consumer referrals.

We provide comparative static results for the optimal policy mix (Proposition 3) and show that the firm chooses to use referrals as long as the referral cost is not too high (Proposition 4). We also explore if a referral policy would be pre-announced by the firm as a part of its advertising message and find that it would not do so because some consumers delay their purchases waiting for recommendations to resolve the uncertainty about the product quality (Proposition 5).

Naturally, consumers may have better information than the firm about other consumers’ valuations for the product. This informational advantage allows them to target their re-
ferrals. To study targeted referrals, we assume that there are two groups of consumers: high-type and low-type. High-type consumers tend to have higher valuations than low-type consumers. Although the firm cannot tell which group consumers belong to, consumers can. If willingness-to-pay distributions are significantly different across groups, then only high-type consumers receive referrals, the price is higher, the ratio of referral fee to profit margin is lower, and the advertising level is lower under targeted referrals than in the case of consumers unable to tell which group others belong to (Proposition 6). Quite intuitively, if consumers have better information, the monopoly relies less on advertising and more on referrals. The proofs not found in the text are in the appendix.

2 Brief Literature Review

A few streams of literature are relevant to our model. First, there is the literature strictly on consumer referrals: Jun and Kim (2008), Byalogorsky et al. (2005), and Galeotti and Goyal (2009). Jun and Kim (2008) assume a finite chain of consumers with i.i.d. random valuations. Consumers are rational and forward-looking - they consider the expected benefit from giving a referral when making their purchase decisions. The authors show that even though the firm sets a common price and referral fee, it effectively price-discriminates between the consumers located early in the chain (who are more valuable to the monopoly) and those later in the chain. Byalogorsky et al. (2005) take the same setup as Jun and Kim (2008), but adopt a behavioral assumption that consumers make referrals whenever the expected utility from making a referral exceeds a critical level of "consumer delight." When consumers are easy to

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3 Arbatskaya and Konishi (2014) justify the tie-breaking rule used in the paper, showing that effective price discrimination is indeed a common feature of the model in the second-best environment (with a common referral fee and same price for all consumers) and the first-best environment (with a sufficient number of policy tools).
delight, a referral program would not be used because referrals would be made even without it. But when consumers are not so easy to delight, the firm would use both a positive referral fee and a lower price. These papers rule out referral congestion.\footnote{The papers do not consider advertising as an alternative communication channel. In contrast, Mayzlin (2006) looks at the case where advertising and word of mouth are both used to influence consumer choices between vertically differentiated products. While in her model consumers cannot distinguish between promotional chat and consumer recommendations, in our model, advertising and referrals are two distinct information channels.} In contrast, Galeotti and Goyal (2009) consider a more complex network model in which consumers make multiple referrals with no cost. They analyze the optimal advertising policy and show that using consumer referrals would unambiguously increase profits. At the same time, an increase in the level of social interaction can increase or decrease the level of advertising and profits. While they concentrate on the relationship between network structure and optimal advertising strategy, we assume a simple complete network and analyze the optimal policy mix for the firm when consumer referral decisions are determined endogenously.

The second stream of literature focuses on advertising and congestion. In his pioneering paper, Butters (1977) formulated a \textit{competitive} model of advertising in which firms send a number of ads to consumers randomly, informing them about the existence of the product and its price. Butters shows that price dispersion occurs in equilibrium. In his model, some portion of ads are wasted due to congestion, but the level of congestion (the number of ads) is socially optimal. Van Zandt (2004), Anderson and de Palma (2009), and Johnson (2013) present alternative information congestion models in which consumers ignore some advertisements they receive.\footnote{Van Zandt (2004) assumes that all consumers can process up to a certain number of ads, while Anderson and de Palma (2009) assume that a consumer’s cost of processing ads depends on the number of ads she receives. Johnson (2013) allows consumers to decide what fraction of ads to block.} They all show that, as the number of ads decreases, both firms and consumers are better off due to a reduction in congestion, though the reasons for
this result differ. In contrast, in our model, referrals are subject to congestion because we endogenize the referral intensity. Despite the presence of referral congestion, referrals are underprovided in our monopoly model.

There is also literature on targeted advertising. Van Zandt (2004) and Johnson (2013) assume that firms sell heterogeneous products and have some information about consumer preferences. They analyze targeted advertising policies in oligopolistic markets (Van Zandt, 2004) and competitive markets (Johnson, 2013). Although the mechanisms are different, both papers show that improved targeting increases firms’ profits and makes consumers better off. Esteban et al. (2001) consider a monopoly choosing between mass and targeted advertising. With targeted advertising, the number of wasted ads is reduced, but the monopoly power increases. The authors show that the latter welfare loss tends to exceed the former benefit. Galeotti and Moraga-Gonzalez (2008) analyze a simple oligopolistic model of targeted advertising and show that market segmentation generates higher profits in equilibrium. These papers assume that firms possess information on consumer types and therefore can conduct targeted advertising. In contrast, we assume that consumers have superior information and the firm uses them as sales agents.

3 The Model of Consumer Referrals

Each of $N$ consumers purchases at most one unit of a product. A consumer’s utility from the product is the sum of a common match (quality) parameter $\mu$ and an idiosyncratic value $v$, net of the price $p$ of the product: $u \equiv \mu + v - p$. Consumers’ idiosyncratic values $v$ follow a known distribution function $G(\cdot)$, with a log-concave survival function $1 - G(\cdot)$ and a continuously differentiable density $g$, defined on $[v, \overline{v}]$ with $\overline{v} \geq 0$. Product quality $\mu$ is ex
ante unknown to consumers. We assume that consumer beliefs about product quality follow a known distribution function $F$, defined on $[\mu, \bar{\mu}]$ with $\mu \geq 0$. We denote by $\mu^e$ the expected quality of the product, $\mu^e \equiv \int_{\mu}^{\bar{\mu}} \mu dF(\mu)$. After purchasing the product, a consumer realizes the value of $\mu$.

The firm is choosing its (non-discriminatory) price $p \geq 0$, its level of advertisement $a \in [0, 1]$ (a fraction of consumers reached by advertisements), and a referral policy characterized by a referral fee $r \geq 0$. The marginal cost of production is $c \geq 0$. Advertisements inform consumers about the existence of the firm’s product and its price, while referrals also inform them about the true quality of the product. Only consumers who are informed of the product through an advertisement ("the informed") or a referral ("the fully informed") can purchase the product. In the base model, we assume that the firm informs consumers about its referral program only after they purchase the product. Then, the informed consumers purchase the product whenever their values $v$ satisfy $v + \mu^e \geq p$, which happens with probability $d(p; \mu^e) = 1 - G(p - \mu^e)$.

Consumers can attempt to collect referral fees by referring other people. The number of referrers is denoted by $n$. The expected value of $n$ is $ad(p; \mu^e) N$ because a consumer needs to receive an ad and purchase the product to be able to refer. Each referral attempt costs

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6To avoid issues with signaling a product’s quality, we assume that the marginal cost of production $c$ is the firm’s private information and that consumers do not update their belief about $\mu$ upon receiving an advertisement. In Section 7.5, we consider the case where the firm itself is not aware of the true quality $\mu$.

7We assume that referrers honestly inform others about the true product quality. In our model, consumers indeed do not have an incentive to oversell the product to others because, in the referral equilibrium, they obtain a zero expected net referral benefit.

8For example, a daycare would send the following message to its current families: "You may or may not be aware that you can earn a free week of childcare by referring a family to our program." This assumption implies that consumers who become informed through ads do not anticipate participating in a referral program. The assumption prevents consumers who receive ads from delaying the purchase until the uncertainty in $\mu$ is resolved by a referral. In an extension to our model, we show that the firm may indeed prefer not to announce its referral policy as a part of its advertising message.
which captures the cost of informing a contact about the product and its quality \( \mu \). On the benefit side, referral attempts can be successful or unsuccessful. If a referrer’s contact has a low willingness-to-pay and/or is already informed, the referral attempt will not be successful. Furthermore, potential referrals may have been contacted by others and may assign credit for the referral to another person.

Referrers simultaneously and independently choose the probability \( q \in [0, 1] \) of sending a referral to other consumers at random, but without contacting the same person more than once.\(^9\) As more referrals are sent out, an increasingly smaller fraction of referrals are successful. The referral reach \( R \) is the fraction of all consumers reached by referrals. When each of \( n \) consumers refers a given consumer (who is not a referrer) with probability \( q \), the reach is described by \( R = 1 - (1 - q)^n \). The per-consumer number of referrals sent by \( n \) referrers is called the referral intensity \( S = nq \). We define referral congestion as the expected ratio of the number of referral messages sent by all referrers to the uninformed to the expected number of referrals registered by them:

\[
\phi(q; n) \equiv \frac{S(q; n)}{R(q; n)} = \frac{nq}{1 - (1 - q)^n} > 1.
\]  

For any \( q > 0 \), there is congestion in referral messages and \( \phi(q; n) > 1 \).

In the symmetric consumer referral equilibrium, each of \( n \) referrers suggests the product to another consumer with probability \( q^E \). To find the equilibrium \( q^E \), we need to look at the incentives of consumers to refer. Denote by \( d^R \) the probability that a consumer who receives a referral buys the good:

\[
d^R = d^R (p; a; \mu, \mu^e) = a \max \{ d(p; \mu) - d(p; \mu^e), 0 \} + (1 - a) d(p; \mu).
\]  

\(^9\)In the base model, we assume that a referrer does not know the willingness-to-pay of other people. We study targeted referrals in Section 6 and Section 7.3.
The first term in (2) corresponds to consumers having received an ad, who did not initially purchase the product based on the expected quality $\mu^e$, but purchased it upon receiving a referral that revealed a higher than expected quality, $\mu > \mu^e$. The second term captures consumers who have not received an ad.

**Proposition 1.** Suppose the firm chooses a policy mix of price $p$, advertising intensity $a$, and referral fee $r$. Then, for any number of referrers $n$, the equilibrium probability of consumer referral $q^E$ is uniquely determined by

$$rd^R(p, a; \mu, \mu^e) = \rho \phi(q^E; n)$$

(3)

for all $r$ above the critical level $r_0 \equiv \rho/d^R$; no referrals are sustained for lower levels of the referral fee. For $r > r_0$, the equilibrium referral probability $q^E$ and referral congestion $\phi$ increase when the referral fee $r$ and product quality $\mu$ increase and when price $p$, advertising intensity $a$, and referral cost $\rho$ decrease.

In the equilibrium, each referring consumer is indifferent between sending and not sending an additional referral. An interesting observation from this fact is that, even if there was another period for referrals to be made after the initial referral market clears, no consumer would make an additional referral. Moreover, due to constant referral cost $\rho$, the expected net benefit from making each referral is zero for any referrer and at the aggregate level. We explore the referral equilibrium in the case of increasing marginal cost of referral as one of the extensions in Section 7.1.

