Enhancing the Efficiency of and Equity in Transplant Organ Allocation via Incentivized Exchange*

Tayfun Sönmez† M. Utku Ünver‡

First Draft: January 2015

Abstract

Within the last decade kidney exchange has become a mainstream paradigm to increase the number of kidney transplants. However, compatible pairs do not participate, and the full benefit from exchange can be realized only if they do. In this paper, we propose a new incentive scheme that relies on incentivizing participation of compatible pairs in exchange via insurance for the patient for a future renal failure. Efficiency and equity analyses of this scheme are conducted and compared with efficiency and equity outcomes of live donation and living donor organ exchange. We also present the potential role of such an incentive scheme to strengthen the national kidney exchange system.

Keywords: Market design, organ allocation, kidney exchange, equity, efficiency, compatible pairs
JEL codes: D47, C78

*Sönmez acknowledges the research support of Goldman Sachs Gives via Dalinc Ariburnu - Goldman Sachs Faculty Research Fund. Sönmez and Ünver acknowledge the research support of NSF via award SES #1426440. We thank the participants at Osaka Market Design Workshop, NBER Market Design Working Group Meeting, NSF/CEME Decentralization Conference, SMU, CIDE in Mexico City, Barcelona JOCS Seminar, CoED at Lund, Brown, Cornell, LSE, and Tel Aviv for their comments.

†Boston College, Department of Economics and Distinguished Research Fellow at Koç University; sonmezt@bc.edu; www2.bc.edu/~sonmezt

‡Boston College, Department of Economics and Distinguished Research Fellow at Koç University; unver@bc.edu; www2.bc.edu/~unver

1 Introduction

The National Organ Transplant Act (NOTA) of 1984 called for an Organ Procurement and Transplantation Network (OPTN) to be created and run by a private, non-profit organization under federal contract. The federal *Final Rule* provides a regulatory framework for the structure and operation of the OPTN: the primary goal of the OPTN is to *increase and ensure the effectiveness, efficiency, and equity of organ sharing* in the national system of organ allocation, and to *increase the supply of donated organs* available for transplantation.[1] As in most resource allocation problems, tension often emerges between the dual objectives of efficiency and equity in the context of organ transplantation. Our ultimate objective in this paper is the introduction and advocacy of a new organ allocation policy that has strong potential not only to increase the supply of organs available for transplantation (thus increasing the efficiency of the organ allocation system), but also to decrease its inequity. To our knowledge, our proposed policy is the first to enhance both the efficiency and equity of of the system. To introduce our policy proposal, it will be helpful to explain two other contributions of our paper.

Our paper makes three main contributions. Our first contribution is the introduction of a new and analytically tractable dynamic large-market model of organ transplantation that can be used to analyze the efficiency and equity implications of various technologies and policies.[2] Unlike former models that focus on a single organ-allocation technology (such as deceased donor organ allocation or living donor organ exchange), our model can be used to analyze the impact of various technologies and policies that are often used together and interact with each other.

Our second contribution is a formal analysis of the efficiency and equity implications of the following three primary organ-transplantation technologies:

1. deceased donor transplantation,

2. living donor transplantation, and

3. living donor organ exchange.

For organs that can be transplanted, the first step in this innovation sequence is deceased donor transplantation, potentially followed by living donor transplantation. Living donor organ exchange becomes a possibility only after the innovation of living donor transplantation. Thus, there is a natural innovation sequence of the primary transplantation technologies. Understanding the implications of each new technology on OPTN's dual goals of efficiency and equity is clearly of paramount importance.

[1] The Final Rule OPTN Charter, retrieved from http://optn.transplant.hrsa.gov/ContentDocuments/OPTN_CHARTER_II_-_NOV_04.pdf on 12/22/2014.

[2] While traditional matching models mostly focus on discrete settings, the use of large market and continuum models had become increasingly common over the last decade, especially in the context of market design applications. These models include Kojima and Pathak (2009), Che and Kojima (2010), Lee (2011), Azevedo and Budish (2012), Azevedo and Leshno (2013),Kojima, Pathak, and Roth (2013), Liu and Pycia (2013), Ashlagi and Roth (2014).

With the introduction of each of these technologies, the supply of donated organs available for transplantation potentially increases. Thus, each innovation potentially increases the efficiency of the organ-allocation system. However, for organs that require blood-type compatibility, the introduction of living donor transplantation can potentially increase the inequity between various patient groups. That is indeed what has been happening in the US for the case of kidneys. Similarly, living donor organ exchange can further increase the inequity between certain patient groups. There is an intuitive explanation for this phenomenon: It is much harder for blood-type O patients to benefit from live donation or living donor organ exchange than patients of other blood types. That is because in the absence of other complications,

1. while a blood-type O patient needs a blood-type O kidney for transplantation,
2. a patient of blood-type A or B can receive a transplant from either a same blood-type donor or a blood-type O donor, and
3. a patient of blood-type AB can receive a transplant from any blood-type donor.

Using our model, we analyze the impact of each new technology on the number of patients of various groups who receive a transplant and characterize the average waiting time for those patients who are fortunate to be able to receive one. Our results support the empirical observation that while living donor transplantation and living donor organ exchange both enhance the overall welfare of the patient population, they are potentially detrimental to equity across patients of different blood types.

To introduce our third and main contribution, it will be helpful to give some background on the current status of living donor organ exchange. This practice is in its infancy with a handful of exchanges in the world for the case of liver transplantation. Moreover, it currently accounts for about 3% of transplants in the US for the case of kidneys. Transplants from kidney exchanges only increased in the last decade, benefiting considerably from a successful collaboration between economists and members of the transplantation community. In the early 2000s, along with Alvin Roth, we formulated kidney exchange as a market design problem. Building on existing practices in kidney transplantation, we analyzed in Roth, Sönmez, and Ünver (2004, 2005b, 2007) how an efficient and incentive-compatible system of exchanges might be organized, and what its welfare implications might be. Through a collaboration with members of the New England transplantation community, we formed the first organized kidney exchange clearinghouse that utilized tools from optimization, matching theory, and market design. The methodology and techniques advocated in our research program provided the backbone of several kidney exchange programs in the US and the rest of the world. Our research program and interactions with the transplantation community revealed that the following five elements are especially important to the design and implementation of a successful kidney exchange program:

1. organization and optimization of the exchange,
2. utilization of gains from larger exchanges,

3. integration of good samaritan donors (a.k.a. non-directed donors) to exchange via kidney chains,

4. inclusion of compatible pairs,

5. utilization of economies of scale via larger kidney exchange programs.

Of these five elements, while the former three have been largely embraced by the transplantation community and successfully utilized by several kidney exchange programs, the success of the latter two elements has so far been limited. For kidney exchange to realize its full promise, it is important to address the failure to include compatible pairs in exchange pools as well as utilizing gains of scope via a unified national exchange program rather than several smaller programs.

We will build on the following observation to introduce an incentives program that will not only encourage participation of compatible pairs, but also support the goal of unification of programs under a large national kidney exchange program: on the one hand countless blood-type O patients with non-O donors are waiting for a potential exchange; on the other hand many O blood-type donors donate directly to their non-O recipients. These non-O recipients thus use up kidneys that are more sought after than they actually need. That is why inclusion of compatible pairs in exchange is so critical. How can compatible pairs be incentivized to participate in kidney exchange, and hence, we avoid the current welfare loss? A natural possibility is offering cash incentives, but cash incentives are currently taboo in much of the world. What we propose instead is the following incentives program.

New Policy Proposal: If an O donor with a compatible non-O patient (or if an AB patient with a compatible non-AB donor) participates in kidney exchange, even though he does not need to, then the patient is given priority in the deceased donor queue in case he needs another kidney in the future.

Under our proposed incentives scheme, participation of compatible pairs is incentivized with an "insurance" for a potential future renal failure. This insurance is of value to patients because transplanted kidneys last well below 20 years on average, and about 15% of kidney transplants are repeat transplants. Our policy proposal might receive wider acceptance in the medical community than cash incentives because such priority is already given to living donors: if a previous living donor needs a kidney transplant in the future, she is prioritized in the deceased donor queue. If adopted, our incentive scheme might confer a major advantage on the US national kidney exchange program run by the United Network for Organ Sharing (UNOS), since UNOS is also in charge of the deceased donor queue. It would not be unrealistic to expect the national kidney exchange program to thrive under this new policy. Using our model we analyze the impact of the introduction of our incentives scheme on the welfare of patient population and analytically show that it increases the welfare of all patient groups. Moreover, for realistic parameters, it also decreases inequity across patients of different blood types. We also consider a model where patients can be listed in multiple exchange programs, and show that at equilibrium the national program that adopts our incentives program emerges as the only major program.

2 A Dynamic Model of Transplant Patients

We consider a comprehensive dynamic organ transplantation model (for organs such as the heart, kidney, liver, and pancreas) in which the deceased donor queue, live donation possibilities for kidneys and livers, and living donor kidney and liver exchange can be incorporated. We consider a continuum flow model in analysis where the cardinality of patients and donors who have arrived at the same time are measured through one dimensional Lebesgue measures at a steady state. We refer to this cardinality per unit time as **measure**.

Consider patients who need a particular organ transplant. Each **patient** is represented by his blood type $X \in \mathcal{T} = \{A, B, AB, O\}$. Suppose p_X refers to the probability of having the X blood type in the population distribution. We refer to the arrival measures of patients or donors as **inflow rates**. We assume that π_X is the inflow rate of new blood-type X patients. Suppose that in the population of new patients, the expected lifetime while living with the disease is distributed with a strictly increasing differentiable distribution function $F(\cdot)$[3] on the interval $[0, T]$.[4] Thus, the measure of X blood-type patients who are alive after t years on is given by $\pi_X[1 - F(t)]$.[5]

In Table 1, survival rates, $1 - F(t)$, for kidneys are listed.[6] At the steady state, when transplantation option is not present, the total mass of X patients is $\int_0^T \pi_X[1 - F(t)]dt$.[7] (cf. Figure 1.)

	Time					
	6 mo.	1 yr.	2 yr.	3 yr.	4 yr.	5 yr.
On dialysis (for kidneys)	84%	75%	61%	50%	42%	34%

Table 1: Survival rates $(1 - F(t))$ for kidney failures in the US.

[3] I.e., the probability density function $f(\cdot)$ is well defined and positive in $(0, T)$.

[4] This expectancy is different for different organs due to disease progression and techniques that can be used to substitute for the deficiency in the body because of the failing organ. For example, kidney patients who can live on dialysis have in general longer survival times.

[5] Hence, $\pi_X dt$ is the two dimensional *Lebesgue measure* of patients who enter in a small time interval dt. By a slight abuse of terminology, throughout the paper we will refer to the two dimensional Lebesgue measures of agent sets, such as $\pi_X dt$, as **mass**.

[6] The kidney data include 2005 estimates for dialysis patients reported in the National Kidney Organization 2012 Annual Report retrieved from http://www.usrds.org/2012/pdf/v2_ch5_12.pdf on 02/25/2012.

[7] Although we assume that inflow rate of patients is constant over time, we could easily make it a function of time as well. For example, population growth is a reason for increase of inflow rate. Increase in longevity is another reason, which not only affects π_X but also F, as older people have a higher tendency to need organ transplantation. These can be incorporated in our model easily. In that case a steady state does not exist. However, we can carry on all of our analyses in this paper and draw similar results in that model as a function of time. For simplicity and transparency of our analyses, we will use a model with constant inflow rates.

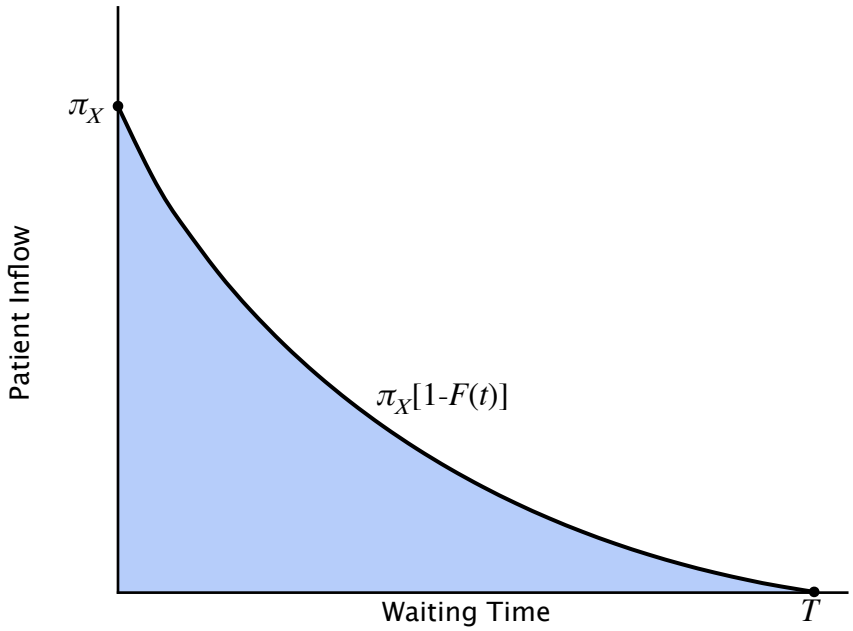

Figure 1: Steady-state X patient distribution over waiting time when organ transplantation is not possible. The shaded area is the mass of patients who are alive at any point in time.

3 Organ Transplantation and Deceased Donor Queue

The best remedy for organ failure is transplantation. A donor must be both blood- and tissue-type compatible with the patient before her organ(s) can be transplanted. O donors are blood-type compatible with all patients. A donors are blood-type compatible with A and AB patients, and B donors are blood-type compatible with B and AB patients. On the other hand, AB donors are blood-type compatible only with AB patients. **Blood-type compatibility** is formally defined through a partial order $\triangleright$ over blood types, such that $X \triangleright Y$ means that X donors are blood-type compatible with Y patients. Blood type distribution among US ethnic groups is reported in Table 2.[8] In general, O blood type is the most common, while AB is the rarest, A is observed more commonly than B, while their rates vary substantially across ethic groups: B has a strong presence among Asian- and African-American groups, while this is not the case for white Americans. The regional blood-type distributions are similar geographically according to the origins of the US ethnic groups.

Once a donor is deemed compatible with a patient, she also has to be tissue-type compatible with the patient. Tissue-type compatibility requires that the patient's body has no pre-formed antibodies against the donor's DNA. Throughout the paper we assume that given a patient and a blood-type compatible donor, **tissue rejection** occurs with a probability $\theta < 1$.[9] For some organs,

[8]Retrieved from http://bloodbook.com on 03/18/2013. The US general population is constructed using the ethnic proportions and could be slightly different from the general distributions reported in other sources.

[9]In real life, tissue rejection probability may be different across the patient population. In those cases, we can

such as the liver, tissue rejection is not an important problem. In those cases, we can assume $\theta \approx 0$. On the the other hand, for other organs, such as the kidney, tissue rejection rate is significant, and hence, $\theta > 0$.

	Blood Types				Pop. %
	O	A	B	AB	— (1992)
African American	49%	27%	20%	4%	12.4%
Asian American	40%	28%	27%	5%	3.3%
Native American	79%	16%	4%	1%	0.8%
White American	45%	40%	11%	4%	83.4%
US all	45.6 %	37.8%	12.6%	4%	

Table 2: Blood Type Distribution in the US.

A common source of donation across organs is deceased donors. The deceased donor queue is governed by a central entity. For example, in the US, for all organ types, UNOS is the federal contractor that is in charge of the queue. We assume throughout the paper that any patient enrolled in the queue remains in the queue until he receives a transplant or he dies.

We denote the inflow rate of the X deceased donors as $\delta_X < \pi_X$ per unit time. Across blood types, the ratio δ_X/π_X need not be constant. For example, it is well known that among minority communities, organ failure is more prominent than among the white American population, even though deceased donation rates are not that significantly different. As the blood-type distribution of minorities is different from the white American population, the ratio δ_X/π_X is not constant across blood types in the US: while a very high percentage of the donors, live or deceased, are white Americans, the patient rate of white Americans is much lower than their donation rate for kidneys and is higher only for lungs. On the other hand, for kidneys and hearts, the patient rate of African-Americans is higher than their donation rate; while for kidneys and livers, the patient rate of Asian-Americans is higher than their donation rate.[10] Although these rates are distorted by many other factors such as live donation possibilities, we can conclude that especially for kidneys the ratio δ_B/π_B is lower than that for other blood types.

When a transplanted organ, i.e., graft, fails, the recipient reenters the deceased donor queue as if he were a new patient. Repeat patients' survival function on the deceased donor queue is "similar to" that of new entrants (for example, that is the case for kidneys), so we assume $1 - F$ is also their survival function. We assume that ϕ^d is the steady-state fraction of the previous recipients whose grafts fail and who reenter the deceased queue per new deceased donor transplant

instead assume, the rejection probability is a random variable $\hat{\theta}$ in $[0, 1)$ with a continuous and integrable density function and a well-defined mean θ, which is independently distributed from other attributes of a patient. This would also work for our purposes.

[10]From the US Department of Health and Human Services - The Office of Minority Health web page for organ donation https://minorityhealth.hhs.gov/templates/browse.aspx?lvl=3 & lvlid=12 retrieved on 02/25/2013.

conducted.[11,12] Thus, if at steady state a ε measure of X patients receive a deceased donor organ at each instance, then a $\phi^d\varepsilon$ measure of previous recipients will reenter the queue at each instance. In 2005, $13.5\%, 7.9\%, 4.1\%, 5.5\%$ of all entering kidney, liver, heart, and lung patients, respectively, were reentrants (Magee et al., 2007). In general, allocation policies do not differentiate primary transplant patients from repeat transplant patients.

3.1 Deceased Donor Allocation Policies

The deceased donor organs are allocated through the points system of UNOS, which is a priority mechanism. When a deceased donor arrives, the point total for each compatible patient is determined. The organ is offered to the patient with the highest point total. If it is rejected by the patient or his doctor for any reason, then the organ is offered to the next patient, and so on. In general, different point schemes are used for different organs. Deceased donor allocation policies usually differ across organs and across geographic transplant regions, although usually a centralized mechanism is used in allocation. For example, for kidneys, at least on paper, *ABO-identical* allocation policies are applied, while for organs for which medical urgency matters more, *ABO-compatible* allocation is more common. That is, in the **ABO-identical** allocation policy, kidneys of blood-type X are offered only to blood-type X patients.[13] On the other hand, in the **ABO-compatible** allocation policy, organs can be offered to any compatible patients. We inspect the welfare and distributional consequences of these two policies separately.

Given a fixed blood-type allocation policy, the waiting time of a patient is often the most significant contributor to the patient's points of in deceased donor allocation for many organs such as kidney, pancreas, or heart. Therefore, we will model deceased donor allocation using **first-in-first-out** (**FIFO** from now on) queues for both the ABO-identical and ABO-compatible allocation schemes.[14]

[11] Fraction ϕ^d is formally calculated as follows: Suppose a measure ε of patients receive transplants at steady state at each instance. If the patient's life with a healthy graft ends, two things could be the reason: either the patient dies, or the patient stays alive but his graft fails. Of the patients who leave the status of "living with a healthy graft," let $h_1(t)$ be the fraction that die after t years from the transplant and $h_2(t)$ be the fraction whose grafts fail after t years from the transplant. Thus, we assume that a random patient's expected lifetime with a healthy graft is distributed with a differentiable distribution function $H(\cdot)$ in some interval $[0, S]$ such that $\frac{dH(t)}{dt} \equiv h(t) \equiv h_1(t) + h_2(t)$ where t refers to the years that passed since the transplant. We assume that this distribution is independent of how long the patient waited initially in the queue to receive his previous transplant. Then the inflow rate of patients reentering the deceased donor queue is given by $\int_0^S \varepsilon h_2(t)dt = \varepsilon \int_0^S h_2(t)dt$. We set $\phi^d = \int_0^S h_2(t)d(t)$. Observe that $\phi^d < \int_0^S h(t)d(t) = 1$.

[12] For simplicity, we assume that it is constant, although it may possibly change as the age distribution of the patients receiving transplants changes in the deceased donor queue, i.e., it may be a function of the waiting time.

[13] In the event that no X patient is available, then the organ is offered to a compatible patient. However, this is the application in the US. On the other hand, Eurotransplant uses full ABO - compatible scheme, and UK Transplant permits O organs to be transplanted to B patients, especially for kidneys (cf. Canadian Council of Transplantation documentation for "Deceased donor allocation in US, Europe, Australia, and New Zealand" released in October 2006).

[14] UNOS has switched to a new deceased donor kidney allocation scheme that will use a quality-based allocation

We use the notation $t_X^{\mathbf{s_1},\mathbf{s_2};s_3}$ for the average waiting times conditional on receiving transplants. In this notation, the subscript refers to the patient's blood type. Moreover:

1. Superscript $\mathbf{s_1}$ refers to the population of patients, with $\mathbf{s_1} = \mathbf{a}$ to denote the waiting time for all deceased and live donation recipients and $\mathbf{s_1} = \mathbf{q}$ to denote the waiting time in the deceased donor queue or for living donor exchange pool for a specific patient group, whichever is appropriate.

2. Superscript $\mathbf{s_2}$ is the transplantation technology we explore with

   - $\mathbf{s_2} = \mathbf{d}$ to denote deceased donor transplantation only, to which we will refer as **deceased donor transplantation** technology for short,
   - $\mathbf{s_2} = \mathbf{l}$ to denote deceased and living donor transplantation, to which we will refer as **living donor transplantation** technology for short,
   - $\mathbf{s_2} = \mathbf{e}$ to denote deceased and living donor transplantation with incompatible pair exchange, to which we will refer as (**regular**) **exchange** technology for short, and
   - $\mathbf{s_2} = \mathbf{i}$ to denote deceased and living donor transplantation with incompatible and incentivized compatible pair exchange, to which we will refer as **incentivized exchange** technology for short.

3. Superscript s_3 refers to deceased donor allocation policy with $s_3 = i$ to denote ABO-identical FIFO allocation and $s_3 = c$ to denote ABO-compatible FIFO allocation.

We state the following lemma, which will help us model the steady state of the deceased donor queue.[15]

Lemma 1 (FIFO matching protocol) *Consider the FIFO matching protocol. Suppose that there is an ordered ω measure of X patients available in the queue and a $\sigma \leq \omega$ measure of blood-type compatible donors arrive. Then*

1. *if $\sigma = \omega$, then all donors, except possibly a finite (and thus of 0 measure) of them, are almost surely matched; and*

2. *if $\sigma < \omega$, then all donors are almost surely matched.*

3.2 Steady State of the Deceased Donor Queue

We are ready to characterize the steady state of the deceased donor queue under the two FIFO allocation policies.

scheme for 20% of all allocation, while 80% of all allocation will continue to be done through its current FIFO-type policy.

[15] This is in spirit similar to the Erdös and Rényi (1960) random graph convergence result. However, in substance it is different, as we are not using the maximal matching policy as in Erdös and Rényi (1960) but rather the FIFO matching policy.

