Partisan and Bipartisan Gerrymandering*

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August 4, 2018

Abstract

This paper analyzes the optimal partisan and bipartisan gerrymandering policies in a model with electoral competitions in policy positions and transfer promises. Party leaders have both office- and policy-motivations. With complete freedom in redistricting, partisan gerrymandering policy generates the most one-sidedly biased district profile, while bipartisan gerrymandering generates the most polarized district profile. In contrast, with limited freedom in gerrymandering, both partisan and bipartisan gerrymandering tend to prescribe the same policy.

Keywords: electoral competition, partisan gerrymandering, bipartisan gerrymandering, policy convergence/divergence, pork-barrel politics

JEL Classification Numbers: C72, D72

*We thank Jim Anderson, Emanuele Bracco, Klaus Demset, Mehmet Ekmekci, Alan Miller, Ron Siegel, and participants of PET 2017 in Paris for their comments and suggestions.

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1 Introduction

It is widely agreed upon that election competitiveness has decreased significantly in recent decades. For example, the re-election rate of the US House of Representatives has increased from 91.82% in 1950 to 98.25% in 2004 (Friedman and Holden 2009). Also, 74 House seats were won by a margin of less than 55% in 2000, but this number decreased to 24 in 2004 (Fiorina et al. 2011). During the same period, Congress has become quite polarized. In the 1960s, the distribution of the representatives' political positions was concentrated more toward the center of the political spectrum, with considerable overlap between Republicans' and Democrats' positions. By the 2000s, the positions became sharply twin-peaked with less overlap.[1] One popular explanation for this phenomenon in US politics is gerrymandering. Fiorina et al. (2011) argue that gerrymandering biased toward *incumbents*, i.e., *bipartisan gerrymandering*, has an effect on the decrease in competitiveness, since both parties try to secure their incumbent seats.[2] They also suggest that this decrease in competitiveness may be one of the causes of the recent political polarization in Congress, since proposing more polarized positions does not jeopardize secured seats.[3]

In contrast, partisan gerrymandering is commonly used to increase the power of a political party, and it may or may not reduce district competitiveness, since the party tries to secure their incumbents but at the same time may create competitive districts to capture the opposing party's districts. Recently, there have been many partisan gerrymandering lawsuits against state legislatures to request redistricting of state congressional maps, including Pennsyl-

[1]It is now standard to use a one-dimensional scaling score (DW-Nominate procedure on economic liberal-conservative, Poole and Rosenthal, 1997) to measure representatives' political positions.

[2]Fiorina et al. (2011) state that "Many (not all) observers believe that the redistricting that occurred in 2001-2002 had a good bit to do with this more recent decline in competitive seats—the party behaved conservatively, concentrating on protecting their seats rather than attempting to capture those of the opposition." (see Fiorina et al., pp. 214-215).

[3]Since polarization has complicated causes, this argument has limitations. Citing that polarization has also been happening in the Senate, Fiorina et al. (2011, pp. 219) suggest that "redistricting is only a minor part of congressional polarization, or that it is important only in combination with other factors such as closed primaries." Alternative explanations for polarization include voters' party sorting and geographical sorting. The former says that voters became sorted into Republican and Democratic parties in the latter half of 1900s due to party elites' polarization (Levendusky 2009). See also Gilroux (2001). The latter says that voters sort themselves into more ideologically homogeneous districts, causing polarization. The rationale behind this is that districts seem to polarize more between redistricting than during them (McCarty, Poole, and Rosenthal 2009).

vania, Maryland, Wisconsin, and North Carolina.[4] In these cases, the courts rely on a measure of vote misrepresentation, the efficiency gap, and heated debates are going on as to whether or not this is an appropriate measure to use (Bernstein and Duchin 2017 and Chambers, Miller, and Sobel 2017).[5] Since these two methods loom large in US politics and public debates, we go one step further to see how different the resulting district maps under partisan and bipartisan gerrymandering are, and how polarized the elected representatives are.

In this paper, we will investigate the difference between optimal partisan and bipartisan gerrymandering and the effects on representative policy positions in a unified framework. For this purpose, we introduce party leaders who are not only office-motivated but also policy-motivated, which is also new to the literature. We set up a two-party political competition model in which party leaders compete with their candidates' (unidimensional) political positions and pork-barrel promises in each electoral district.[6] We assume that there are minimum units of indivisible localities with the same population, and that a gerrymanderer partitions the set of localities freely to create electoral districts. Each locality has a voter distribution, and we say that the gerrymanderer has more freedom in redistricting if the distribution is concentrated on a point on the political spectrum. With pork-barrel politics, the party leader understands that pork-barrel policies in competitive districts are costly, and therefore she has strong incentives to collect her supporters in the winning districts in order to avoid large pork-barrel promises.

Traditionally, the literature on gerrymandering often discusses two tactics in partisan gerrymandering: one is to concentrate or "*pack*" those who support the opponent in losing districts, and the other is to evenly distribute or "*crack*" supporters in winning districts. Packing serves to waste the opposing party's strong supporters' votes, while cracking utilizes the votes of party supporters as effectively as possible. Owen and Grofman (1988) show that

[4] In February 2018, the Pennsylvania Supreme Court blamed the states' district map on partisan gerrymandering, and told state lawmakers to redraw the state's 18 House districts, which currently favor Republicans.

[5] The efficiency gap and the vote-seat curve are measures based on only two numbers, the vote shares and the seat shares of the two parties, which may not contain sufficient information to appropriately evaluate partisan gerrymandering. As Chambers et al. (2017) correctly recognize, the presence of extremists and election uncertainty should be taken into account. For example, consider the outcome of conservative bipartisan gerrymandering. It is possible that the resulting redistricting plan generates extremely polarized policy positions for the elected representatives but is still perfectly desirable under criteria like the efficiency gap or the seat-vote curve.

[6] We assume that party leaders can choose their candidates' policy positions freely.

a *pack-and-crack* policy is optimal when a partisan gerrymanderer has limited freedom in redistricting (a constant-average constraint: see the literature review). In contrast, Friedman and Holden (2008) argue that advances in computing technologies and the availability of big data sets allow gerrymanderers higher degrees of freedom in redistricting, and they obtain a very different optimal policy from pack-and-crack: the *slice-and-mix* policy, in which districts are created by first mixing the strongest opposition group of voters and the strongest supporter group, then mixing the second strongest opposition and supporting groups, and so on. This policy wastes opposition groups' votes, generating the most one-sided allocation from the most extreme to the most moderate districts.

Assuming that party leaders are also policy-motivated, we show that the optimal policies are not the same as the ones in the literature. We show that the slice-and-mix policy is still optimal for the party leaders in charge of partisan gerrymandering when they can redistrict with complete freedom. In contrast, when the freedom on gerrymandering is limited by the constraint in redistricting imposed by Owen and Grofman (1988), we obtain a different result: when voters and party leaders are highly policy-sensitive (roughly speaking), a consecutive partition of localities is also the optimal policy for partisan gerrymandering (*order-and-partition*).[7] We will also systematically compare the optimal policies under partisan and bipartisan gerrymandering when the gerrymanderer(s) face different levels of freedom in redistricting. We show that when the gerrymanderer can redistrict with complete freedom, the resulting outcomes in partisan and bipartisan gerrymandering are very different: bipartisan gerrymandering results in the polarized electoral districts without leaving moderate and competitive ones, while partisan gerrymandering results in a one-sided allocation, leaving some competitive districts. In contrast, when the gerrymandering freedom is limited, a consecutive partition of localities is the optimal *order-and-partition* policy for both partisan and bipartisan gerrymandering.[8]

The rest of the paper is organized as follows. Section 2 discusses related literature. In Section 3, we introduce our model and analyze electoral competition in each district. In Section 4, we investigate the optimal gerrymandering strategy when the party leader has complete freedom. In Section 5, we proceed to cases where the gerrymanderer's freedom is limited. Section 6 concludes the study. All proofs are collected in Appendix A.

[7]The optimal policy is different from pack-and-crack in our model, since party leaders are assumed to be policy-motivated, unlike in Owen and Grofman (1988).

[8]See the the Section 2 for an empirical observation in Friedman and Holden (2009).

2 Related Empirical and Theoretical Literature

Recent empirical studies show that the effects of gerrymandering may be inconclusive. Friedman and Holden (2009) investigate whether or not the House-incumbent re-election rate depends on gerrymandering being partisan or bipartisan.[9] In partisan gerrymandering cases, the majority party may try to oust the opposing party's incumbents, and this may be reducing the incumbent re-election rate. In contrast, in bipartisan gerrymandering cases, both parties try to secure their incumbents' re-election, maximizing safe seats.[10] Interestingly, Friedman and Holden (2009) study the data up to 2004 and do not find significant differences between bipartisan and partisan gerrymandering effects on the rising incumbent re-election rate.[11] As they mention, this result suggests that partisan gerrymandering may not be as effective as popularly thought. In another interesting paper, Grainger (2010) finds that legislatively drawn districts have been less competitive with more extreme voting positions (polarization) than panel-drawn districts by using a quasi-natural experiment that alternates between legislatively drawn and panel-drawn districts in California.[12] McCarty et al. (2006, 2009) document that the political polarization of the House of Representatives has increased in recent decades, using data on roll call votes, but they find only a minimal relation between polarization and gerrymandering.[13,14]

The pioneering paper that laid out the analytical framework of gerrymandering problems is Owen and Grofman (1988). They introduce uncertainty in each district's median voter's position and consider the situation where a

[9] Redistricting in the US is usually conducted by state legislatures (partisan gerrymandering), but in Arizona, Hawaii, Idaho, Montana, New Jersey, and Washington it is conducted by bipartisan redistricting commissions. In California and Iowa, redistricting lines are drawn by nonpartisan redistricting committees.