The comparative statics results of Proposition 1 are intuitive. The factors that increase the benefit of making referrals (a higher referral fee $r$, higher product quality $\mu$, lower price $p$, or lower advertising intensity $a$) or reduce referral cost (lower referral cost $\rho$) must
increase referral probability and congestion in order for consumers to remain indifferent between referring and not referring. In the proof of Proposition 1, we show that, quite intuitively, congestion increases in $q$ and $n$; therefore, the equilibrium referral intensity $q^E$ is also negatively affected by the number of referring consumers $n$.

4 Monopoly Choice of Price, Advertising, and Referral Policy

In this section, we characterize the optimal (profit-maximizing) monopoly policy mix and derive conditions under which a firm would choose to support consumer referrals. The cost of advertising per consumer is described by function $C(a)$, which increases at an increasing rate in the fraction $a$ of consumers reached, $C'(a) > 0$ and $C''(a) > 0$. To guarantee the interior solution for advertising intensity in the presence of referrals, we additionally assume that $C'(0) < \rho$ and $C(a)$ is sufficiently convex: $C''(a) > \rho / (1 - a)^2$.

From this section on, we assume that $N$ is a large number, which allows us to obtain a very useful approximation argument (Judd, 1985). By the law of large numbers, there are $n = ad(p; \mu^E) N$ consumers who purchase the product and make referrals. As $N$ grows large, $n$ grows large as well, and $q^E$ goes down to zero.\textsuperscript{10} Thus, with a large $N$, we can use an approximation $R^E = 1 - (1 - q^E)^n \approx 1 - e^{-nq^E} = 1 - e^{-S^E}$ because $\ln(1 - q)n = n \ln(1 - q) \approx -nq$ for large $N$ and $S^E = nq^E$. This approximation argument is used in the presentation of the advertisement model by Butters (1977) in Tirole (1988). Inverting this relationship, we can then write referral intensity and congestion as functions of $R$ only:

\textsuperscript{10}To see that $q^E \to 0$ as $n \to \infty$, note that from Proposition 1, the equilibrium congestion $\phi^E = \phi(q^E; n) = nq^E(n)/ (1 - (1 - q^E(n))^n)$ remains constant as $n$ changes. From $\phi^E \geq nq^E(n)$, it follows that $q^E(n) \leq \phi^E/n$. Since $\phi^E/n$ converges to zero as $n$ goes to infinity, so does $q^E(n)$. The (expected) number of referrals each consumer sends is $k^E \simeq Nq^E$, which does not approach zero as $q^E$ approaches zero.
\[ S = S(R) = -\ln(1 - R) \text{ and } \varphi = \varphi(R) = \frac{S}{R} = -\left(\ln(1 - R)\right)/R. \]

Summarizing the above, we have the following useful lemma.

**Lemma 1.** Suppose that \( N \) is large. The referral congestion can then be written as a function of referral reach only: \( \varphi(R) = -\left(\ln(1 - R)\right)/R \), where \( \varphi'(R) > 0 \).

Throughout the rest of the paper we will assume that \( N \) is a large number and, therefore, the equilibrium referral congestion is independent of the number of referrers \( n \). In particular, for any firm’s policy mix \((p, a, r)\), the equilibrium referral congestion \( \phi(q^E; n) \) can be written as a function of only the equilibrium referral reach \( R^E \): \( \varphi(R^E) \).

We can then use Proposition 1 to describe the equilibrium referral reach \( R^E \) as a function of policy variables \( p, a, \) and \( r \): \( R^E = R^E(p, a, r) \). The firm can achieve a higher equilibrium referral reach when it sets a lower price, advertises less, offers a higher referral fee, and has a higher product quality. That is, \( \frac{\partial R^E}{\partial p} < 0, \frac{\partial R^E}{\partial a} < 0, \frac{\partial R^E}{\partial r} > 0, \text{ and } \frac{\partial R^E}{\partial \mu} > 0 \). The referral reach is also higher when the referral cost is lower: \( \frac{\partial R^E}{\partial \mu} < 0 \).

The firm’s per-consumer profit is:

\[ \Pi(p, a, r; \mu, \mu^e) = a(p - c)d(p; \mu^e) + R^E(p - c - r)d^R(p, a; \mu, \mu^e) - C(a), \quad (4) \]

where \( d^R \) is defined in (2) and \( R^E = R^E(p, a, r) \). The first term captures profits from consumers who purchase after receiving an ad and the second one from consumers who purchase the product by referrals.

Let \( \pi(p, \mu^e) = (p - c)d(p; \mu^e) \) be the profitability of a sale to a consumer who expects product quality \( \mu^e \). We can prove the following useful properties of this function under the assumption of log-concave survival function \( 1 - G(\cdot) \).
Lemma 2. Function $\pi(p; \mu)$ has a unique profit-maximizing price $p^m(\mu)$ that is increasing in $\mu$. Moreover, $\frac{\partial \pi(p; \mu)}{\partial p} \geq 0$ for $p \leq p^m(\mu)$ holds for all $\mu$.

Let $p^m(\mu^e)$ and $p^m(\mu)$ denote the monopoly prices that maximize the per-consumer profits $\pi(p, \mu^e)$ and $\pi(p, \mu)$ from the informed and the fully informed consumers. The equilibrium referral condition (3) of Proposition 1 implies that

$$rd^R R^E = \rho S(R^E).$$

(5)

This permits us to rewrite the firm’s profit as:

$$\Pi(p, a, r; \mu, \mu^e) = a\pi(p; \mu^e) + R^E \pi^R(p, a; \mu, \mu^e) - \rho S(R^E) - C(a),$$

(6)

where

$$\pi^R = \pi^R(p, a; \mu, \mu^e) = (p - c) a^R(p, a; \mu, \mu^e) = (1 - a) \pi(p; \mu) + a \max \{\pi(p; \mu) - \pi(p; \mu^e), 0\}$$

(7)

is the profitability of a referral consumer. Absent referrals, the firm’s profit is $\Pi^0(p, a; \mu^e) = a\pi(p; \mu^e) - C(a)$. The firm sets its price $p^m(\mu^e)$ and advertising level $a^0*$, where $a^0*$ is the solution to the first-order condition: $\frac{d\Pi^0}{da} = \pi(p; \mu^e) - C'(a) = 0$.

We next investigate the optimal policy mix for the firm that uses referrals. Note that the firm’s profit in (6) is affected by referral $r$ only through referral reach $R^E = R^E(p, a, r)$. The first order condition with respect to $r$ is, therefore,

$$\frac{d\Pi}{dr} = \frac{\partial \Pi}{\partial R} \frac{\partial R^E}{\partial r} = 0,$$

(8)

with

$$\frac{\partial \Pi}{\partial R} = \pi^R(p, a; \mu, \mu^e) - \rho S'(R).$$

(9)
Since \( \frac{\partial R^E}{\partial r} > 0 \) for \( r > r_0 \), the referral reach \( R \) is optimized under the optimal \( r^* \): \( \frac{\partial \Pi}{\partial R} = 0 \). Then, the indirect effects of \( p \) and \( a \) on profits through their effects on \( R^E \) are zero. This dramatically simplifies our analysis: as long as the referral fee is optimally set, the first order conditions for \( p \) and \( a \) set partial derivatives with respect to \( p \) and \( a \) to zero: \( \frac{\partial \Pi}{\partial p} = \frac{\partial \Pi}{\partial a} = 0 \) and \( \frac{\partial \Pi}{\partial a} = \frac{\partial \Pi}{\partial a} = 0 \).

The profitability of a referred consumer depends on whether consumers are optimistic about product quality (\( \mu \leq \mu^e \)) or pessimistic about it (\( \mu > \mu^e \)). Using (6) and (7), we can write the firm’s profit as:

\[
\Pi(p, a, r; \mu, \mu^e) = \begin{cases} 
    a \pi(p; \mu^e) + (1 - a) R^E \pi(p; \mu) & \text{if } \mu \leq \mu^e \\
    a (1 - R^E) \pi(p; \mu^e) + R^E \pi(p; \mu) & \text{if } \mu > \mu^e
\end{cases}
\]

We arrive at the following Proposition.

**Proposition 2.** Provided that the firm chooses its referral fee optimally, it sets its price \( p^* \) between the monopoly price for the informed consumer \( p^m(\mu^e) \) and the fully informed consumer \( p^m(\mu) \). The firm advertises less when it uses referrals.

This proposition says that the optimal price \( p^* \) is pulled away from the monopoly price for the informed consumers \( p^m(\mu^e) \) towards the monopoly price for the fully informed consumers \( p^m(\mu) \). The key observation here is that the optimal price is essentially the monopoly price for the mix of consumers the firm faces. It is significant that we can separate the pricing decision from the consumer referral considerations when the referral fee is optimally set. In such a case, the firm can ignore the effect of its price on referral reach and simply use the price to maximize the profitability of sales to consumers who are informed by ads and/or referrals. If we assume no uncertainty in product quality (\( \mu^e = \mu \)), then we have the following statement.
Corollary 1. Suppose that the firm sets the referral fee optimally. If \( \mu^e = \mu \), then the optimal price \( p^* \) is exactly the same as the monopoly price \( p^m(\mu) \).

What if for some reason \( r \) cannot be optimally set? Suppose that there is a cap on \( r \), but the cap is higher than \( r_0 \) so that referrals still occur. Then, the resulting referral reach is less than the optimal level, i.e. \( \frac{\partial R}{\partial R} > 0 \). Since \( \frac{\partial p^E}{\partial p} < 0 \) holds, the optimal price \( p \) is set below the monopoly price in this case. Similarly, the optimal advertising level \( a \) is lower than the one when there is no cap on \( r \). In Section 7.1, we analyze the case where the marginal cost of making referrals increases with an increase in the number of referrals made. In this case, the optimal price would also be less than the monopoly price as a lower price implies more referrers and lower total referral costs.