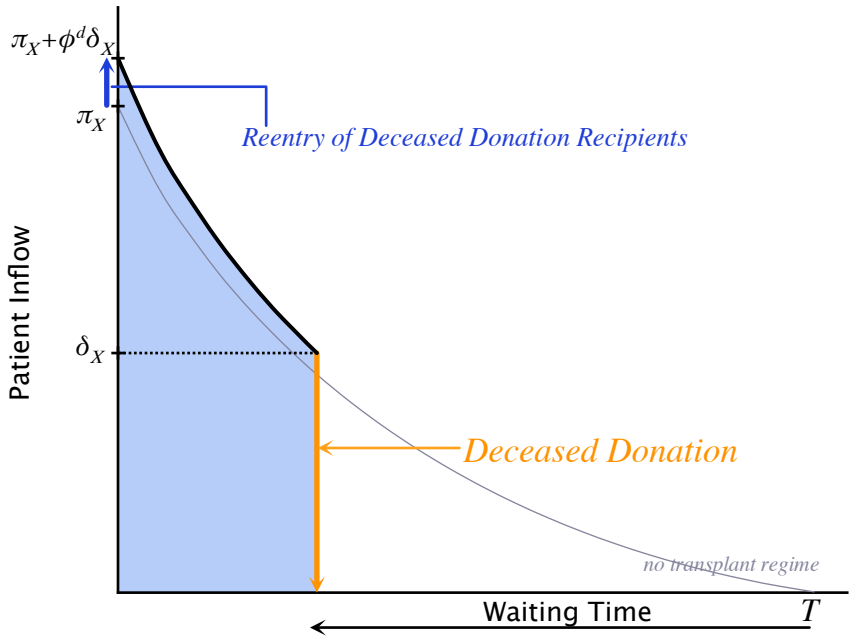

Figure 2: Steady-state X deceased donor queue under the ABO-identical **deceased donor transplantation** technology: Incoming deceased donors, of a δ_X measure, are matched with a δ_X measure of the longest waiting patients at each time.

3.2.1 ABO-Identical Deceased Donor Transplantation

Consider any blood type X. In the steady state, as $\delta_X < \pi_X$, there will always be a positive mass of X patients available in the deceased donor queue. Moreover, as FIFO protocol is used, the organs of the δ_X measure will be allocated to the longest-waiting X patients in the queue. Thus, by Lemma 1, these donors will be almost surely matched to the longest waiting cohort of δ_X measure of patients. We make the following observation regarding reentries to the queue:

Observation 1 *Under the ABO-identical deceased donor allocation policy, as a δ_X measure of X patients receive transplants per unit time, a $\phi^d\delta_X$ measure of previous recipients reenter the deceased donor queue per unit time due to graft failure.*

Let the receiving cohort have arrived t_X years before the current point in time. As there is a $[\pi_X + \phi^d\delta_X][1 - F(t_X)]$ measure of patients in this cohort including reentries and new arrivals, we should have

$$[\pi_X + \phi^d\delta_X][1 - F(t_X)] = \delta_X.$$

Hence, at steady state, the time spent on the X queue by the receiving cohort can be found through $t_X = F^{-1}(1 - \frac{\delta_X}{\pi_X+\phi^d\delta_X}) < T = F^{-1}(1)$. This is also the waiting time for X patients conditional on survival. Based on this analysis, we state the following characterization of the deceased donor queue at steady state. (cf. also Figure 2.)

Theorem 1 (ABO-identical deceased donor transplantation) *Under the ABO-identical deceased donor transplantation technology, at steady state, the waiting time for X patients conditional on receiving a transplant is*

$$t_X^{\mathbf{q},\mathbf{d};i} = F^{-1}\big(1 - \tfrac{\delta_X}{\pi_X + \phi^d \delta_X}\big), \tag{1}$$

which is also the average waiting time conditional on getting a transplant. Moreover, $\frac{\delta_X}{\pi_X+\phi^d\delta_X}$ is the probability of a patient ever receiving a transplant. The mass of the patients in the deceased donor queue is

$$\int_0^{t_X^{\mathbf{q},\mathbf{d};i}} [\pi_X + \phi^d \delta_X][1 - F(t)]dt.$$

Proof. Immediately follows from the analysis preceding the theorem.[16] ■

3.2.2 ABO-Compatible Deceased Donor Transplantation

The following lemmata analyze the role of blood-type compatibility in the waiting times of different blood types under the ABO-compatible deceased donor transplantation technology.

Lemma 2 *Let $X \neq Y$ be two blood types such that $X \triangleright Y$. Then under the ABO-compatible deceased donor transplantation technology, waiting times of X and Y patients at steady state satisfy $t_Y^{\mathbf{q},\mathbf{d};c} \leq t_X^{\mathbf{q},\mathbf{d};c}$.*

We make the following formal definition of *pooled* blood types:

Definition 1 *If blood types in some $\mathcal{S} \subseteq \mathcal{T}$ donate organs only to the blood types in $\mathcal{S}$ and they receive organs only from blood types in $\mathcal{S}$ at steady state, and there is no proper subset of $\mathcal{S}$ with this property, then we say that blood types in $\mathcal{S}$ are **pooled**.*

For example, if O organs are transplanted to A and B patients besides O, and A and B organs are only transplanted to A and B patients, respectively, then $\{O, A, B\}$ is a pooled set. On the other hand, neither $\{O, A\}$ is pooled (as O organs are also transplanted to B patients) or $\{A, B\}$ is pooled (as both A and B patients also receive O organs). The whole blood type set $\mathcal{T} = \{O, A, B, AB\}$ is not pooled, either, as its proper subset $\{O, A, B\}$ is pooled. Lemma 3 characterizes the waiting times of pooled blood types:

Lemma 3 *For two distinct blood types X and Y, if Y patients receive X organs at steady state under the ABO-compatible deceased donor transplantation technology, then $t_X^{\mathbf{q},\mathbf{d};c} = t_Y^{\mathbf{q},\mathbf{d};c}$.*

Moreover, if blood types in $\mathcal{S} \subseteq \mathcal{T}$ are pooled together, then the waiting time of each $X \in \mathcal{S}$ is given by

$$t_X^{\mathbf{q},\mathbf{d};c} = t_{\mathcal{S}} \equiv F^{-1}\left(1 - \frac{\sum_{X\in\mathcal{S}} \delta_X}{\sum_{X\in\mathcal{S}} (\pi_X + \phi^d \delta_X)}\right) \tag{2}$$

[16] Average and deceased donor waiting times are identical, as the only means of transplantation is deceased donors under this technology.

Observe that $t_X^{\mathbf{q},\mathbf{d};i} = t_{\{X\}}$ as defined in Equation 2 for all blood types X.

Using Lemmata 2 and 3 together with the FIFO feature of the deceased donor allocation policy and the partial order structure of the blood-type compatibility relationship, we can determine which types will be pooled together under the ABO-compatible deceased donor transplantation technology:

Theorem 2 (ABO-compatible deceased donor transplantation) *At steady state, suppose Y blood type has the longest ABO-identical allocation waiting time and X blood type has the shortest ABO-identical allocation time among all blood types W with $W \triangleright Y$. Suppose further that $t_X^{\mathbf{q},\mathbf{d};i} < t_Y^{\mathbf{q},\mathbf{d};i}$. Then X and Y patients will be pooled together (possibly with other types) under ABO-compatible allocation. Moreover, we can treat X and Y together as a* composite blood type *$\{X,Y\}$ with deceased donor inflow rate $\delta_{\{X,Y\}} = \delta_X + \delta_Y$ and patient inflow rate $\pi_{\{X,Y\}} = \pi_X + \pi_Y$ such that $W \triangleright \{X,Y\}$ for all blood types W with $W \triangleright Y$, and $\{X,Y\} \triangleright Z$ for all blood types Z with $X \triangleright Z$.*

Theorem 2 can be used iteratively to determine the ABO-compatible deceased donor transplantation waiting times for all blood types with the simple mathematical fact that for all $a, b, c, d > 0$ whenever $\frac{a}{b} < \frac{c}{d}$ we have $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$:

Pooling procedure for blood types for ABO-compatible deceased donor transplantation:

1. Find all waiting times t_X as defined in Equation 2 for all $X \in \mathcal{T}$.[17]
2. Suppose X has the longest t_W among all $W \in \mathcal{T}$. Let Y have the shortest t_W among all $W \in \mathcal{T}$ with $W \triangleright X$.
   (a) If $Y = X$ then X is not pooled with any other blood type and $t_X^{\mathbf{q},\mathbf{d};c} = t_X$. Repeat Step 1 for the remaining blood types $\mathcal{T} \setminus \{X\}$.
   (b) If $Y \neq X$ then X is pooled with Y (possibly together with other types). Replace the two blood types X and Y with the composite blood type $\mathcal{S} = X \cup Y$ and update the blood-type compatibility partial order $\triangleright$ as defined in Theorem 2. Repeat Step 1 for the new blood type set $\mathcal{T} := \big(\mathcal{T} \setminus \{X,Y\}\big) \cup \{\mathcal{S}\}$.

4 Living Donor Transplantation

Organs such as the kidney, liver, and lung have live donation possibilities. Live donation is especially common for kidneys. In 2011, 34% of all kidney transplants in the US were from living donors.

We will refer to a living donor as a **paired donor**. We will assume that each patient has at most one paired donor. We assume that a $\lambda \in [0,1]$ fraction of incoming patients have a paired donor. We also assume that the blood types of the patient and the donor are independent and

[17] With a slight abuse of notation, even if X is not a set, it also refers to the set $\{X\}$.

uncorrelated. We will refer to a patient with a paired donor as a **paired patient** and a patient without a paired donor as a **single patient**.[18] The patient and his paired donor are represented as a **pair**. The blood types of the pair, $X - Y \in \mathcal{T} \times \mathcal{T}$, X being the patient's and Y being the donor's blood type, determine the **type of the pair**.

If the paired donor of a patient is both blood- and tissue-type compatible, we refer to the pair as a **compatible** pair, and otherwise as an **incompatible** pair. Recall that by assumption there is a θ probability chance that a blood-type-compatible donor is tissue-type incompatible with a patient. Given a paired patient, let p_Y be the probability of his paired donor to be blood type Y.

Transplanted organs from living donors can also fail, as in the case of transplants from deceased donors. As in the case of deceased donors, we assume that reentering patients have the same survival function $1 - F$ as new patients. However, it is well known that living donor grafts survive longer than deceased donor grafts. We assume that $\phi^l \leq \phi^d$ is the fraction of live donation recipients reentering the deceased donor queue per each living donor organ transplant at steady state. We further assume that the reentrants (who received a graft previously from either a deceased donor or a living donor) are single (and no longer paired) upon reentry.

Consistent with the donation rates throughout the world, in the rest of the paper we assume the following:

Assumption 1 *There is a shortage of deceased donor organs even in the absence of paired patients, i.e.,* $(1 - \lambda)\pi_X + \phi^d \delta_X \geq \delta_X$ *for all* $X \in \mathcal{T}$.

We can calculate the inflow rates of compatible and incompatible pair types:

- An O patient needs an O donor. Thus, $(1 - \theta)p_O \lambda \pi_O$ is the inflow rate of O patients with a compatible paired donor. On the other hand, $\theta p_O \lambda \pi_O$ is the measure of incompatible $O - O$ pairs, $p_Y \lambda \pi_O$ is the measure of $O - Y$ pairs with $Y \in \{A, B, AB\}$, who are all incompatible.

- An $X \in \{A, B\}$ blood-type patient can get an organ from O or X donor. Thus, given $Y \in \{X, O\}$, $(1 - \theta)p_Y \lambda \pi_X$ is the inflow rate of X patients with a compatible Y living donor; on the other hand, $\theta p_Y \lambda \pi_X$ is the measure of incompatible $X - Y$ pairs. We have $p_Y \lambda \pi_X$ as the inflow rate of $X - Y$ pairs with $Y \in \{A, B, AB\} \setminus \{X\}$. The latter are incompatible pairs.

- An AB patient can get an organ from all blood-type donors. Thus, $(1-\theta)p_Y \lambda \pi_{AB}$ is the inflow rate of compatible $AB - Y$ pairs, and $\theta p_Y \lambda \pi_{AB}$ is the inflow rate of incompatible $AB - Y$ pairs for all $Y \in \mathcal{T} = \{O, A, B, AB\}$.

For a paired patient of blood type X, let p_X^l denote the probability that his paired donor is compatible with the patient. Thus, $p_X^l \lambda \pi_X$ is the inflow rate of X patients with compatible living donors.

[18] In reality, if the paired donor is a blood relative of the patient, the blood types of the patient and donor are correlated through degree of relation and laws of genetics. Hence, potentially figuring out the exact correlation can be complicated. For our purposes, we simply assume that the blood types of the patient and his paired donor are uncorrelated to make our arguments.

These patients receive organs from their paired donors upon entry, and they do not wait in the deceased donor queue. We make the following observation regarding the allocation and reentry measures of deceased and living donor organ recipients:

Observation 2 *At steady state,*

- *a $p^l_X \lambda \pi_X$ measure of X patients receive live donation per unit time without waiting in the deceased donor queue, and hence, a $\phi^l p^l_X \lambda \pi_X$ measure of previous live donation recipients reenter the deceased donor queue per unit time; and*
- *a δ_X measure of X patients receive deceased donor organs per unit time under ABO-identical FIFO allocation policy, and hence, a $\phi^d \delta_X \pi_X$ measure of previous deceased donation recipients reenter the queue per unit time.*

Hence, the total inflow rate of patients entering or reentering the X deceased donor queue under the ABO-identical FIFO allocation policy is given as

$$\pi_X^{\mathbf{q},\mathbf{l};i} = \underbrace{\pi_X}_{\text{new patients}} + \underbrace{\phi^d \delta_X}_{\text{reentry / deceased}} + \underbrace{\phi^l p^l_X \lambda \pi_X}_{\text{reentry / live}} - \underbrace{p^l_X \lambda \pi_X}_{=\mathbf{l}_X\text{: compatible pairs}}. \tag{3}$$

Above, "reentry / deceased" and "reentry / live" refer to the reentering previous deceased and living donor organ recipients, respectively. Equation 3 and Observation 2 imply that the ABO-identical allocation waiting time conditional on survival in the X deceased donor queue is given by (cf. Figure 3)

$$t_X^{\mathbf{q},\mathbf{l};i} = F^{-1}\left(1 - \frac{\delta_X}{\pi_X^{\mathbf{q},\mathbf{l};i}}\right). \tag{4}$$

The average waiting time for patients under living donor transplantation technology conditional on receiving a transplant is substantially less for all blood types than those under the deceased donor transplantation. Many patients have compatible living donors, and they immediately receive a transplant without waiting. Hence, the average waiting time is

$$t_X^{\mathbf{a},\mathbf{l};i} = \frac{\delta_X t_X^{\mathbf{q},\mathbf{l};i}}{\delta_X + p^l_X \lambda \pi_X} \tag{5}$$

conditional on receiving a transplant under the ABO-identical deceased donor allocation policy.

The analysis in Theorem 2 can be used to find which blood types are pooled together under the ABO-compatible deceased donor allocation policy by using $\pi_X + \phi^l p^l_X \lambda \pi_X - p^l_X \lambda \pi_X$ instead of π_X for all X. This analysis also helps us pin down the waiting times in the deceased donor queue under ABO-compatible allocation. In particular, we will make use of the following lemma:

Lemma 4 *Fix a blood type X. Under living donor transplantation, the ABO-compatible deceased donor allocation waiting time for every blood type Y, $t_Y^{\mathbf{q},\mathbf{d};c}$, is continuous and weakly increasing in π_X and continuous and weakly decreasing in δ_X; moreover, $t_X^{\mathbf{q},\mathbf{d};c}$ is strictly increasing in π_X and strictly decreasing in δ_X.*

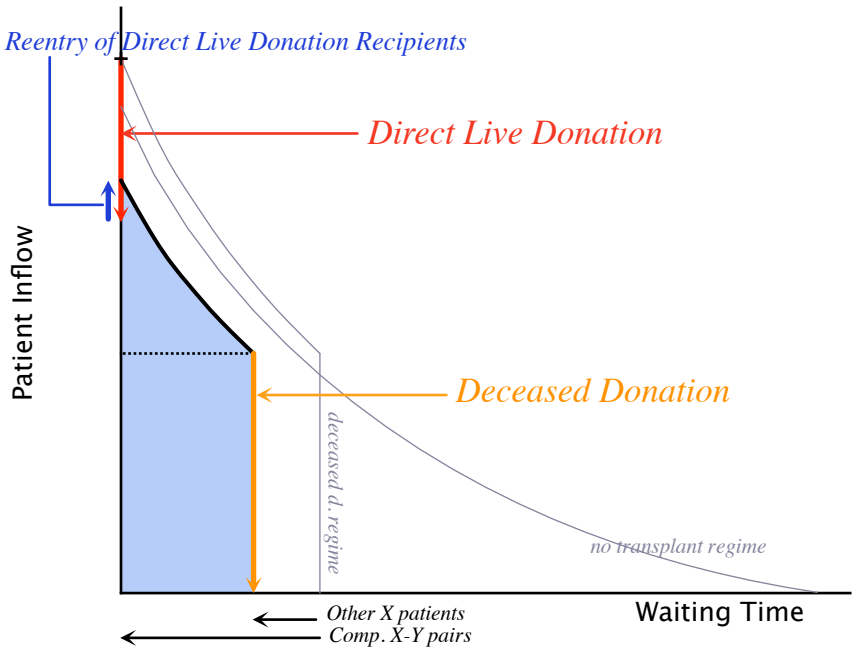

Figure 3: Steady-state X patient deceased donor queue under the **living donor transplantation** technology with ABO-identical deceased donor allocation: Inflow rate π_X of patients increases by the inflow rate of reentering previous deceased and live donation recipients, $\phi^d \delta_X$ and $\phi^l p^l_X \lambda \pi_X$, respectively; and decreases by $p^l_X \lambda \pi_X$, the outflow of paired patients who immediately receive an organ from their compatible donors.

We are ready to make a more detailed analysis of how different blood types are affected by the availability of live donation. Due to the partial-order structure of blood-type compatibility across blood types, not all blood types will be affected equally when live donation is possible. For example, O blood-type paired patients are at a disadvantage in finding a compatible paired donor. In general, A blood type is more prominent in the population than B. Therefore, at random, A blood-type paired patients will have a higher chance of finding a compatible donor than B types, given that they can all receive from O blood-type donors as well as their own types. Finally, AB blood-type paired patients have the highest chance of a compatible paired donor.

However, depending on the exact shape of the survival function, $1-F$ and the deceased-donor-to-new-patient inflow rate ratios across blood types, δ_X/π_X, O blood type does not necessarily experience the lowest decrease in waiting time, and AB blood type does not necessarily experience the greatest improvement.

On the other hand, for the benchmark case, where δ_X/π_X, the deceased donor to new patient inflow rate ratio, is the same for each blood type, we can make unambiguous predictions.[19]

Theorem 3 (Living donor transplantation and inequity in waiting times) *Suppose Assumption 1 holds. Living donor transplantation will unambiguously decrease the steady state ABO-identical and ABO-compatible deceased donor and overall average donor waiting times for all blood types.*

Consider the benchmark case that the ratio δ_X/π_X is constant across all $X \in \mathcal{T}$. Then, under living donor transplantation, no blood types pool under ABO-compatible deceased donor allocation. Furthermore, both for ABO-identical or ABO-compatible policies and both for deceased and living donor transplantation technologies, the following hold:

- *O patients have the lowest waiting time decrease;*
- *AB patients have the highest waiting time decrease; and*
- *provided that $p_A > p_B$, A patients have a higher waiting time decrease than B patients.*

In particular, if $p_A > p_B$, then the deceased donor queue waiting times (conditional on receiving a transplant) satisfy $t_O^{\mathbf{q},\mathbf{l};c} = t_O^{\mathbf{q},\mathbf{l};i} > t_B^{\mathbf{q},\mathbf{l};c} = t_B^{\mathbf{q},\mathbf{l};i} > t_A^{\mathbf{q},\mathbf{l};c} = t_A^{\mathbf{q},\mathbf{l};i} > t_{AB}^{\mathbf{q},\mathbf{l};c} = t_{AB}^{\mathbf{q},\mathbf{l};i}$, and the average waiting times (conditional on receiving a transplant) satisfy $t_O^{\mathbf{a},\mathbf{l};c} = t_O^{\mathbf{a},\mathbf{l};i} > t_B^{\mathbf{a},\mathbf{l};c} = t_B^{\mathbf{a},\mathbf{l};i} > t_A^{\mathbf{a},\mathbf{l};c} = t_A^{\mathbf{a},\mathbf{l};i} > t_{AB}^{\mathbf{a},\mathbf{l};c} = t_{AB}^{\mathbf{a},\mathbf{l};i}$.

Another simpler metric we can use to measure efficiency and inequity of different transplantation technologies is to compare the ratio of the transplant measure to the new patient inflow rate. We refer to this metric as the *transplant ratio* of a technology. It will be a crucial metric for

[19] Although these conclusions seem to have been reached with the help of our assumption that blood types of patients are uncorrelated with their paired donors, a version of this result will also hold true even if there is positive correlation in a pair's blood types; however, the magnitude of the difference in eventual waiting times will not be as extreme.

comparison for marginal inequality caused by different transplantation technologies. This will be also independent of additional assumptions on the shape of survival rate function $1 - F$.

Let $\mathbf{l}_X = p_X^l \lambda \pi_X$ be the inflow rate of X patients with compatible donors. Then $\mathbf{l}_X/\pi_X$ is the **live donation transplant ratio** for X patients, and δ_X/π_X is the blood type X **deceased donation transplant ratio** under the ABO-identical policy. We have the following result:

Theorem 4 (Living donor transplantation and inequity in transplant ratios) *Suppose Assumption 1 holds. Living donor transplantation unambiguously increases transplant ratios for all blood types. Moreover, live donation transplant ratios satisfy* $\mathbf{l}_O/\pi_O < \mathbf{l}_A/\pi_A, \mathbf{l}_B/\pi_B < \mathbf{l}_{AB}/\pi_{AB}$, *i.e., O patients benefit marginally the least and AB patients marginally benefit the most from living donor transplantation technology. Additionally, if* $p_A > p_B$, *then* $\mathbf{l}_B/\pi_B < \mathbf{l}_A/\pi_A$, *i.e., A patients marginally benefit more than B patients.*

5 Living Donor Exchange

In this section we analyze the effect of having a living donor exchange program on waiting times of different patient groups. In practice, a paired donor usually donates directly to her paired patient, and the patient leaves the pool before he ever enters the deceased donor queue. For the incompatible pairs, we assume that a living donor exchange program operates in parallel with the deceased donor queue. Incompatible pairs are listed in the exchange program. While waiting for a deceased donor organ in the queue, patients also wait for an exchange to be conducted with another incompatible pair.

Formally, a two-way **exchange** matches two pairs where the patient of the first pair is compatible with the donor of the second pair and the patient of the second pair is compatible with the donor of the first pair. We refer to such pairs as **mutually compatible** pairs.[20] We also say that if the donor of the first pair is blood-type compatible with the patient of the second pair and vice versa, then these pairs are **mutually blood-type compatible**. We refer to the queue of the pairs in the exchange program as the **exchange pool**. An **exchange matching** is a set of exchanges between mutually compatible pairs such that each pair is matched in at most one exchange. For a given pair type $X - Y$, we refer to $Y - X$ as its **reciprocal** type.

We will assume that the donor exchange is conducted in an *optimal manner* by matching the most measures of pairs at each point in time.[21]

Not all incompatible pairs are relatively abundant. For example, far fewer measures of incompatible $A - O$ patient-donor pairs exist in the exchange pool than $O - A$ pairs. $A - O$ pairs are

[20] We can also think of exchanges that can match more than two pairs, such as three-way, four-way, etc. For simplicity, we focus on two-way exchanges in our analysis. However, our results can easily be extended to cover larger exchange sizes as in Roth, Sönmez, and Ünver (2007). Any sizes of exchanges greater than four will not change the results as reported in that paper.