[10] According to Cain (1985), the goal of bipartisan gerrymanding is to protect incumbents of both parties, whereas partisan gerrymanding seeks to provide an advantage to one party.

[11] After 2008, the incumbent re-election rate went down significantly.

[12] Grainger (2010) provides a detailed history of Californian redistricting: in 1970s and 1990s, district lines were drawn by independent panels of judges, whereas in the 1960s, 1980s, and 2000s, redistricting was done legislatively. He uses this quasi-natural experiment to test the hypotheses. Interestingly, the 1960s and 2000s redistrictings were bipartisan, whereas those in the 1980s were partisan and led by the Democrats.

[13] Krasa and Polborn (2015) argue that their answer may be incomplete if the political positions of district candidates are mutually interdependent.

[14] As early evidence, Ferejohn (1977) finds little support for gerrymandering being the cause of the decline in competitiveness of congressional districts in the 1960s.

partisan gerrymanderer redesigns districts in order to maximize the expected number of seats in a partisan gerrymandering setup. They assume that the uncertainty in the median voter's political position is local and independent across districts when the objective is expected number of seats. Assuming that the average position of district median voters must stay the same after redistricting (a *constant-average constraint*), they show that the optimal strategy is "packing" the opponents in losing districts, and "cracking" the rest of voters evenly across the winning districts *with substantial margins*, so that the party can win districts even in cases of negative shocks.[15,16] Friedman and Holden (2008), on the other hand, assume that a partisan gerrymanderer has *full freedom* in allocating the population over a finite number of districts, and that she maximizes the expected number of seats when there is only valence uncertainty in median voters' utilities (thus, there is no uncertainty in the median voter's political position). In this idealized situation, they find that the optimal strategy is "*slice-and-mix,*" which is similar to our optimal strategy under a different model. Thus, theoretically, the levels of freedom in gerrymandering can affect the optimal policy.

In bipartisan gerrymandering, Gul and Pesendorfer (2010) extend Owen and Grofman (1988) by introducing a continuum of districts and voters' party affiliations. Here, bipartisan gerrymandering means that the two parties own their territories and redistrict exclusively within each territory. They assume that each party leader can redistrict her party's territory (the districts with her party's seats) independently, maximizing the probability of winning the majority of seats.[17] They show that the optimal policy is again a version of "pack-and-crack." However, these papers do not compare the optimal partisan and bipartisan gerrymandering policies. They also do not model spatial competition in policy positions, and the elected representatives' positions are implicitly assumed to be the district median voters' positions (Downsian competition).

Our model is related to vote-buying models like the one in Dekel, Jackson, and Wolinsky (2008), in which the political competition is deterministic and

[15]They also consider the case where the partisan gerrymanderer maximizes the probability to win a working majority of seats for her party by assuming that the uncertainty is global. They again get pack-and-crack policy as the optimal policy.

[16]The original "cracking" tactics are to create the maximum number of winning districts *with the smallest margins*. In the traditional literature, some argue that gerrymandering will increase political competition for this reason. In this paper, we use the "cracking" tactics from Owen and Grofman (1988).

[17]They consider two feasibility constraints. The first is the constant mean of median voters' positions which is the same as the one in Owen and Grofman (1988). The second is that the status quo needs to be a *mean-preserved spread* of a feasible redistricting plan.

modeled as an auction process with a discrete bidding increment. This paper adopts the traditional continuous policy proposal and adds a gerrymandering stage. Although our model seem to be related to the so-called *probabilistic voting model* in, for example, Lindbeck and Weibull (1987) and Dixit and Londregan (1996), the voting decision in our model is deterministic, as in Dekel et al. (2008). Moreover, we introduce parties' platform decisions in addition to pork-barrel politics.[18]

Although we take the positive view, researchers also analyze gerrymandering from a normative perspective. The focus is on how gerrymandering affects the relation between seats and the vote shares won by a party, the so-called "*seat-vote curve.*" Coate and Knight (2007) identify the socially optimal seat-vote curve and the conditions under which the optimal curve can be implemented by a districting plan. With fixed and extreme parties' policy positions, they find that the optimal seat-vote is biased toward the party with larger partisan population. However, Bracco (2013) shows that when parties strategically choose their policy positions, the direction of the seat-vote curve bias should be the opposite. Besley and Preston (2007) construct a model similar to Coate and Knight's and show the relation between the bias of the seat-vote curve and parties' policy choices. They further empirically test the theory and the result shows that reducing the electoral bias can make parties' strategies more moderate.

3 The Model

We consider a two-party (L and R) multidistrict model. There are many (possibly infinite) localities in the state, each of which is considered the minimal unit in redistricting (a locality, e.g., a street block, cannot be divided into smaller groups in redistricting). Let $\mathcal{L}$ denote the set of those discrete localities, each of which has population $\frac{1}{|\mathcal{L}|}$. The state has K districts, and $|\mathcal{L}|$ is a multiple of K. To comply with the equal population requirement, the party in power needs to create those K districts by combining $\frac{|\mathcal{L}|}{K} = n$ localities in each one. Locality $\ell = 1, ..., |\mathcal{L}|$ has a voter distribution function $F_\ell : (-\infty, \infty) \to [0, 1]$, where $(-\infty, \infty)$ is the one-dimensional ideology (or political) spectrum and $F_\ell(\theta)$ is non-decreasing with $F_\ell(-\infty) = 0$ and

[18] Dixit and Londregan (1998) propose a pork-barrel model with strategic ideological policy decision based on their previous work. However, the ideology policy in their paper is the equality-efficiency concern engendered by parties' pork-barrel strategies. Therefore, the ideology decision in their work is a consequence of pork-barrel politics, instead of an independent policy dimension.

$F_\ell(\infty) = 1$. Ideology $\theta < 0$ is regarded as left, and $\theta > 0$ is right. A *redistricting plan* $\pi = \{D^1, ..., D^K\}$ with $|D^k| = n$ for all $k = 1, ..., K$, is a partition of $\mathcal{L}$.[19] The gerrymandering party's leader chooses the optimal district partition π from the set of all possible partitions Π.[20] In each district k, the voter distribution function F^k is an average of distribution functions of n localities: $F^k(\theta) = \frac{1}{n}\sum_{\ell \in D^k} F_\ell(\theta)$. District k's median voter is denoted by $x^k = x^k(D^k) \in (-\infty, \infty)$ with $F^k(x^k) = \frac{1}{2}$. We assume the uniqueness of x^k in each districting plan. Although x^k is solely determined by D^k, we can write $x^k = x^k(D^k(\pi)) = x^k(\pi)$ for all $k = 1, ..., K$ with a slight abuse of notation. Finally, let $F(\theta) = \frac{1}{|\mathcal{L}|}\sum_\ell F_\ell(\theta)$ be the state population distribution, and let θ_m, *the state median voter*, be determined by $F(\theta_m) = \frac{1}{2}$.

We will consider two cases later: one with complete freedom in redistricting as in Friedman and Holden (2008), and another is with limited ability in the line as in Owen and Grofman (1988). Throughout the paper, we order localities by the political positions of the median voter.

We also introduce uncertainty in the position of the median voter after redistricting is done. At each election time, the current economic and social state and the current party in power affect voters' political positions in the same direction: i.e., the voter distribution is shifted by common shocks. Formally, let y be a realization of the uncertain shock term. The median voter of the actual election in district k is denoted by $\hat{x}^k = x^k + y$. We assume that y follows a probabilistic distribution function $G : [-\bar{y}, \bar{y}] \to [0, 1]$, where $\bar{y} > 0$ is the largest value of relative economic shock and $G(0) = \frac{1}{2}$. We assume that electoral competition occurs after y is realized: the resulting median voter's position after the shock realization is $\hat{x}^k$.

We model pork-barrel elections in a similar manner as Dixit and Londregan (1996). A type θ voter in district k evaluates party j according to the utility function with two arguments: one is the policy position of the candidate representing the corresponding party, $\beta_j^k \in \mathbb{R}$, and the other is the party's *pork-barrel* transfer $t_j^k \in \mathbb{R}_+$. We interpret this pork-barrel transfer as a *promise* of local public good provision (measured by the amount of monetary spending) in the case where the party's candidate is elected. Formally, a voter θ in district

[19] A partition π of $\mathcal{L}$ is a collection of subsets of $\mathcal{L}$, $\{D^1, ..., D^K\}$, such that $\cup_{k=1}^K D^k = \mathcal{L}$ and $D^k \cap D^{k'} = \emptyset$ for any distinct pair k and k'.

[20] In reality, there are many restrictions on what can be done in a redistricting plan. For example, a district is required to be connected geographically. Despite this complication, our analysis can still be extended to cases with geographic restrictions by introducing the set of admissible partitions $\Pi^A \subseteq \Pi$ (see Puppe and Tasnadi, 2009)

k evaluates party j's offer by

$$U_{\theta}(j) = t_j^k - c(|\theta - \beta_j^k|) \tag{1}$$

where $c(d) \geq 0$ is a *voter's ideology cost function*, which is increasing in the distance between a candidate's position and her own position. We assume that $c(\cdot)$ is continuously differentiable, and satisfies $c(0) = 0$, $c'(0) = 0$, and $c'(d) > 0$ and $c''(d) > 0$ for all $d > 0$ (strictly increasing and strictly convex).