Proposition 3 describes the comparative static responses of the optimal monopoly policy mix \((p^*, a^*, r^*)\), assuming that \( \frac{\partial^2 \pi}{\partial p \partial \mu} > 0 \) holds at \((p^*, \mu)\) and \((p^*, \mu^e)\).¹¹

Proposition 3. Suppose that \( \frac{\partial^2 \pi}{\partial p \partial \mu} > 0 \) holds at \((p^*, \mu)\) and \((p^*, \mu^e)\). Assuming the regular optimum, the comparative statics results on the optimal policy \((p^*, a^*, r^*)\) are summarized in Table 1:

| \( \mu \leq \mu^e \) | \( \frac{dp^*}{dp} \) | \( \frac{da^*}{dp} \) | \( \frac{dR^*}{dp} \) | \( \frac{d\left( \frac{\pi^*}{p^* - \pi} \right)}{dp} \) | \( \frac{dp^*}{d\mu} \) | \( \frac{da^*}{d\mu} \) | \( \frac{dR^*}{d\mu} \) | \( \frac{d\left( \frac{\pi^*}{p^* - \pi} \right)}{d\mu} \) | \( \frac{dp^*}{da^*} \) | \( \frac{da^*}{da^*} \) | \( \frac{dR^*}{da^*} \) | \( \frac{d\left( \frac{\pi^*}{p^* - \pi} \right)}{da^*} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \mu > \mu^e \) | + | + | - | - | ? | ? | ? | + | + | + | + | + |

It is natural that when referral cost \( \rho \) goes up, the resulting equilibrium referral reach \( R^* \) decreases. It is also natural that the optimal advertising intensity \( a^* \) increases since referrals

¹¹Note that the log concavity of \( 1 - G(\cdot) \) implies \( \frac{\partial^2 \pi(p; \mu)}{\partial p \partial \mu} > 0 \) only at \((p, \mu)\), satisfying \( p = p^m(\mu) \) (see the proof of Lemma 2). We need more than that to obtain the comparative statics results of Table 1.
and ads are alternative information channels, even though the content of advertisements and referrals is not the same.

The response of the optimal price $p^*$ to an increase in $\rho$ needs more explanation. From (10), the marginal profit $\frac{\partial \pi}{\partial p}$ is a weighted sum of the marginal profit from an informed consumer $\frac{\partial \pi(p;\mu^e)}{\partial p}$ and a fully informed consumer $\frac{\partial \pi(p;\mu)}{\partial p}$. If $\mu < \mu^e$, then by Lemma 2 and Proposition 2, $p^m_\mu(p) < p^* < p^m_\mu(\mu^e)$, $\frac{\partial \pi(p;\mu)}{\partial p} < 0$, and $\frac{\partial \pi(p;\mu^e)}{\partial p} > 0$. Since a higher $\rho$ results in a lower $R$ and higher $a$, the firm faces relatively more informed consumers than fully informed consumers (10), and the optimal price must increase. By similar arguments, the price decreases in case $\mu > \mu^e$.

Notice also that if price were fixed, then a higher $\mu$ or lower $\mu^e$ would have comparative statics results that do not depend on the relationship between $\mu$ and $\mu^e$. Intuitively, the higher the product quality is relative to the expected quality, the more the firm relies on referrals rather than advertising. Adjustments of price, which depends on whether $\mu$ is higher than $\mu^e$ and on the signs of cross-partial derivatives of profit function with respect to $p$ and $\mu$, complicate matters through the interactions of price $p$ with $a$ and $R$. The ambiguity in the effects of changes in $\mu$ makes it harder for consumers to guess the quality of good $\mu$ based on the observed firm’s choices. The sign of the comparative statics results with respect to the marginal cost of production $c$ also cannot be determined in general.

Proposition 4 provides a necessary and sufficient condition for the firm to use consumer referrals.

**Proposition 4.** For any given $p$ and $a > 0$, the firm supports consumer referrals if and only if $\rho < \overline{\rho}$, where $\overline{\rho} = \pi^R(p; a; \mu, \mu^e)$. 

The profitability of the referral consumer, $\pi^R = \pi^R (p, a; \mu, \mu^e)$ is determined in (7). Note that the threshold level for referral cost $\overline{p}$ is higher when advertising $a$ is low, product quality $\mu$ is high and the expected quality $\mu^e$ is low provided $\mu > \mu^e$. That is, referral policy adoption is more likely when the product is of higher quality, the firm advertises less, and consumers are pessimistic about product quality.

**Corollary 2.** The firm supports consumer referrals if the referral cost is sufficiently small, $\rho < \rho_0$, where $\rho_0 = \pi^R (p^m (\mu^e), a^0; \mu, \mu^e)$.

The result is intuitive. A firm that has no referral policy can improve its profits by introducing a referral policy with a referral fee $r \in (r_0, p^m (\mu^e) - c)$, while keeping its price $p^m (\mu^e)$ and advertising $a^0$ at the same level. Such a referral fee exists when $\rho$ is sufficiently low: $\rho < \rho_0$. The firm then earns additional profits from consumers purchasing by referral.

5 Advertising Referral Program: Dynamic Considerations

So far we have assumed that consumers are not aware of the firm’s referral program until they purchase the product. However, some firms openly advertise their referral programs. This generates an entirely new type of consumer behavior: consumers may delay purchases while waiting for a referral that would resolve uncertainty about product quality.

To analyze the consumers’ strategic incentives to wait for recommendations, we assume that there are two periods, and consumers and the firm discount future benefits and costs using a common discount factor $\delta \in (0, 1)$. In the beginning of period 1, a subset of consumers receive ads containing information about the firm’s price and its referral policy. Out of these consumers, some purchase the product and then make referrals to other consumers. In the
beginning of period 2, consumers who receive referrals become fully informed. Then, all 
consumers who are aware of the product and have not yet purchased it have a chance to 
purchase the product.

Suppose that a consumer with value \( v \geq p - \mu^e \) receives an ad. If she purchases the 
product right away, her expected utility is:

\[
U^p(v) = v + \mu^e - p = E(u),
\]

(11)

where \( \mu^e = E(\mu) \).

If she delays purchasing until period 2, she can make a more informed decision if she 
obtains a referral. Her expected utility is then:

\[
U^d(v) = \delta (1 - R) E(u) + \delta R \times \Pr(u \geq 0) \times E(u|u \geq 0).
\]

(12)

The difference in the expected utility from purchasing and waiting is:

\[
U^p(v) - U^d(v) = (1 - \delta) (v + \mu^e - p) - \delta R \times \Pr(\mu \geq p - v) \times [E(\mu|\mu \geq p - v) - \mu^e],
\]

(13)

where the first term is due to the discounting of future purchases and the second term is due 
to the information gained from obtaining a referral.

As we show in Lemma 3, there exists a unique \( v^* = v^*(p, \delta, R) \), such that consumers 
with values \( v > v^* \) buy immediately because \( U^p(v) > U^d(v) \), and consumers with values 
\( v < v^* \) delay their purchase decisions because \( U^p(v) < U^d(v) \). The value of the person who is 
indifferent between purchasing and waiting \( v^* \) is implicitly determined by \( U^p(v^*) - U^d(v^*) = 0 \).

**Lemma 3.** Suppose that \( \mu^e + \bar{v} \leq p \leq \mu^e + \tilde{v} \). Then, there exists a unique consumer value 
\( v^* \in (p - \mu^e, p) \), such that consumers whose value \( v < v^* \) delay their purchases and \( \frac{\partial v^*}{\partial p} = 1 \),
\[ \frac{\partial v^*}{\partial \delta} > 0, \quad \frac{\partial v^*}{\partial R} > 0, \quad \text{and} \quad \frac{\partial v^*}{\partial \mu} < 0. \]

Quite intuitively, as the discount factor \( \delta \) increases, more consumers choose to wait for referrals. As referral reach \( R \) increases, consumers have a higher chance of receiving a referral, which encourages them to wait. A price increase lowers the expected utility of buying the product and discourages purchases. And as \( \mu^c \) increases, fewer consumers wait to receive referral information.

We will now show that it is not profitable for the firm to announce its referral program to consumers. We denote by \( \Pi^d = \Pi^d(p, a, r) \) the profit of the firm when it advertises its referral program and by \( (p^{dx}, a^{dx}, r^{dx}) \) the associated profit-maximizing policy mix. The firm’s profit comes from four consumer segments: i) consumers who receive an ad and purchase in period 1; ii) consumers who receive an ad, delay their purchase, receive no referral, but buy anyway in period 2; iii) consumers who receive an ad, delay their purchase, receive a referral, and buy by referral in period 2; and iv) consumers who do not receive an ad, receive a referral, and buy by referral in period 2.

We first prove the following lemma. Suppose that the firm chooses an arbitrary policy combination and announces its referral program as a part of its advertising message. We show that the firm can earn more profit by not advertising the referral policy. If \( \mu \leq \mu^c \), then not advertising is simply better for the firm without a change in the policy combination. However, if \( \mu > \mu^c \), then consumers’ purchase delays make the referral market more vital, thus increasing the referral reach \( R^E \), and it may increase the firm’s sales in comparison with not advertising the referral program. Therefore, if the policy mix is kept the same, the firm may earn more profit by advertising the referral program. However, even in this case, the
firm can achieve the same effect by merely reducing \( a \) and it can earn more profit by not advertising the referral program.

**Lemma 4.** Suppose that the firm chooses a policy combination and advertises its referral policy. There is another policy combination for which the firm can earn a higher profit while not advertising its referral policy.

The arbitrary policy combination can be \( (p^{d*}, a^{d*}, r^{d*}) \). Thus, we have the following statement.

**Proposition 5.** Announcement of a referral program as part of an advertising message reduces the firm’s profit:

\[
\Pi(p^*, a^*, r^*) > \Pi^d(p^{d*}, a^{d*}, r^{d*}). \tag{14}
\]

Intuitively, a pre-announcement of a referral program by the firm generates delays in consumer purchases. This is not in the firm’s best interest since the firm needs to pay referral fees and discounts future payments.\(^{12}\)

### 6 The Two-group Model: Targeted Referrals

In the base model, we assume that, unlike the firm, consumers can credibly transmit information about product quality to other consumers. In this section, we assume that consumers have another type of advantage over the firm. Consumers can tell what group other con-

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\(^{12}\)In our framework, some consumers have an incentive to wait for referrals to reveal true product quality. However, in reality, these consumers may actively search for such information from the users of the product, provided their search costs are not too high. Therefore, the firm may have less to lose from announcing its referral program.
sumers belong to, while the firm cannot distinguish between consumer groups. For simplicity, in this section we assume that there is no product quality uncertainty and \( \mu = \mu^c = 0 \).

There are two groups of consumers: \( H \) and \( L \) with fractions \( \lambda^H \) and \( \lambda^L \), respectively \((\lambda^H + \lambda^L = 1)\). Group \( H \) consumers tend to have a higher willingness-to-pay in the sense of the \textit{hazard-rate dominance} than group \( L \), i.e., for all \( p \), \( \frac{g^H(p)}{1-G^H(p)} < \frac{g^L(p)}{1-G^L(p)} \) holds, where \( G^H(v) \) and \( G^L(v) \) are the cumulative distribution functions of values for groups \( H \) and \( L \). The supports of the distribution functions overlap, so that some consumers who belong to group \( L \) have a higher willingness-to-pay than some consumers in group \( H \). The general distribution \( G(v) \) is a weighted average of \( G^H(v) \) and \( G^L(v) \): \( G(v) = \lambda^H G^H(v) + \lambda^L G^L(v) \) for all \( v \). Let \( \pi^\theta(p) = (p - c) (1 - G^\theta(p)) \) for \( \theta \in \{H, L\} \) and \( \pi(p) = (p - c) (1 - G(p)) \).