[21] This myopic exchange method turns out to be also dynamically optimal. While selecting among a particular pair from a given type $X - Y$, organ exchange is also performed on a FIFO basis (cf. Theorem 5).

incompatible only if there is tissue incompatibility between the A patient and O donor, while $O-A$ pairs are always incompatible.

Based on this observation, we make the following assumption:

Assumption 2 *For any incompatible pair type $X-Y$ such that $X \neq Y$ and $X \triangleright Y$, its inflow rate to the exchange pool is not less than the inflow rate of its reciprocal type $Y-X$, i.e., $\theta p_X \pi_Y \leq p_Y \pi_X$.*[22]

Another assumption concerns the prevalence of $A-B$ and $B-A$ types. This assumption is made for notational convenience; a symmetric version of the results would hold if we did not make this assumption, without loss of generality.

Assumption 3 *$A-B$ pairs do not inflow any slower than $B-A$ pairs to the exchange pool, i.e., $p_A \pi_B \leq p_B \pi_A$.*[23]

To give an idea of how easily this assumption is satisfied, recall that for kidneys, we have $\theta \approx 0.1$ and for livers, $\theta = 0$. For all organs with exchange programs, this inequality holds with a good deal of slack for all populations.

Through Assumptions 2 and 3, all incompatible $X-Y$ pairs with $Y \triangleright X$ and $X-Y = B-A$ pairs can be matched immediately with $Y-X$ pairs, as $Y-X$ pairs will always be more in mass than $X-Y$ pairs in the exchange pool. Observe that the probability of mutual compatibility between an $X-Y$ pair and and a $Y-X$ pair is $(1-\theta)^2 > 0$. We state a slightly different version of Lemma 1 for exchange:

Lemma 5 (Exchange matching protocol) *Consider an ω measure of pairs denoted by the set M and a $\sigma \leq \omega$ measure of pairs denoted by set N (possibly intersecting with M), that are mutually blood-type compatible with the pairs in M. Suppose these sets are formed randomly using the governing population distributions. Then, there almost surely exists an exchange matching that matches all pairs in N with pairs in M.*

Proof. It follows from the random graph convergence theorem of Erdös and Rényi (1960). ■

Using the terminology in Ünver (2010), we classify the pairs into several categories, based on their desirability in exchange.

Overdemanded pair types are the ones with a blood type donor who can donate to her patient's blood type yet who is not of the same blood type. These are $A-O, B-O, AB-A, AB-B, AB-O$ types. **Underdemanded pair types** are those with a blood type donor who cannot feasibly donate to her patient's blood type, excluding types A-B and B-A. That is,

[22] A simple requirement that would make the assumption hold is that the ratio of live donation and patient inflow rates are similar across blood types; i.e., $p_X/\pi_X \approx p_Y/\pi_Y$ for all blood types X, Y. This would be ensured if live donation and illness rates are not too different for different blood types.

[23] On a separate note, for kidneys Terasaki, Gjertson, and Cecka (1998) report that $A-B$ pairs make up of 5% of all pairs while $B-A$ pairs make up of 3%. However, our assumption has nothing to do with this observation.

underdemanded types are reciprocals of overdemanded types, i.e., $O-A, B-O, A-AB, B-AB, O-AB$. **Reciprocally demanded pair types** are $A-B$ and $B-A$, as they can be matched with each other in a donor exchange, when tissue incompatibility does not exist. Finally, **self-demanded pair types** are those with the same blood-type donor and patient: $O-O, A-A, B-B, AB-AB$.

The names associated with these classes will be more meaningful after our analysis. The following lemma shows the role of overdemanded types in exchange (similar results were also reported in Roth, Sönmez, and Ünver, 2007; Ünver, 2010):

Lemma 6 (Exchange blood-type feasibility) *An underdemanded type pair can be matched only with an overdemanded type pair in an exchange. An overdemanded type pair can be matched with an overdemanded, underdemanded, reciprocally demanded, or self-demanded type pair. A reciprocally demanded type pair can be matched with a (reciprocal of its type) reciprocally demanded or overdemanded type pair. A self-demanded type pair can be matched with a same type or overdemanded type pair. In particular:*

- *An underdemanded $O-A$ (or $O-B$) pair can be matched only with a pair from overdemanded types $A-O$ (or $B-O$) or $AB-O$. An underdemanded $A-AB$ (or $B-AB$) pair can be matched only with a pair from overdemanded types $AB-A$ (or $AB-B$) or $AB-O$. An underdemanded $O-AB$ pair can be matched only with an overdemanded $AB-O$ pair.*

- *A reciprocally demanded $A-B$ (or $B-A$) pair can be matched only with a pair from the other reciprocally demanded type $B-A$ (or $A-B$) or overdemanded types $AB-A$ (or $AB-B$) or $AB-O$.*

- *A self-demanded $X-X$ pair can be matched with a same type pair. Additionally, an $O-O$ pair can be matched only with a pair from overdemanded types $A-O, B-O$, or $AB-O$; an $A-A$ (or $B-B$) pair can be matched only with a pair from overdemanded types $AB-A$ (or $AB-B$) or $AB-O$; and an $AB-AB$ pair can be only matched with a pair from overdemanded types $AB-A, AB-B$, or $AB-O$.*

Next, we model how the exchange pool and deceased donor queue evolve at steady state. In this section, we focus on ABO-identical deceased donor allocation. Living donor exchange is mostly prevalent for kidneys, and kidney deceased donor allocation is mostly ABO-identical. Recall that only incompatible pairs participate in exchange. It turns out that we can conduct *optimal* two-way exchanges in an ABO-identical manner as well. We can match $X-Y$ pairs with $Y-X$ pairs as they become available. We show that this is **optimal** in the sense that the measure of exchange transplants at each instance is maximized. We discuss ABO-compatible policies for deceased donor allocation together with optimal exchange in Appendix A as it requires substantially different tools.

We characterize the FIFO ABO-identical exchange as an optimal policy as follows:[24]

[24] A result that is similar but logically independent form ours was proven in Ünver (2010) for discrete problems with waiting costs.

Theorem 5 (FIFO ABO-identical exchange is optimal) *Suppose Assumptions 2 and 3 hold. Then the exchange policy, immediately matching each arriving pair with the longest-waiting mutually compatible pair of its reciprocal type, is optimal.*

Moreover, this policy maximizes the mass of pairs that can be matched within any time interval. In particular, it matches a larger mass of pairs than the alternative policy of running the exchange only once at the end of the time interval.

With the availability of exchange, we separate patients into different **groups** based on their blood type and donor status as single, paired with a compatible donor, or paired with an incompatible donor. We can measure the efficiency and equity effects of each technological regime change on these groups. There are 29 patient groups based on these two criteria.

As we will prove (in Theorem 6 below) that compatible and incompatible pairs of blood-type compatible types receive transplants at time 0 (under the ABO-identical optimal exchange), we do not distinguish them in our discussion. Therefore, we denote each patient group by the pair type $X - Y$ if the patient is paired and by the blood type X if the patient is single.

Through Theorem 5, we compute the **measure of X patients matched through exchange** under the above-described optimal exchange policy, denoted as $\mathbf{e}_X$ for all $X \in \mathcal{T}$:

$$\begin{aligned}
\mathbf{e}_O &= \underbrace{\theta p_O \lambda \pi_O}_{O-O \text{ pairs}} + \underbrace{\theta p_O \lambda (\pi_A + \pi_B + \pi_{AB})}_{O-A,\, O-B,\, O-AB \text{ pairs}}, \\
\mathbf{e}_A &= \underbrace{\theta p_A \lambda \pi_A}_{A-A \text{ pairs}} + \underbrace{\theta p_O \lambda \pi_A}_{A-O \text{ pairs}} + \underbrace{p_A \lambda \pi_B}_{A-B \text{ pairs}} + \underbrace{\theta p_A \lambda \pi_{AB}}_{A-AB \text{ pairs}}, \\
\mathbf{e}_B &= \underbrace{\theta p_B \lambda \pi_B}_{B-B \text{ pairs}} + \underbrace{\theta p_O \lambda \pi_B}_{B-O \text{ pairs}} + \underbrace{p_A \lambda \pi_B}_{B-A \text{ pairs}} + \underbrace{\theta p_B \lambda \pi_{AB}}_{B-AB \text{ pairs}}, \text{ and} \\
\mathbf{e}_{AB} &= \underbrace{\theta p_{AB} \lambda \pi_{AB}}_{AB-AB \text{ pairs}} + \underbrace{\theta (p_O + p_A + p_B) \lambda \pi_{AB}}_{AB-O,\, AB-A,\, AB-B \text{ pairs}}.
\end{aligned} \tag{6}$$

We use these measures to analyze how the availability of exchange affects the waiting time in the deceased donor queue. We continue to focus on ABO-identical deceased donor allocation. As more patients receive living donor transplants under exchange technology than under living donor transplantation technology, the waiting times of patients improve across all blood types. Some of these pairs are matched immediately as they enter the pool. These belong to overdemanded or self-demanded types, or the less abundant reciprocal type $B - A$. And some pairs are matched only after waiting in the pool. As a result, not all of them receive transplants, since some of their paired patients die while waiting. These pairs belong to underdemanded types or the more abundant reciprocal type $A - B$. They wait in the exchange pool and the deceased donor queue simultaneously, and either

- are "pooled" with single patients of the same blood type in the deceased donor queue, so that simultaneously some of them will receive deceased donor organs and some will participate in exchange; or

- wait for less time than their cohort of single patients and participate exclusively in exchange.

To determine the waiting times, for each blood type X, let

$$\pi^e_{X-Y} = \begin{cases} \theta p_Y \lambda \pi_X & \text{if } Y \triangleright X \\ p_Y \lambda \pi_X & \text{otherwise} \end{cases} \tag{7}$$

refer to the **exchange pool $X-Y$ inflow rate**, and let

$$\pi^d_X = \underbrace{(1-\lambda)\pi_X}_{\text{new w/o living donors}} + \underbrace{\phi^d \delta_X}_{\text{reentry / deceased}} + \underbrace{\phi^l p^l_X \lambda \pi_X}_{\text{reentry / live}} + \underbrace{\phi^l \mathbf{e}_X}_{\text{reentry / exchange}} \tag{8}$$

be the **single X patient inflow rate** for reentrants and new single patients. We calculate the following ratios for each blood type X:

1. The ratio of deceased donor inflow rate to single patient inflow rate (for new patients and reentrants):
$$r^d_X = \frac{\delta_X}{\pi^d_X} = \frac{\delta_X}{(1-\lambda)\pi_X + \phi^d \delta_X + \phi^l p^l_X \lambda \pi_X + \phi^l \mathbf{e}_X}.$$
2. For each underdemanded type $X-Y$ (i.e., $Y \neq X$ and $Y \triangleright X$), the ratio of incompatible $Y-X$ inflow rate to $X-Y$ inflow rate :
$$r_{X-Y} = \frac{\pi^e_{Y-X}}{\pi^e_{X-Y}} = \frac{\theta p_X \lambda \pi_Y}{p_Y \lambda \pi_X}.$$
3. For reciprocal type $A-B$,
$$r_{A-B} = \frac{\pi^e_{B-A}}{\pi^e_{A-B}} = \frac{p_A \lambda \pi_B}{p_B \lambda \pi_A}.$$

Ratio $r^d_X = \frac{\delta_X}{\pi^d_X}$ would be relevant if we wanted to allocate all X deceased donors to only X single patients. For an underdemanded type $X-Y$ or $X-Y = A-B$, ratio $r_{X-Y} = \frac{\pi^e_{Y-X}}{\pi^e_{X-Y}}$ would be relevant if we did not want $X-Y$ pairs to receive deceased donation, but only to participate in ABO-identical optimal exchange. In these cases, conditional on survival, the waiting time of single X patients would be $t^d_X = F^{-1}\big(1 - \frac{\delta_X}{\pi^d_X}\big)$, and the waiting time of $X-Y$ pairs would be $t_{X-Y} = F^{-1}\big(1 - \frac{\pi^e_{Y-X}}{\pi^e_{X-Y}}\big)$.[25]

However, underdemanded or reciprocally demanded $X-Y$ pairs have another option besides waiting for their reciprocal type pairs. If available deceased donors arrive earlier, they can receive deceased donor transplants. We assume that patients accept the first donor who is offered to them

[25]The waiting time of $B-A$ is 0 as this type is on the shorter side of the market when compared to $A-B$, by Assumption 3.

through deceased donor allocation or exchange.[26] Hence, the patient of an $X-Y$ type pair will never wait for a $Y-X$ pair for exchange if a deceased organ comes first, i.e. if $t_{X-Y} < t^d_X$. As time is decreasing in r ratios, all we need to do is to compare these ratios in an iterative manner to decide whether any underdemanded type or $A-B$ type will receive a deceased donor transplant:

Exchange technology pooling procedure for single and paired patients under ABO-identical deceased donor allocation:

1. Let $X-Y_1, ..., X-Y_k$ be the ordered list of underdemanded or reciprocally demanded types ascending in r_{X-Y} ratio. Define for each $\ell = 0, ..., k$:

$$r^d_{X,X-Y_1,...,X-Y_\ell} = \frac{\delta_X + \pi^e_{Y_1-X} + ... + \pi^e_{Y_\ell-X}}{\pi^d_X + \pi^e_{X-Y_1} + ... + \pi^e_{X-Y_\ell}}. \quad (9)$$

2. For $\ell \in \{0, ..., k-1\}$, suppose types $X-Y_1, ..., X-Y_\ell$ have already been deemed to be receiving both deceased donor and exchange transplants.

   - If $r_{X-Y_{\ell+1}} < r^d_{X,X-Y_1,...,X-Y_\ell}$ then $X-Y_{\ell+1}$ pairs receive both exchange transplants and deceased donor transplants with the rest of the X single patients and $X-Y_1, ..., X-Y_\ell$ pairs. We continue with Step 2 with $\ell := \ell + 1$.
   - If $r_{X-Y_{\ell+1}} \geq r^d_{X,X-Y_1,...,X-Y_\ell}$ then all types $X-Y_{\ell+1}, ..., X-Y_k$ only receive exchange transplants, but no transplants from deceased donors. We terminate the procedure.[27]

Based on this procedure, we state the following theorem:

Theorem 6 (Welfare effects of ABO-identical optimal exchange) *Suppose Assumptions 1-3 hold. Consider the ABO-identical deceased donor allocation and optimal exchange policies. Consider a blood type X. Conditional on survival, the waiting time and the measure receiving donation are given for subgroups of X patients as follows:*

1. X paired patients with compatible donors immediately receive their donor's organ upon entry.

[26] This assumption can be rationalized by the risk associated with dying while waiting for an organ and high risk aversion. To model this choice explicitly under a wider class of preferences, we can introduce additional structure regarding the cardinal preferences of the patients and the shape of the survival distribution $1-F(t)$. The patients could be willing to wait more for a living donor than a deceased one since a transplant from the former survives longer. On the other hand, there is an associated trade-off since longer waits could result in death and with inferior life quality to living with a functioning graft. The patients will be willing to wait as long as the second disutility does not outweigh the first utility marginally. When $1-F(t)$ is concave (i.e., for $t < t'$ dying at time t' is more likely than at time t), an incentive compatibility constraint would lead to a waiting time gap between willingness to wait for exchange and for deceased donors: at steady state when patients can receive a deceased donor organ t years after entry, each patient will be willing to wait at most $\tau(t)$ years additionally for a living donor organ. All our calculations can be modified to include this time gap function without much change.

[27] When some $X-Y$ pairs receive deceased donor transplants and later reenter the pool, whether the patient of such a pair reenters as part of a new pair or he reenters without a living donor does not have any impact on waiting times. As $X-Y$ pairs will be pooled with X single patients, what matters is the total inflow rate of new and reentering $X-Y$ pairs and X single patients, which is the same under either assumption.

2. *X paired patients who are part of incompatible overdemanded or self-demanded type pairs and, if $X = B$, then of $B - A$ type pairs, immediately participate in an exchange upon entry.*

3. *Suppose patients of underdemanded and reciprocally demanded types $X - Y_1, ..., X - Y_\ell$ receive both deceased donor and exchange transplants while patients of underdemanded and reciprocally demanded types $X - Y_{\ell+1}, ..., X - Y_k$ receive only exchange transplants. Then:*

   - *Conditional on survival, X single patients and patients of $X - Y_1, ..., X - Y_\ell$ pairs wait for a deceased donor or exchange transplant for*

$$t_X^{\mathbf{q},\mathbf{e};i} = F^{-1}\left(1 - \frac{\delta_X + \pi^e_{Y_1-X} + ... + \pi^e_{Y_\ell-X}}{\pi^d_X + \pi^e_{X-Y_1} + ... + \pi^e_{X-Y_\ell}}\right). \tag{10}$$

   - *Conditional on survival, for all $m \in \{\ell + 1, ..., k\}$, patients of $X - Y_m$type pairs wait for an exchange transplant for*

$$t_{X-Y_m}^{\mathbf{q},\mathbf{e};i} = F^{-1}\left(1 - \frac{\pi^e_{Y_m-X}}{\pi^e_{X-Y_m}}\right). \tag{11}$$

   - *The average waiting time for all X patients conditional on receiving a transplant is given as*

$$t_X^{\mathbf{a},\mathbf{e};i} = \frac{\left[\delta_X + \sum_{m=1}^{\ell} \pi^e_{Y_m-X}\right] t_X^{\mathbf{q},\mathbf{e};i} + \left[\sum_{m=\ell+1}^{k} \left[\pi^e_{Y_m-X} t_{X-Y_m}^{\mathbf{q},\mathbf{e};i}\right]\right.}{\delta_X + \mathbf{e}_X + p^l_X \lambda \pi_X} \tag{12}$$

Proof. It follows from the procedure discussed before the statement of the theorem. For Equation 12 X patient inflow rate with compatible living donors, $p^l_X \lambda \pi_X$ and X patient inflow rate with incompatible but blood-type compatible donors have 0 waiting time. ■

We are ready to state some inequity consequences of exchange. Although all blood types benefit from exchange, O and AB patients benefit the least, and B blood types benefit the most under mild conditions. We use the **exchange transplant ratios**, $\{\frac{\mathbf{e}_X}{\pi_X}\}$, for this comparison. However, when we consider **living donor and exchange transplant ratios**, $\{\frac{\mathbf{l}_X + \mathbf{e}_X}{\pi_X}\}$, we see that O benefit the least, A and B patients benefit more than O, and AB patients benefit the most. This results hold in a benchmark model where no blood type is more likely to donate live than to get sick, i.e., when live donation propensities are independent of blood type. Thus, although B is behind A in living donor transplant ratio (provided that $p_B < p_A$ as in the general population in the US and most of the world; cf. Theorem 4), the increase coming through B's exchange transplant ratio makes its living donor and exchange transplant ratio level with that of A's.

Theorem 7 (Living donor transplantation and exchange and inequity in transplant ratios) *Suppose Assumption 1 holds. Consider a benchmark model where the ratio of living donation rate to patient inflow rate is the same among blood types, i.e., $\frac{p_X}{\pi_X}$'s is the same among all $X \in \mathcal{T}$. Then transplant ratios satisfy:*

- *For exchange only:* $\frac{\mathbf{e}_O}{\pi_O} = \frac{\mathbf{e}_{AB}}{\pi_{AB}} < \frac{\mathbf{e}_A}{\pi_A}, \frac{\mathbf{e}_B}{\pi_B}$. *If additionally* $p_A > p_B$, *then* $\frac{\mathbf{e}_A}{\pi_A} < \frac{\mathbf{e}_B}{\pi_B}$
- *For living donor transplantation and exchange together:* $\frac{\mathbf{l}_O+\mathbf{e}_O}{\pi_O} < \frac{\mathbf{l}_A+\mathbf{e}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B}{\pi_B} < \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}}{\pi_{AB}}$.

The intuition behind the first result comes from the fact that A and B have the additional advantage of exchange from two tissue-type-compatible pairs that are blood-type incompatible, i.e. exchanges between $A-B$ and $B-A$ pairs. In exchanges including AB or O patients, at least one pair should be tissue-type incompatible, and this pair becomes available for exchange with $\theta < 1$ probability. Additionally, if $p_A > p_B$, then $\pi_A > \pi_B$ holds as well in the benchmark model. Although $A-B$ and $B-A$ pair types participate in exchanges in equal measures, such exchanges are percentage-wise more beneficial for B patients, and thus, B has the highest exchange transplant ratio.

However, the exchange technology's contribution by itself is not sufficient to change the inequity caused by living donor transplantation in transplant ratios, as indicated by the second part of the theorem. One additional remark: the transplant ratios of A and B come very close to that of AB as a result of the exchange technology. To see this, observe that the added benefit for AB over A or B of living donor transplantation and exchange is that AB patients get direct live donation from AB donors while A or B patients cannot. As the AB blood type is rare in the population, the aforementioned transplant ratios are very close.

6 A New Proposal: Incentivizing Compatible Pairs To Participate in Exchange

One shortcoming of the current living donor exchange practices is that they utilize almost exclusively *incompatible* pairs. As a result, many non-O patients receive transplants from an O donor, effectively utilizing O organs in a highly inefficient way. However, if *compatible* pairs can be incentivized to participate in exchange, then the lack of balance between reciprocal type pairs will be mitigated. One sensible way of incentivizing compatible pairs to participate is to give their patients priority in the deceased donor queue if their transplanted graft fails in the future. As noted earlier in the Introduction, living donors are already incentivized in a similar manner. If a living donor's organ fails in the future, he will get priority in the deceased donor queue. A similar practice of prioritizing not only the donor but also the patient of a compatible pair may face little resistance in the medical community.

In this section, using the tools we developed in the earlier sections, we analyze the efficiency and equity effects of such an incentive scheme. Thus, when a paired patient with a compatible donor receives a transplant through exchange and this graft later fails, we assume that the FIFO structure of deceased-allocation policy has been altered. In particular, such reentrants, who we refer to as **prioritized reentrants**, are placed at the front of the queue. In this section, we analyze the welfare effects of incentivized exchange with respect to its alternative, regular exchange.