Therefore, voter θ votes for party L if and only if

$$U_{\theta}(L) - U_{\theta}(R) = [c(|\theta - \beta_R^k|) - c(|\theta - \beta_L^k|)] + t_L^k - t_R^k > 0 \tag{2}$$

Since the (after shock) median voter's type in district k is $\hat{x}^k = x^k + y$, given β_L^k, β_R^k, t_L^k and t_R^k, L wins in district k if and only if

$$U_{\hat{x}^k}(L) - U_{\hat{x}^k}(R) = [c(|\hat{x}^k - \beta_R^k|) - c(|\hat{x}^k - \beta_L^k|)] + t_L^k - t_R^k > 0 \tag{3}$$

Each party leader in the state (composed of these K districts) cares about (i) the influence or status within her party based on the number of winning districts in her state, (ii) the candidate's policy position in each district, and (iii) the district-specific pork-barrel spending. We assume that the party leader prefers to win a district with a candidate whose position is closer to her own ideal ideological position and with a *smaller* pork-barrel promise. The former is regarded as the "policy-motivation" in the literature. By formulating the latter, we consider a situation where the leader bears some costs when implementing the promised local public spending, as in the example of the bargaining efforts needed to push for federal funding. To simplify the analysis, we assume that the negative utility by pork-barrel is measured by the amount of money promised. We denote the ideal political positions of the leaders of party L and R by θ_L and θ_R, respectively, with $\theta_L < \theta_R$. Without loss of generality, we set $\theta_L = -\theta_R$, but we will stick to notations θ_L and θ_R until the gerrymandering analysis starts to help the reader comprehend the model more easily. Formally, by winning in district k, party j's leader gets utility

$$V_j^k = Q_j - t_j^k - C(|\beta_j^k - \theta_j|),$$

where $Q_j > 0$ is the fixed payoff that party j's leader obtains from each winning district, and $C(d)$ is a *party leader's ideology cost function* with $C(0) = 0$, $C'(0) = 0$, $C'(d) > 0$ and $C''(d) > 0$ (strictly increasing and strictly convex). This cost function C can be different from the voter's cost function c. Thus, in general, party j's payoff under policy position vector $(x_j^k, t_j^k)_{j\in\{L,R\}}^K$ and winning districts being $W_j \subseteq \{1, ..., K\}$ is

$$\sum_{k\in W_j} \left(Q_j - t_j^k - C(\left|\beta_j^k - \theta_j\right|)\right) + \sum_{k\notin W_j} \sigma \left(-C(\left|\beta_i^k - \theta_j\right|)\right) \tag{4}$$

where $\sigma \in [0, 1]$ is a parameter that discounts party j's policy disutilities from losing districts (policies are determined by the opponent party). Since the performance of electoral equilibrium is not sensitive to σ, we will assume for the sake of simplicity that $\sigma = 0$ throughout the paper except for our robustness discussion in the conclusion, where we show that all of our results hold if σ is small enough.[21]

Assumption 1. ***(No Disutility from Losing Districts)*** $\sigma = 0$.

An implicit assumption is that all districts are equal for party leaders. This is justified because the national party elites are ultimately interested in the number of seats their party gets, so the number of seats a state party leader wins is important in recognizing her contribution to the national party. Also, since we are considering a state's gerrymandering problem, it may be reasonable to assume that the benefit from winning a district does not depend on which district is won.

We impose a simple sufficient condition that assures interior solutions for both parties.

Assumption 2. ***(Relatively Strong Office Motivation)*** For all feasible $\hat{x}^k$, $Q_j \geq \min_{\beta}\{C(|\theta_j - \beta|) + c(|\beta - \hat{x}^k|)\}$ holds for $j = L, R$.

Notice that if the party leader gets 0 (or some fixed) utility, she must offer a pork-barrel promise equal to $Q_j - C(|\theta_j - \beta|)$. Therefore, the median voter gets utility $U_{\hat{x}^k} = Q_j - C(|\theta_j - \beta|) - c(|\beta - \hat{x}^k|)$ if party j wins. This assumption means that the payoff from winning a district, Q_j, is large enough so that for any $\hat{x}^k$, both parties can offer the median voter positive utility, which is a sufficient condition for the candidate selection problem to have an interior solution. Note that the set of feasible $\hat{x}^k$ is not the entire real line. The model only allows bounded finite median voters' positions and a finite $\bar{y}$. Therefore, there must exist a Q_j to satisfy this assumption. Moreover, the implication of this assumption is that it guarantees that in equilibrium both parties promise positive pork-barrels. We will see this more clearly in the next section.

[21] In Appendix C, we relax this assumption by assuming the losing payoff is $V_j^k = -\sigma C(|\beta_i^k - \theta_j|)$ where $\sigma \in [0, 1]$ is a preference parameter for the policy implemented by the other party. Our main results hold if σ is not too large. See details in Appendix C.

The state redistricting may be decided by one or both parties. It is straightforward that, in the first case, one party leader chooses π. In the later one, we assume that K_L districts belong to L and the remaining $K_R = K - K_L$ districts belong to R. Without loss of generality, we assume L chooses $\{D^1, ..., D^{K_L}\}$ and R chooses $\{D^{K_L+1}, ..., D^K\}$. Stage 0 applies only to bipartisan gerrymandering. We will discuss the bipartisan case in more detail in a later section.

The timing of the game is as follows:[22]

0. Two parties L and R swap localities in their territories if they can agree (bipartisan case only).

1. One party, say L, (or each party in the bipartisan case) chooses a redistricting plan $\pi = (D^k)_{k=1}^K$ (or $\pi_L = (D^k)_{k=1}^{K_L}$ and $\pi_R = (D^k)_{k=K_L+1}^K$), and thus a median voter vector $(x^1, ..., x^k, ..., x^K)$.

2. The common shock $y^k \in [-\bar{y}, \bar{y}]$ is realized.

3. Given the districting plan in stage 1 and the realized median voter $\hat{x}^k = x^k + y$ in stage 2, party leaders L and R simultaneously choose local policy positions and pork-barrel promises $(\beta_L^k, t_L^k)_{k=1}^K$ and $(\beta_R^k, t_R^k)_{k=1}^K$, respectively.

4. All voters vote sincerely. The winning party is committed to its policy position and its pork-barrel promise in each district $k = 1, ..., K$. All payoffs are realized.

We will employ **weakly undominated subgame perfect Nash equilibrium** as the solution concept. We require that in stage 3, party leaders play weakly undominated strategies so that the losing party leader does not make cheap promises to the district median voters.[23] We will call a weakly undominated subgame perfect Nash equilibrium simply an equilibrium.

[22] We can separate stage 3 into two substages: policy position choices followed by pork-barrel promises. If we do, the loser of a district k will get zero payoff in every subgame, and thus the loser becomes indifferent among policy positions. Thus, we need equilibrium refinement to predict the same allocation. By assuming that the loser party chooses the policy position that minimizes the opponent party leader's payoff, we can obtain exactly the same allocation in SPNE.

[23] This game is the first price auction under complete information. In general, there is a continuum of pure strategy equilibria. The losing party does not suffer from cheap promises, since she gets zero utility in losing districts anyway. The winning party needs to match the offer as long as she can get a positive payoff by doing so. By demanding that players play weakly undominated strategies, we can eliminate these unreasonable equilibria. We can also justify our equilibrium refinement by a mixed strategy equilibrium concept. There is a

3.1 Stage 3: Electoral Competition with Pork-Barrel Politics

We solve the equilibria of the game by backward induction. We start with stage 3, knowing that voters vote sincerely in stage 4. Notice that the key player is the median voter in the voting stage. Given that the median voter is decisive, the median voter is indifferent between two candidates' policy positions in any equilibrium of stage 3. However, the median voter always chooses the deeper pocket candidate, i.e., the candidate with higher payoff. This is because the candidate with the higher payoff can pay more to change the voter's mind if the voter does not vote for him with probability 1.

Thus, when the leader of winning party j makes her policy decisions in district k in equilibrium, she needs to match i's offer in terms of median voter's utility. Without loss of generality, we consider the case where party L wins. Formally, the party leader L's problem is described by

$$\begin{gathered}\max_{\beta_L^k, t_L^k}\{Q_L - t_L^k - C(\left|\theta_L - \beta_L^k\right|)\} \\ \text{subject to } t_L^k - c(|\hat{x}^k - \beta_L^k|) \geq \bar{U}_R^k,\ t_L^k \geq 0, \text{ and} \\ Q_L - t_L^k - C(\left|\theta_L - \beta_L^k\right|) \geq 0,\end{gathered} \tag{5}$$

where $\bar{U}_R^k$ is the median voter's utility level from R's offer. Notice that $t_L^k \geq 0$ and $Q_L - t_L^k - C(\left|\theta_L - \beta_L^k\right|) \geq 0$ may or may not be binding while $t_L^k - c(|\hat{x}^k - \beta_L^k|) \geq \bar{U}_R^k$ must be binding. The solution for this maximization problem is straightforward. Define $\hat{\beta}_j(\hat{x}^k, \theta_j)$ by the following equation

$$c'(|\hat{x}^k - \hat{\beta}_j(\hat{x}^k, \theta_j)|) = C'(\left|\theta_j - \hat{\beta}_j(\hat{x}^k, \theta_j)\right|). \tag{6}$$

Notice that (5) is simply the first-order condition of optimization problem (4) after substituting $t_L^k = c(|\hat{x}^k - \beta_L^k|) + \bar{U}_R^k$ into the objective function. Also, the optimal policy is $\beta_L^{k^*} = \hat{\beta}_L(\hat{x}^k, \theta_L)$ when $-c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|) \leq \bar{U}_R^k$. That is, it is not enough for the winning party to win just by using the policy platform. In this case, it is clear that the optimal pork-barrel promise is

$$t_L^{k^*}(\bar{U}_R^k) = \bar{U}_R^k + c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|).$$

Although it may appear unclear at first whether $-c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|) \leq \bar{U}_R^k$ holds or not, it turns out this condition always holds. This is because a similar

unique mixed strategy equilibrium in which the winning party plays a pure strategy while the losing party plays a mixed strategy equilibrium. The outcome of this mixed strategy equilibrium coincides with the weakly undominated Nash equilibrium in pure strategies.

optimization problem applies for the losing party (and Assumption 2 assures that the winning party's budget is not binding).