We assume the log concavity of \( 1 - G^\theta(\cdot) \) for \( \theta \in \{H, L\} \). This assures the uniqueness of profit-maximizing prices: \( p^\theta \equiv \arg \max_p \pi^\theta(p) \) and \( p^m \equiv \arg \max_p \pi(p) \). The hazard-rate dominance condition and Lemma 2 imply \( p^H > p^m > p^L \).

Consumer \( i \) who receives the firm’s advertisement can choose \( q_i^H \) and \( q_i^L \) as referral intensities for two different groups because she can distinguish which of her friends belong to \( H \) and \( L \) groups. We have the same referral equilibrium as before, but the condition applies for each group \( \theta = H, L \). The equilibrium referral reach for group \( \theta \) consumers \( R^\theta = R^\theta(p, a, r) \) is defined implicitly by

\[
(1-a)(1-G^\theta(p))rR^\theta = \rho S(R^\theta)
\]

for all \( r > r_0^\theta \equiv \frac{\rho}{(1-a)(1-G^\theta(p))} \), and the equilibrium referral intensity is higher when referral fee \( r \) is higher and price \( p \), advertising intensity \( a \), and referral cost \( \rho \) are lower. Note that

\[\text{\footnotesize 13 The model can also be interpreted as a social circles model. Consumers in a group know each other and referrals are made only within that group.}\]
\( \lambda^\theta \) has no effect in determining the consumer referral intensity and reach in each group.

The firm’s per-consumer profit in this environment is:

\[
\Pi(p, a, r) = \sum_{\theta \in \{H, L\}} \lambda^\theta (a + (1 - a)R^\theta) \pi^\theta (p) - \rho \sum_{\theta \in \{H, L\}} \lambda^\theta S^\theta - C(a),
\]

(16)

where \( R^\theta = R^\theta (p, a, r) > 0 \) for \( r > r^\theta_0 \) and \( R^\theta = 0 \) otherwise.

Equation (16) is clearly a natural extension of (6), but there is an important difference. The firm can no longer control \( R^H \) and \( R^L \) independently by using a single referral fee \( r \). We cannot use the technique we used in the base model to simplify \( \frac{d\Pi}{dp} \) and \( \frac{d\Pi}{da} \) because \( (1 - a)\pi^\theta (p) - \rho S^\theta (R^\theta) = 0 \) is not assured for either \( \theta \). For this reason, calculating the optimal monopoly price under active referrals for both groups is no longer simple. There is no dichotomy in the firm’s decision problem, where \( p \) is used to maximize profit per consumer and \( r \) is used to control \( R^E \). However, we can show that the firm chooses to increase its price after the introduction of consumer referrals when only group \( H \) gets consumer referrals (i.e., when \( r^H_0 < r \leq r^L_0 \)). In this case, \( r \) needs to control only \( R^H \), and we can apply the same technique as before.

We compare the optimal policies under random referrals \((p^*, r^*, a^*)\) and targeted referrals \((p^T, a^T, r^T)\). We will assume the following sufficient condition for no referrals to be extended to type-\( L \) consumers under targeted referrals: \( \pi^L (p^m) \leq \rho \).

**Proposition 6.** Suppose that \( \pi^L (p^m) \leq \rho \) holds. Under targeted referrals, the firm’s optimal policy \((p^T, a^T, r^T)\) is such that group-\( L \) consumers receive no referrals, and the firm advertises less under targeted referrals than under random referrals, \( a^T < a^* \). Moreover,

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\[14\] Of course, if the firm could use differentiated referral fees \((r^H \text{ and } r^L)\), the optimal referral reach \( R^H \) and \( R^L \) can be set for each group separately: \( (1 - a)\pi^\theta (p) - \rho S^\theta (R^\theta) = 0 \). However, it is unreasonable to assume that the firm can set type-dependent referral fees because the whole point of this extension is to examine how the firm may use consumer referrals to utilize superior consumer information.
the optimal price $p^{T*}$ is higher than the standard monopoly price $p^m$ and $p^{H*} > p^{T*} > p^m > p^{L*}$ holds. The equilibrium referral reach is higher, while the ratio of referral fee to profit margin is lower under targeted referrals than under random referrals: $R^{T*} > R^*$ and $r^{T*}/(p^{T*} - c) < r^*/(p^m - c)$.

Proposition 6 shows that if consumers possess superior information about who would be likely to purchase the product, then the firm would reduce its reliance on mass advertising and shift to using more consumer referrals. Interestingly, consumers can be better off or worse off by the firm’s use of referrals when consumers have an information advantage. Under no referrals, every consumer has an equal probability of receiving information about the product. However, with targeted referrals, consumers who belong to a low willingness-to-pay type are less likely to receive the information, although some of them may have high valuations for the product. Thus, the impact of targeted referrals on consumers may depend on consumer type.

7 Extensions

We can extend our base model in various ways.

7.1 Increasing Marginal Referral Cost

The constant marginal referral cost assumption is important for establishing the monopoly pricing result of Corollary 1.\footnote{We owe this insight to the Editor, Ben Hermalin.} If the marginal referral cost $\rho(k)$ is increasing in the number
of referrals $k$ each consumer makes ($\rho'(k) > 0$), the referral equilibrium formula becomes:

\[ r d^R (p, a; \mu, \mu^e) = \rho(k^E) \phi(q^E; n(p, a)). \] (17)

The proof is available upon request.

The number of referrals each referring consumer makes $k^E$ is now affected by the number of referring consumers $n = ad(p; \mu^e) N$. The optimal referral fee $r^*$ still eliminates the impacts of $p$ and $a$ on profit due to changes in $R^E$, but changes in $p$ and $a$ have a direct impact on $k^E$. An increase in $p$ reduces $n$, and this results in an increase in the marginal referral cost. Thus, the optimal price in this case is lower. Similarly, the optimal advertisement level under increasing $\rho(k)$ is higher than the one with constant $\rho$. This is an intuitive result: if $\rho(k)$ is increasing in $k$, the firm has an incentive to reduce the equilibrium $k$. Price cuts and higher advertising intensity increase the number of referrers. Therefore, in order to achieve the same level of referral reach $R$, fewer referrals per referrer are made and referral costs are lower.

### 7.2 Private Referral Benefits

We can allow for consumer referrals to be motivated by reasons other than monetary payoffs. For example, suppose that each time a successful referral is made, a referrer receives not only a referral fee $r$ but also a non-monetary private benefit $B > 0$. Then, the consumer referral equilibrium is $(r + B) d^R = \rho \varphi(R^E(p, a, r))$ and the firm’s profit can be written as:

\[ \Pi(p, a, r; \mu, \mu^e) = a(p - c)d(p; \mu^e) + (p - c + B)R^E d^R (p, a; \mu, \mu^e) - \rho S(R^E) - C(a). \] (18)

16For the analysis of this subsection, we assume that the referrers choose the number of referrals $k$ instead of referral probability $q$, treating $k$ as a real number. When $N$ is large, these two formulations of our referral model are essentially the same.
The private referral benefit $B$ effectively reduces the marginal cost of selling by referrals. Not surprisingly, we find that the firm supports more referrals and advertises less. Its price is lower if consumers are optimistic ($\mu \leq \mu^e$) and higher when they are pessimistic ($\mu > \mu^e$) about the product quality.

7.3 Consumers Know Valuations of Others

Suppose consumers know other consumers’ valuations for a product (or know that the valuations are high enough for consumers to buy the product). Then, consumers target referrals to individuals whose valuations are sufficiently high. In this case, the firm chooses a higher price than in the base model. A price increase has an additional benefit of a more precisely targeted referrals. Although referral messages are not wasted on unlikely prospects, the savings are not fully captured by either consumers or the firm because of a higher congestion level. To reduce congestion, the monopoly sets a lower referral fee relative to its profit margin when referrals are targeted than when they are random. Less advertising is sustained in this case than in the base model because referrals are more targeted and are therefore cheaper to use. An additional reason for less advertising is that due to the price distortion, the profitability of each sale is lower.

7.4 Price Discrimination

Consider the possibility of a monopoly offering different prices to consumers who come by advertising $p^*_a$ and referral $p^*_R$. If $\mu \leq \mu^e$, then the firm would choose to discriminate in favor of referral consumers: $p^*_R = p^m(\mu) \leq p^m(\mu^e) = p^*_a$. If $\mu > \mu^e$, then $p^*_R > p^*_a$, and in this case, the referrals would not be used because referred consumers would not acknowledge referrals and would purchase using the advertised price. Similarly, we could also allow the
firm to offer a referral benefit to both the referrer and the person they refer or for the referral partners to split the referral benefit.

7.5 Profit Maximization when $\mu$ is Unknown to the Firm

We next consider the case where the firm ex ante does not know the true quality of the product $\mu$. In the base model, the firm has more information than consumers, and so the firm’s policy $(p,a,r)$ may serve as a signal of true quality. This issue disappears when $\mu$ is unknown to the firm.\footnote{This comment especially applies if the marginal cost of production $c$ is known to consumers. Consumers know that the firm charges the monopoly price, and they can identify the true quality $\mu$ from the price $p$ in the base model. If consumers do not know $c$, it is harder for them to infer the value of $\mu$.} Assume that the firm is also uncertain about the quality of the product until the firm receives feedback from buyers. In this case, we can minimally modify our analysis by assuming that the firm decides on its referral fee $r$ after the value of $\mu$ is realized. That is, the firm chooses $(a,p)$ before the value of $\mu$ is realized and then picks $r$ to optimize the referral reach for the realized value of $\mu$ using equations (8) and (9). This second stage optimization gives us an explicit formula for $R^*(\mu)$, and the firm’s expected profit can be written explicitly. We can show that the first order condition for profit maximization is similar to the one in the base model, but with relatively lower weights on the demand from referral consumers. (The derivations are available from the authors upon request.)