We focus on an ABO-identical FIFO deceased donor allocation policy (except the prioritized reentrants), as this is the primary policy adopted for kidney allocation and the kidney is the most

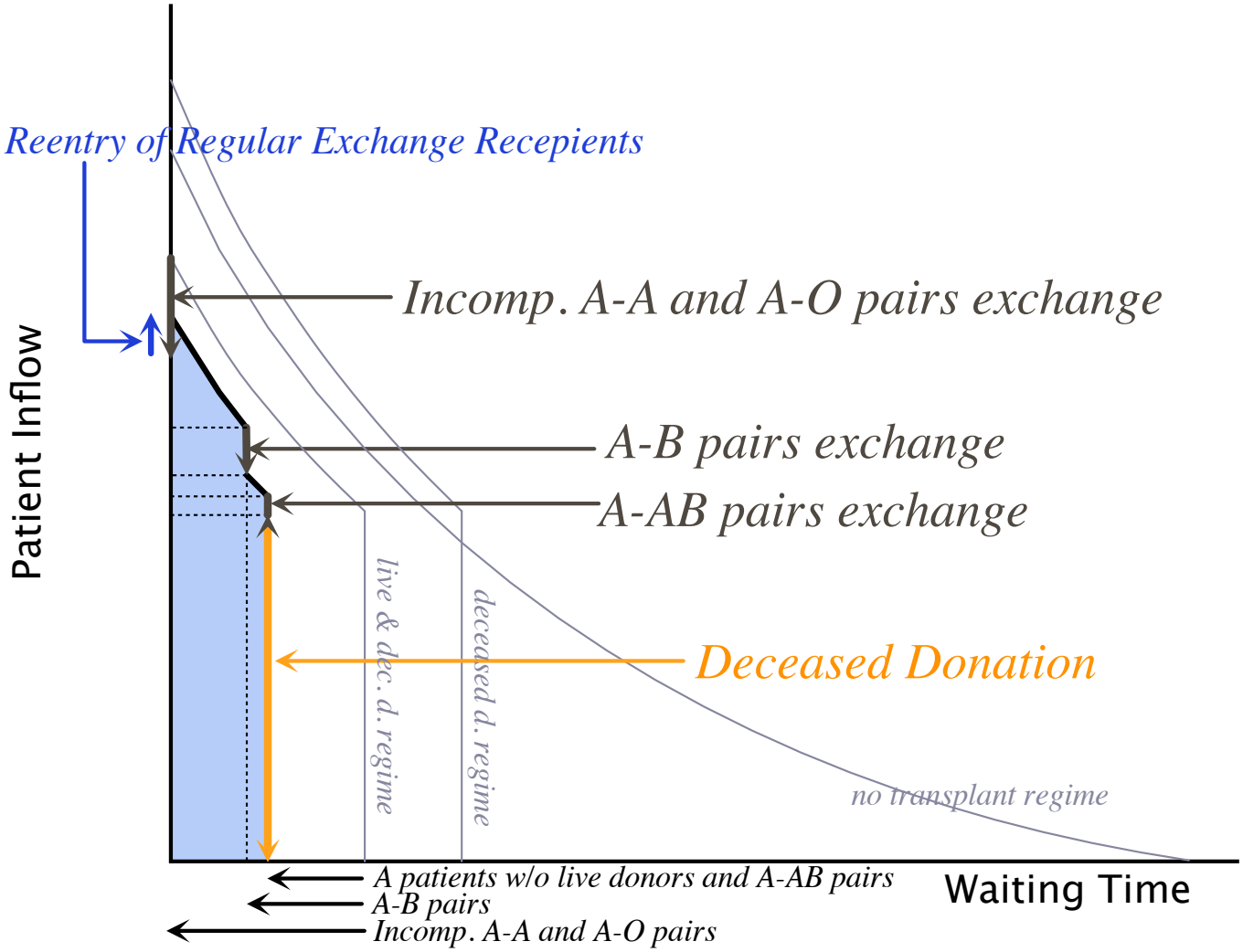

Figure 4: An example of a steady-state A deceased donor queue under the ABO-identical **regular exchange** technology (also with ABO-identical deceased donor allocation): Inflow rate π_A of patients decreases by $p_A^l \lambda \pi_A = (1-\theta)(p_A + p_O)\lambda \pi_A$ as a result of living donor transplantation and a further $\mathbf{e}_A^{od} + \mathbf{e}_A^{sd} = \theta(p_A + p_O)\lambda \pi_A$ as a result of exchange for $A-O$ and $A-A$ types at time 0. Under Assumption 2, $p_A \pi_B \geq p_A \pi_B$, and assuming $A-B$ type pairs do not end up receiving any deceased donor transplants, a measure of $p_A \pi_B$ of $A-B$ pairs participate in exchange at time deceased donor at time $t_{A-B}^{\mathbf{q},\mathbf{e};i}$. Assuming that $A-AB$ type pairs both receive deceased donor transplants and participate in exchange, a measure of $\theta p_A \lambda \pi_{AB}$ of $A-AB$ pairs participate in exchange at time $t_A^{\mathbf{q},\mathbf{e};i}$. As a result, the waiting time of single A patients and pooled $A-AB$ pairs decreases in the deceased donor queue with respect to that under the living donor transplantation technology.

common organ transplanted through living donor exchange.

Suppose an endogenous proportion ρ of all compatible pairs takes up the incentivized exchange option. We will maintain the following assumption in this and the next sections.

Assumption 4 *Compatible pairs may join the exchange pool only if an exchange is immediately available, and thus exchange does not involve a waiting cost; that is, the inflow rate of any underdemanded type $X-Y$ (i.e., $X \triangleright Y$ and $X \neq Y$) and its reciprocal overdemanded type $Y-X$ satisfy $[\rho(1-\theta)+\theta]p_X\pi_Y \leq p_Y\pi_X$.*

This assumption ensures that the inflow rate of any underdemanded type is greater than the inflow rate of its reciprocal type pairs, who are either incompatible or compatible and willing to use the incentivized exchange option. This is a simplification. If this is not the case, the excess inflow of paired patients with compatible donors will not wait for exchange, but will instead receive transplants from their donors immediately. As a result, compatible pairs never wait.[28]

We assume that we give precedence in exchange to incompatible pairs of a type over its compatible pairs (if they exist).

Under this assumption, we first show that ABO-identical exchange is also optimal for incentivized exchange technology (the analogue of Theorem 6, which was proved for regular exchange).

Theorem 8 (FIFO ABO-identical incentivized exchange is optimal) *Suppose Assumptions 1, 3, and 4 hold. Under incentivized exchange technology, the following policy is optimal:*

- *For any self-demanded type, immediately match incompatible pairs of this type with each other whenever feasible, and*
- *for any underdemanded type or type $B-A$, match the longest waiting pairs of this type with their reciprocal incompatible or willing compatible pairs whenever feasible.*

Moreover, this policy maximizes the mass of pairs that can be matched within any closed time interval, and in particular, matches a larger mass of pairs than waiting for the pairs to arrive and running the exchange once at the end of the time interval.

The following theorem outlines the predictable differences of the outcomes under exchange with incentivized compatible pairs with respect to regular exchange.

Theorem 9 (Incentivized exchange and its efficiency and equity consequences) *Suppose Assumptions 1, 3, and 4 hold. Under the ABO-identical incentivized exchange technology (with ABO-identical deceased donor allocation), with respect to regular exchange,*

[28] This assumption also endogenizes ρ to some degree. In a general equilibrium of this model, ρ would be endogenously maximized to match the maximum possible number of underdemanded pairs through exchange, so that if a non-participating compatible pair were to try to participate in incentivized exchange, it would not be able to participate in exchange immediately and had to wait, contradicting equilibrium conditions. Hence, a version of Assumption 4 would hold endogenously.

1. *a weakly higher measure of patients is matched for each patient group. In particular, underdemanded type pairs are matched with a strictly higher measure.*

2. *No compatible pairs of type $X-X$ participate in incentivized exchange for any $X \in \mathcal{T}$ (since incompatible $X-X$ blood types are matched with each other through regular exchange);*

3. *No O reentrants are prioritized; however, A, B, and AB reentrants from compatible pairs that participated in exchange are prioritized.*

4. *Waiting times for underdemanded types strictly decrease. Waiting times for O, A, and B single patients and their pooled pair types may increase or decrease. Waiting time for AB single patients increases. Waiting time for other pair types was 0 under regular exchange and does not change.*

The proof of this theorem, especially of Statement 4, is also of independent interest. It quantifies the conflicting effects that affect waiting times when we switch from regular exchange to incentivized exchange. Additionally, Figure 5 provides an example for A patients under the incentivized exchange technology illustrating these effects.

We also inspect the equity consequences of incentivized ABO-identical exchange with compatible pairs in terms of transplant ratios.

We state the marginal measures of transplants (in addition to regular exchange technology) due to the new technology:

$$\begin{aligned} \mathbf{i}_O &= \rho(1-\theta)p_O\lambda(\pi_A+\pi_B+\pi_{AB}) \\ \mathbf{i}_A &= \rho(1-\theta)p_A\lambda\pi_{AB} \\ \mathbf{i}_O &= \rho(1-\theta)p_B\lambda\pi_{AB} \\ \mathbf{i}_{AB} &= 0 \end{aligned} \tag{13}$$

Thus, $\mathbf{i}_X/\pi_X$ is the **marginal incentivized exchange transplant ratio** for blood type X.

We have the following theorem under a benchmark model:

Theorem 10 (Incentivized exchange and decrease of inequity in transplant ratios) *Suppose Assumption 1 holds and live donation rates are equal among blood types, i.e., p_X/π_X is a constant among all $X \in \mathcal{T}$. Then, incentivized exchange benefits O patients most, followed by A and B equally, and does not benefit AB patients at all. That is, $0 = \frac{\mathbf{i}_{AB}}{\pi_{AB}} < \frac{\mathbf{i}_A}{\pi_A} = \frac{\mathbf{i}_B}{\pi_B} < \frac{\mathbf{i}_O}{\pi_O}$. Moreover, overall transplant ratios under incentivized exchange except deceased donor transplants satisfy*

$$\frac{\mathbf{l}_O+\mathbf{e}_O+\mathbf{i}_O}{\pi_O} \le \frac{\mathbf{l}_A+\mathbf{e}_A+\mathbf{i}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B+\mathbf{i}_B}{\pi_B} \le \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}+\mathbf{i}_{AB}}{\pi_{AB}},$$

where weak inequalities all hold with equality if and only if $\rho = 1$.

Thus, incentivized exchange – to some degree – reverses the increasing inequity caused by the previous technologies in waiting times and transplant ratios for O patients.

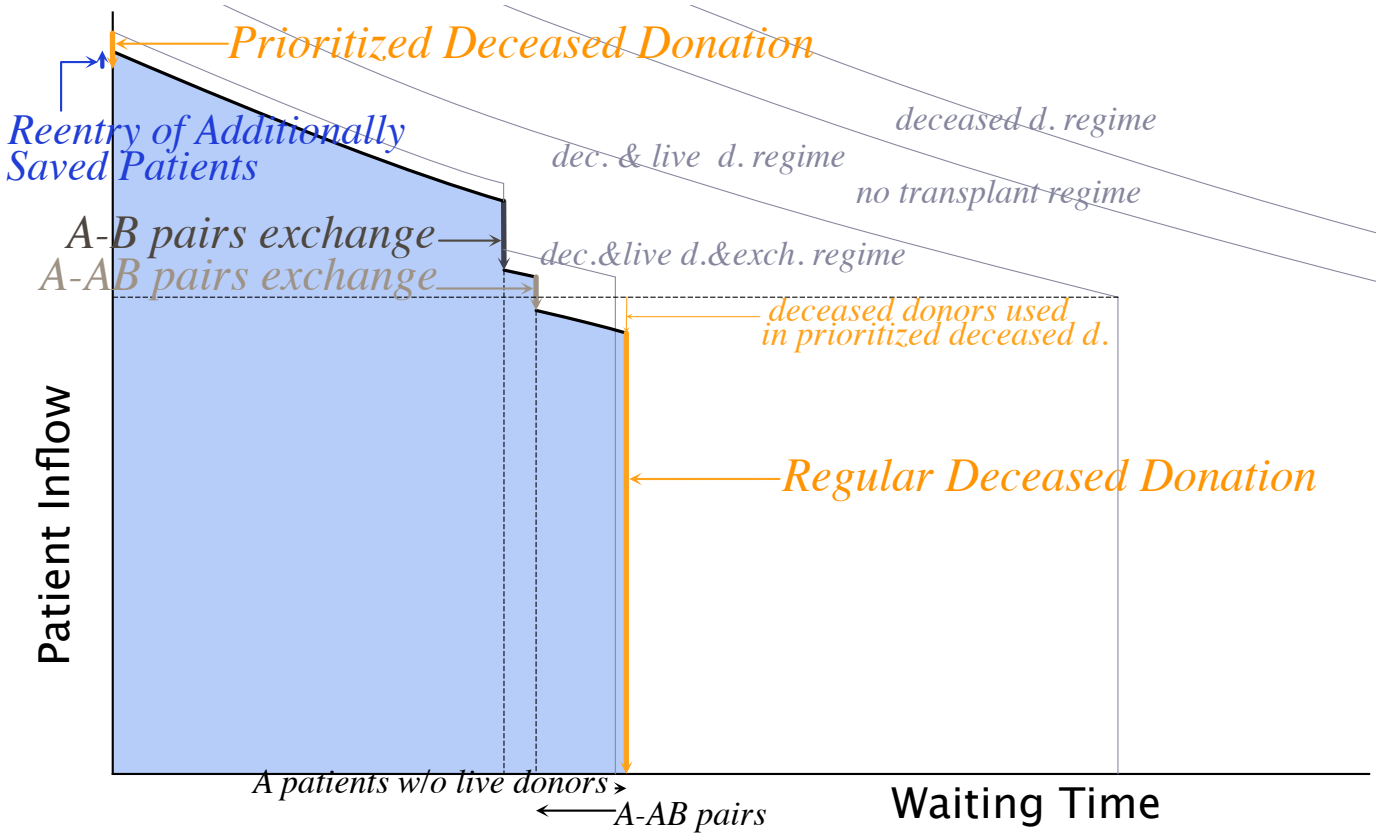

Figure 5: An example of a steady-state A deceased donor queue under the ABO-identical **incentivized exchange** technology (with also ABO-identical deceased donor allocation): A $\rho \in (0,1]$ fraction of all compatible pairs participate in exchange. A $\phi^l \rho(1-\theta)p_O \lambda \pi_A$ measure of reentrants, who were previously paired with compatible O donors, are prioritized to receive deceased donor transplants upon reentry. $A-AB$ type is no longer pooled with A single patients, and a measure of $[\theta+\rho(1-\theta)p_A]\lambda\pi_{AB}$ of $A-AB$ pairs are matched through exchange (through all incompatible and ρ fraction of compatible $AB-A$ pairs). Single patients who are not prioritized could be negatively or positively affected depending on the underlying parameters. We show in this figure a scenario that makes them wait slightly longer. This figure is scaled differently with respect to the previous ones to show the marginal effects of our proposal in detail.

7 Numerical Policy Experiments

In this section, we report the results and predictions of numerical policy experiments. These are based on our model to estimate the potential effects of the transplantation technologies discussed. We especially inspect how our new proposal, incentivized exchange, affects efficiency and equity. We gave theoretical predictions throughout the paper, the sharpest of which were obtained using two benchmark models. In these models, we assume that deceased or live donation rates are equal across all blood types.

We use the US OPTN kidney data for the year 2011. Our model's backbone parameters such as inflow rates for blood types, $\{\pi_X\}$, have to be estimated from the data using our model (cf. Table 3).[29] Living donor transplant data include both direct donation and transplants through exchange, which were not widespread in 2011. We make two assumptions, which lead to lower and upper bounds in our estimates. Assuming that all pairs that participated in exchange arrived in 2011, we find an upper bound on inflow rates. And assuming that all pairs that participated in exchange arrived before 2011, we find a lower bound. We also uncover the paired donor rate λ using our model based on unobserved intended live donations that did not materialize. For example, an $O-A$ pair is not detectable from the data, as the A donor could not donate to O, and hence, there is no recorded evidence for the existence of the pair. We had to calculate the numerical predictions regarding this censored data using our model.

One immediate observation is that the benchmark model with equal deceased donation rates does not fit very well for the US example (as mentioned before in the deceased donor transplantation section). We find $\delta_B/\pi_B = 28.75\% < \delta_{AB}/\pi_{AB} = 33.19\% \simeq \delta_O/\pi_O = 34.33\% < \delta_A/\pi_A = 37.84\%$ for the upper bound calculation (lower bound is similar). Hence, we conclude that B blood types get end-stage renal disease more often. Minorities are known to be more prone to kidney disease. Moreover, B blood type is more common among minorities such as African-Americans and Asian-Americans. Thus, this finding is not very surprising (as predicted in Section 3). Observe that the deceased donor number distribution is almost consistent with the population blood type distribution.

Under living donor transplantation technology, waiting times for ABO-identical deceased donor allocation are estimated as 5.33 years for B, 4.59 years for O and AB, and 3.90 years for A (see Table 5, left pane, middle row).[30] Deceased donor allocation is done on a more regional basis than national. Moreover, a graft can easily go bad if a suitable patient is not found in time. Thus, in practice, it turns out that on many occasions AB patients benefit from these and receive transplants from other blood types (the same observation goes for A and B patients, who receive from O deceased donors more often than necessary). Hence, in the guidelines, the ultimate decision

[29] Data source and relevant calibration parameters are summarized in Table 3.

[30] The previous OPTN policy (page 76, at http://optn.transplant.hrsa.gov/ContentDocuments/OPTN_Policies.pdf retrieved on 11/09/2014) states that B and O blood type organs should be given to their own blood types only. The new policy adopted in December, 2014 explicitly calls first for ABO-identical allocation for majority of kidneys (see throughout the policy at http://optn.transplant.hrsa.gov/ContentDocuments/Policy8_Update_KAS_12-2014.pdf retrieved on 11/10/2014).

	Blood Types				Total
	O	**A**	**B**	**AB**	
			OPTN 2011 Data		
Additions to the Queue	16,240	11,237	4,832	1,260	33,568
Living Donor Recipients	2,544	2,256	743	227	5,770
- through exchange	199	167	58	18	442
Deceased Donor Recipients (σ_X)	5,050	3,964	1,420	592	11,026
Reentrants	905	690	256	80	1,931
Median Waiting Years	5.07	3.31	5.30	2.34	
			Estimates		
Deceased Donor Organs (δ_X)	5,290	4,026	1,319	392	11,026
Reentry Rate (ϕ)	11.92%	11.09%	11.84%	9.77%	11.50%
New Entrants (π_X) high	15,607	10,805	4,645	1,198	32,254
low	15,408	10,638	4,587	1,180	31,812
Living Donor Rate (λ) low	31.11%	22.54%	24.86%	17.91%	26.85%
high	34.53%	24.98%	27.55%	19.90%	29.79%

Table 3: Arrivals to and transplants from the kidney deceased donor queue in 2011 in the US. Data obtained from Organ Transplantation and Transplant Network (OPTN) using the "national data" option from http://www.http://optn.transplant.hrsa.gov on 02/25/2013. Deceased donor numbers are reported for each blood type separately, but not the actual number of grafts transplanted. Using the empirical fact that 1.48 kidneys are harvested from each deceased donor on average, we found the number of deceased donor grafts available for each blood type.

is left to the physician. We refer to this actual allocation policy as **de facto deceased allocation**. Since the AB blood type is seen in only $3-4\%$ of the population, even a few violations of FIFO cause dramatic decreases in AB's waiting time. As a result (Table 3), the actual median waiting times in the deceased donor queue for AB is the shortest (2.34 years), while B is the longest, but very close to O (5.30 versus 5.07 years). And A's is less than these two blood types (at 3.31 years). Hence, in our policy discussion we will mostly ignore AB and focus on A, B, and O. Moreover, we will use ABO-identical policies to approximate actual deceased donor allocation and optimal exchange policies.[31]

A second observation in Table 3 is noteworthy. As actual waiting times differ vastly across blood types, certain blood-type patients appear to be "looking for" paired donors more intensely than others. In our model, we expect the paired donor rate λ to be constant for all blood types. However, in the data this is different across blood types as 20% for AB, 25% for A, 27.5% for B, and 35% for O. As we know, the O blood type is at a disadvantage; it looks like they try hardest

[31]In the paper, we report the upper-bound waiting times, which are all less than .1 years different from the lower-bound times.

Average Waiting Time Est. for All Recipients (in years)	Blood Types			
	O	A	B	AB
Deceased Donor Transplantation				
De Facto Deceased Donor Allocation	5.36	4.84	5.56	3.36
ABO-Identical	5.18	4.78	5.82	5.29
ABO-Compatible	5.32	4.83	5.32	4.83
Living Donor Transplantation				
De Facto Deceased Donor Allocation	3.45	2.74	3.55	1.90
ABO-Identical	3.35	2.69	3.67	3.08
ABO-Compatible	3.40	2.68	3.38	2.72
Regular Exchange ($\rho = 0\%$)				
De Facto Deceased Donor Allocation	3.30	2.44	2.92	1.83
ABO-Identical	3.20	2.37	3.01	2.97
ABO-Compatible	3.27	2.44	3.02	2.63

Average Waiting Time Est. for All Recipients (in years)	Blood Types			
	O	A	B	AB
Incentivized Exchange ($\rho = 25\%$)				
De Facto Deceased Donor Allocation	3.14	2.44	2.91	1.85
ABO-Identical	3.03	2.40	3.02	2.96
ABO-Compatible	3.15	2.44	2.84	2.62
Incentivized Exchange ($\rho = 50\%$)				
De Facto Deceased Donor Allocation	2.96	2.43	2.90	1.86
ABO-Identical	2.85	2.40	3.01	2.97
ABO-Compatible	3.02	2.44	2.67	2.60
Incentivized Exchange ($\rho = 100\%$)				
De Facto Deceased Donor Allocation	2.53	2.41	2.89	1.89
ABO-Identical	2.43	2.37	2.98	2.96
ABO-Compatible	2.58	2.42	2.58	2.61

Table 4: Numerical policy experiment reflecting average waiting time conditional on receiving a transplant. These are the average waiting time estimates for all patients who receive transplants, including those who receive (1) transplants immediately through exchange, direct live donation, or prioritized deceased donation, (2) living donor transplants after waiting some time through exchange, and (3) deceased donor transplants after waiting in the deceased donor queue under different technologies.

Waiting Time Est. for Deceased Donation Recipients (in years)	Blood Types			
	O	A	B	AB
Deceased Donor Transplantation				
De Facto Deceased Donor Allocation	5.36	4.84	5.56	3.36
ABO-Identical	5.18	4.78	5.82	5.29
ABO-Compatible	5.32	4.83	5.32	4.83
Living Donor Transplantation				
De Facto Deceased Donor Allocation	4.79	3.98	5.04	2.51
ABO-Identical	4.59	3.90	5.33	4.59
ABO-Compatible	4.76	3.97	4.76	3.97
Regular Exchange ($\rho = 0\%$)				
De Facto Deceased Donor Allocation	4.67	3.85	4.61	2.49
ABO-Identical	4.47	3.73	4.90	4.57
ABO-Compatible	4.57	3.85	4.57	3.85

Waiting Time Est. for Deceased Donation Recipients (in years)	Blood Types			
	O	A	B	AB
Incentivized Exchange ($\rho = 25\%$)				
De Facto Deceased Donor Allocation	4.36	3.87	4.64	2.53
ABO-Identical	4.16	3.79	4.95	4.62
ABO-Compatible	4.32	3.87	4.32	3.87
Incentivized Exchange ($\rho = 50\%$)				
De Facto Deceased Donor Allocation	4.04	3.88	4.66	2.56
ABO-Identical	3.84	3.80	4.99	4.67
ABO-Compatible	4.08	3.89	4.08	3.89
Incentivized Exchange ($\rho = 100\%$)				
De Facto Deceased Donor Allocation	3.94	3.96	4.77	2.64
ABO-Identical	3.70	3.88	5.09	4.78
ABO-Compatible	4.03	3.97	4.03	3.97

Table 5: Numerical policy experiment reflecting deceased donor queue waiting time conditional on receiving a transplant. These are the waiting time estimates for patients who receive deceased donor transplants under different technologies. The prioritized reentrants are *excluded* from the calculation for incentivized exchange technologies.

to find a compatible paired donor.[32] We use these rates for each blood type in what follows.[33]

[32]Also cultural issues, such as family composition among different ethnic groups, can play a role in paired donor

Table 4 reflects the average waiting time estimates for our numerical policy experiment. Table 5 is for the deceased donor queue waiting times.

We summarize our findings using the average waiting times for all patients conditional on receiving a transplant. Similarly, Table 7 gives waiting times for different pair types under exchange policies with different rates of compatible pairs. Table 6 reflects the predicted number of patients receiving transplants for each blood type.