It is obvious that the winning party's pork-barrel promise is related to what the losing party proposes in equilibrium. The following lemma shows that the losing party cannot lose with a nonzero surplus.

Lemma 1. *Suppose R is the losing party in district k. In equilibrium, (1) the median voter is indifferent between L and R and votes for L with probability 1. (2) R proposes the policy pair $(\beta_R^{k^*}, t_R^{k^*})$, which is the solution to the following problem*

$$\max_{\beta_R^k, t_R^k} U_{\hat{x}^k}(R) = t_R^k - c(|\hat{x}^k - \beta_R^k|)$$
$$\text{subject to } t_R^k \geq 0 \text{ and } Q_R - t_R^k - C(\left|\theta_R - \beta_R^k\right|) \geq 0.$$

That is, the losing party leader offers a policy position and a pork-barrel promise that leaves herself zero surplus in equilibrium, and

$$\beta_R^{k^*} = \hat{\beta}_R(\hat{x}^k, \theta_R)$$
$$t_R^{k^*} = Q_R - C(\left|\theta_R - \hat{\beta}_R(\hat{x}^k, \theta_R)\right|).$$

Moreover, this policy pair is the best she can offer for the realized median voter $\hat{x}^k$.

The intuition of this lemma is straightforward. If the losing party R does not offer the median voter (β_R^{k*}, t_R^{k*}), then the winning party L will provide the median voter the same utility level, and the losing party R can always offer the median voter something better than her original offer to win the district. This cannot happen in equilibrium. Therefore, the equilibrium strategy is $\beta_R^{k^*} = \hat{\beta}_R(\hat{x}^k, \theta_R)$ and $t_R^{k^*} = Q_R - C(\left|\theta_R - \beta_R^{k^*}\right|)$ for the losing party R. The policy pair provides the median voter with the utility $\bar{U}_R^{k^*} = Q_R - C(\left|\theta_R - \beta_R^{k^*}\right|) - c(|\hat{x}^k - \beta_R^{k^*}|)$. Using this $\bar{U}_R^{k^*}$, one can solve the winning party's equilibrium pork-barrel promise $t_L^{k^*} = Q_R - C(\left|\theta_R - \beta_R^{k^*}\right|) - c(|\hat{x}^k - \beta_R^{k^*}|) + c(|\hat{x}^k - \beta_L^{k^*}|)$.

One thing left to decide is which party should be the winning party. Notice that, by Lemma 1, the losing party always proposes the best offer by depleting all of her surplus. Therefore, the party that can *potentially* provide the median voter with a higher utility level is the winner. Notice that j party's pork-barrel promise is bounded above by the party leader's payoff evaluated at β_j^{k*} (otherwise, the leader gets a negative utility):

$$Q_j - C(\left|\theta_j - \hat{\beta}_j(\hat{x}^k, \theta_j)\right|).$$

Substituting this into the median voter's utility, we obtain

$$W_R^k = Q_R - C(\left|\theta_R - \hat{\beta}_R(\hat{x}^k, \theta_R)\right|) - c(|\hat{x}^k - \hat{\beta}_R(\hat{x}^k, \theta_R)|).$$

Similarly, for party L, we obtain

$$W_L^k = Q_L - C(\left|\theta_L - \hat{\beta}_L(\hat{x}^k, \theta_L)\right|) - c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|),$$

where W_R^k and W_L^k are the (potential) maximum utilities that the median voter gets from the corresponding party's offer. Therefore, party L wins in the third stage if and only if

$$\begin{aligned} &Q_L - Q_R > \\ &\qquad \left[c(|\hat{x}^k - \hat{\beta}_L|) + C(\left|\theta_L - \hat{\beta}_L\right|)\right] - \left[c(|\hat{x}^k - \hat{\beta}_R|) + C(\left|\theta_R - \hat{\beta}_R\right|)\right] \end{aligned} \tag{7}$$

If $Q_L = Q_R$, then L wins if and only if

$$\left|\theta_L - \hat{x}^k\right| < \left|\theta_R - \hat{x}^k\right|. \tag{8}$$

Summarizing the above, we have the following results in stages 3 and 4.

Lemma 2. *Suppose that Assumption 2 is satisfied. Define $\hat{\beta}_j(\hat{x}^k, \theta)$ by (5). We have*

1. *For the losing party j, the optimal choice is $\beta_j^{k^*} = \hat{\beta}_j(\hat{x}^k, \theta_j)$ which lies in the interval $(\hat{x}^k, \theta_j)$ (or $(\theta_j, \hat{x}^k)$) and $t_j^{k^*} = Q_j - C(|\theta_j - \beta_j^{k^*}|)$*

2. *For the winning party i, the optimal choice is $\beta_i^{k^*} = \hat{\beta}_i(\hat{x}^k, \theta_i)$, which lies in the interval $(\hat{x}^k, \theta_i)$ (or $(\theta_i, \hat{x}^k)$), and $t_i^{k^*} = Q_j - C(|\theta_j - \beta_j^{k^*}|) - c(|\hat{x}^k - \beta_j^{k^*}|) + c(|\hat{x}^k - \beta_i^{k^*}|)$.*

3. *Irrespective of $\hat{x}^k \gtrless \theta_i$, we have $\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} = \frac{c_i''}{C_i'' + c_i''}$, where $c_i'' = c''(|\hat{x}^k - \hat{\beta}_i(\hat{x}^k, \theta_i)|)$ and $C_i'' = C''(\left|\theta_i - \hat{\beta}_i(\hat{x}^k, \theta_i)\right|)$.*

4. *Party i wins in the kth district if and only if*

$$Q_i - Q_j > \mathcal{C}(\left|\theta_i - \hat{x}^k\right|) - \mathcal{C}(\left|\theta_j - \hat{x}^k\right|),$$

where $\mathcal{C}(\left|\theta_i - \hat{x}^k\right|) \equiv C(\left|\theta_i - \hat{\beta}_i(\hat{x}^k, \theta_i)\right|) + c(|\hat{x}^k - \hat{\beta}_i(\hat{x}^k, \theta_i)|)$.

The above lemma directly implies that *if party i wins*, party i's leader's realized payoff from district k given $\hat{x}^k = x^k + y$ is written as:

$$\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j) = (Q_i - Q_j) - \left(\mathcal{C}(\left|\theta_i - \hat{x}^k\right|) - \mathcal{C}(\left|\theta_j - \hat{x}^k\right|)\right)$$

Using $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j)$, when party i wins in district k the expected payoff from district k for the party leader is written as:

$$E\tilde{V}_i^k(x^k, \theta_i, \theta_j) = \int_{-\bar{y}}^{\bar{y}} \max\left\{\tilde{V}_i^k(x^k + y, \theta_i, \theta_j), 0\right\} g(y)dy$$

Note that due to the additive separability of the payoff function, party leader i's expected payoff under partition π (district median voters' profile $\left(x^k(\pi)\right)_{k=1}^K$)) is written as

$$\begin{aligned} E\tilde{V}_i(\pi, \theta_i, \theta_j) &\equiv \int_{-\bar{y}}^{\bar{y}} \sum_{k=1}^{K} max\left\{\tilde{V}_i^k(x^k(\pi) + y, \theta_i, \theta_j), 0\right\} g(y)dy \\ &= \sum_{k=1}^{K} \int_{-\bar{y}}^{\bar{y}} max\left\{\tilde{V}_i^k(x^k(\pi) + y, \theta_i, \theta_j), 0\right\} g(y)dy \\ &= \sum_{k=1}^{K} E\tilde{V}_i^k(x^k(\pi), \theta_i, \theta_j) \end{aligned}$$

Since we assume that $\theta_L = -\theta_R$ without loss of generality, we can prove the following properties.[24]

Lemma 3. *The following properties are satisfied for $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j)$:*

1. *The realized winning payoff for party L (R), $\tilde{V}_L^k$ ($\tilde{V}_R^k$) is decreasing (increasing) in $\hat{x}^k$.*
2. *The realized winning payoff for party i, $\tilde{V}_i^k$, is strictly convex in $\hat{x}^k$, if $C'''(\cdot) > 0$ and $c'''(\cdot) > 0$, and $Q_L = Q_R$.*

The next lemma is presented in preparation for the Stage 2 analysis.

Lemma 4. *The following properties are satisfied for $E\tilde{V}_i^k(x^k, \theta_i, \theta_j)$:*

[24] The readers may feel the assumption that the third derivatives of the cost functions are positive is rather strong. We use this assumption in some of our formal results, but we show that it can be relaxed in some situations (see Example 1 below).

1. *The expected winning payoff for party L (R), in location k, $E\tilde{V}_L^k$ ($E\tilde{V}_R^k$) is decreasing (increasing) in x^k.*

2. *The expected winning payoff for party L (R) in location k, $E\tilde{V}_L^k$ ($E\tilde{V}_R^k$) is decreasing (increasing) and strictly convex in x^k, if $C'''(\cdot) > 0$ and $c'''(\cdot) > 0$, and $Q_L = Q_R$.*

3. *The expected winning payoff for party L (R), $E\tilde{V}_L$ ($E\tilde{V}_R$) is decreasing (increasing) and strictly convex, if $C'''(\cdot) > 0$ and $c'''(\cdot) > 0$, and $Q_L = Q_R$.*

We are now ready to discuss the setup of partisan and bipartisan gerrymandering problems.