7.6 Cap on the Number of Referrals

We explore the implications of a cap on the number of referrals each referrer can make. Let us assume that consumers’ (marginal) referral cost is constant at $\rho$ up to $K$ referrals, but they cannot make more referrals than $K$. This modification requires the model to have multiple periods in which consumer referrals are made. For simplicity, we assume that
advertising is done only initially (at period 0); in subsequent periods, consumer referrals spread the information about the product. In order to get rid of the delay incentive discussed in Section 5, we assume that the consumer referral program is not announced in advertising period 0.\footnote{This assumption does not mean that consumers will not know about the consumer referral program from period 1 on because, from period 1, product information spreads only through referrals and a consumer who sends referrals already knows both $\mu$ and the existence of the referral program. She tells her contacts about the quality of the product $\mu$ and a consumer referral program.} Therefore, from period 1 on, referrals are the only medium used to transmit product information to other consumers. This model bridges our model with the models of referral chains in industrial organization and marketing, in which consumers are located on a line and each consumer can make at most one referral without congestion (Jun and Kim, 2008, Byalogorsky et al., 2008, and Arbatskaya and Konishi, forthcoming). Notice that with the cap on the number of referrals, the initial success probability of a referral is high and the net benefit of the referral is positive because many consumers are not aware of the product. As time goes by, more and more consumers become aware of the product, and the net referral benefit goes down. Depending on the size of $K$, two things can happen. If $K$ is small, then the referral chain may fall short of achieving the referral reach for which the net referral benefit is zero. If $K$ is large enough, then the referral chain terminates in finite number of periods, thus achieving the level of awareness for which additional referrals are no longer beneficial. In either case, consumer referrals and advertising are no longer substitutes. A larger number of referring consumers speeds up the process of consumer referrals. The firm may have an incentive to increase advertising intensity (and lower its price), especially if the firm is not very patient. This extension seems worthwhile to pursue.
8 Conclusion

Several information channels are available to sellers who market their products to consumers. These include traditional mass advertising on TV and in newspapers and consumer referral policies. In the base model, we assume that, unlike mass advertising, consumer referrals can provide accurate information on the quality of the product $\mu$ for an experience good. We look at the optimal advertising, referral, and price policies for the monopoly. Under correct consumer expectations ($\mu = \mu^e$), we find that the profit-maximizing price is the standard monopoly price, provided that the referral fee is optimally chosen. Intuitively, a monopoly does not use its price to manage consumer referrals, but instead directly uses a referral fee. We also argue that the firm would not inform consumers about its referral program as a part of its advertising message. A consumer referral program can improve the firm’s profit when referral cost is relatively low. The firm relies more heavily on referrals when consumers have superior information about other consumers’ preferences and when they derive non-monetary private benefits from making helpful recommendations.

The welfare effects of referrals tend to be positive. Referrals increase consumer awareness about the product and help solve the adverse selection problem of uncertain product quality. For any given level of advertising and price, referrals are underprovided because of the non-appropriability of consumer surplus. Consumers who are informed through referrals are not worse off. Whether consumers informed through advertisements benefit as well depends on the price adjustment. For optimistic or rational consumers ($\mu \leq \mu^e$), the firm’s price is lower (or the same) when referrals are used. We also show that increasing marginal referral costs and caps on referral rewards incentivize the firm to reduce its price in an attempt to stimulate
referrals. Hence, in all the cases where referral cost is sufficiently small for the firm to use referrals and where the price does not increase, referrals result in a Pareto improvement. No consumer is worse off and some are better off because they are better informed. It follows that in such a case, if the firm supports consumer referrals, it is socially optimal to do so. However, in the case of pessimistic consumers \( (\mu > \mu^c) \) or with targeted referrals, the price is higher than in the absence of referrals, and the ex ante welfare change would depend on the relative magnitudes of the price change, which depends on the distribution of consumer valuations.
References


Appendix A: Proofs

Proof of Proposition 1. There are $n$ potential referrers who have purchased the product and are fully informed. Focusing on a symmetric equilibrium, suppose that $n-1$ referrers are choosing referral probability $q$, while the remaining referrer $i$ chooses $q_i$. Referral attempts are made randomly. With probability $1-a$ referral attempts reach uninformed consumers. We assume that if a consumer receives $h$ referral attempts, then she chooses one with equal probability $1/h$. Then, the probability that a given consumer (who is not one of referrers) registers a referral from $i$ is:

$$\psi_i(q_i, q) = \sum_{h=0}^{n-1} \frac{1}{h+1} q_i (1-q)^{n-1-h} q^h \times C(n-1, h), \quad (19)$$

where $C(n-1, h) = (n-1)!/(n-1-h)!h!$. Note that the term $(1-q)^{n-1-h} q^h \times C(n-1, h)$ denotes the probability that the given consumer receives $h$ referral attempts from other $n-1$ referrers. By rearranging the formula, we obtain:

$$\psi_i(q_i, q) = \sum_{h=1}^{n} \frac{1}{h} q_i (1-q)^{n-h} q^{h-1} \times C(n-1, h-1) \quad (20)$$

$$= \sum_{h=1}^{n} \frac{1}{h} q_i (1-q)^{n-h} q^{h-1} \times \frac{(n-1)!}{(h-1)!(n-h)!}$$

$$= \frac{1}{n} \sum_{h=1}^{n} q_i (1-q)^{n-h} q^{h-1} \times \frac{n!}{h!(n-h)!}$$

$$= \frac{1}{n} \times \frac{q_i}{q} \sum_{h=1}^{n} (1-q)^{n-h} q^h \times C(n, h)$$

$$= \frac{q_i}{nq} [1 - (1-q)^n] = q_i \frac{1}{\phi(q; n)}, \quad \text{for } k > 0, \text{ where } \phi(q; n) = \frac{n \times q}{1-(1-q)^n}; \psi_i(q_i, 0) = q_i \text{ for } q = 0.$$

Assuming $q > 0$, the expected referral reward to referrer $i$ from each referral she makes is $\frac{d^R_r}{\phi(q; n)}$, where $d^R$ is the probability
that a referred consumer purchases the product. Using (2),
\[
d^{R} = d^{R}(p, a; \mu, \mu^{e}) = \begin{cases} 
(1 - a)d(p; \mu) & \text{if } \mu \leq \mu^{e} \\
 d(p; \mu) - ad(p; \mu^{e}) & \text{if } \mu > \mu^{e}
\end{cases}
\] (21)
The cost of making a referral is \(\rho\). Therefore, referrer \(i\)'s objective function is linear in \(q_{i}\). Note that \(\phi(q; n) > 1\) for \(q \in (0, 1]\) and \(n \geq 2.19\)

In a symmetric interior equilibrium, consumers are indifferent between which \(q_{i}\) to choose. The symmetric equilibrium \(q^{E}\) is implicitly calculated as a solution to
\[
d^{R}r = \rho\phi(q^{E}; n).
\] (22)
Since \(\phi(q; n) > 1\) for \(k > 0\), this equation has an interior solution only if \(r > \frac{\rho}{\sigma^{n}}.20\) Then, given that others are choosing \(q^{E}\), consumer \(i\) obtains a zero payoff for any strategy, and she might as well choose \(q^{E}\). Thus, \(q^{E}\) is the symmetric referral equilibrium when \(r > \frac{\rho}{\sigma^{n}}\). The equilibrium referral strategy \(q^{E}\) is unique when it exists because \(\phi(q; n)\) is strictly increasing in \(q\). To see this, we use \((1 + q)^{n} > 1 + nq\) in the following:

\[
\frac{\partial \phi(q; n)}{\partial q} = n\frac{(1 - (1 - q)^{n} - nq(1 - q)^{n-1})}{(1 - (1 - q)^{n})^{2}} > n\frac{1 - (1 - q)(1 + (n - 1)q)}{(1 - (1 - q)^{n})^{2}} > n\frac{1 - (1 - q^{2})^{n-1}}{(1 - (1 - q)^{n})^{2}} > 0.
\] (23)

The equilibrium referral congestion \(\phi^{E} = \phi(q^{E}; n)\) equals \(\frac{r^{dR}r}{\rho}\). From (21), it is increasing in \(r\) and \(\mu\) and decreasing in \(\rho\), \(p\), and \(a\); it is independent of \(\mu^{e}\) when \(\mu \leq \mu^{e}\) and decreases

---

19 We can show by induction that \((1 - q)^{n} > 1 - nq\) for \(q \in (0, 1]\) and \(n \geq 2\). When \(n = 2\), \((1 - q)^{2} = 1 - 2q + q^{2} > 1 - 2q\), and the result is true for \(n = 2\). Suppose it is true for some \(n\): \((1 - q)^{n} > 1 - nq\). We need to show that it is then true for \(n + 1\): \((1 - q)^{n+1} > 1 - (n + 1)q\). Note that \((1 - q)^{n+1} = (1 - q)(1 - q)^{n} > (1 - q)(1 - nq)\) by the inductive hypothesis, and therefore \((1 - q)^{n+1} > 1 - (n + 1)q + nq^{2} > 1 - (n + 1)q\).

20 When \(q = 0\), the net benefit of making a referral is \(d^{R}r - \rho\). Hence, for \(r \leq \frac{\rho}{\sigma^{n}}\), the only symmetric equilibrium is \(q^{E} = 0\).
in \( \mu^e \) when \( \mu > \mu^e \). From \( \frac{\partial \phi(q;n)}{\partial n} = \frac{q(1-q)^n}{(1-q)^n - 1} \left[ \ln (1 - q)^n - 1 + (1 - q)^{-n} \right] \) and \( \ln x > 1 - \frac{1}{x} \)
for \( x \neq 1 \), it follows that \( \phi(q;n) \) is strictly increasing in \( n \) for \( q > 0 \). Hence, the equilibrium referral strategy \( q^E = q^E(p,a,r;\mu,\mu^e,\rho,n) \) increases in \( r \) and \( \mu \) and decreases in \( p, a, \mu^e, \rho, \) and \( n \) for \( r > \frac{\partial \pi}{\partial r} \).

**Proof of Lemma 2.** Since \( 1 - G(\cdot) \) is log concave, we have

\[
\frac{d^2 \ln (1 - G(p - \mu))}{dp^2} = \frac{-g'(p - \mu) (1 - G(p - \mu)) - (g(p - \mu))^2}{(1 - G(p - \mu))^2} < 0
\]

or

\[
g'(p - \mu) (1 - G(p - \mu)) + (g(p - \mu))^2 > 0.
\]

The first-order condition for maximizing \( \pi(p;\mu) \) with respect to \( p \) is

\[
\frac{\partial \pi(p;\mu)}{\partial p} = 1 - G(p - \mu) - (p - c) g(p - \mu) = 0
\]

or

\[
p - c = \frac{1 - G(p - \mu)}{g(p - \mu)}.
\]

The corresponding second-order condition is

\[
\frac{\partial^2 \pi(p;\mu)}{\partial p^2} = -2g(p - \mu) - (p - c)g'(p - \mu)
\]

\[
= -\frac{1}{g(p - \mu)} \left[ g'(p - \mu) (1 - G(p - \mu)) + 2 (g(p - \mu))^2 \right] < 0,
\]

assuming the first-order condition holds. Then, log concavity assures that the second-order condition holds for prices satisfying the first order condition. The uniqueness of the profit-maximizing price \( p^m(\mu) \) for each \( \mu \) follows from the index theorem.