In terms of waiting time, we observe that each new technology decreases the average waiting time for O patients from 5.18 years under deceased donor transplantation to 3.35 with a transition to living donor transplantation, to 3.20 years with a transition to regular exchange. It further falls to 3.03, 2.85, and 2.43 years with transitions to incentivized exchange with $\rho = 25\%$, $\rho = 50\%$, and $\rho = 100\%$, respectively.

Estimates of Patients Receiving Transplants (in numbers)
upon reentry, patient of a compatible $X - Y$ pair
participating in exchange receives an X deceased donor kidney

	Blood Types			
	O	**A**	**B**	**AB**
Total Living Donor Transplants				
Deceased Donor Transplantation	0	0	0	0
Living Donor Transplantation	1,945.40	1,779.51	590.63	188.13
Regular Exchange ($\rho = 0\%$)	2,373.91	2,310.06	968.37	211.38
Incentivized Exchange ($\rho = 25\%$)	2,754.33	2,327.84	974.29	211.38
Incentivized Exchange ($\rho = 50\%$)	3,134.75	2,345.62	980.22	211.38
Incentivized Exchange ($\rho = 100\%$)	3,895.59	2,381.17	992.06	211.38
Deceased Donor Transplants				
De Facto Deceased Donor Allocation	5,050.00	3,964.00	1,420.00	592.00
ABO-Identical	5,289.75	4,025.91	1,318.73	391.61
ABO-Compatible	4,994.20	3,962.75	1,508.09	437.04
	- 5,100.30	- 3,977.62	- 1,614.28	- 454.77

Table 6: Numbers of patients estimated to receive transplants under various policies.

In terms of both efficiency and equity consequences of the step-wise changes across the 4 technologies, from **d** (deceased donor transplantation) to **l** (living donor transplantation), from **l** to **e** (regular exchange), and finally from **e** to **i** (incentivized exchange with $\rho = 100\%$ participation), we observe the following: O patients are predicted to experience 37%, 6%, and 20% increases in number of transplants, respectively. These numbers are 45%, 20%, and 1% for B; 44%, 9% and 1% for A (and finally for AB, 48%, 4%, and 0), respectively. Thus:

- AB patients are predicted to experience the highest gain from **d** to **l**, and O patients the lowest.

rates. This contributes to the observed disparity. For example, consider the B blood type. Although its waiting time is as long as O's and even longer, its patients' pairing rate is not as high.

[33] As waiting times decrease across the board under different policies, the paired donor rates can also equate among the blood types. We assume this is not the case.

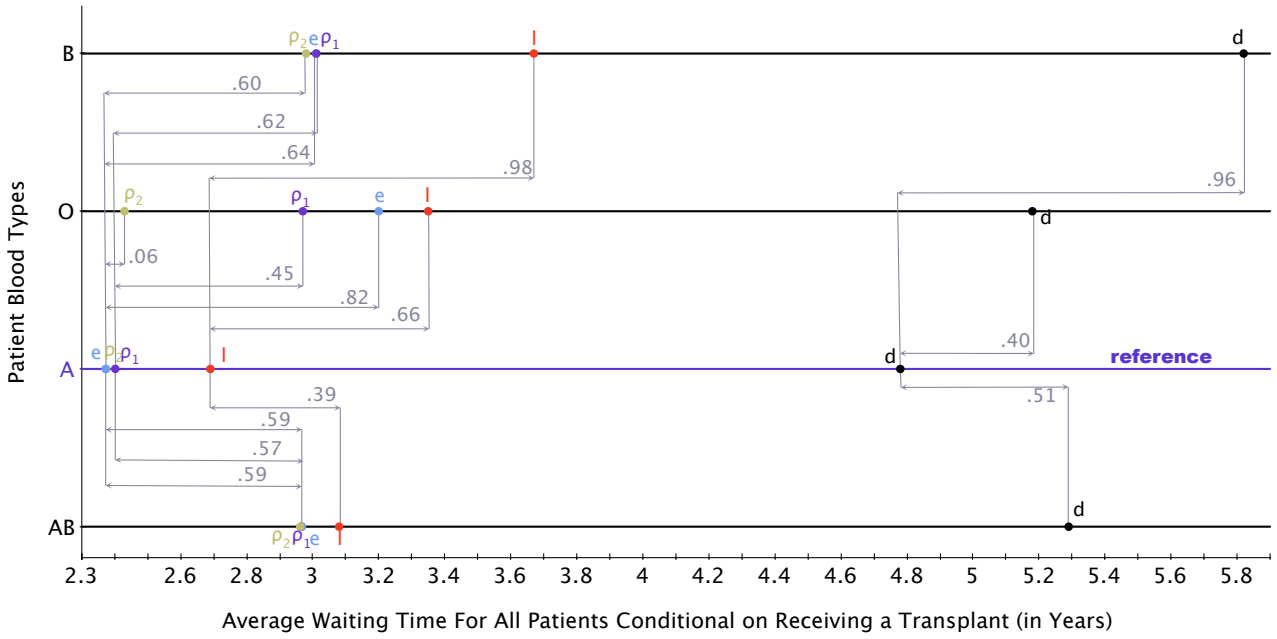

Figure 6: Differences of A patient predicted average waiting times from those of other blood types under various transplantation technologies with ABO-identical deceased donor allocation. Glossary: **d**: deceased donor transplantation, **l**: living donor transplantation, **e**: regular exchange, ρ_1: incentivized exchange with $\rho = 50\%$, ρ_2: incentivized exchange with $\rho = 100\%$.

- B patients are predicted to experience the highest gain from **l** to **e**, and AB and O patients the lowest.
- O patients are predicted experience the highest gain from **e** to **i**, and AB patients the lowest.

Observe that these estimates are consistent with our theoretical predictions even though our theory does not assume any heterogeneity among behavioral and medical characteristics of different blood-type patients and donors as the data reflect (cf. Theorems 4, 7, 9, and 10).

Further policy implications of Table 4 are summarized in Figure 6 in terms of average waiting time for all patients of each blood type (conditional on receiving a transplant). Under **d**, the waiting time for B is .96 years longer than that of A, and the waiting time for O is .4 years longer than that of A. We take A patients' waiting time as the reference. This difference increases to .98 years for B and .66 years for O under **l**: O blood types benefit the least, and A benefits from live donation more than B and O do. Once **e** becomes available though, B decreases this gap to .64 years, while the gap between O and A goes up to .82 years. However, under **i**, O starts to close the gap with A. First the difference falls to .45 years with $\rho = 50\%$ and then to .06 years with $\rho = 100\%$. Both A and B have prioritized reentrants who receive a deceased donor kidney as soon as they enter. However, deceased donor queue waiting times increase for both A and B under **i** (cf. Table 5). Thus, their *average* waiting times can increase or decrease depending on which of these effects dominate. In

Waiting Time Est. for Blood-Type-Incompatible Pairs When Compatible Pairs are Prioritized (in years) upon reentry, patient of a compatible $X-Y$ pair participating in exchange receives an X deceased donor kidney							
% of Comp.	**Pair Types**						
Pairs in	**O – A**	**O – B**	**O – AB**	**A – B**	**A – AB**	**B – A**	**B – AB**
$\rho = 0\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.36	pooled w **B**
$\rho = 25\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.36	pooled w **B**
$\rho = 50\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.36	pooled w **B**
$\rho = 100\%$	2.06	0.43	2.98	0	0.55	1.36	2.25

Table 7: Numerical policy experiment reflecting regular and incentivized exchange waiting times for pairs conditional on receiving a transplant. Here "pooled" means that some $X-Y$ pairs receive deceased donor transplants along with single X patients while some other $X-Y$ pairs simultaneously participate in exchange. Their waiting times are reflected in Table 5 in the columns regarding the X blood type.

the end, both B's and A's average waiting times slightly change from **e** to different **i** technologies.[34]

Policy makers may worry that prioritized reentrants from AB, A, and B blood types would cause a more substantial negative effect on the waiting times of single patients (indeed, deceased donor waiting queue time increases from 3.73 years under **e** to 3.88 years under **i** with $\rho = 100\%$ for A; cf. Table 5), then these negative effects can be alleviated at a small decrease of the improvement in O blood-type patients.

Consider the following **modified incentivized exchange** (**m**) technology: If the patient of a compatible $X-Y$ type pair reenters the pool after the pair participated in exchange, then this patient is prioritized and given a Y deceased donor graft, but not an X deceased donor graft. So an X reentrant of an $X-O$ pair would get an O kidney but not an X kidney.

As we already save so many O patients under **i**, we project that **m** would lead to a Pareto improvement in terms of waiting times for all patient types. Indeed the results of such a policy experiment are given in Table 8. We observe that deceased donor queue waiting times improve across all blood types, and average waiting times decrease more dramatically for A and AB patients under a transition to **m** from **e**.

[34] Although we assumed throughout the theoretical analysis that $\pi^e_{A-B} \geq \pi^e_{B-A}$ through Assumption 3, our estimates reflect an opposite trend. This is because B patients "work harder" to find paired donors than A patients who wait less in the deceased donor queue. Our numerical analysis, as mentioned before, assumes that this disparity in donor recruitment will continue to hold even with increasing transplant possibilities for B patients. Therefore, we find the opposite of the assumption holding. As noted in the text, this assumption was made for notational ease, mostly. Our results apply symmetrically in the situation $\pi^e_{A-B} < \pi^e_{B-A}$. This is reflected in Table 7 with $B-A$ pairs waiting for 1.36 years for exchange while $A-B$ pairs participate in exchange as soon as they arrive.

Average Waiting Time Est. for All Recipients
(in years)
upon reentry, patient of a compatible $X-Y$ pair
participating in exchange receives a Y deceased donor kidney

	Blood Types			
	O	**A**	**B**	**AB**
Modified Incentivized Exchange ($\rho = 25\%$)				
De Facto Deceased Donor Allocation	3.16	2.41	2.89	1.81
ABO-Identical	3.05	2.37	2.99	2.93
ABO-Compatible	3.16	2.41	2.71	2.58
Modified Incentivized Exchange ($\rho = 50\%$)				
De Facto Deceased Donor Allocation	3.01	2.38	2.85	1.79
ABO-Identical	2.89	2.34	2.96	2.90
ABO-Compatible	3.04	2.39	2.55	2.53
Modified Incentivized Exchange ($\rho = 100\%$)				
De Facto Deceased Donor Allocation	2.60	2.35	2.85	1.81
ABO-Identical	2.50	2.28	2.89	2.92
ABO-Compatible	2.61	2.31	2.60	2.51

Waiting Time Est. for Deceased Donation Recipients
(in years)
upon reentry, patient of a compatible $X-Y$ pair
participating in exchange receives a Y deceased donor kidney

	Blood Types			
	O	**A**	**B**	**AB**
Modified Incentivized Exchange ($\rho = 25\%$)				
De Facto Deceased Donor Allocation	4.39	3.82	4.58	2.47
ABO-Identical	4.20	3.74	4.90	4.55
ABO-Compatible	4.34	3.82	4.34	3.82
Modified Incentivized Exchange ($\rho = 50\%$)				
De Facto Deceased Donor Allocation	4.12	3.78	4.55	2.45
ABO-Identical	3.92	3.70	4.87	4.53
ABO-Compatible	4.12	3.79	4.12	3.79
Modified Incentivized Exchange ($\rho = 100\%$)				
De Facto Deceased Donor Allocation	4.12	3.76	4.53	2.41
ABO-Identical	3.87	3.68	4.85	4.49
ABO-Compatible	4.10	3.76	4.10	3.76

Table 8: Numerical policy experiment reflecting overall average waiting time and deceased donor queue waiting time conditional on receiving a transplant under **modified** incentivized exchange. In the right pane, prioritized reentrants are *excluded* from the calculation for deceased donation waiting times.

8 Multiple Exchange Platforms and Exchange with Incentivized Compatible Pairs

Although in our model we assume that there is a unique central living donor organ exchange authority, in reality many parallel platforms compete with each other in the case of kidney exchange in the US. Given the vagueness of the original National Organ Transplant Act of 1984 regarding the legality of living donor exchanges, it had to be amended in 2007. As a result, the US national kidney exchange program started under the provision of UNOS later, in 2010. UNOS is originally the federal contractor that oversees deceased donor allocation in the US. On the other hand, regional kidney exchange programs had started in early 2000s. For example, the New England Program for Kidney Exchange was founded in 2004, while the Ohio Solid Organ Consortium has conducted ad-hoc kidney exchanges since early 2000s. Currently most kidney exchanges are done in smaller non-profit programs rather than the UNOS national program. However, the pairs with difficult-to-match patients due to severe tissue sensitivity have a much higher chance of being matched in a large pool of pairs rather than in a small pool. And these smaller programs match internally easier-to-match pairs. Thus, the left-over, difficult-to-match pairs form the majority of the national program pair pool. Therefore, such pairs have a very small chance of being matched under the current realm of market formation. The advantage of a large kidney exchange program is that it will provide a more efficient system than several smaller programs (for example, see the simulations reported in RSÜ 2005a; 2007).

The consolidation of multiple programs in a single large kidney exchange program is difficult. RSÜ (2005c) showed that there is no incentive-compatible exchange mechanism that would make smaller programs reveal all their pairs to the centralized program. This result models programs as decision makers trying to maximize the number of transplants received by their registered pairs. Hence, it is an often-debated challenge how to create a single exchange pool with voluntary participation.

It turns out that our proposal of incentivized exchange can also help us create a single large exchange pool. Although there are multiple programs for exchange, only one of them is also in charge of the administration of the deceased donor queue (i.e., UNOS). Hence, we can give the right of incentivizing compatible pairs *only* to UNOS.

In this section, we show that such a policy design, which will cause compatible pairs to register *only* at the UNOS national exchange program, will attract most of the other critical pairs to UNOS. Therefore, at equilibrium there will be a large exchange pool containing the most pairs – namely, the UNOS national program.

8.1 The Exchange Participation Game for Pairs

Consider the following dynamic game. Suppose there are $n+1$ living donor kidney exchange platforms $P_0, P_1, ..., P_n$. Platform P_0 is the UNOS national exchange program.

Exchange with incentivized compatible pairs is available only in the UNOS program, P_0, which also oversees deceased donor allocation. Hence, only the UNOS program gives priority to the reentering patient of a compatible pair who previously participated in an exchange conducted through its program.

Each platform uses an *ABO-compatible* FIFO optimal exchange policy to maximize the measure of pairs matched,[35] while the national program uses the optimal policy by incentivizing compatible pair participation with prioritized deceased donor allocation. In the ABO-compatible FIFO policy, ties among pairs who arrive at the same time are broken through an even lottery as long as it does not affect efficiency, as explained in Appendix A. Hence, a $B-O$ pair can be matched with an excess $A-B$ pair (i.e., one that remains unmatched after all arriving $B-A$ pairs are matched) or an $O-B$ pair with equal probability if they have waited the longest and either matching would result with the same efficiency outcome in terms of maximizing the measure of pairs matched.

We assume that an exogenously determined ρ–fraction of compatible pairs (satisfying Assumption 4) from overdemanded types automatically register for exchange at platform P_0, and they are not strategic agents. It is straightforward to extend our results to the case when compatible pairs are strategic agents and ρ is endogenously determined through their own risk attitudes.

We assume that compatible overdemanded pairs are always immediately matched, whether they participate in exchange or not. If there is no available pair in the exchange platform, the compatible pair's donor donates to her paired patient immediately and the compatible pair leaves the pool.

[35] It is easier to find the implications of the ABO-compatible policy for the equilibria of this game.

Single patients are not strategic agents, either. All patients simultaneously wait in the deceased donation queue.

On the other hand, each patient with an incompatible donor is a strategic agent who would like to receive a transplant as soon as possible.[36] For simplicity, we assume that an incompatible pair immediately accepts the first offered donor, deceased or living. Our results do not change if we explicitly model the utility functions of patients over time and living versus deceased donors (such as, using a measure of expected survival of the transplant).

An incompatible pair can opt in or out of the exchange pool at any time after it arrives.

A paired patient without a compatible donor who has not registered in any exchange platform waits to receive a deceased donor under the ABO-identical FIFO allocation policy.

We inspect the Nash equilibria of this game. The first lemma is obvious, so we just state it without proving:

Lemma 7 *At any Nash equilibrium of the participation game, if $X-Y$ type pairs register for exchange at two distinct exchange platforms with positive probability, then their expected waiting times are the same at these platforms.*

In this game, any strategy that tells pairs not to participate in exchange at any platform is weakly dominated. For different ρ, there may exist equilibria in dominated strategies. For example, there exists an equilibrium in which no pair participates in exchange when $\rho = 0$. When $\rho > 0$, there are equilibria in which no self-demanded or reciprocally demanded pair participates in exchange. Hence, we focus on equilibria in undominated strategies:

Proposition 1 *In the participation game, there are pure strategy Nash equilibria in undominated strategies. The total measure of patients matched through exchange or deceased donor transplantation is the same and maximal across all such equilibria for the given ρ; moreover, this total measure strictly increases in ρ.*

There are indeed multiple pure strategy equilibria where different measures of pairs register at different programs. Some of these equilibria can be constructed straightforwardly: Denote one equilibrium by σ^* where all pairs register at P_0. Consider another strategy profile σ' where a sufficiently small fraction ϵ of each pair type registers at platform P_1, while the rest of the pairs continue to register at P_0. Now, P_1 works as a mini-version of P_0 with the same ratio of different pair types registering. Hence, all pairs are matched at the same time at both P_0 and P_1 through exchange and (if needed) deceased donor transplantation. Thus, σ' is also an equilibrium, as no pair has any incentive to deviate.

On the other hand, this kind of an equilibrium allows only a *sufficiently small* fraction of pairs to register at platforms other than P_0. Otherwise, there will be excess compatible pairs registering at P_0. Hence, underdemanded pairs registering at other platforms will have unilateral incentives to

[36] As a paired patient may die while waiting for a transplant, we assume that receiving an earlier transplant is preferable to receiving a later transplant.

deviate and register at P_0. Thus, this maximum fraction, which we denote as ϵ, is inversely related to ρ.

Our model does not consider explicitly *difficult-to-match* pairs, and assumes that each pair has the same tissue-type-incompatibility probability. In reality, there exist positive measures of highly sensitized pairs, and their chances of being matched are much smaller when the size of the exchange pool is small. Hence, from a practical view point, a large exchange platform will be more desirable than several small platforms, although all equilibria in undominated strategies are efficient.

Moreover, it does not matter for efficiency purposes whether or not some pair types participate in exchange at a particular platform. For example, any positive measure of incompatible $O-O$ pairs can participate at any platform, and they will all be matched with each other without affecting efficiency. On the other hand, if a positive measure of incompatible $A-O$ pairs participated at a platform where there are no underdemanded pairs, this would decrease the efficiency of the exchange. Hence, we will refer to all pair types that are not self-demanded as **efficiency-critical types**.

Thus, it is important to create a large exchange platform with efficiency-critical pair types. Our main result of this section states the conditions that guarantee the creation of a large program:

Theorem 11 *Suppose Assumptions 1, 3, and 4 hold. Then, in the participation game,*

- *ϵ, the maximum total equilibrium measure of registrants at platforms other than the national exchange program P_0, decreases with increasing ρ; and*
- *if*

$$\rho > \frac{\sum_{X-Y\in\mathbf{O}\setminus\{B-A\}} \theta p_Y \pi_X + p_A \pi_B}{\sum_{X-Y\in\mathbf{O}\setminus\{B-A\}} (1-\theta) p_Y \pi_X},$$

then the total measure of efficiency critical pairs participating at P_0 is more than the sum of their respective participation rates in other platforms in every undominated pure strategy equilibrium.

Assumption $\rho > \frac{\sum_{X-Y\in\mathbf{O}\setminus\{B-A\}} \theta p_Y \pi_X + p_A \pi_B}{\sum_{X-Y\in\mathbf{O}\setminus\{B-A\}} (1-\theta) p_Y \pi_X}$ in the theorem ensures that the measure of compatible pairs participating in exchange are relatively high. In particular using the numerical policy experiment data of the previous section, we need $\rho > 32\%$ when $\theta = 0.11$ for this condition to hold.

9 Conclusion

Over the last decade living donor organ exchange emerged as an important transplantation technology, in addition to the two primary technologies deceased donor transplantation and living donor transplantation. While analyzing efficiency and equity implications of individual technologies have been an important focus for researchers, doctors, and policy makers in health care, there has been no study to date that assesses the interaction between them and their collective implications. As the share of transplants from living donor transplantation and from organ exchange increased over

the years, a need for a model that studies the interaction of these organ transplantation technologies as well as their collective implications has arisen. Our model is a first attempt to fill this gap in the literature.

The natural innovation sequence of transplantation technologies is (1) deceased donor transplantation, (2) living donor transplantation, and (3) living donor organ exchange. As expected, each new technology has increased the overall welfare of the transplant organ patient population. However, the subsequent innovations have also increased the inequity between patients of the two primary blood types A and O. In the US, more than 95 percent of the population is of blood types A, B, and O. Blood types O and A are especially common with 45 percent and 38 percent representation respectively, whereas less than 13 percent is of blood type B. While blood type O is the most common blood type for all ethnicities, the distribution of blood types differ considerably across different ethnicities. Blood type O is especially common among African Americans and Native Americans, blood type A is considerably more common among white Americans than minorities, and conversely blood type B is considerably more common among minorities than White Americans. Since the need for organ transplantation differ across different ethnicities, understanding the impact of transplantation technologies on patients of different blood types is important. For example, for kidneys, the most common organ used for transplantation, minorities are more prone to kidney disease.[37] As such, starting with the base line technology of deceased donor transplantation alone, patients of blood types B and O (i.e. blood types more common among minorities), have experienced longer waits in deceased donor queues. While members of the transplantation community have been in constant search of policies to overcome this inequity, ironically the subsequent two transplantation technologies, living donor transplantation and living donor organ exchange have mostly increased this inequity. That is, while both technologies have increased the overall welfare of the patient population, they also resulted in an increased difference between the wait times across patients of different blood types. Taking the wait times in deceased donor queue as a metric, both living donor transplantation and living donor organ exchange have increased the inequity between patients of blood types O and A. That is not unexpected since it is harder for O patients not only to find a compatible living donor, but also to take part in organ exchange. For patients of blood type B the effect of these technologies has been somewhat different. While living donation increased the inequity between patients of blood types B and A, living donor organ exchange decreased it mitigating some of this adverse equity implications on blood type B patients. Our paper is the first paper to analyze the welfare and equity implications of existing organ transplantation technologies, shedding light on these empirical patterns.

Our final major contribution is the introduction a new policy that has the potential not only to

[37] African Americans are almost four times as likely as Whites to develop kidney failure. Compared to non-Hispanic whites, Hispanics are almost 1.5 times more likely to be diagnosed with kidney failure. Compared to Whites, American Indians are about 1.8 times more likely to be diagnosed with kidney failure. See http://nkdep.nih.gov/learn/are-you-at-risk/race-ethnicity.shtml and U.S. Renal Data System, USRDS 2010 Annual Data Report: Atlas of Chronic Kidney Disease and End-Stage Renal Disease in the United States, National Institutes of Health, National Institute of Diabetes and Digestive and Kidney Diseases, Bethesda, MD, 2010.

increase the overall patient welfare, but also to decrease the above mentioned inequity across patients of different blood types. Currently compatible pairs very rarely participate in organ exchange and their lack to do so results in considerable welfare loss. Our proposed policy is based on incentivizing participation of compatible pairs in exchange by prioritizing patients of such pairs at the deceased donor queue for possible future organ failures of the transplanted organ. Our proposed policy is the first one with a potential to decrease the inequity across various patient populations while at the same time increasing the overall welfare.