3.2 The Partisan Gerrymandering Problem

Without loss of generality, we formalize *the partisan gerrymandering party leader's optimization problem* as the case where $K_L = K$ and L is in charge of redistricting. Lemma 2 shows that $x^k = x^k(\pi)$ is the sufficient statistic to determine the outcome of the kth district. Notice that the indirect utility of L, $\tilde{V}_L^k(\hat{x}^k, \theta_L, \theta_R)$, is relevant only when party L wins in district k. The choice of $\pi = \left(D^1, ..., D^K\right)$ affects the party leader L's payoff $E\tilde{V}_L$ through $\left(x^1(D^1), ..., x^K(D^K)\right)$ represented by its indirect utility $\tilde{V}_L^k(x^k(\pi) + y, \theta_L, \theta_R)$ conditional on L winning.

From now on, we suppress θ_L and θ_R in indirect utilities $\tilde{V}_L^k$, $E\tilde{V}_L^k$, and $E\tilde{V}_L$. We can rewrite the party leader L's gerrymandering choice to be the result of the following maximization problem

$$\pi^* \in \arg\max_{\pi \in \Pi} E\tilde{V}_L(\pi)$$

The SPNE of this game is $(\pi^*, (\beta_L^{k^*})_{k=1}^K, (\beta_R^{k^*})_{k=1}^K, (t_L^{k^*})_{k=1}^K, (t_R^{k^*})_{k=1}^K)$.

3.3 The Bipartisan Gerrymandering Problem

Since bipartisan gerrymandering requires negotiation between the two parties, there can be many possible formulations. As mentioned before, one way is to assume that each party has preexisting "territory" as in Gul and Pesendorfer (2010).

In this paper, we will take this slightly further. We will assume that, before redistricting, parties L and R can rearrange localities that belong to

$\{1, ..., K_L\}$ and $\{K_L + 1, ..., K\}$ by negotiating which localities belong to their own territory. This is because it may be beneficial for both parties to swap some of the localities in their territories, if the original allocation of localities in each district is arbitrary.

We will model these negotiations in the simplest manner. If both parties can (strictly) Pareto-improve their payoffs by swapping localities, then they will agree to do so in order to achieve the maximal improvement. With this formulation, we can show that bipartisan gerrymandering generates the most polarized districts under reasonably mild assumptions (Propositions 2 and 3) when party leaders have complete freedom in gerrymandering, and that bipartisan gerrymandering always generates exactly the same "order-and-partition" allocation as partisan gerrymandering does (Proposition 5) when it is subject to the "*constant-average-constraint,*" as in the spirit of Owen and Grofman (1988).

4 Gerrymandering with Complete Freedom

As a limit case, let us consider the *ideal situation* for the gerrymanderer (Friedman and Holden, 2008). This situation occurs when there is a large number of infinitesimal localities with politically homogeneous population: for all position $x \in (-\infty, \infty)$, there are localities ℓs with $F_\ell(x - \delta) = 0$ and $F_\ell(x + \delta) = 1/|\mathcal{L}|$ for a small $\delta > 0$.[25] That is, the gerrymanderer can freely create any kind of population distributions for K districts as long as they sum up to the total population distribution. We ask what strategy the gerrymanderer should take. By Lemma 4-1, she is better off by making the (ex ante) median voter's allocation as far from the other party leader's position as possible. This strategy increases the winning payoff and the probability of winning the district. Thus, the gerrymanderer tries to create the furthest district structure from the opponent party leader's position.

4.1 Partisan Gerrymandering

In partisan gerrymandering cases, the party leader in charge of gerrymandering will try to make district medians as far away as possible from the other

[25] To be exact, we can assume that there is a finite number of political positions $\Theta \equiv \{\theta^1, ..., \theta^H\}$ such that for all ℓ, $F_\ell(\theta) = \frac{1}{|\mathcal{L}|}$ for some $\theta \in \Theta$: i.e., population is homogenous in each locality, and that $n = \frac{|\mathcal{L}|}{K}$ is an odd number. Then, we will have well defined x^k for any D^k. We use a continuous approach in order to make the comparison with Friedman and Holden (2008) easier.

party leader's position.[26] Without loss of generality, we assume that party L is in charge of gerrymandering. To create the most extreme district, its median voter position x_L^1 should satisfy $F(x_L^1) = \frac{1}{2K}$ (the most extreme district achievable with population $\frac{1}{K}$). Although the remaining half population to the right of x_L^1 can be anything in district 1, it would be a good idea to waste the other party's strong supporters by combining them. Thus, party L's leader will create district 1 by combining sets $\left\{\theta \leq x_L^1 : F(x_L^1) = \frac{1}{2K}\right\}$ and $\left\{\theta \geq z_L^1 : 1 - F(z_L^1) = \frac{1}{2K}\right\}$. Strictly speaking, in this case the median voter can be anyone in the set $[x_L^1, z_L^1]$, but by making the measure of the former set slightly higher than the latter, we can pin down the median voter's position at x_L^1. Of course, this is not exactly an optimal policy, but we use this approach in order to compare our results with those in Friedman and Holden (2008).[27] Similarly, we create districts $2, ..., K$ sequentially. Let x_L^k be such that $F(x_L^k) = \frac{k}{2K}$ for all $k = 1, ..., K$, and let z_L^k be such that $1 - F(z_L^k) = \frac{k}{2K}$. We have

$$-\infty = x_L^0 < x_L^1 < ... < x_L^K = z_L^K < ... < z_L^1 < z_L^0 = \infty.$$

We call this redistricting plan a **party-L-slice-and-mix** policy, which is proposed in Friedman and Holden (2008). Under the slice-and-mix policy, the resulting district median voter allocation is $x_L \equiv (x_L^1, ..., x_L^K)$ with x_L^k is such that $F(x_L^k) = \frac{k}{2K}$ for each $k = 1, ..., K$. We will show that this is the optimal policy for the leader of party L. Figure 1 is an example of party-L-slice-and-mix strategy when $K = 4$. District $k = 1, ..., 4$ is composed of two slices numbered by k. District median voter allocation is $x_L \equiv (x_L^1, ..., x_L^4)$.

The following result is straightforward: in order for x^k to be the median voter in district $k = 1, ..., K$, x^k must satisfy $F(x^k) \geq \frac{k}{2K}$ and $1 - F(x^k) \geq \frac{k}{2K}$. (We define $x_R = (x_R^1, ..., x_R^K)$ in a perfectly symmetric way.)

Lemma 5. *There is no (ex ante) median voter allocation $x = (x^1, ..., x^K)$ with $x^1 \leq x^2 \leq ... \leq x^K$ such that $x^k < x_L^k$ for any $k = 1, ..., K$. Symmetrically, there is no median voter allocation $x = (x^1, ..., x^K)$ with $x^1 \geq x^2 \geq ... \geq x^K$ such that $x^k > x_R^k$ for any $k = 1, ..., K$.*

[26] As long as there are positive winning probabilities in all districts (if $\bar{y}$ is large enough), this is true. If not, party L's leader may need to create unwinnable districts, but she would be indifferent as to how to draw lines for these districts. Even in this case, however, the slice-and-mix below is one of the optimal strategies.

[27] If we use the exactly finite setup described in footnote 25, then we can have a perfectly consistent model with a well-behaved optimal strategy with exact district median voters. We thank the associate editor for pointing this out. We decided to stick to a continuum approximation, just to make the comparison with Friedman and Holden (2008) easier.

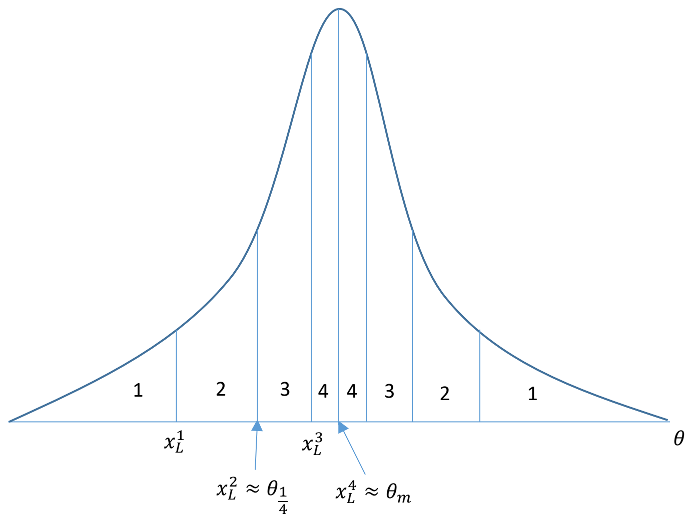

Figure 1: Party-L-slice-and-mix when $K = 4$.

Clearly, these district median voter allocations x_L and x_R are the most biased district median voter allocations toward the left and the right, respectively. Under x_L, redistricting the first and the second districts does not make two districts with intermediate medians. With this lemma and Lemma 4-1, we derive the following result.

Proposition 1. *Suppose that the gerrymanderer can create districts with complete freedom and that party L (R) is in charge of gerrymandering. Then the party-L (R)-slice-and-mix policy is an optimal gerrymandering policy. The resulting district median voter allocation in district k is approximately x_L^k (x_R^k).*

Another interesting observation from this proposition is that even when $Q_L = Q_R$, if party L is the majority party in terms of the state population (that is $\theta_m < 0$ where $F(\theta_m) = \frac{1}{2}$), then it can win all seats with a probability of 50% or higher ($x_L^K < 0$). Also, one can observe that the median of x_L^k's is around $\theta_{\frac{1}{4}}$ where $F(\theta_{\frac{1}{4}}) = \frac{1}{4}$. Therefore, complete freedom in gerrymandering means the minority's impact on the election will be completely diluted. However, it is rare in US politics for one party to monopolize all districts, partly because of the presence of majority-minority district requirements (see Shotts, 2001).[28] The majority-minority requirement forces the gerrymanderer to seek the second-best districting plan as a result even when she has complete free-

[28] In fact, even though either one of the two parties must be the majority in a state, the majority party usually does not win all districts. This can be attributed to Section 2 of the Voting Rights Act (accompanied by other United States Supreme Court cases), which

dom. It is worthwhile to note that the slice-and-mix strategy is identical to the optimal policy analyzed in Friedman and Holden (2008), although Friedman and Holden do not include competition with political positions nor pork-barrel politics.