By the second order condition at \( p = p^m(\mu) \), we know that there is \( \epsilon > 0 \) such that

\( (p^m(\mu) - \epsilon, p^m(\mu)) \) with \( \frac{\partial \pi(p;\mu)}{\partial p} > 0 \) and \( (p^m(\mu), p^m(\mu) + \epsilon) \) with \( \frac{\partial \pi(p;\mu)}{\partial p} < 0 \). Since \( \frac{\partial \pi(p;\mu)}{\partial p} = 0 \)

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holds only at \( p = p^m(\mu) \), the sign of \( \frac{\partial \pi(p;\mu)}{\partial p} \) stays positive for \( p < p^m(\mu) \) and negative for \( p > p^m(\mu) \). Thus, we conclude that \( \frac{\partial \pi(p;\mu)}{\partial p} \geq 0 \) if and only if \( p \leq p^m(\mu) \).

Turning our attention to the cross-partial derivative of \( \pi(p;\mu) \), we have

\[
\frac{\partial^2 \pi(p;\mu)}{\partial p \partial \mu} = g(p-\mu) + (p-c)g'(p-\mu)
\]

\[
= \frac{1}{g(p-\mu)} \left[ g'(p-\mu) (1-G(p-\mu)) + (g(p-\mu))^2 \right] > 0
\]

at \( p = p^m(\mu) \) and \( \frac{d\pi^u(\mu)}{d\mu} = -\left( \frac{\partial^2 \pi(p^u(\mu);\mu)}{\partial p \partial \mu} \right) / \left( \frac{\partial^2 \pi(p^u(\mu);\mu)}{\partial p^2} \right) > 0. \]

**Proof of Proposition 2.** By Lemma 2, we know that \( \frac{\partial \pi(p;\mu)}{\partial p} \geq 0 \) if and only if \( p \leq p^m(\mu) \), and \( p^m(\mu) \) is increasing in \( \mu \). Thus, if \( p^* > \max \{p^m(\mu), p^m(\mu^e)\} \) or \( p^* < \min \{p^m(\mu), p^m(\mu^e)\} \), then \( \frac{\partial \pi(p^*;\mu)}{\partial p} \) and \( \frac{\partial \pi(p^*;\mu^e)}{\partial p} \) have the same sign, and \( p^* \) cannot satisfy the first-order condition for \( p \). This is a contradiction. This proves that \( p^m(\mu^e) \geq p^* \geq p^m(\mu) \).

In the rest of the proof, we show that consumer referrals reduce the level of advertising. For any given policy mix \((p, a, R)\), the difference in the firm’s profit with and without referrals is:

\[
\Pi(p, a, r; \mu, \mu^e) - \Pi^0(p, a; \mu^e) = R^E \pi^R (p, a; \mu, \mu^e) - \rho S \left( R^E \right).
\]

If \( \mu \leq \mu^e \), then \( \frac{d(\Pi-\Pi^0)}{da} = -R^E \pi (p; \mu) < 0 \) and if \( \mu > \mu^e \), then \( \frac{d(\Pi-\Pi^0)}{da} = -R^E \pi (p; \mu^e) < 0 \). Hence, there is less advertising under referrals, given any \( p \) and any \( r > r_0 \). Since \( \pi (p^*; \mu^e) < \pi (p^m (\mu^e); \mu^e) \), there is less advertising under referrals for optimally chosen \( p \) and any \( r > r_0 \) (including the optimal \( r^* \)).

**Proof of Proposition 3.**

The following shorthand notations are used in the proof: \( \pi = \pi (p; \mu), \pi^e = \pi (p; \mu^e), \)

\( \pi_p = \frac{\partial \pi(p;\mu)}{\partial p}, \pi_{\mu} = \frac{\partial \pi(p;\mu)}{\partial \mu}, \pi_{pp} = \frac{\partial^2 \pi(p;\mu)}{\partial p^2}, \pi_{p\mu} = \frac{\partial^2 \pi(p;\mu)}{\partial p \partial \mu}, \pi_{p}^{e} = \frac{\partial \pi(p;\mu^e)}{\partial p}, \pi_{\mu}^{e} = \frac{\partial \pi(p;\mu^e)}{\partial \mu}, \)
\( \pi^e_{pp} = \frac{\partial^2 \pi(p, \mu^e)}{\partial p^2} \), and \( \pi^e_{pp} = \frac{\partial^2 \pi(p, \mu^e)}{\partial p \partial \mu^e} \).

**Case I (\( \mu \leq \mu^e \)):** Totally differentiating the system of first-order conditions for \( R, p, \) and \( a \):

\[
\begin{align*}
\left\{ \begin{array}{l}
(1 - a)\pi = \rho S'(R) \\
\frac{d\mu}{dp} = a\pi^e_p + (1 - a) R\pi_p = 0 \\
\frac{d\mu}{da} = \pi^e - R\pi - C''(a) = 0
\end{array} \right.,
\end{align*}
\]

we find that

\[
\begin{pmatrix}
-\rho S'' \\
(1 - a)\pi_p \\
-\pi
\end{pmatrix}
\begin{pmatrix}
\pi^e_p + (1 - a) R\pi_{pp} \\
\pi^e_p - R\pi_p \\
\pi^e - R\pi_p
\end{pmatrix}
\begin{pmatrix}
(1 - a)\pi \\
\pi^e \\
-\pi
\end{pmatrix}
\begin{pmatrix}
dR \\
dp \\
da
\end{pmatrix}
\begin{pmatrix}
-(1 - a)\pi^e_p \\
(1 - a) R\pi_{pp} \\
R\pi_p
\end{pmatrix}
\begin{pmatrix}
S' \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
d\mu \\
d\mu^e \\
d\rho
\end{pmatrix}
\]

at \((p, a, r) = (p^*, a^*, r^*)\). Since the firm maximizes its profit, the matrix in the LHS is negative semi-definite. Since the optimum is regular, we have the determinant \( D \) and the principal minors of the matrix in the LHS satisfy:

- \( D = (\rho S'' C'' - \pi^2) (a\pi^e_{pp} + (1 - a) R\pi_{pp}) - 2 (\pi^e_p - R\pi_p) (1 - a)\pi_p + \rho S'' (\pi^e_p - R\pi_p)^2 + C'' ((1 - a)\pi_p)^2 < 0, \)

- \(-\rho S'' (a\pi^e_{pp} + (1 - a) R\pi_{pp}) - (1 - a)^2 (\pi_p)^2 > 0, \)

- \( \rho S'' C'' - \pi^2 > 0, \)

- \(-C'' (a\pi^e_{pp} + (1 - a) R\pi_{pp}) - (\pi^e_p - R\pi_p)^2 > 0, \)

- \(-\rho S'' < 0, a\pi^e_{pp} + (1 - a) R\pi_{pp} < 0, \) and \(-C'' < 0.\)

From \( \mu \leq \mu^e \) and \( a\pi_{p}(p; \mu^e, c) + (1 - a) R\pi_p (p; \mu, c) = 0, \) we have \( \pi^e_p \geq 0 \) and \( \pi_p \leq 0. \)

Thus, the first term in \( D \) is negative, but all others are positive.

The impacts of an increase in \( \rho \) on the optimal policies are:
\[
\frac{dR}{d\rho} = \frac{S'}{D} \left[-C'' \left(a\pi_{pp}^e + (1 - a)R\pi_{pp}^e\right) - (\pi_{p}^e - R\pi_p)^2\right]
\]
\[
\frac{dp}{d\rho} = \frac{S'}{D} \left[(1 - a)\pi_p C'' - \pi \left(\pi_p^e - R\pi_p\right)\right]
\]
\[
\frac{da}{d\rho} = \frac{S'}{D} \left[(1 - a)\pi_p \left(\pi_p^e - R\pi_p\right) + \pi \left(a\pi_{pp}^e + (1 - a)R\pi_{pp}^e\right)\right].
\]

Thus, we conclude \(\frac{da}{d\rho} > 0\), \(\frac{dp}{d\rho} > 0\), and \(\frac{dR}{d\rho} < 0\).

The impacts of an increase in \(\mu^e\) on the optimal policies are:

\[
\frac{dR}{d\mu^e} = \frac{1}{D} \left[a\pi_{pp} \left\{-C''(1 - a)\pi_p + \pi \left(\pi_p^e - R\pi_p\right)\right\} - \pi^e \left\{(1 - a)\pi_p \left(\pi_p^e - R\pi_p\right) + \pi \left(a\pi_{pp}^e + (1 - a)R\pi_{pp}^e\right)\right\}\right]
\]
\[
\frac{dp}{d\mu^e} = \frac{1}{D} \left[-a\pi_{pp} \left(\rho S'' C'' - \pi^2\right) + \pi^e \left\{-\rho S'' \left(\pi_p^e - R\pi_p\right) + \pi(1 - a)\pi_p\right\}\right]
\]
\[
\frac{da}{d\mu^e} = \frac{1}{D} \left[a\pi_{pp} \left\{-\rho S'' \left(\pi_p^e - R\pi_p\right) + \pi(1 - a)\pi_p\right\} - \pi^e \left\{-\rho S'' \left(a\pi_{pp}^e + (1 - a)R\pi_{pp}^e\right) - (1 - a)^2 (\pi_p)^2\right\}\right].
\]

Thus, \(\frac{dR}{d\mu^e} < 0\), \(\frac{dp}{d\mu^e} > 0\) and \(\frac{da}{d\mu^e} > 0\) hold. We cannot determine the signs of the impacts of an increase in \(\mu\) on the optimal policies.