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A Appendix: ABO-Compatible Exchange and Deceased Donor Allocation

For some organs, such as the liver, a patient in the deceased donor queue can get precedence in receiving any ABO-compatible deceased donor organ. Moreover, for the kidney, similar practices are in effect in many other countries (cf. Footnote 13). If an egalitarian concern is in place, a similar practice can be adopted for exchange: $AB-O$ type pairs can be used to match $A-AB$ $B-AB$ or $O-AB$ pairs, not just $O-AB$ pairs. If two-way exchange is the only available exchange policy, saving any of them would be efficient in Pareto sense. However, a FIFO policy can also be adopted. Then, an $AB-O$ type pair can be matched with the longest waiting pair of these three types. However, these underdemanded types can also receive organs from the deceased donor queue. They will determine which source to use, either exchange or deceased donor, according to their waiting time. Sorting out what patient group receives from what source leads to a seemingly complex graph-theory problem. However, thanks to techniques from combinatorial optimization theory, we can solve this cumbersome problem.[38]

Consider the following two-sided graph (cf. Figure 7), with sides labeled **O** and **U**. Side **O** consists of 4 nodes O, A, B, AB, representing the deceased donor blood types, 5 nodes representing overdemanded pair types $A-O, B-O, AB-O, AB-A, AB-B$, and type $B-A$, which is on the short side among the two reciprocally demanded types, $A-B$ and $B-A$, by Assumption 3:

$$\mathbf{O} = \{O, A, B, AB, A-O, B-O, AB-O, AB-A, AB-B, B-A\}. \tag{14}$$

The other side also consists of 10 nodes, 4 representing the blood types of single patients, 5 for the underdemanded pair types, and 1 for the $A-B$ pair type:

$$\mathbf{U} = \{O, A, B, AB, O-A, O-B, O-AB, A-AB, B-AB, A-B\}. \tag{15}$$

We refer to each node $i \in \mathbf{U} \cup \mathbf{O}$ as a *patient group* by a slight abuse of terminology, as some of these nodes may refer to pair types. The nodes in both sides are connected with *blood-type feasibility* links, when possible. These links are represented by a matrix of 0's and 1's, $C = [c_{i,j}]_{i \in \mathbf{U}, j \in \mathbf{O}}$. Two types $i \in \mathbf{U}$ and $j \in \mathbf{O}$ are linked, i.e. $c_{i,j} = 1$, when

1. types i and j are mutually blood-type compatible if both i and j are pair types,

[38] The same technique can be adopted to determine which blood types will be pooled when exchange is not possible. Instead, we gave an explicit simple algorithm for that case in Section 3.2.2.

2. an i blood-type patient is blood-type compatible with a j blood-type donor, if both i and j are blood types, and

3. patient of a type i pair is blood-type compatible with a j blood-type donor, if j is a blood type and i is a pair type.

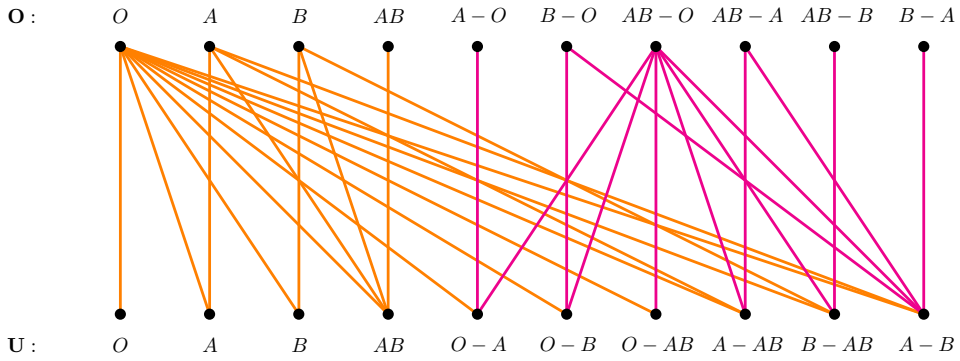

Figure 7: ABO-compatible exchange and deceased donation feasibility graph $(\mathbf{O}, \mathbf{U}, C)$. Lighter links correspond to deceased donation possibilities and darker links correspond to exchange possibilities.

Each node $h \in \mathbf{O} \cup \mathbf{U}$ is assigned a weight w_h such that w_h is the inflow rate for the single patient / deceased donor / pair type h:

$$w_h = \begin{cases} \delta_h & \text{if } h \in \mathbf{O} \cap \mathcal{T} \\ \pi^d_h & \text{if } h \in \mathbf{U} \cap \mathcal{T} \\ \pi^e_h & \text{if } h \in (\mathbf{O} \cup \mathbf{U}) \cap \mathcal{T} \times \mathcal{T} \end{cases} \tag{16}$$

where π^e_h is defined in Equation 7 and π^d_h is defined in Equation 8.

Now, we determine the *least privileged* node subset of $\mathbf{U}$ as follows: For any $\mathbf{V} \subseteq \mathbf{U}$ and $\mathbf{P} \subseteq \mathbf{O}$, define

$$\mathbf{C_V}(\mathbf{P}) = \{j \in \mathbf{P} \mid c_{i,j} = 1 \text{ for some } i \in \mathbf{V}\}, \tag{17}$$

$$\mathbf{C_P}(\mathbf{V}) = \{i \in \mathbf{V} \mid c_{i,j} = 1 \text{ for some } j \in \mathbf{P}\}, \text{ and} \tag{18}$$

$$r^d_{\mathbf{V}}(\mathbf{P}) = \frac{\sum_{j \in \mathbf{C_V}(\mathbf{P})} w_j}{\sum_{i \in \mathbf{V}} w_i}. \tag{19}$$

Here, $\mathbf{C_V}(\mathbf{P})$ ($\mathbf{C_P}(\mathbf{V})$) is the set of deceased donor and pair types in $\mathbf{P} \subseteq \mathbf{O}$ (patient and pair types in $\mathbf{V} \subseteq \mathbf{U}$) that can be feasibly matched through deceased donation or exchange with some type in set $\mathbf{V} \subseteq \mathbf{U}$ ($\mathbf{P} \subseteq \mathbf{O}$). $\mathbf{C_P}(\mathbf{V})$ will be used in subsequent proofs in the appendices. On the other hand, $r^d_{\mathbf{V}}(\mathbf{O})$ is the supply-to-demand ratio for $\mathbf{V}$. This ratio is the generalization of the r^d ratio

defined in Equation 9. Now we can find the subset of $\mathbf{U}$ which minimizes r^d:[39]

$$\mathbf{V}_1 = \arg\min_{\mathbf{V}\subseteq\mathbf{U}} r^d_{\mathbf{V}}(\mathbf{O}); \quad \text{and} \tag{20}$$

$$\mathbf{P}_1 = \mathbf{C}_{\mathbf{V}_1}(\mathbf{O}). \tag{21}$$

Then we iteratively construct the partition $\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_k$ of $\mathbf{U}$ as follows:

$$\mathbf{V}_\ell = \arg\min_{\mathbf{V}\subseteq\mathbf{U}\setminus\cup_{m=1}^{\ell-1}\mathbf{V}_m} r^d_{\mathbf{V}}(\mathbf{O}\setminus\cup_{m=1}^{\ell-1}\mathbf{P}_m); \quad \text{and} \tag{22}$$

$$\mathbf{P}_\ell = \mathbf{C}_{\mathbf{V}_\ell}(\mathbf{O}\setminus\cup_{m=1}^{\ell-1}\mathbf{P}_m) \tag{23}$$

This means that the patient and pair groups belonging to $\mathbf{V}_\ell$ are the least fortunate, i.e. **bottleneck**, after serving the groups in $\mathbf{V}_1, ..., \mathbf{V}_{\ell-1}$. Moreover, we can assign all deceased donors and pairs belonging to types in $\mathbf{P}_\ell$ exclusively to patients and pairs of types in $\mathbf{V}_\ell$, since it is feasible to do so as designated by matrix C. This result follows from the minimum cut - maximum flow theorem of Ford and Fulkerson (1956) in combinatorial optimization theory.[40] Even when we do that, their waiting time in the exchange and deceased donor queues will not be lower than the other groups in $\mathbf{U}\setminus\cup_{m=1}^{\ell}\mathbf{V}_m$, as r^d ratio is lowest for $\mathbf{V}_\ell$ once $\mathbf{V}_1, ..., \mathbf{V}_{\ell-1}$ are fixed. Moreover, their waiting time will be given as

$$t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_\ell} = F^{-1}\Big(1 - r^d_{\mathbf{V}_\ell}(\mathbf{P}_\ell)\Big). \tag{24}$$

Theorem 12 (Optimal ABO-compatible exchange) *Suppose Assumptions 1, 2, and 3 hold. Under ABO-compatible exchange, the waiting time for each group of patients $i \in \mathbf{U}$, i.e., blood-type i single patients if i is a blood type, and type i pairs if i is a pair type, the waiting time is characterized by $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_\ell}$ in Equation 24 where $i \in \mathbf{V}_\ell$ and $\mathbf{V}_\ell$, $\mathbf{P}_\ell$ are defined as in Equations 16-23.*

Proof of Theorem 12. We prove the theorem using the concept of *flow networks* developed in the combinatorial optimization and graph theory literature (see for example Korte and Vygen, 2002 for an excellent survey).

This tool will be used to show that, for each $\ell \in \{1, ..., k\}$, for each patient group $i \in \mathbf{V}_\ell$ (as defined in Equations 20 and 22), we can feasibly serve deceased donors / pairs belonging to groups in $\mathbf{P}_\ell$ (as defined in Equations 21 and 23) to patients of group i at a rate $w_i\, r^d_{\mathbf{V}_\ell}(\mathbf{P}_\ell)$ (as defined in Equations 16, 19, and 24).

A *flow network* in our context is the directed graph with *nodes* $\mathbf{N} = \{\sigma, \tau\} \cup \mathbf{U} \cup \mathbf{O}$ such that σ is referred to as the *source* and τ is referred to as the *sink*. An *edge* of the flow network originating from node i and pointing at node j is denoted by (i, j). In particular, each $\mathbf{U}$ node is pointed at

[39] If there is more than one such set, then we take the largest of them, which is uniquely defined.

[40] For example, see Katta and Sethuraman (2006), Yılmaz (2009), Athanassoglou and Sethuraman (2011), Budish, Che, Kojima, and Milgrom (2013), and Che, Kim, and Mierendorff (2013) for uses of the minimum cut - maximum flow theorem in the probabilistic matching framework.

by σ. Hence, for each $i \in \mathbf{U}$, (σ, i) is in the network. Also each node in $\mathbf{O}$ points at τ. Hence, for each $j \in \mathbf{O}$, $(j, \tau) \in \mathbf{N}$. Moreover, there are edges starting from each node in $\mathbf{U}$ and ending at some nodes in $\mathbf{O}$: for each $i \in \mathbf{U}$ and $j \in \mathbf{O}$, $(i, j) \in \mathbf{N}$ if and only if $c_{i,j} = 1$. Let E be the set of edges of the network.

We will send flows from the source σ through the edges of the graph, and these flows will reach the sink. For this purpose, each edge $(i, j) \in E$ has also a *capacity* $q(i, j) > 0$ denoting the maximum flow it can carry. For all other pairs of nodes $(i, j) \notin E$, let $q(i, j) = 0$. Let $q = (q(i, j))_{i,j \in \mathbf{N}}$ denote the capacity vector for all the edges. A *flow network* is denoted by the pair $(\mathbf{N}, q)$. Fix a flow network $(\mathbf{N}, q)$.

A *flow function* $f : \mathbf{N} \times \mathbf{N} \to \mathbb{R}$ is a mapping such that for each $i, j \in \mathbf{N}$ we have (i) if $q(i, j) > 0$ then $0 \leq f(i, j) \leq q(i, j)$ and if $q(i, j) = 0$ then $f(i, j) \leq 0$, (ii) $f(j, i) = -f(i, j)$, and (iii) if $i \notin \{\sigma, \tau\}$ then $\sum_{h \in \mathbf{N}} f(i, h) = 0$. Property (i) says that an edge cannot carry a flow higher than its capacity. In particular, for existing directed edges, the flow cannot be negative; and if there is no directed edge from one node to another, then the flow cannot be positive. Property (ii) is a technical one and used for ease of notation, making sure that the flow is a directed quantity but not scalar: the flow of the reverse of an edge is the negative of the flow of the edge. Property (iii) says that for any node other than the source and the sink, the flows from it and into it cancel out, i.e., all flows entering it also leave the node. Let $\mathcal{F}$ be the set of flow functions. We refer to $f(i, j)$ as the *flow from node i to j under f*. For a subset of nodes $\{\sigma\} \subseteq \mathbf{S} \subseteq \mathbf{N} \setminus \{\tau\}$, the *flow from* $\mathbf{S}$ *(to* $\mathbf{N} \setminus \mathbf{S}$*) under* f is denoted by $f(\mathbf{S}) = \sum_{i \in \mathbf{S}, j \in \mathbf{N} \setminus \mathbf{S}} f(i, j)$. Such a subset of nodes $\mathbf{S}$ is referred to as a *cut*.

The *total capacity of a cut* $\mathbf{S}$ is defined as $q(\mathbf{S}) = \sum_{i \in \mathbf{S}, j \in \mathbf{N} \setminus \mathbf{S}} q(i, j)$, i.e., it is the sum of the capacities of edges originating from a node in $\mathbf{S}$ and ending at a node in $\mathbf{N} \setminus \mathbf{S}$. A *minimum cut* $\mathbf{S}$ is a cut such that $q(\mathbf{S}) = \min_{\{\sigma\} \subseteq \mathbf{S}' \subseteq \mathbf{N} \setminus \{\tau\}} q(\mathbf{S}')$, i.e. a cut with the minimum total capacity.

The *flow of* f is its flow from cut $\mathbf{N} \setminus \{\tau\}$ (to $\{\tau\}$), which is also equal to its flow from cut $\{\sigma\}$ (to $\mathbf{N} \setminus \{\sigma\}$). The *maximum flow* over the flow network $(\mathbf{N}, q)$ is defined as $\max_{f \in \mathcal{F}} f(\mathbf{N} \setminus \{\tau\})$.

The following is the fundamental theorem that relates the capacities of the edges to the maximum flow that can be carried over a flow network:

Minimum Cut - Maximum Flow Theorem (Ford and Fulkerson (1956)): The maximum flow over a flow network is equal to the total capacity of one of its minimum cuts.

One direction of the theorem's statement, i.e., the maximum flow cannot exceed the total capacity of a minimum cut, is obvious by the definition of a flow function. The other direction is proven through this theorem.

For our flow network used in the proof of our theorem, we define the capacities as follows (cf. Figure 8, where the edges are denoted by lines with arrows and their capacities are written on the lines; it defines a flow network using the feasible exchange and deceased donation graph given in Figure 7):

For an edge (i, j) such that $i \in \mathbf{U}$ and $j \in \mathbf{O}$, we set its capacity to $q(i, j) = +\infty$. Hence, it can carry any flow. On the other hand, for edge (j, τ) for each $j \in \mathbf{O}$, we set its capacity $q(j, \tau) = w_j$,

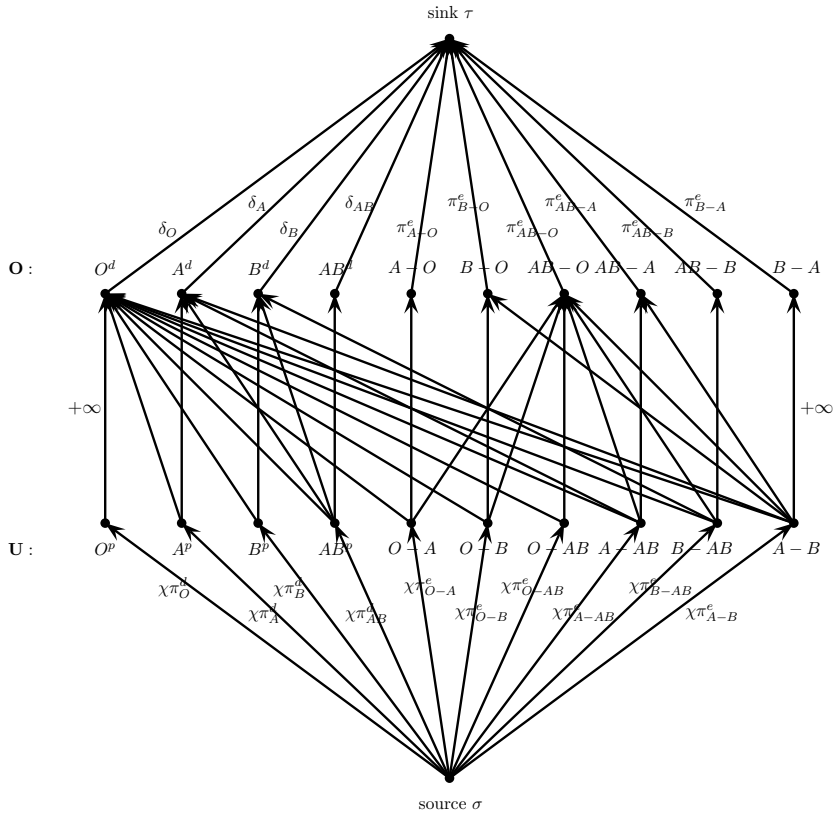

Figure 8: The χ–parametric flow network for the proof of Theorem 12, using the exchange and deceased donation feasibility graph in Figure 7. To prevent confusion, the nodes representing single patients (i.e., blood types in $\mathbf{U}$ as defined in Equation 15) are superscripted by p and the nodes representing deceased donors (i.e., blood types in $\mathbf{O}$ as defined in Equation 14) are superscripted by d.

the inflow rate of the deceased donor / pair type j to the pool, as defined in Equation 16. For edge (σ, i) for each $i \in \mathbf{U}$, we set its capacity $q^\chi(\sigma, i) = \chi w_i$, where w_i is the inflow rate of the single patient / pair type i to the pool, as defined in Equation 16, and $\chi \in \mathbb{R}_+$ is a parameter that will be changed in our construction. We refer to such a flow network as a *χ–parametric flow network*.

The idea behind this construction is as follows: as we increase χ continuously starting from 0, the flows carried from the source to the rest of the network are set to be equal to the capacities of the edges from the source for an appropriately defined flow function $f^\chi \in \mathcal{F}$. As χ is close to zero, all the flows can be carried over the network and hence, $\{\sigma\}$ is a minimum cut. We will be able to increase these continuously until a break point occurs at $\chi_1 < 1$, i.e., the minimum cut becomes a proper superset of $\{\sigma\}$. To see that, suppose to the contrary that $\chi_1 \geq 1$. We have the total capacity of cut $\mathbf{N} \setminus \{\tau\}$ equal to $q^{\chi_1}(\mathbf{N} \setminus \{\tau\}) = \sum_{j \in \mathbf{O}} w_j$, which should be greater than or equal to maximum flow over the network. On the other hand, the total capacity of cut $\{\sigma\}$ is equal to $q^{\chi_1}(\{\sigma\}) = \chi_1 \sum_{i \in \mathbf{U}} w_i$. We increase χ to χ_1 so that the flow of f^{χ_1} is equal to $q^{\chi_1}(\{\sigma\})$. However, this is a contradiction by Assumptions 1, 2, and 3, as the flow of f^{χ_1}, $q^{\chi_1}(\{\sigma\}) > q^{\chi_1}(\mathbf{N} \setminus \{\tau\})$, the

maximum flow over the network at χ_1.

Hence, at $\chi = \chi_1 < 1$ there will be a minimum cut larger than $\{\sigma\}$, such that we will not be able to carry all the flows if we exceed χ above χ_1. Let $\{\sigma\} \subsetneq \mathbf{N}_1$ be this minimum cut. If there are multiple such cuts, let $\mathbf{N}_1$ be the largest of them. It is straightforward to see that there is a minimum cut, which includes all minimum cuts as subsets.

What are the properties of this minimum cut? Suppose $i \in \mathbf{N}_1 \cap \mathbf{U}$. Then observe that all $j \in \mathbf{O}$ such that $c_{i,j} = 1$ is also in $\mathbf{N}_1$. As otherwise the edge (i, j) with capacity $q(i, j) = +\infty$ would make the total capacity of the minimum cut equal to $+\infty$. However, this is a contradiction to $\mathbf{N}_1$ being a minimum cut, as the cut $\{\sigma\}$ has always a finite total capacity (see Figure 9 for an example of a possible minimum cut at some χ_1). Hence, whenever $i \in \mathbf{N}_1 \cap \mathbf{U}$ then all $j \in \mathbf{O}$ with

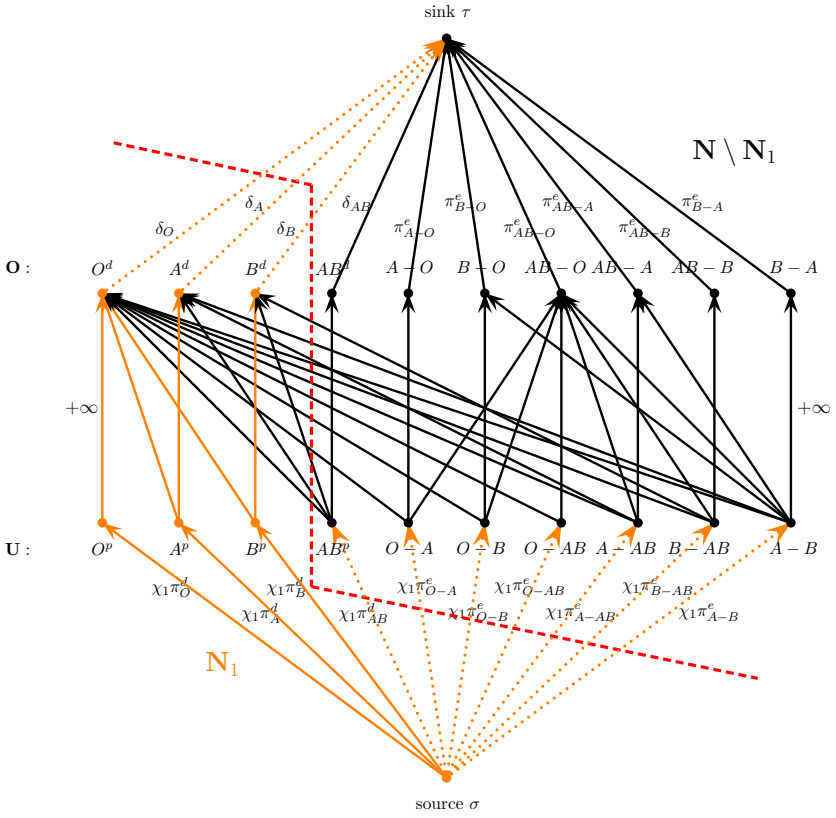

Figure 9: Example of a possible minimum cut $\mathbf{N}_1$ at some $\chi = \chi_1$ with $\mathbf{N}_1 \cap \mathbf{U} = \mathbf{V}_1 = \{O^p, A^p, B^p\}$. Hence, $\mathbf{N}_1 \cap \mathbf{O} = \mathbf{P}_1 = \mathbf{C}_{\mathbf{V}_1}(\mathbf{O}) = \{O^d, A^d, B^d\}$. The edges from $\mathbf{N}_1$ to $\mathbf{N} \setminus \mathbf{N}_1$ are denoted by dotted pointed lines. This cut's total capacity is $q^{\chi_1}(\mathbf{N}_1) = \delta_A + \delta_B + \delta_{AB} + \chi_1(\pi^d_{AB} + \pi^e_{O-A} + \pi^e_{O-B} + \pi^e_{O-AB} + \pi^e_{A-AB} + \pi^e_{B-AB} + \pi^e_{A-B})$.