4.2 Bipartisan Gerrymandering

The preexisting territories of parties L and R are the districts they won in the previous election, described by K_L and $K_R = K - K_L$, respectively, and their territory-wise population distributions are described by $F_L(\theta)$ and $F_R(\theta)$, respectively, with (i) $F(\theta) = F_L(\theta)+F_R(\theta)$ for all θ, (ii) $F_L(\infty) = K_L \times \frac{n}{|\mathcal{L}|}$, and (iii) $F_R(\infty) = K_R \times \frac{n}{|\mathcal{L}|}$.[29] Through minor abuse of notation, let $x_L^0 = \infty$ and θx_L^k be such that $F_L(x_L^k) = \frac{k}{2K}$, and let z_L^k be such that $F_L(\infty) - F_L(z_L^k) = \frac{k}{2K}$ for $k = 1, ..., K_L$ by applying the same method for territories as in the previous section. Similarly, let $z_R^0 = -\infty$ and z_R^k be such that $F_R(z_R^k) = \frac{k-K_L}{2K}$, and let x_R^k be such that $F_R(\infty) - F_R(x_R^k) = \frac{k-K_L}{2K}$ for $k = K_L + 1, ..., K$.

We call this bipartisan policy a (K_L, K_R)**-territory-wise-slice-and-mix** policy, and the resulting median voter profile is $(x_L^1, ..., x_L^{K_L}, x_R^{K_L+1}, ..., x_R^K)$. By Lemma 5 again, $(x_R^{K_L+1}, ..., x_R^K)$ is the K_R right-most median voter profile, and $(x_L^1, ..., x_L^{K_L})$ is the K_L left-most median voter profile given their territories. Figure 3 is an example of (K_L, K_R)-bipartisan-slice-and-mix policy when $K_L = K_R = 2$ and $\bar{\theta} = \theta_m$. In this case, both parties use slice-and-mix to create (x_L^1, x_L^2) and (x_R^3, x_R^4). Thus, this is one of the most polarized district median voter allocations, and is very different from a partisan gerrymandering median voter allocation, which has some more competitive districts. Thus, by the same logic as in Proposition 1, we have the following proposition since territorial redistricting is done independently.

Proposition 2. *Suppose that the both parties can create districts independently and with complete freedom for their territories. Then the* (K_L, K_R)*-territory–wise-slice-and-mix policy is an optimal gerrymandering policy. The resulting district median voter allocation is* $(x_L^k)_{k=1}^{K_L}$ *and* $(x_R^k)_{k=K_L+1}^{K}$.

Note that if party R has some significantly left-oriented population in its territory in the sense of $F_R(x_L^{K_L}) > 0$, party L may benefit by getting this group of voters in its territory, since it can move at least one of its districts' median voters to the left. Recall that party L is better off making the median voter's allocation as far from party R's position as possible by Lemmas

essentially prevents the minority votes from being diluted in the voting process in a manner similar as our slice-and-mix strategy.

[29] This is the setup adopted by Gul and Pesendorfer (2010).

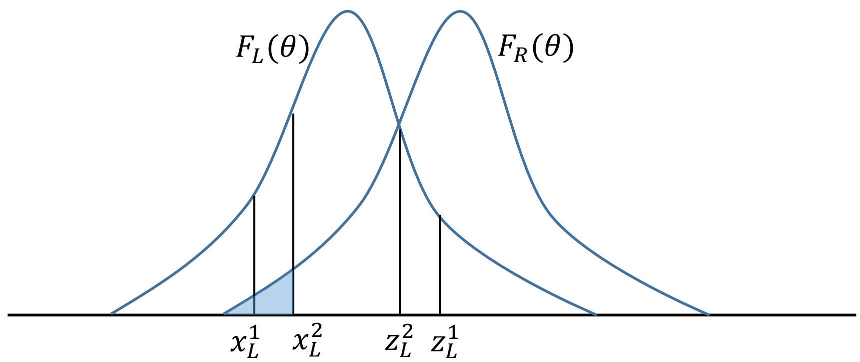

Figure 2: $f_L(\theta)$ and $f_R(\theta)$ are territory-wise locality distribution densities. Given these distributions, L applies slice-and-mix strategy, and x_L^1 and x_L^2 are the median voters in district 1 and 2. Notice that $F_R(x_L^2)$ (represented by the shadowed area) allows L to create more extreme median voters in districts 1 and 2. The same incentive exists for R in this example. So, swapping localities is Pareto-improving.

3-1 and 4-1. See Figure 2 for an example. The same is true for party R if $F_L(\infty) - F_L(x_R^{K_L+1}) > 0$.[30] If uncertainty $\bar{y}$ is small, then there is no uncertainty in district elections, and bipartisan gerrymandering benefits from swapping these localities. The following proposition illustrates this mutually beneficial negotiation between the two parties since party leaders do not care about the opponent party's policy positions in losing districts.

Proposition 3. *Suppose that the resulting district median voter allocation is approximately $(x_L^k)_{k=1}^{K_L}$ and $(x_R^k)_{k=K_L+1}^{K}$ under the (K_L, K_R)-territory–wise-slice-and-mix policy. Suppose that (i) $F_R(x_L^{K_L}) > 0$ and $F_L(\infty) - F_L(x_R^{K_L+1}) > 0$ hold, and (ii) $\bar{y}$ satisfies $\bar{y} \leq \min\left\{\left|x_L^{K_L}\right|, \left|x_R^{K_L+1}\right|\right\}$. Then, the two parties benefit from swapping these localities in a bipartisan negotiation between them.*

Condition (ii) says that party L wins in districts $k = 1, ..., K_L$, and party R wins in districts $k = K_{L+1}, ..., K$ with certainty. Then, both parties are better off having more extreme polarizations in their winning districts, as long as parties do not care about policies in losing districts. If this swapping incentive exists, both parties would continue to do so until $F_R(x_L^{K_L}) = 0$ and

[30]Note that the median voters in created districts are determined only by F_L in interval $(-\infty, x_L^{K_L})$ and F_R in interval $(x_R^{K_L+1}, \infty)$. The rest of the F_L and F_R distributions do not make any difference under slice-and-mix.

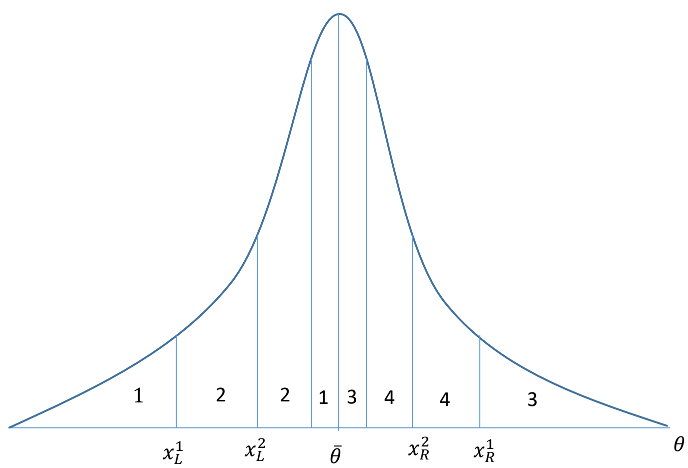

Figure 3: (K_L, K_R)-slice-and-mix when $K_L = K_R = 2$.

$F_L(\infty) - F_L(x_R^{K_L+1}) = 0$. Figure 3 depicts an example of the equilibrium bipartisan gerrymandering in this situation.

5 Gerrymandering with Limited Freedom

In this section, we will explore how the "slice-and-mix" result would be modified if we drop the "complete freedom" in gerrymandering. In the spirit of Owen and Grofman (1988) and Gul and Pesendorfer (2010), we say a gerrymandering problem is subject to a *constant-average-constraint* if the resulting $(x^1(\pi), ..., x^K(\pi))$ must satisfy

$$\frac{\sum_{k=1}^{K} x^k(\pi)}{K} = \bar{\mu} \tag{9}$$

for some fixed $\bar{\mu}$. Owen and Grofman (1988) analyzed the optimal partisan gerrymandering policy by imposing the same constraint. They obtained the famous pack-and-crack result when the office-motivated party leader maximizes the number of seats under this constraint.

To apply the above constraint to our locality setup, we will focus on the case where the political position is normally distributed in all localities. With normality, any feasible redistricting plan satisfies exactly this constraint (8) (the proof is obvious when we note that the median is equivalent to the mean under normality).

Lemma 6. *Suppose that the voter distribution in each locality is normally distributed, i.e., $F_\ell \sim N(\mu_\ell, \sigma_\ell)$ for each $\ell \in \mathcal{L}$. Then, the median of district k is*

$$x^k(\pi) = \frac{1}{n} \sum_{\ell \in D^k(\pi)} \mu_\ell.$$

Moreover, for all $\pi \in \Pi$, $\frac{\sum_{k=1}^{K} x^k(\pi)}{K} = \theta_m = \bar{\mu}$.