**Case II (\(\mu > \mu^e\))**: Totally differentiating the system of first-order conditions for \(R\), \(p\), and \(a\)

\[
\begin{align*}
\pi - a\pi_p^e &= \rho S'(R) \\
\frac{d\pi}{dp} &= a \left(1 - R\right) \pi_p^e + R\pi_p = 0 \\
\frac{d\pi}{da} &= \left(1 - R\right) \pi_p^e - C'(a) = 0
\end{align*}
\]

we find that

\[
\begin{pmatrix}
-\rho S'' \\
\pi_p - a\pi_p^e \\
-\pi^e
\end{pmatrix}
\begin{pmatrix}
\frac{dR}{dp} \\
\frac{dp}{dp} \\
\frac{da}{dp}
\end{pmatrix}
= \begin{pmatrix}
-\pi^e \\
-a\pi_{pp}^e + R\pi_{pp}^e \\
-(1 - R)\pi_p^e - C''
\end{pmatrix}
\begin{pmatrix}
\frac{d\mu}{dp} \\
\frac{d\mu^e}{dp}
\end{pmatrix}
\]

\[(36)\]
at \((p, a, r) = (p^*, a^*, r^*)\). Since the firm maximizes its profit, the matrix in the LHS is negative semi-definite. Since the optimum is regular, the determinant \(D\) and the principal minors of the matrix in the LHS satisfy:

\[
\begin{align*}
D &= (\rho S'' C'' - (\pi^e)^2) \left( a (1 - R) \pi_{pp}^e + R \pi_{pp} \right) - 2 \left( \pi_p - a \pi_p^e \right) \pi^e (1 - R) \pi_{pp}^e + \rho S'' ((1 - R) \pi_{pp}^e)^2 + C'' \left( \pi_p - a \pi_p^e \right)^2 < 0, \\
-\rho S'' \left( a (1 - R) \pi_{pp}^e + R \pi_{pp} \right) - (\pi_p - a \pi_p^e)^2 &> 0, \\
\rho S'' C'' - (\pi^e)^2 &> 0, \\
-C'' \left( a (1 - R) \pi_{pp}^e + R \pi_{pp} \right) - (1 - R)^2 (\pi_p^e)^2 &> 0, \\
-\rho S'' &< 0, \ a (1 - R) \pi_{pp}^e + R \pi_{pp} < 0, \text{ and } -C'' < -0.
\end{align*}
\]

From \(\mu > \mu^e\) and \(a (1 - R) \pi_p(p; \mu^e, c) + R \pi_p(p; \mu, c) = 0\), we have \(\pi_p > 0\) and \(\pi_p^e < 0\).

The impacts of an increase in \(\rho\) on the optimal policies are:

\[
\begin{align*}
\frac{dR}{d\rho} &= \frac{S'}{D} \left[ -C'' \left( a (1 - R) \pi_{pp}^e + R \pi_{pp} \right) - (1 - R)^2 (\pi_p^e)^2 \right], \\
\frac{dp}{d\rho} &= \frac{S'}{D} \left[ C'' \left( \pi_p - a \pi_p^e \right) - \pi_p^e (1 - R) \pi_p^e \right] \\
\frac{da}{d\rho} &= \frac{S'}{D} \left[ \left( \pi_p - a \pi_p^e \right) (1 - R) \pi_p^e + \pi_p^e (a (1 - R) \pi_{pp}^e + R \pi_{pp}) \right].
\end{align*}
\]

Thus, \(\frac{dR}{d\rho} < 0, \frac{da}{d\rho} > 0, \text{ and } \frac{dp}{d\rho} < 0\) hold.

The impact of an increase in \(\mu\) on the optimal policies are:
\[
\frac{dR}{d\mu} = \frac{1}{D} \left\{ -\pi' \left\{ -C'' (a (1 - R) \pi^e + R\pi_{pp}) - ((1 - R)\pi^e_p)^2 \right\} 
+ R\pi_{pp} \left\{ -C'' (\pi_p - a\pi^e_p) + \pi^e (1 - R)\pi^e_p \right\} \right\} 
\]

\[
\frac{dp}{d\mu} = \frac{1}{D} \left[ \pi \left\{ -C'' (\pi_p - a\pi^e_p) + \pi^e (1 - R)\pi^e_p \right\} - R\pi_{pp} \left\{ \rho S'' C'' - (\pi^e)^2 \right\} \right] 
\]

\[
\frac{da}{d\mu} = \frac{1}{D} \left[ -\pi \left\{ (\pi_p - a\pi^e_p) (1 - R)\pi^e_p + \pi^e (a (1 - R)\pi^e_{pp} + R\pi_{pp}) \right\} 
+ R\pi_{pp} \left\{ -\rho S'' (1 - R)\pi^e_p + \pi^e (a - a\pi^e_p) \right\} \right] 
\]

Thus, \(\frac{dR}{d\mu} > 0, \frac{dp}{d\mu} > 0\), and \(\frac{da}{d\mu} < 0\) hold. We cannot determine the signs of the impact of an increase in \(\mu^e\) on the optimal policies.\]

**Proof of Proposition 4.** For any fixed \(p\) and \(a > 0\), referral policy adoption is profitable if and only if the firm can introduce a referral policy with a referral fee \(r \in (r_0, p - c)\), where \(r_0 \equiv \frac{p}{\rho}\). Such \(r\) exists if and only if \(\rho < \bar{\rho}\) because \(r_0 < p - c\) is equivalent to \(\rho < (1 - a)\pi(p; \mu)\) if \(\mu \leq \mu^e\), and \(\rho < \pi(p; \mu) - a\pi(p; \mu^e)\) if \(\mu > \mu^e\). From Proposition 1, the firm that chooses \(r > r_0\) supports referrals. Since \(r < p - c\), it would receive positive additional profits from consumers buying by referral without altering its profits from the informed consumers. Hence, for any fixed \(a\) and \(p\), the firm can increase its profits by an introduction of a referral program if and only if \(\rho < \bar{\rho}\).\]

**Proof of Corollary 2.** When the firm does not use referrals, it chooses price \(p^m(\mu^e)\) and advertising level \(a^0^*\) such that \(\frac{dp^m}{da} = \pi (p; \mu^e) - C'(a^0^*) = 0\). By Proposition 4, the firm benefits from introducing referrals while keeping \(p^m(\mu^e)\) and \(a^0^*\) if \(\rho < \rho_0 \equiv \bar{\rho} (p^m(\mu^e), a^0^*) = \pi^R (p^m(\mu^e), a^0^*, \mu, \mu^e)\). At the optimal policy mix \((p^*, a^*, R^*)\), the profits \(\pi^R (p^*, a^*, \mu, \mu^e)\) will be even higher. Thus, the firm’s profits are higher with referrals.\]
Proof of Lemma 3. We need to show that $U^p(v) > U^d(v)$ for all $v > v^*$ and $U^p(v) < U^d(v)$ for all $v < v^*$. It is easy to see that $\frac{\partial(U^p - U^d)}{\partial v} > 0$, $\frac{\partial(U^p - U^d)}{\partial v} < 0$, $\frac{\partial(U^p - U^d)}{\partial \mu} > 0$, and $\frac{\partial(U^p - U^d)}{\partial \theta} < 0$, and that if $v = p - \mu^e$, then $U^p(p - \mu^e) = 0 < U^d(p - \mu^e)$; and if $v = p$, then $U^p(p) = \mu^e > \delta \mu^e = U^d(p)$. Then, there exists a unique $v^* \in (p - \mu^e, p)$ with $U^p(v^*) - U^d(v^*) = 0$. If $\mu^e + \nu \leq p \leq \mu^e + \bar{\nu}$, then $v^* \in (\nu, \bar{\nu})$. Totally differentiating $U^p(v^*) - U^d(v^*) = 0$, we obtain the desired results for the signs of partial derivatives of $v^*$.

Proof of Lemma 4. Suppose the firm chooses a policy mix $(p, a, r)$. Using $\mu^* = p - v^*$, the firm’s profit when the referral program is advertised is:

$$
\Pi^d(p, a, r) = (p - c) a \left[ d(p; \mu^*) + \delta (1 - R^E_1) \{d(p; \mu) - d(p; \mu^*)\} \right]
+ \delta (p - c - r) R^E_1 d^R - C(a),
$$

where

$$
d^R_1 = d^R_1(p, a; \mu, \mu^e) = a \max \{d(p; \mu) - d(p; \mu^*), 0\} + (1 - a) d(p; \mu)
$$

is the probability that a consumer who receives a referral buys the good and $R^E_1$ is the equilibrium referral reach when the referral program is advertised.

Note that by Proposition 1 and Lemma 1, $R^E_1 = R^E_1(p, a, r)$ is determined by:

$$
rR^E_1 d^R_1 = \rho S(R^E_1).
$$

In contrast, if the program is not announced, the equilibrium referral reach $R^E = R^E(p, a, r)$ is determined by:

$$
rR^E d^R = \rho S(R^E).
$$

As before, we will use the equilibrium referral condition (41) to simplify the firm’s profit.
function:

\[
\Pi^d(p, a, r) = a\pi(p, \mu^e) + a(1 - R_1^E)\delta (\pi(p, \mu^e) - \pi(p, \mu^*)) + aR_1^E\delta \max \{\pi(p, \mu) - \pi(p, \mu^*), 0\} + (1 - a)R_1^E\delta \pi(p, \mu) - \delta \rho S(R_1^E) - C(a).
\]

From \( \mu^* < \mu^e \), it follows that \( \pi(p, \mu^e) > \pi(p, \mu^*) \).

**Case 1.** Consider Case 1: \( \mu^* < \mu \). Since \( \mu^* < \mu^e \), we have two subcases: (i) \( \mu^* < \mu < \mu^e \) and (ii) \( \mu^* < \mu^e < \mu \). In both subcases, the firm’s profit if it announces its referral policy is:

\[
\Pi^d(p, a, r) = a(1 - \delta)\pi(p, \mu^*) + a(1 - R_1^E)\delta \pi(p, \mu^e) + R_1^E\delta \pi(p, \mu) - \delta \rho S(R_1^E) - C(a). \quad (43)
\]

We start with subcase (i): \( \mu^* < \mu < \mu^e \). In this case, \( \pi(p, \mu^*) < \pi(p, \mu) < \pi(p, \mu^e) \). The firm’s profit without the referral policy announcement is

\[
\Pi(p, a, r) = a\pi(p, \mu^e) + (1 - a)R_1^E\delta \pi(p, \mu) - \delta \rho S(R_1^E) - C(a), \quad (44)
\]

and the difference in profits is:

\[
\Pi(p, a, r) - \Pi^d(p, a, r) = a \left[ (1 - \delta(1 - R_1^E)) \pi(p, \mu^e) - R_1^E\delta \pi(p, \mu) - (1 - \delta)\pi(p, \mu^*) \right] - \delta \left[ a \left( R_1^E - R_1^E \right) \pi(p, \mu^e) + \rho \left( S(R_1^E) - S(R_1^E) \right) \right] > a \left[ (1 - \delta(1 - R_1^E)) \pi(p, \mu^e) - R_1^E\delta \pi(p, \mu) - (1 - \delta)\pi(p, \mu^*) \right] \quad (45)
\]

Substituting \( \pi(p, \mu) \) for \( \pi(p, \mu^* \), we obtain:

\[
\Pi(p, a, r) - \Pi^d(p, a, r) > a \left( 1 - \delta(1 - R_1^E) \right) (\pi(p, \mu^e) - \pi(p, \mu)) > 0. \quad (46)
\]

Thus, by employing the same policy mix \((p, a, r)\), the firm can make more profit by simply not advertising its referral program.
Second, consider subcase (ii): $\mu^* < \mu^e < \mu$. In this case, $\pi(p, \mu^*) < \pi(p, \mu^e) < \pi(p, \mu)$.