$c_{i,j} = 1$ also satisfy $j \in \mathbf{N}_1$. Let $\mathbf{V}_1 = \mathbf{N}_1 \cap \mathbf{U}$, and $\mathbf{P}_1 = \mathbf{N}_1 \cap \mathbf{O}$. By the above construction $\mathbf{P}_1 = \mathbf{C}_{\mathbf{V}_1}(\mathbf{O})$ (as defined in Equation 21).

The total capacity of $\mathbf{N}_1$ is equal to

$$q^{\chi_1}(\mathbf{N}_1) = \sum_{i \in \mathbf{N} \setminus \mathbf{V}_1} q^{\chi_1}(\sigma, i) + \sum_{j \in \mathbf{P}_1} q(j, \tau) = \chi_1 \sum_{i \in \mathbf{N} \setminus \mathbf{V}_1} w_i + \sum_{j \in \mathbf{P}_1} w_j.$$

On the other hand, the flow of f^{χ_1} over the network at χ_1 is given as

$$f^{\chi_1}(\{\sigma\}) = \sum_{i \in \mathbf{U}} f^{\chi_1}(\sigma, i) = \chi_1 \sum_{i \in \mathbf{U}} w_i.$$

This is the maximum as all the capacity of the edges from σ are used, i.e., $f^{\chi_1}(\sigma, i) = q^{\chi_1}(\sigma, i)$ for all $i \in \mathbf{U}$.

As $\mathbf{N}_1$ is a minimum cut, by the Minimum Cut-Maximum Flow Theorem,

$$q^{\chi_1}(\mathbf{N}_1) = f^{\chi_1}(\{\sigma\}).$$

Hence,

$$\chi_1 \sum_{i \in \mathbf{U} \setminus \mathbf{V}_1} w_i + \sum_{j \in \mathbf{P}_1} w_j = \chi_1 \sum_{i \in \mathbf{U}} w_i,$$

leading to

$$\chi_1 = \frac{\sum_{j \in \mathbf{P}_1} w_j}{\sum_{i \in \mathbf{V}_1} w_i} = r^d_{\mathbf{V}_1}(\mathbf{P}_1)$$

where r^d was defined in Equation 19.

Observe that even if we increase χ beyond χ_1, the flow over the edges $\left\{(\sigma, i)\right\}_{i \in \mathbf{V}_1}$ will not increase and no additional flow through the increased χ will flow through the nodes $j \in \mathbf{P}_1$. Therefore, we can remove the nodes in $\mathbf{V}_1$ and $\mathbf{P}_1$ from the network and repeat the above exercise iteratively. As result, we determine a number of minimum cuts $\mathbf{N}_1, ..., \mathbf{N}_k$ with corresponding node sets in $\mathbf{U}$ as $\mathbf{V}_1, ..., \mathbf{V}_k$ and node sets in $\mathbf{O}$ as $\mathbf{P}_1, ..., \mathbf{P}_k$ with breakpoints $\chi_1 < ... < \chi_k < 1$ such that $\mathbf{P}_\ell = \mathbf{C}_{\mathbf{V}_\ell}(\mathbf{O} \setminus \cup_{\ell'=1}^{\ell-1} \mathbf{P}_{\ell'})$ and $\chi_\ell = \frac{\sum_{j \in \mathbf{P}_\ell} w_j}{\sum_{i \in \mathbf{V}_\ell}} = r^d_{\mathbf{V}_\ell}(\mathbf{P}_\ell)$ for each $\ell \in \{1, ..., k\}$.

This proves that for each patient group $i \in \mathbf{V}_\ell$, we can feasibly serve deceased donors / pairs belonging to groups in $\mathbf{P}_\ell$ to group i at a rate $f^{\chi_\ell}(\sigma, i) = \chi_\ell w_i = w_i r^d_{\mathbf{V}_\ell}(\mathbf{P}_\ell)$. Hence, we can feasibly match a measure $f^{\chi_\ell}(\sigma, i)$ of patients belonging to group i with arriving deceased donors (through deceased donor transplantation) and pairs (through exchange) in $\mathbf{P}_\ell$ by Lemmas 1 and 5, respectively.

Define $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_\ell} = F^{-1}\Big(1 - r^d_{\mathbf{V}_\ell}(\mathbf{P}_\ell)\Big)$ for each ℓ. Observe that $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_k} < t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_{k-1}} < ... < t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_1}$.

At time $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_k}$ after entry, the measure of live patients belonging to the groups in $\mathbf{V}_k$ is exactly equal to $\sum_{j \in \mathbf{P}_k} w_j$, the inflow rate of deceased donors / pairs belonging to groups in $\mathbf{P}_k$. None of the other patients belonging to groups in $\mathbf{V}_1, ..., \mathbf{V}_{k-1}$ can be matched through deceased donor transplantation or exchange using deceased donors / pairs belonging to $\mathbf{P}_k$. Hence, they have to wait longer than $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_k}$. Moreover, none of the patients of groups in $\mathbf{V}_k$ will be matched with deceased donors / pairs of groups in $\mathbf{P}_1, ..., \mathbf{P}_{k-1}$, as this will decrease their waiting time at the cost of increasing the waiting time of other groups, contradicting the FIFO protocol. We also proved above that all remaining live patients/pairs in V_k after $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_k}$ years of entry (that is $r^d_{\mathbf{V}_k}(\mathbf{P}_k)$ fraction of the arriving rate) can be matched with all arriving deceased donors / pairs belonging to groups in $\mathbf{P}_k$. Hence, remaining live patients belonging to groups in $\mathbf{V}_k$ will be matched after $t^{\mathbf{q},\mathbf{e};c}_{\mathbf{V}_k}$ years of entry. We repeat the above argument for each of the remaining sets $\ell = k-1, ..., 1$, concluding the proof of the theorem. ■

B Appendix: Remaining Proofs

Proof of Lemma 1. We prove it by contradiction: If $\sigma = \omega$, then suppose an infinite or uncountable number of donors are unmatched, and if $\sigma < \omega$, then suppose a donor is unmatched with a positive probability under the FIFO policy. In either case, an infinite or uncountable number of patients is unmatched. But then, take a donor who is unmatched. There exists almost surely a compatible unmatched patient, as the probability of finding no tissue-type compatible patient is $\lim_{n\to\infty} \theta^n = 0$. ■

Proof of Lemma 2. Since $X \neq Y$ and $X \triangleright Y$, we have $Y \not\triangleright X$. Moreover, $W \triangleright Y$ for all blood types W such that $W \triangleright X$.

Suppose to the contrary, $t_Y^{\mathbf{q},\mathbf{d};c} > t_X^{\mathbf{q},\mathbf{d};c}$. Then the longest-waiting Y patients would receive the maximum measure of organs that would otherwise go to X patients under the FIFO policy, as they are waiting more than the longest-waiting X patients. Hence, either Y patients do not wait at all, i.e., $t_Y^{\mathbf{q},\mathbf{d};c} = 0$, or X patients never receive transplant, i.e., $t_X^{\mathbf{q},\mathbf{d};c} = T$. Either case contradicts the assumption. ■

Proof of Lemma 3. Suppose Y patients receive X organs at steady state under the ABO-compatible FIFO allocation policy. By Lemma 2, $t_Y^{\mathbf{q},\mathbf{d};c} \leq t_X^{\mathbf{q},\mathbf{d};c}$. Suppose the inequality is strict. Then either all X organs would go to the longest-waiting X patients, which would contradict the fact that X organs are transplanted to Y patients, or X patients would not be waiting at all in the deceased donor queue, which would contradict the assumption that $t_Y^{\mathbf{q},\mathbf{d};c} < t_X^{\mathbf{q},\mathbf{d};c}$. Hence, $t_Y^{\mathbf{q},\mathbf{d};c} = t_X^{\mathbf{q},\mathbf{d};c}$.

Next, suppose that blood types in some $\mathcal{S} \subseteq \mathcal{T}$ are pooled together. Then there is a chain of blood types $X_1, ..., X_k$ where $\mathcal{S} = \{X_1, ..., X_k\}$ such that X_1 receives from X_1 and X_2, ..., X_{k-1} receives from X_{k-1} and X_k. By the previous paragraph, all types in $\mathcal{S}$ have the same waiting time under the ABO-compatible allocation scheme. Moreover, the supply-demand equations for these types are given as, for all $X \in \mathcal{S}$,

$$\sigma_X = [\pi_X + \phi^d \sigma_X][1 - F(t_{\mathcal{S}})]$$

where $t_{\mathcal{S}}$ is the common waiting time and σ_X is the measure of organs supplied to X patients. At steady state, we observe an inflow rate $\phi^d \sigma_X$ of reentrants. Moreover, $\sum_{X\in\mathcal{S}} \sigma_X = \sum_{X\in\mathcal{S}} \delta_X$. Hence summing up the left-hand sides and right-hand sides of these equations, respectively, we get $\sum_{X\in\mathcal{S}} \delta_X = [\sum_{X\in\mathcal{S}}(\pi_X + \phi^d \delta_X)][1 - F(t_{\mathcal{S}})]$. The solution for $t_{\mathcal{S}}$ is as in Equation 2. ■

Proof of Theorem 2. By Lemma 2, $t_Y^{\mathbf{q},\mathbf{d};c} \leq t_X^{\mathbf{q},\mathbf{d};c}$. As $t_X^{\mathbf{q},\mathbf{d};i}$ is the shortest among $t^{\mathbf{q},\mathbf{d};i}$ for types that Y can receive from, the only way $t_Y^{\mathbf{q},\mathbf{d};c} \leq t_X^{\mathbf{q},\mathbf{d};c}$ can happen is that Y patients receive X organs at steady state or X pools with another type, which has a higher $t^{\mathbf{q},\mathbf{d};i}$ than Y. However, the latter is not correct by assumption. Therefore, Y and X patients are pooled (possibly together with other types). By Lemma 3, $t_Y^{\mathbf{q},\mathbf{d};c} = t_X^{\mathbf{q},\mathbf{d};c}$. Moreover, by transferring some of the X organs that the Y and X patients receive to other compatible patients, the waiting time of Y and X patients can be adjusted above $t_{\{X,Y\}}$ but no higher than $t_Y^{\mathbf{q},\mathbf{d};i}$. Similarly, by transferring some of the X

organs that Y patients are receiving to X patients, and substituting those with other compatible organs for Y, the waiting time of Y and X patients can be adjusted below $t_{\{X,AB\}}$ but no lower than $t_X^{\mathbf{q},\mathbf{d};i}$. Observe that the waiting time of no donor blood type that is compatible with Y patients can be made less than $t_X^{\mathbf{q},\mathbf{d};i}$ or more than $t_Y^{\mathbf{q},\mathbf{d};i}$, at steady state, under the constraint of Lemma 3, which says that all donating blood types to Y patients will have the same waiting time. Hence, the composite type of X and Y behaves like Y when it is receiving organs and behaves like X when it is donating organs with deceased donor inflow rate $\delta_X + \delta_Y$ and patient inflow rate $\pi_X + \pi_Y$, by Lemma 3. ■

Proof of Lemma 4. Suppose that for a given X, the non-negative real line can be divided into a sequence of open intervals marked by $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < ...$ such that for any k, for any $\pi_X \in (\varepsilon_k, \varepsilon_{k+1})$ the sets of pooled types do not change. And the sets of pooled types do change in transition from ε_k^- to ε_k^+ for each k.

For any $\pi_X \in (\varepsilon_k, \varepsilon_{k+1})$, Equation 2 gives the waiting time of any pooled set $\mathcal{S}$. Moreover, $t_{\mathcal{S}}^{\mathbf{q},\mathbf{d};c}$ strictly increases in π_X for the pooled set $\mathcal{S}$ that includes X. The waiting times of other types do not change.

Moreover, waiting times are continuous in π_X and bounded in this open interval. Hence, left- and right-hand limits exist at each ε_k. Next, for some k suppose at $\pi_X = \varepsilon_k$ for some blood type left-hand limit is higher than its value at $\pi_X = \varepsilon_k$ for the waiting time, i.e., $\lim_{\pi_X \to \varepsilon_k^-} t_Z^{\mathbf{q},\mathbf{d};c} > t_Z^{\mathbf{q},\mathbf{d};c} \mid_{\pi_X = \varepsilon_k}$ for some $Z \in \mathcal{T}$. Suppose, at ε_k, Z is pooled in $\mathcal{S}_1 \in 2^{\mathcal{T}}$. However, as $\sum_{Y \in \mathcal{T}} \pi_Y$ at $\pi_X \to \varepsilon_k^-$ can be made arbitrarily close to its value at $\pi_X = \varepsilon_k$, for some types of a pooled set $\mathcal{S}_2 \in 2^{\mathcal{T}} \setminus \{\mathcal{S}_1\}$ at $\pi_X \to \varepsilon_k^-$ we necessarily have $\lim_{\pi_X \to \varepsilon_k^-} t_{\mathcal{S}_2} < t_{\mathcal{S}_2} \mid_{\pi_X = \varepsilon_k}$. This can happen only if some $Y \in \mathcal{S}_1 \cap \mathcal{S}_2$ that donates to a blood type in $\mathcal{S}_2$ at $\pi_X \to \varepsilon_k^-$, which is no longer pooled within $\mathcal{S}_2$ but within $\mathcal{S}_1$ at $\pi_X = \varepsilon_k$. But then, this contradicts the definition of ABO-compatible FIFO policy as some deceased donors of Y blood type, which is no longer pooled in $\mathcal{S}_2$ at $\pi_X = \varepsilon_k$, could be given to the patients of one or more blood types in $\mathcal{S}_2$ and their waiting time can be decreased without making it smaller than the waiting time for $\mathcal{S}_1$ at $\pi_X = \varepsilon_k$.

The cases where $\lim_{\pi_X \to \varepsilon_k^-} t_Z^{\mathbf{q},\mathbf{d};c} < t_Z^{\mathbf{q},\mathbf{d};c} \mid_{\pi_X = \varepsilon_k}$, $\lim_{\pi_X \to \varepsilon_k^+} t_Z^{\mathbf{q},\mathbf{d};c} > t_Z^{\mathbf{q},\mathbf{d};c} \mid_{\pi_X = \varepsilon_k}$, and $\lim_{\pi_X \to \varepsilon_k^+} t_Z^{\mathbf{q},\mathbf{d};c} < t_Z^{\mathbf{q},\mathbf{d};c} \mid_{\pi_X = \varepsilon_k}$ are handled in a symmetric manner, leading to a contradiction. Hence, this shows that all blood types' ABO-compatible waiting times are continuous in π_X.

Since each waiting time $t_Y^{\mathbf{q},\mathbf{d};c}$ is continuous at each $\pi_X = \varepsilon_k$ for all $Y \in \mathcal{T}$ and it is weakly (and strictly for $Y = X$) decreasing at each open interval $\pi_X \in (\varepsilon_k, \varepsilon_{k+1})$, then it is weakly (and strictly for $Y = X$) decreasing in π_X.

The proof for "decreasing and continuous in δ_X" is analogous to the proof for "increasing and continuous in π_X" and follows the above proof. ■

Proof of Theorems 3 and 4. Observe that we have $p_O^l = p_O(1-\theta)$, $p_A^l = (p_O + p_A)(1-\theta)$, $p_B^l = (p_O + p_B)(1-\theta)$, and $p_{AB}^l = 1-\theta$. Hence, $p_O^l < p_A^l, p_B^l < p_{AB}^l$. Since $\mathbf{l}_X/\pi_X = p_X^l \lambda$ (recall that $\mathbf{l}_X = p_X^l \lambda \pi_X \in (0, \pi_X)$ is the inflow rate of compatible pairs with X patients), we obtain Theorem 4.

For Theorem 3, first, consider the ABO-identical deceased donor allocation policy. By Equation

4, for any X,

$$t_X^{\mathbf{q},\mathbf{l};i} \;= F^{-1}\big(1 - \tfrac{\delta_X}{(\pi_X-(1-\phi^l)\mathbf{l}_X)+\phi^d\delta_X}\big). \tag{25}$$

As $t_X^{\mathbf{q},\mathbf{l};i}$ is increasing in net patient inflow rate, comparing Equation 1 with Equation 25 we conclude for all X, $t_X^{\mathbf{q},\mathbf{l};i} < t_X^{\mathbf{q},\mathbf{d};i}$.

Next, consider the ABO-compatible deceased donor allocation policy. Assume that we introduce patient–living donor pairs for each blood type one at a time. The net effect of having patients with living donors is a decrease in the new patient inflow rate π_X by $\mathbf{l}_X$ for each X (as in the case of ABO-identical allocation policy). Hence, using Lemma 4 for all four blood types consecutively, we conclude that $t_X^{\mathbf{q},\mathbf{l};c} < t_X^{\mathbf{q},\mathbf{d};c}$ for all X.

In the rest of the proof, we analyze the benchmark case where δ_X/π_X is constant across all blood types X. Then $\mathbf{l}_O \le \mathbf{l}_X$ for all X and $\mathbf{l}_{AB} \ge \mathbf{l}_X$ for all X. These in turn imply that $t_O^{\mathbf{q},\mathbf{l};i} \ge t_X^{\mathbf{q},\mathbf{l};i}$ for all X and $t_{AB}^{\mathbf{q},\mathbf{l};i} \le t_X^{\mathbf{q},\mathbf{l};i}$ for all X, respectively, since $t_X^{\mathbf{q},\mathbf{l};i}$ is decreasing in $\mathbf{l}_X$. We also have

$$\frac{\delta_O}{\pi_O-(1-\phi^l)\mathbf{l}_O} \le \frac{\delta_A}{\pi_A-(1-\phi^l)\mathbf{l}_A}\,,\ \frac{\delta_B}{\pi_B-(1-\phi^l)\mathbf{l}_B} \le \frac{\delta_{AB}}{\pi_{AB}-(1-\phi^l)\mathbf{l}_{AB}}.$$

Then by Theorem 2 and the procedure following this theorem, using $\pi_X - \mathbf{l}_X\delta_X$ instead of π_X for all X, we observe that none of the blood types are pooled together when live donation is possible under the ABO-compatible deceased donation policy. Thus, we also have $t_X^{\mathbf{q},\mathbf{l};c} = t_X^{\mathbf{q},\mathbf{l};i}$ for all X. Further assume that $p_A > p_B$. Then $p_A^l > p_B^l$. Therefore, $\mathbf{l}_A > \mathbf{l}_B$, which in turn implies $\frac{\delta_B}{\pi_B-(1-\phi^l)\mathbf{l}_B} < \frac{\delta_A}{\pi_A-(1-\phi^l)\mathbf{l}_A}$, and hence, $t_A^{\mathbf{q},\mathbf{l};i} < t_B^{\mathbf{q},\mathbf{l};i}$.

Given this result, comparing Equation 5 across blood types together with the fact that $p_O^l < p_A^l, p_B^l < p_{AB}^l$ leads to the analogous result for the overall average waiting times for deceased and living donors. If $p_B^l < p_A^l$, then we get the required result in Theorem 3. ■

Proof of Theorem 5. Under the proposed policy, by Lemma 6 all self-demanded pairs can be matched with their own type pairs as soon as they arrive, and all pairs of type $B-A$ that has the lower inflow rate (by Assumption 3) than $A-B$ pairs will be matched as soon as they arrive with their reciprocal-type pairs. Hence, under this policy only $A-B$ pairs will remain in the exchange pool at any point in time. These pairs can only be matched with overdemanded pairs by Lemma 5, as $B-A$ pairs are already committed to other $A-B$ pairs.

Next consider underdemanded type pairs. These are $Y-X$ type pairs such that $Y \neq X$ and $Y \triangleright X$. By Assumption 2, we have $\theta p_Y \pi_X \le p_X \pi_Y$. By Lemma 5, they can only be matched with overdemanded types. Recall that the inflow rate of each $Y-X$ type pair to the exchange pool is $p_Y\lambda\pi_X$. Their reciprocal type $X-Y$, which is overdemanded, has the inflow rate $\theta p_X\lambda\pi_Y < p_Y\lambda\pi_X$. Hence, we can match all such overdemanded pairs $X-Y$ (by Lemma 6) as soon as they enter the pool under the proposed policy with their reciprocal type pairs. As all overdemanded, self-demanded, and type $B-A$ reciprocally demanded pairs are matched as soon as they arrive, by Lemma 6, the proposed policy achieves the maximum measure of pairs matched. At steady state, as no incompatible overdemanded, self-demanded, and $B-A$ type pair waits in the pool, gets

immediately matched, and saves one additional pair, the maximum mass of possible exchanges is also conducted in this manner in any closed time interval.

On the other hand, if we do not conduct the exchanges immediately whenever they become available, but only after a closed time interval, then some of the patients of overdemanded, self-demanded, and $B-A$ type pairs who have arrived earlier will not survive. Hence, when we conduct the exchanges at the end of the time interval, we will match a strictly smaller mass of possible pairs than we would have matched under the proposed policy. ■

Proof of Theorem 7. Consider $\{\mathbf{e}_X\}_{X\in\mathcal{T}}$, the overall measures of pairs with X blood type pairs participating in exchange for each $X \in \mathcal{T}$ reported in Equation 6. Observe that

$$\begin{aligned}
\frac{\mathbf{e}_O}{\pi_O} =& \theta p_O\lambda + \theta p_O\lambda\frac{\pi_A+\pi_B+\pi_{AB}}{\pi_O} = \theta(p_O+p_A+p_B+p_{AB})\lambda\\
\frac{\mathbf{e}_A}{\pi_A} =& \theta p_A\lambda + \theta p_O\lambda + p_A\lambda\frac{\pi_B}{\pi_A} + \theta p_A\lambda\frac{\pi_{AB}}{\pi_A} = (\theta p_O+\theta p_A+p_B+\theta p_{AB})\lambda\\
\frac{\mathbf{e}_B}{\pi_B} =& \theta p_B\lambda + \theta p_O\lambda + p_A\lambda + \theta p_B\lambda\frac{\pi_{AB}}{\pi_B} = (\theta p_O+p_A+\theta p_B+\theta p_{AB})\lambda\\
\frac{\mathbf{e}_{AB}}{\pi_{AB}} =& \theta(p_{AB}+p_O+p_A+p_B)\lambda
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that p_X/π_X is a constant among all $X \in \mathcal{T}$. Since $\theta < 1$, we have $\mathbf{e}_O/\pi_O = \mathbf{e}_{AB}/\pi_{AB} < \mathbf{e}_A/\pi_A, \mathbf{e}_B/\pi_B$. With the additional assumption $p_A > p_B$, we obtain $\mathbf{e}_A/\pi_A < \mathbf{e}_B/\pi_B$.