With this lemma, we can identify the optimal gerrymandering strategy for a special case. Starting from a district partition π with $h, k \in \{1, ..., K\}$ and $x^k(\pi) \le x^h(\pi)$, consider swapping a pair of localities $\ell \in D^k(\pi)$ and $\tilde{\ell} \in D^h(\pi)$ with $\mu_{\tilde{\ell}} < x^k(\pi) \le x^h(\pi) < \mu_\ell$. Let the outcome of this redistricting be π': i.e., $D^k(\pi') = D^k(\pi) \cup \{\tilde{\ell}\} \backslash \{\ell\}$, $D^h(\pi') = D^h(\pi) \cup \{\ell\} \backslash \{\tilde{\ell}\}$, and $D^{\tilde{k}}(\pi) = D^{\tilde{k}}(\pi)$ for all $\tilde{k} \ne k, h$. From the above formula, it is clear that $x^k(\pi') < x^k(\pi) \le x^h(\pi) < x^h(\pi')$.

Which plan should the party leader (say, party L's leader) choose between π and π'? The answer depends on the shape of $\tilde{EV_i}$. It is obvious that if $\tilde{EV_i}$ is a convex function in the ex ante median voter's position x^k, the party leader would prefer π' to π as long as $k \in K_L$. As we have seen in Lemma 4-3, if the third derivatives of cost functions are positive, party leaders have convex expected payoff functions, and they prefer more heterogenous districts in median voters' positions to homogenous districts.[31]

In this case, it is easy to characterize the optimal partisan gerrymandering policy with limited freedom. Let a district partition $\pi = \{D^1(\pi), ..., D^K(\pi)\}$ such that for all $k = 1, ..., K'$ and all $h = 2, ..., K$ with $k < h$, all $\ell \in D^k(\pi)$ and $\tilde{\ell} \in D^h(\pi)$, $\mu_\ell \le \mu_{\tilde{\ell}}$ holds, where K' is such that for all districts $k > K'$, there is absolutely no chance for party L to win: $x^{K'}(\pi) - \bar{y} < 0$ but $x^k(\pi) - \bar{y} > 0$ for all $k > K'$. We call this allocation π an order-and-partition gerrymandering policy. This policy packs the most opposing localities in unwinnable districts, and partitions the rest of the localities into consecutive locality districts. By Lemma 4-2, we have the following result.

Proposition 4. *Suppose that the voter distribution is normal in each locality and $Q_L = Q_R$. In addition, suppose that $C'''(\cdot) \ge 0$ and $c'''(\cdot) \ge 0$ hold. Then, the optimal partisan gerrymandering policy is order-and-partition $(\bar{x}^k)_{k=1}^{K'}$ with packing in the unwinnable districts. In particular, $(\bar{x}^k)_{k=1}^{K}$ is one of the optimal*

[31] The relevant case is $h \in K_L$. If $h \notin K_L$, it is obvious that party L prefers π' to π.

partisan gerrymandering policies. If $K' = K$, the unique optimal policy is order-and-partition $(\bar{x}^k)_{k=1}^K$.

Thus, cracking is not necessarily a good strategy here, unlike in Owen and Grofman (1988). The difference between the current paper and theirs is that our party leaders are also policy-motivated.[32] What about the case where $C'''(\cdot) \geq 0$ and $c'''(\cdot) \geq 0$ do not hold? Actually, we can show that $\tilde{V}_i^k$ is concave if $C'''(\cdot) \leq 0$ and $c'''(\cdot) \leq 0$, so it appears that pack-and-crack is the way to go. Indeed, it is true for the deterministic case ($\bar{y} = 0$) or for cases where $\bar{y}$ is small enough. However, even if the third derivatives are negative, $E\tilde{V}_L^k$ can be convex, as is seen in the following example (see also Appendix B).

Example 1. We introduce a convenient special ideology cost function such that both voters' and party leaders' cost functions have common constant elasticity. Let $C(d) = a^C d^\gamma$ and $c(d) = a^c d^\gamma$, where $\gamma > 1$, $a^C > 0$, and $a^c > 0$ are parameters. In this case, both party leaders and voters have the same elasticity that is constant γ. This conveniently yields the following formula. Denote $A = A(a^C, a^c) = a^C \left(\frac{\alpha}{1+\alpha}\right)^\gamma + a^c \left(\frac{1}{1+\alpha}\right)^\gamma > 0$ where $\alpha = \left(\frac{a^C}{a^c}\right)^{\frac{1}{\gamma-1}}$. We can choose a^C and a^c to set $A = 1$ for each γ: then we have $\mathcal{C}(d) = Ad^\gamma = d^\gamma$. In this case, $\tilde{V}_L^k$ is concave (convex) in $\hat{x}^k$ if $\gamma \leq 2$ ($\gamma \geq 2$). Suppose that $\theta_L = -1$, $\theta_R = 1$ (thus L wins if and only if $\hat{x}^k < 0$), and $g(y) = \frac{1}{2\bar{y}}$ if and only if $y \in [-\bar{y}, \bar{y}]$ (uniform distribution). Also, suppose that all possible x^k are in $[-1, 1]$ and $\left(\frac{Q}{A}\right)^{\frac{1}{\gamma}} \geq 2 + \bar{y}$ holds to satisfy Assumption 2. If $\bar{y} > 1$, there is always a chance to win the election: we have $x^k - \bar{y} < 0$ and $x^k + \bar{y} > 0$. In this case,

$$E\tilde{V}_L^{k\prime\prime} = \frac{\gamma}{2\bar{y}}[(1 - x^k + \bar{y})^{\gamma-1} - (-1 - x^k + \bar{y})^{\gamma-1}] > 0.$$

For all $\gamma > 1$, $(1-x^k+\bar{y})^{\gamma-1} - (-1-x^k+\bar{y})^{\gamma-1} > 0$ holds. Thus, the expected utility is convex in x^k, even if $\gamma < 2$ or $\mathcal{C}'''(d) < 0$. This example shows that even without positive third derivatives, the order-and-partition strategy is optimal for the constrained case with the constant average constraint.□

We now set our sights on bipartisan gerrymandering. In Section 4.2, we argue that when $\bar{y}$ is not large enough to change the district winners, parties

[32] Without the policy motivation, the payoff function is only related to winning probability, and pack-and-crack is optimal under a mild assumption on g. See also Gul and Pesendorfer (2010).

have incentives to swap localities as long as the swap can increase the ability of both parties to create extreme districts in the redistricting stage. A similar idea applies to the constrained gerrymandering case. Suppose that $E\tilde{V}^k$ is convex (it is assured if the third derivatives of cost functions are positive). Suppose also that there exist ℓ and $\tilde{\ell}$ with $\ell \in D^k$, $\tilde{\ell} \in D^h$, and $\mu_\ell > \mu_{\ell'}$, and that districts k and h belong to parties L and R, respectively. By swapping ℓ and $\tilde{\ell}$, parties L and R can make x^k and x^h decrease and increase by the same magnitude, respectively. This is strictly Pareto-improving for both parties. Therefore, the two parties should agree to form consecutive territories in stage 0.

Then, in stage 1, both party leaders adopt order-and-partition in their own consecutive territory; it does not matter whether redistricting is conducted by a partisan or bipartisan committee. This observation shows that for the gerrymandering problem with limited freedom, bipartisan gerrymandering does not create a more polarized allocation than partisan gerrymandering, and incumbents' re-election rates would be the same.

Proposition 5. *Suppose that the voter distribution is normal in each locality and $Q_L = Q_R$. In addition, suppose that $C'''(\cdot) \geq 0$ and $c'''(\cdot) \geq 0$ hold. Then, the optimal bipartisan gerrymandering policy is order-and-partition $(\bar{x}^k)_{k=1}^K$, which is identical to the partisan policy.*

The constant average constraint forbids a gerrymanderer from diluting supporters of the other party by mixing in his own supporters. Notice that while gerrymanderer L can pull the median of district medians to $\theta_{\frac{1}{4}}$ in the complete freedom case, the median of medians has to remain as θ_m, the population median, when the constant average constraint applies. Propositions 4 and 5 echo Friedman and Holden (2009), in which the insignificant difference between partisan and bipartisan gerrymandering is a possible consequence of the Voting Rights Act of 1982, which significantly limits the gerrymanderer's ability to dilute votes.

6 Conclusion

In this paper, we propose a gerrymandering model in which candidates compete with their policy positions and pork-barrel politics, and analyze the relationship between gerrymandering and policy polarizations. In the context of gerrymandering, this is the first paper to introduce policy motivations as well as office motivations. The model's tractability allows us to compare the performance of partisan and bipartisan gerrymandering under different constraints.

We show that gerrymandering may play some role in polarizing candidates' policy positions.

In the main text, we assumed that party leaders do not care about policies chosen by the opponent in their losing districts. Although this assumption may make sense in the context of gerrymandering, it would be interesting to see how our results are affected by weakening this assumption, since many models in electoral competition assume that losing candidates also care about policy realizations, as in a seminal paper by Wittman (1983). In Appendix C, we conduct a robustness check for analyzing the case of $\sigma > 0$ in (4). Although election equilibria are affected by this modification, the modifications are just parallel reductions of equilibrium winning payoffs. This is because losing payoffs are changed from zero to the disutility levels from the opponent parties' policy choices. When σ is large, both Lemma 4.1 and Lemmas 4.2 and 4.3 can be violated, which can cause problems for our results. However, if σ is not large, then all the results including Example 1 hold. In numerical calculations, it seems safe to say that our results hold for significantly wide range of σ (see Appendix C).

There are some potentially interesting yet difficult avenues for further research. First, one may want to introduce uncertainty in election results (e.g., uncertainty in the median voter's position after policy proposal) into our model. If uncertainty is infinitesimal (e.g., if the gerrymander can only observe that the median voter's position belongs to the interval $[\hat{x}^k - \epsilon, \hat{x}^k + \epsilon]$ for ϵ being a (small) preference perturbation) and if the gerrymanderer has complete freedom in redistricting, the slice-and-mix strategy may still be optimal, à la Friedman and Holden (2008). However, with significant uncertainty in median voters' positions, as in Gul and Pesendorfer (2010), we cannot predict what will happen.