Consider an advertising level $a_1$ such that $R^E(p, a_1, r) = R^E_1(p, a, r)$; i.e., set $a_1$ to equate $d^R_1$ and $d^R$ when $\mu^e < \mu$:

$$a_1 = \frac{d(p; \mu^*)}{d(p; \mu^e)} a = \frac{\pi(p; \mu^*)}{\pi(p; \mu^e)} a. \quad (47)$$

We find that $a_1 < a$. The firm’s profit from a policy mix $(p, a_1, r)$ without the referral policy announcement is:

$$\Pi(p, a_1, r) = a_1 (1 - \delta R^E_1) \pi(p, \mu^e) + R^E_1 \delta \pi(p, \mu) - \delta \rho S(R^E_1) - C(a_1), \quad (48)$$

and the difference in profits is:

$$\Pi(p, a_1, r) - \Pi^d(p, a, r) \quad (49)$$

$$= a_1 (1 - \delta R^E_1) \pi(p, \mu^e) - a (1 - \delta) \pi(p, \mu^*) - a \delta (1 - R^E_1) \pi(p, \mu^e) - C(a_1) + C(a)$$

$$> a (1 - \delta R^E_1) \pi(p, \mu^e) - a (1 - \delta) \pi(p, \mu^*) - a \delta (1 - R^E_1) \pi(p, \mu^e)$$

$$= a (1 - \delta) \pi(p, \mu^e) - a (1 - \delta) \pi(p, \mu^*) = a (1 - \delta) (\pi(p, \mu^e) - \pi(p, \mu^*)) > 0. $$

Thus, choosing $(p, a_1, r)$ without advertising the referral program achieves a higher profit than choosing $(p, a, r)$ and advertising the referral program.

**Case 2.** Consider Case 2: $\mu < \mu^*$. This implies $\mu < \mu^* < \mu^e$ and $\pi(p, \mu) < \pi(p, \mu^*) < \pi(p, \mu^e)$. In this case, $R^E = R^E_1$ holds since no consumer who delays purchase purchases the product, even if a referral is received. The firm’s profit with the announcement is

$$\Pi^d(p, a, r) = a (1 - \delta (1 - R^E_1)) \pi(p, \mu^*) + a (1 - R^E_1) \delta \pi(p, \mu^e) + (1 - a) R^E_1 \delta \pi(p, \mu) - \delta \rho S(R^E_1) - C(a), \quad (50)$$

and using (44) we have

$$\Pi(p, a, r) - \Pi^d(p, a, r) = a (1 - \delta (1 - R^E_1)) (\pi(p, \mu^e) - \pi(p, \mu^*)) > 0. \quad (51)$$

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Proof of Proposition 6. By the assumption $\pi^L(p^m) \leq \rho$, the marginal cost of extending referrals to low-type consumers exceeds the marginal benefit if $p \geq p^m$.

Consider $p \geq p^m$. In the referral equilibrium under targeted consumer referrals,

$$(1 - a)(1 - G^H(p))rR^*_T = \rho S^*(R^*_T), \quad (52)$$

if $r_0^H < r \leq r_0^L$. The firm’s profit function is:

$$\Pi(p,a,r) = a\pi(p) + \lambda^H(1 - a)\pi^H(p)R^*_T - \lambda^H \rho S^*(R^*_T) - C(a). \quad (53)$$

The first-order condition for the referral fee $r$ is

$$\frac{d\Pi}{dr} = \lambda^H \left[(1 - a)\pi^H(p) - \frac{\rho}{1 - R^*_T}\right] \frac{\partial R^*_T}{\partial r} = 0, \quad (54)$$

and, therefore, the expression in the square brackets is zero at the optimal $r^*_T$ (that is, the marginal net benefit of extending referral reach among group-$H$ consumers is zero). It follows that

$$(1 - a) \pi^H(p)(1 - R^*_T) = \rho. \quad (55)$$

First, consider pricing. Conditional on the optimal choice of $r^*_T$, the first-order condition for price $p$ is

$$\frac{d\Pi}{dp}\bigg|_{r=r^*_T} = \frac{\partial \pi(p)}{\partial p} a + \lambda^H(1 - a) \frac{\partial \pi^H(p)}{\partial p} R^*_T = 0. \quad (56)$$

The standard monopoly price $p^m$ satisfies $\frac{\partial \pi(p^m)}{\partial p} = 1 - G(p^m) - (p^m - c) g(p^m) = 0$. Since $\frac{g^H(p)}{1 - G^H(p)} < \frac{g(p)}{1 - G(p)}$ holds for all $p$, we have $\frac{p^m g^H(p^m)}{1 - G^H(p^m)} < \frac{p^m g(p^m)}{1 - G(p^m)} = 1$. Thus, $\frac{\partial \pi^H(p^m)}{\partial p} = 1 - G^H(p^m) - p^m g^H(p^m) > 0$ holds. By applying the same argument as in the proof of Lemma 2
(log concavity of $1 - G$ and $1 - G^H$), we know that $\frac{\partial \pi^H(p)}{\partial p} > 0$ for all $p < p^H$ and $\frac{\partial \pi(p)}{\partial p} < 0$ for all $p > p^m$. This argument proves that $p^T > p^m$.

Second, we consider advertising. Under the optimally chosen $r^T$, the derivative of the profit with respect to $a$ can be written as:

$$\frac{d\Pi}{da} \bigg|_{r=r^*} = \pi(p^T) - \lambda^H \pi^H(p^T) R^T - C'(a^T)$$
$$= \lambda^L \pi L(p^T) + \lambda^H \pi^H(p^T) (1 - R^T) - C'(a^T)$$
$$< \lambda^L \pi(p^m) + \lambda^H \frac{\rho}{1 - a^T} - C'(a^T)$$
$$\leq \lambda^L \rho + \lambda^H \frac{\rho}{1 - a^T} - C'(a^T)$$
$$< \frac{\rho}{1 - a^T} - C'(a^T). \quad (57)$$

In the benchmark model of random referrals, the profit-maximizing level of advertising $a^*$ (conditional on the optimal choice of $r^*$) is described by

$$\pi(p^m) (1 - R^*) - C'(a^*) = \frac{\rho}{1 - a^*} - C'(a^*) = 0. \quad (58)$$

Thus, we have

$$\frac{\rho}{1 - a^T} - C'(a^T) > \frac{\rho}{1 - a^T} - C'(a^*) = 0.$$
These equations imply $R^T > R$ because we know that $p^H > p^T > p^m$, $a^T < a^*$, and
\[(p^T - c) (1 - G^H (p^T)) > (p^m - c) (1 - G^H (p^m)) > (p^m - c) (1 - G(p^m)).\] To see that, note that $\frac{\partial G^H}{\partial p} > 0$ holds for $p \in (p^m, p^H)$ by the log concavity of $1 - G^H(p)$ and $1 - G^H(p) > 1 - G(p)$.

Finally, we will compare the ratios of referral fees to profit margins in the case of random and targeted referrals:
\[(1 - a^*)(1 - G(p^m))r^* = \rho \phi (R^*) \quad (61)\]
and
\[(1 - a^T)(1 - G^H (p^T))r^T = \rho \phi (R^T) . \quad (62)\]

Using equations (59), (60), (61), and (62), we obtain:
\[
\frac{r^*}{p^m - c} = \varphi (R^*) (1 - R^*) \quad (63)
\]
\[
\frac{r^T}{p^T - c} = \varphi (R^T) (1 - R^T). \quad (64)
\]

Since $R^T > R^*$, to show that $\frac{r^*}{p^m - c} > \frac{r^T}{p^T - c}$, we only need to prove that $\zeta(R) \equiv \varphi (R) (1 - R)$ is a decreasing function, where $\varphi (R) = -(\ln(1 - R)) / R$. To see this, differentiate $\zeta(R)$ to obtain $\zeta' (R) = \frac{1}{R^2} \ln(1 - R) + \frac{1}{R}$. Note that $\ln x$ is a strictly concave function with $\ln(1) = 0$ and $(\ln x)' = 1$ at $x = 1$. Thus, $\ln(x) < x - 1$ for all $x \neq 1$. This implies $\ln(1 - R) < -R$. Therefore, for all $R \in (0, 1)$ we have $\zeta' (R) < -\frac{1}{R} + \frac{1}{R} = 0$.■
Appendix B: Technical Appendix

Convex Cost of Referral

Suppose that the cost of making referral is not linear. We continue to assume that the referral policy is not pre-announced by the firm. If that is the case, the net benefit of making referrals is positive, but consumers do not anticipate these net benefits when making a decision whether to buy the product upon receiving an advertisement. Let $\rho(k)$ be the marginal cost to a consumer from sending $k$th referral.

**Proposition 1’.** *The equilibrium consumer referral number $k^E$ is implicitly defined by:*

$$d^R r = \rho(k^E) \phi(k^E; n)$$  \hspace{1cm} (65)

*for $r > r_0 \equiv \frac{\rho(0)}{d^R}$, and no referrals are sustained for lower levels of the referral fee.*

**Proof of Proposition 1’.** Suppose each of $n = (1 - G(p))aN > 0$ referrers, except for referrer $i$, makes referrals to a fraction $\frac{k}{N}$ of all consumers. As in the proof of Proposition 1, a proportion of uninformed consumers who use referrals from $i$ is $\psi_i(\frac{k_i}{N}, \frac{k}{N}) = \frac{k_i}{nN} \left[1 - \left(1 - \frac{k}{N}\right)^n\right]$ for $\frac{k}{N} > 0$. Referrer $i$’s optimal referral choice $k_i^N$ is obtained by maximizing her per-consumer net benefit of referral

$$\beta \left( \frac{k_i}{N}, \frac{k}{N}; n \right) = d^R r \times \psi_i \left( \frac{k_i}{N}, \frac{k}{N} \right) - \rho \left( k_i \right),$$  \hspace{1cm} (66)

where $d^R$, defined in (2), is the probability that a referred consumer purchases the product.

For any $r > r_0 \equiv \frac{\rho(0)}{d^R}$, we have $\frac{\partial \psi_i \left( \frac{k_i}{N}, \frac{k}{N} \right)}{\partial \left( \frac{k}{N} \right)} = \frac{R}{S}$. The unique equilibrium $k^E$ satisfies $d^R r = \rho \left( k^E \right) \phi \left( k^E; n \right)$. Since $\rho \left( k^E \right) \phi \left( k^E; n \right)$ is increasing in $k^E$ and $d^R r > \rho \left( 0 \right) \lim_{k \to 0} \phi \left( k; n \right) = \rho \left( 0 \right)$ for $r > r_0$, there exist a unique $k^E > 0$. ■