Next consider $\{\mathbf{l}_X + \mathbf{e}_X\}_{X\in\mathcal{T}}$, direct living donor and exchange transplants. We have

$$\begin{aligned}
\frac{\mathbf{l}_O+\mathbf{e}_O}{\pi_O} =& (1-\theta)p_O\lambda + \theta(p_O+p_A+p_B+p_{AB})\lambda = (p_O+\theta p_A+\theta p_B+\theta p_{AB})\lambda\\
\frac{\mathbf{l}_A+\mathbf{e}_A}{\pi_A} =& (1-\theta)(p_O+p_A)\lambda + (\theta p_O+\theta p_A+p_B+\theta p_{AB}) = (p_O+p_A+p_B+\theta p_{AB})\lambda\\
\frac{\mathbf{l}_B+\mathbf{e}_B}{\pi_B} =& (1-\theta)(p_O+p_A)\lambda + (\theta p_O+p_A+\theta p_B+\theta p_{AB})\lambda = (p_O+p_A+p_B+\theta p_{AB})\lambda\\
\frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}}{\pi_{AB}} =& (1-\theta)(p_{AB}+p_O+p_A+p_B)\lambda + \theta(p_{AB}+p_O+p_A+p_B)\lambda = (p_{AB}+p_O+p_A+p_B)\lambda
\end{aligned}$$

Since $\theta < 1$, we have, $\frac{\mathbf{l}_O+\mathbf{e}_O}{\pi_O} < \frac{\mathbf{l}_A+\mathbf{e}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B}{\pi_B} < \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}}{\pi_{AB}}$. ■

Proof of Theorem 8. Using Assumption 4 instead of Assumption 2, the proof follows verbatim the proof of Theorem, after noting that no self-demanded type can be used to save additional underdemanded or $A-B$ type pair (by Lemma 5). ■

Proof of Theorem 9. Let $\psi^{\mathbf{i},i}$ be the ABO-identical optimal policy explained in Theorem 8 under incentivized exchange, and $\varphi^{\mathbf{e},i}$ be the ABO-identical optimal policy explained in Theorem 5 under regular exchange. Recall that any reentrant is classified as a single patient. Under $\psi^{\mathbf{i},i}$, no unwilling compatible pairs and compatible self-demanded pairs participate in exchange. And under $\varphi^{\mathbf{e},i}$, no compatible pairs participate in exchange. Such compatible pairs' patients immediately receive transplants from their paired donors. All willing overdemanded type pairs are matched

through exchange with their reciprocal types under both $\psi^{\mathbf{i},i}$ and $\varphi^{\mathbf{e},i}$ upon entry (by Assumption 4). We first prove Statement 1 and then the rest.

Proof of Statement 1: First consider underdemanded pairs. Suppose that an underdemanded $X-Y$ pair type is not pooled with X single patients for deceased donation under $\varphi^{\mathbf{e},i}$. Under $\psi^{\mathbf{i},i}$, that type of pairs is matched at the rate

$$\mu_{\mathbf{i}}^{X-Y} = [\rho(1-\theta)+\theta]p_X\lambda\pi_Y, \tag{26}$$

at each point in time while under $\varphi^{\mathbf{e},i}$, they are matched at the rate

$$\mu_{\mathbf{e}}^{X-Y} = \theta p_X\lambda\pi_Y, \tag{27}$$

which is strictly smaller.

Next, suppose that pair types $X_1-Y_1,...,X_\ell-Y_\ell$ are pooled altogether for deceased donation, and suppose among these pair types, $X_{\ell^*}-Y_{\ell^*}$ is underdemanded. Note that all of these pair types are either underdemanded or $A-B$. Each X_k-Y_k is matched at the rate $\mu_{\mathbf{e}}^{X_k-Y_k}+\varepsilon_{\mathbf{e}}^{X_k-Y_k}$ under $\varphi^{\mathbf{e},i}$, where the rate $\varepsilon_{\mathbf{e}}^{X_k-Y_k}>0$ is the measure of X_k-Y_k pairs whose patients receive deceased donation and $\mu_{\mathbf{e}}^{X_k-Y_k}$ is defined as in Equation 27. Under $\psi^{\mathbf{i},i}$, $\mu_{\mathbf{i}}^{X_k-Y_k}$ is the measure of the reciprocal X_k-Y_k pairs (who are on high demand) willing to participate in exchange, which is strictly larger than $\mu_{\mathbf{e}}^{X_k-Y_k}$, while the rate of deceased donation does not change. Hence, while $\mu_{\mathbf{i}}^{X_k-Y_k}-\mu_{\mathbf{e}}^{X_k-Y_k}$ more of X_k-Y_k pairs participate in exchange under $\psi^{\mathbf{i},i}$, fewer of such pairs may receive deceased donation. Suppose that $\varepsilon_{\mathbf{i}}^{X_k-Y_k}$ is the rate of X_k-Y_k pairs receiving deceased donation under $\psi^{\mathbf{i},i}$. We will show that $\xi_k=[\mu_{\mathbf{i}}^{X_k-Y_k}+\varepsilon_{\mathbf{i}}^{X_k-Y_k}]-[\mu_{\mathbf{e}}^{X_k-Y_k}+\varepsilon_{\mathbf{e}}^{X_k-Y_k}]>0$ for all k. Suppose not for some k. In particular, if there are multiple such k, let k be chosen with the smallest $\xi_k\leq 0$. Hence, as waiting time of all pairs $X_1-Y_1,...,X_\ell-Y_\ell$ is the same under $\varphi^{\mathbf{e},i}$, X_k-Y_k's waiting time increases the most among all pairs or stays the same and no other pair's waiting time increases under $\psi^{\mathbf{i},i}$. Hence, X_k-Y_k continues to be pooled with X_k single patients under $\psi^{\mathbf{i},i}$. As $\mu_{\mathbf{i}}^{X_{\ell^*}-Y_{\ell^*}}-\mu_{\mathbf{e}}^{X_{\ell^*}-Y_{\ell^*}}>0$, and for all $k^*\neq\ell^*$ we have, $\mu_{\mathbf{i}}^{X_{k^*}-Y_{k^*}}-\mu_{\mathbf{e}}^{X_{k^*}-Y_{k^*}}\geq 0$, then a higher share of deceased donors should go to X_k-Y_k pairs under $\psi^{\mathbf{i},i}$ with respect to $\varphi^{\mathbf{e},i}$. Hence, $\varepsilon_{\mathbf{i}}^{X_k-Y_k}-\varepsilon_{\mathbf{e}}^{X_k-Y_k}>0$ implying that $\xi_k>0$, a contradiction.

Hence, unless $A-B$ is pooled by itself with A single patients under $\varphi^{\mathbf{e},i}$, any pooled paired group with X single patients has a strictly higher measure of being matched at each point in time under $\psi^{\mathbf{i},i}$.

We continue with other patient groups. All overdemanded pairs and self-demanded pairs receive live donation under both $\psi^{\mathbf{i},i}$ and $\varphi^{\mathbf{e},i}$ immediately after their arrival. We already showed that underdemanded pairs strictly benefit from $\psi^{\mathbf{i},i}$. Moreover, by Assumption 3, and Theorems 5 and 8, all $B-A$ pairs are matched with $A-B$ pairs through exchange as soon as they enter the exchange pool. This and the proof for underdemanded pairs imply that $A-B$ pairs either benefit under $\psi^{\mathbf{i},i}$ (if they are pooled with an underdemanded type for deceased donation under $\varphi^{\mathbf{e},i}$) or they remain indifferent between the two technologies (otherwise).

Next consider any $X\in\mathcal{T}$ blood-type single patients. As more underdemanded-type pairs are matched through exchange and the same measure of $A-B$ pairs participate in exchange under

$\psi^{\mathbf{i},i}$, overall fewer underdemanded-type and $A-B$ type pairs will be left from the same cohort for deceased donation. Hence, weakly more X single patients receive deceased donation under $\psi^{\mathbf{i},i}$.

Proof of Statement 2: Under $\psi^{\mathbf{i},i}$, by Theorem 9 $X-X$ pairs are matched in exchange only with $X-X$ pairs. Moreover, all incompatible $X-X$ pairs are almost surely matched through exchange with each other as soon as they arrive. Hence, no compatible $X-X$ pair is used to match them.

Proof of Statement 3: Patient blood type O can form 4 types of pairs: $O-O$, $O-A$, $O-B$, and $O-AB$. None of them can form compatible pairs except $O-O$. By Statement 3, no compatible $O-O$ pairs participate in exchange. Hence, upon possible reentry under $\psi^{\mathbf{i},i}$, no O patients are prioritized. On the other hand, positive measures of compatible overdemanded pairs with A, B, AB patients participate in exchange. Therefore, a positive measure of these patients reenters at steady state and gets prioritized.

Proof of Statement 4: First observe that the waiting time of underdemanded types strictly decreases by Statement 1. The waiting times of reciprocally demanded $B-A$ type pairs and $A-B$ type pairs do not increase by Statement 1. Moreover, self-demanded and overdemanded type pairs do not wait and get immediately matched under both technologies. Finally, we consider single patients. To see how their waiting times are affected, we consider the change of exchange rates for compatible and incompatible pairs first. We do this analysis for all blood types separately.

1. O patients:

Compatible pairs: $O-O$ is the only compatible type with O patients. However, incompatible $O-O$ pairs are already matched immediately with each other in exchange. Hence, a

$$\kappa_O = 0$$

measure of compatible pairs with O patients participates in exchange.

Incompatible pairs: A measure of $[\rho(1-\theta)+\theta]p_O\lambda[\pi_A+\pi_B+\pi_{AB}]$ incompatible pairs with O patients is matched through exchange with their reciprocal type pairs at each point in time. This is a net increase of

$$\xi_O = \rho(1-\theta)p_O\lambda[\pi_A+\pi_B+\pi_{AB}]$$

with respect to regular exchange. If some of these pair types are pooled for deceased donation under exchange with incentivized compatible pairs, then they are also pooled for deceased donation under regular exchange.

Single patients:

* *Prioritized reentrants*: As no O reentrants are prioritized, all O deceased donors are still given to O single patients, and there is a

$$\phi^l\kappa_O = 0$$

measure of prioritized O reentrants per unit time.

* *Regular single patients*: On the other hand, some additional O patients are saved through exchange, an additional measure of

$$\phi^l\xi_O = \phi^l[\rho(1-\theta)]p_O\lambda[\pi_A+\pi_B+\pi_{AB}]$$

of O patients reenters with respect to regular exchange. These reentrants join the regular deceased donor queue. However, if some underdemanded pairs with O patients receive deceased donation

under exchange, then some of these fall from competition for deceased donation under incentivized exchange. Depending the size of this fallout, the net effect on the net inflow rate of O single patients can be negative or positive, but this additional inflow rate to the regular deceased donation queue will be no more than

$$\phi^l \xi_O.$$

Depending on which of the above effects dominates, the waiting time for regular O single patients can slightly increase or decrease under incentivized exchange.

2. A patients:

Compatible pairs: A measure

$$\kappa_A = \rho(1-\theta)p_O\lambda\pi_A$$

of $A-O$ type compatible pairs participates in exchange to save $O-A$ type pairs. Self-demanded $A-A$ type compatible pairs do not participate in exchange.

Incompatible pairs: A measure $[\rho(1-\theta)+\theta]p_A\lambda\pi_{AB}$ of underdemanded type pairs $A-AB$ is matched through exchange in every point in time. This is a net increase of

$$\xi_A = \rho(1-\theta)p_A\lambda\pi_{AB}$$

with respect to regular exchange. If some of these pair types are pooled for deceased donation under incentivized exchange, then they are also pooled for deceased donation under regular exchange.

The reciprocally demanded pair type $A-B$ continues to run a deficit as $B-A$ inflow is – by Assumption 3 – lower than $A-B$ inflow. If $A-B$ type pairs wait both for $B-A$ type pairs and decease donors under incentivized exchange, see the case for single patients to understand the effect of incentivized exchange on their waiting times below. On the other hand if they are waiting exclusively for $B-A$s under incentivized exchange policies, then $A-B$ types wait for the same time under both regular and incentivized exchange, and exactly the same measure of them gets matched.

Single patients:

* *Prioritized reentrants*: Patients of some of the $A-O$ type compatible pairs that previously participated in exchange reenter as their grafts fail. Their inflow is

$$\phi^l\kappa_A = \phi^l\rho(1-\theta)p_O\lambda\pi_A.$$

These A reentrants, who no longer have living donors, go directly to the top of the A deceased donor queue instead of going to the bottom as under regular exchange. We will refer to this as *incentivized exchange burden*. This is also the rate of the decease donors reserved for these patients.

* *Regular single patients*: An additional ξ_A measure of $A-AB$ pairs are saved by $AB-A$ types through exchange. A measure of

$$\phi^l\xi_A = \phi^l\rho(1-\theta)p_A\lambda\pi_{AB}$$

A patients reenters and joins in the regular queue with the single A patients. However, if some $A-AB$ pairs receive deceased donation under regular exchange, then some of these fall from competition for deceased donation under incentivized exchange. Depending the size of this fallout, the net effect on the net inflow of A single patients for the regular queue can be negative or positive, but this additional inflow will be no more than

$$\phi^l \xi_A - \phi^l \kappa_A.$$

As a result, the waiting time for new A single patients can slightly increase or decrease under exchange with incentivized compatible pairs (cf. Figure 5 for an example of the overall impact of this new exchange policy on A patients).

3. B patients: Symmetric version of A patients, except that $B-A$'s are immediately matched with $A-B$'s when they enter the pool by the assumption that $B-A$'s are on the short side.

4. AB patients:

Compatible pairs: A total measure of

$$\kappa_{AB} = \rho(1-\theta)[p_O + p_A + p_B]\lambda\pi_{AB}$$

compatible $AB-O$, $AB-A$, and $AB-B$ type pairs participate in exchange to save their reciprocals at each point in time. Self-demanded compatible $AB-AB$ type pairs do not participate in exchange.

Incompatible pairs: All incompatible pairs with AB patients are either self-demanded or overdemanded. Hence, they are matched immediately when they arrive through exchange with their reciprocal types under both regular exchange and exchange with incentivized compatible pairs. Hence, additionally a

$$\xi_{AB} = 0$$

measure of incompatible pairs with AB patients is matched under the new regime.

Single patients:

* *Prioritized reentrants*: The reentry burden of AB patients from previous compatible pairs that participated in exchange is

$$\phi^l \kappa_{AB} = \phi^l \rho(1-\theta)[p_O + p_A + p_B]\lambda\pi_{AB},$$

which is the rate of prioritization for AB reentrants to the deceased donor queue. This is also the rate of the decease donors reserved for these patients.

* *Regular single patients*: On the other hand, the same measure of AB patients reenters at each point in time under both regular exchange and exchange with incentivized compatible pairs. No pairs with AB patients are pooled for deceased donation under either regular exchange or exchange with incentivized compatible pairs. Hence, a

$$\phi^l \xi_{AB} = 0$$

measure of additional AB reentrants from previous incompatible pairs reenters the deceased donor queue. Net increase of rate of entry to the regular AB deceased donor queue is negative and equal to

$$-\phi^l \kappa_{AB}.$$

As a result, the waiting time for regular AB single patients unambiguously slightly increases under exchange with incentivized compatible pairs. This holds as all of the prioritized AB patients receive deceased donation under exchange with incentivized compatible pairs, while some patients from the same population would have died and not received deceased donation under the alternative regime, regular exchange. ■

Proof of Proposition 1. Fix $\rho \in [0,1]$ such that Assumption 4 holds. Consider the following strategy profile σ^*: all pairs register at P_0, the national program, with probability 1. As an optimal

exchange mechanism is used, then under this profile the maximal measure of pairs are matched. Moreover, σ^* is a pure strategy equilibrium in undominated strategies: as no pairs register in any other platform, it is a best response to register at P_0.

Consider an arbitrary pure strategy equilibrium profile σ in undominated strategies. Each pair registers at a unique exchange platform with probability one as soon as it arrives.

We prove that all pairs belonging to overdemanded pair types and pair type $B-A$ are matched with pairs belonging to underdemanded types or pair type $A-B$ immediately when they arrive under σ. To the contrary, suppose there is a platform P_a where a positive measure of pairs of a type $X-Y \in \mathbf{O} \cap \mathcal{T} \times \mathcal{T}$, i.e., overdemanded or type $B-A$, is not matched with pairs of types in $\mathbf{U} \cap \mathcal{T} \times \mathcal{T}$, i.e., either underdemanded or type $A-B$, at σ when they arrive with a positive probability (using the notation in Appendix A).

Consider any pair type $W_1 - Z_1$ in set $\mathbf{U}$ that has $c_{W_1-Z_1,X-Y} = 1$ (i.e., that is mutually blood-type compatible with an $X-Y$ type pair using the same notation). All pairs of type W_1-Z_1 should be matched immediately at σ. Otherwise such a pair x can register at P_a and can be immediately matched with probability 1 with one of the $X-Y$ pairs at σ. The reason for this is as follows: As pair x is of measure 0 and a positive measure of $X-Y$ pairs are either being matched with other overdemanded pairs or not being matched at all, the platform P_a, which is using an optimal exchange policy with randomization when there are multiple possible pairs to match, will match pair x immediately with probability 1. This implies that all $W_1 - Z_1$ type pairs are matched with probability 1 through exchange when they arrive at σ by Lemma 7.

Suppose $\mathbf{P}_1' \subseteq \mathbf{O} \cap \mathcal{T} \times \mathcal{T}$ is the set of overdemanded pair types or type $B-A$ with which $W_1 - Z_1$ type pairs are mutually blood-type compatible: that is, $\mathbf{P}_1' = \mathbf{C}_{\{W_1-Z_1\}}(\mathbf{O} \cap \mathcal{T} \times \mathcal{T})$ (as defined in Equation 17). Observe that $AB - O \in \mathbf{P}_1'$. Let $\mathbf{V}_1' \subseteq \mathbf{U} \cap \mathcal{T} \times \mathcal{T}$ be the set of underdemanded pair types or $A-B$ that are mutually blood-type compatible with the types in $\mathbf{P}_1'$: that is, $\mathbf{V}_1' = \mathbf{C}_{\mathbf{P}_1'}(\mathbf{U} \cap \mathcal{T} \times \mathcal{T})$ (as defined in Equation 18). As $AB - O \in \mathbf{P}_1'$, we have $\mathbf{V}_1' = \mathbf{U} \cap \mathcal{T} \times \mathcal{T} = \{O-A, O-B, O-AB, A-AB, A-AB, B-AB, A-B\}$ (cf. Figure 7).

All pairs belonging to types in $\mathbf{V}_1'$ should be matched immediately with probability 1 at σ, as otherwise, one pair that does not get matched immediately with positive probability can register at a platform where a positive measure of $W_1 - Z_1$ type pairs register at σ. As all $W_1 - Z_1$ pairs are matched immediately with pairs of types in $\mathbf{P}_1'$ and this one pair is of measure 0, it would be guaranteed to be matched immediately as well.

Pairs of types in $\mathbf{U} \cap \mathcal{T} \times \mathcal{T}$ can only be matched with pairs of types in $\mathbf{O} \cap \mathcal{T} \times \mathcal{T}$. We have a measure of $e_1 = \sum_{X-Y \in \mathbf{U} \cap \mathcal{T} \times \mathcal{T} \setminus \{A-B\}} [\theta + \rho(1-\theta)] p_Y \lambda \pi_X + p_B \lambda \pi_A$ underdemanded and $A-B$ pairs being matched through exchange at every moment in time at σ. However, the total measure of overdemanded and $B-A$ pairs arriving at each moment is only $e_2 = \sum_{Y-X:\mathbf{O} \cap \mathcal{T} \times \mathcal{T} \setminus \{B-A\}} [\theta + \rho(1-\theta)] p_X \lambda \pi_Y + p_A \lambda \pi_B$. By Assumptions 2 and 4, $e_2 > e_1$. Hence, a positive measure of underdemanded pairs should wait under any feasible exchange scheme, contradicting the fact that all pairs of types in $\mathbf{U} \cap \mathcal{T} \times \mathcal{T}$ are matched immediately.

Thus, we showed that all pairs of types in $\mathbf{O} \cap \mathcal{T} \times \mathcal{T}$ matched to pairs of types in $\mathbf{U} \cap \mathcal{T} \times \mathcal{T}$

under equilibrium. As any positive measure of self-demanded types can be matched with each other at any platform, equilibrium σ maximizes the total measure of pairs being matched through exchange, and hence, through deceased donation, as well.

As ρ goes up, the measure of pairs of types in $\mathbf{O} \cap \mathcal{T} \times \mathcal{T}$ goes up. Hence, more underdemanded and overdemanded pairs are matched at any equilibrium, while the measure of reciprocally demanded pairs matched in exchange stays constant or goes up. On the other hand, if single patients are pooled with some types in $\mathbf{U} \cap \mathcal{T} \times \mathcal{T}$ before ρ goes up, the measure of such patients being matched also increases. ■

Proof of Theorem 11. By Proposition 1, as the maximal measure of pairs are matched at pure Nash equilibria in undominated strategies, the worst equilibrium in undominated strategies for P_0 is the best equilibrium for other platforms. The measure of pairs matched in every moment in time through exchange in any pure strategy equilibrium in undominated strategies is given as

$$\sum_{X-X \in \mathcal{T} \times \mathcal{T}} \theta p_X \lambda \pi_X + 2 \sum_{X-Y \in \mathbf{O} \setminus \{B-A\}} [\theta + \rho(1-\theta)] p_Y \lambda \pi_X + 2 p_A \lambda \pi_B$$

where $[\rho(1-\theta)]p_Y \lambda \pi_X$ is the measure of compatible pairs participating in exchange, which also save the same amount of the underdemanded or $A-B$ pairs through exchange. In the worst equilibrium for P_0 only the compatible pairs participate in exchange at P_0 among all overdemanded and $B-A$ pairs. Hence no $B-A$ pair participates at P_0. Among the underdemanded pairs and $A-B$ type pairs, for such a type $X-Y$, the participation at P_0 is such that exactly $\rho(1-\theta)]p_Y \lambda \pi_X$ survive and get matched with the compatible pairs. Contrary to the claim, suppose that as ρ increases, the participation of overall pairs decreases or stays the same at P_0 under the worst equilibrium. Then, more compatible pairs are available of each (feasible) type. Thus, the waiting time for the underdemanded pairs registered at P_0 decreases while the waiting time at other programs for the same types stays the same or increases. This leads to a contradiction.

Now, if

$$\rho \;>\; \frac{\sum_{X-Y \in \mathbf{O} \setminus \{B-A\}} \theta p_Y \pi_X + p_A \pi_B}{\sum_{X-Y \in \mathbf{O} \setminus \{B-A\}} (1-\theta) p_Y \pi_X},$$

then

$$2 \sum_{X-Y \in \mathbf{O} \setminus \{B-A\}} \rho(1-\theta) p_Y \lambda \pi_X \;>\; 2 \sum_{X-Y \in \mathbf{O} \setminus \{B-A\}} \theta p_Y \lambda \pi_X + 2 p_A \lambda \pi_B,$$

where the left-hand side denotes the least measure of pairs matched at P_0 at each point in time at any equilibrium with undominated pure strategies and the right-hand side denotes the maximum total measure of efficiency critical pairs matched outside of P_0 at an equilibrium.

Therefore, more pairs of types in $\mathbf{O}$ register at P_0 at any pure undominated equilibrium, as half of the above measures belong to pairs of types in $\mathbf{O}$ registering at P_0 (left-hand side) and other platforms in total (right-hand side), respectively. To the contrary, suppose that a less or equal

measure of pairs of types in $\mathbf{U}$ registers at P_0. Then as more pairs of types in $\mathbf{U}$ are matched within P_0 than at all other platforms combined, some pair type $X - Y \in \mathbf{U}$ will have a lower waiting time at P_0 than at some other platform, leading to a contradiction to Lemma 7. ■