Second, in this paper we concentrated on one type of pork-barrel politics: candidates' "promise" transfers contingent on their winning of the districts. These kinds of promises are different from campaign expenditures. In the latter case, even if a candidate loses in a district, the spent campaign expenditure will not come back (an all-pay auction). In some circumstances, such a model may be more realistic if there is uncertainty in the election results. However, introducing uncertainty in election results is not trivial, as we mentioned before. These issues are left for future research.

Appendix A: Proofs

Proof of Lemma 1. First, by Assumption 2, the non-negativity constraint of t_R^k is not needed. Second, in any equilibrium, the median voter is indifferent to L and R's proposals and votes for the candidate who yields higher payoffs. Suppose not, without loss of generality, let $Q_L - t_L^k - C(|\beta_L^k - \theta_L|) > Q_R - t_R^k - C(|\beta_R^k - \theta_R|) = 0$ and the median voter votes for L with probability less than 1. Then, L can win for sure by promise $t_L^k + \epsilon$. Then, voting for L with probability less than 1 cannot be an equilibrium.

Then, there are three cases: if R loses with $Q_R - t_R^{k^*} - C(\left|\theta_R - \beta_R^{k^*}\right|) > 0$ and its offer gives the median voter utility equal to $\bar{U}$ in equilibrium, it must be that L wins with positive indirect utility and also provides the median voter with the utility level $\bar{U}$. However, this means that R can win the election by providing, say, ϵ more pork-barrel promises. This contradicts the equilibrium condition. The second case is that $Q - t_R^{k^*} - C(\left|\theta_R - \beta_R^{k^*}\right|) = 0$ but $U_{x^k}(R)$ is not maximized. In this case, there must exist some points (t', β') that satisfy $Q - t' - C(|\theta_R - \beta'|) = 0$, but the pair provides the median voter strictly higher utility. In this case, any point on the segment connecting (t', β') and $(t_R^{k^*}, \beta_R^{k^*})$ is strictly better off for both R and the median voter x^k by the strict convexity of the preferences. Again, this contradicts the equilibrium condition. The third case, $Q - t_R^{k^*} - C(\left|\theta_R - \beta_R^{k^*}\right|) < 0$, cannot happen, since the strategy that generates a negative payoff is a weakly dominated strategy for R's leader. $\square$

Proof of Lemma 2. We only need to prove Lemma 2-3. We consider two cases: **(Case-1)** $\hat{x}^k > \theta_i$, and **(Case-2)** $\hat{x}^k < \theta_i$.

(Case-1): In this case, $\hat{\beta}_i = \hat{\beta}(\hat{x}^k, \theta_i)$ is determined implicitly by the first-order condition

$$C'(\hat{\beta}_i - \theta_i) = c'(\hat{x}^k - \hat{\beta}_i)$$

Totally differentiating with respect to $\hat{x}^k$ and $\hat{\beta}_i$, we obtain

$$(C'' + c'')d\hat{\beta}_i = c'' d\hat{x}^k$$

(Case-2): In this case, $\hat{\beta}_i = \hat{\beta}(\hat{x}^k, \theta_i)$ is determined implicitly by the first-order condition

$$C'(\theta_i - \hat{\beta}_i) = c'(\hat{\beta}_i - \hat{x}^k)$$

Totally differentiating with respect to $\hat{x}^k$ and $\hat{\beta}_i$, we obtain

$$(C'' + c'')d\hat{\beta}_i = c'' d\hat{x}^k$$

Thus, we get the same condition either way. We have completed the proof.$\square$

Proof of Lemma 3. We will focus on the case of $i = L$. When $i = R$, we can apply the same procedure. We will first show the following claim.

Claim. $\mathcal{C}_i' = c_i'$ when $\theta_i < \hat{x}^k$, $\mathcal{C}_i' = -c_i'$ when $\theta_i > \hat{x}^k$ and $\mathcal{C}_i'' = \frac{c_i'' C_i''}{c_i'' + C_i''}$, where $\mathcal{C}_i = \mathcal{C}(\left|\hat{x}^k - \theta_i\right|)$, $c_i = c\left(\left|\hat{x}^k - \hat{\beta}_i\left(\hat{x}^k, \theta_i\right)\right|\right)$, and $C_i = C\left(\left|\hat{\beta}_i\left(\hat{x}^k, \theta_i\right) - \theta_i\right|\right)$.

Proof of Claim. So, there are two cases: **(Case-a)** $\theta_i < \hat{x}^k$, and **(Case-b)** $\theta_i > \hat{x}^k$.
(Case-a): Taking the first derivative, we have

$$\mathcal{C}'(\hat{x}^k - \theta_i) = C'(\hat{\beta}_i - \theta_i)\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} + c'(\hat{x}^k - \hat{\beta}_i)(1 - \frac{\partial \hat{\beta}_i}{\partial \hat{x}^k}) = c'(\hat{x}^k - \hat{\beta}_i),$$

Here, we used the first-order condition $C' = c'$, which must hold at the optimum. Taking the second-order derivative, we have

$$\begin{aligned}
\mathcal{C}''(\hat{x}^k - \theta_i) &= c''(\hat{x}^k - \hat{\beta}_i)(1 - \frac{\partial \hat{\beta}_i}{\partial \hat{x}^k}) \\
&= c''(\hat{x}^k - \hat{\beta}_i)\left(1 - \frac{c''(\hat{x}^k - \hat{\beta}_i)}{c''(\hat{x}^k - \hat{\beta}_i) + C''(\hat{\beta}_i - \theta_i)}\right) \\
&= \frac{c''(\hat{x}^k - \hat{\beta}_i)C''(\hat{\beta}_i - \theta_i)}{c''(\hat{x}^k - \hat{\beta}_i) + C''(\hat{\beta}_i - \theta_i)}
\end{aligned}$$

(Case-b): Taking the first-order derivative, we have

$$\mathcal{C}'(\theta_i - \hat{x}^k) = -C'(\theta_i - \hat{\beta}_i)\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} + c'(\hat{\beta}_i - \hat{x}^k)(\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} - 1) = -c'(\hat{\beta}_i - \hat{x}^k),$$

Taking the second-order derivative, we have

$$\begin{aligned}
\mathcal{C}''(\theta_i - \hat{x}^k) &= -c''(\hat{\beta}_i - \hat{x}^k)(\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} - 1) \\
&= c''(\hat{\beta}_i - \hat{x}^k)\left(1 - \frac{c''(\hat{x}^k - \hat{\beta}_i)}{c''(\hat{x}^k - \hat{\beta}_i) + C''(\theta_i - \hat{\beta}_i)}\right) \\
&= \frac{c''(\hat{\beta}_i - \hat{x}^k)C''(\theta_i - \hat{\beta}_i)}{c''(\hat{\beta}_i - \hat{x}^k) + C''(\theta_i - \hat{\beta}_i)}
\end{aligned}$$

We have completed the proof of the Claim.$\blacksquare$

We start with Lemma 3-1. First, we consider **(Case-1):** $\hat{x}^k \in (\theta_L, \theta_R)$, then $\tilde{V}_L^k = (Q_L - Q_R) - \big(\mathcal{C}(x^k + y - \theta_L) - \mathcal{C}(\theta_R - x^k - y)\big)$. Thus, we have

$$\frac{d\tilde{V}_L^k}{d\hat{x}^k} = -\left(\mathcal{C}'(\hat{x}^k - \theta_L) + \mathcal{C}'(\theta_R - \hat{x}^k)\right) < 0.$$

This implies that $\tilde{V}_L^k$ is decreasing in $\hat{x}^k$. In the case of $\tilde{V}_R^k$, $\frac{d\tilde{V}_R^k}{d\hat{x}^k} > 0$ and $\tilde{V}_R^k$ is increasing in $\hat{x}^k$.

There are two more cases: **(Case-2)** $\hat{x}^k < \theta_L$, and **(Case-3)** $\hat{x}^k > \theta_R$.
(Case-2): $\frac{d\tilde{V}_L^k}{d\hat{x}^k} = \mathcal{C}'(\theta_L - \hat{x}^k) - \mathcal{C}'(\theta_R - \hat{x}^k) < 0$, since $\mathcal{C}''(d) > 0$. Thus, $\tilde{V}_L^k$ is decreasing in $\hat{x}^k$.
(Case-3): $\frac{d\tilde{V}_L^k}{d\hat{x}^k} = -\mathcal{C}'(\hat{x}^k - \theta_L) + \mathcal{C}'(\hat{x}^k - \theta_R) < 0$, since $\mathcal{C}''(d) > 0$. Thus, $\tilde{V}_L^k$ is decreasing in $\hat{x}^k$.

For the convexity, again we have three cases: **(Case-1)** $\hat{x}^k \in (\theta_L, \theta_R)$, **(Case-2)** $\hat{x}^k < \theta_L$, and **(Case-3)** $\hat{x}^k > \theta_R$. In each case, we have the same second-order derivatives:
(Case-1): $\frac{d^2\tilde{V}_L^k}{d(\hat{x}^k)^2} = -\mathcal{C}''(\hat{x}^k - \theta_L) + \mathcal{C}''(\theta_R - \hat{x}^k)$.
(Case-2): $\frac{d\tilde{V}_L^k}{d\hat{x}^k} = \mathcal{C}'(\theta_L - \hat{x}^k) - \mathcal{C}'(\theta_R - \hat{x}^k)$ and $\frac{d^2\tilde{V}_L^k}{d(\hat{x}^k)^2} = -\mathcal{C}''(\theta_L - \hat{x}^k) + \mathcal{C}''(\theta_R - \hat{x}^k)$.
(Case-3): $\frac{d\tilde{V}_L^k}{d\hat{x}^k} = -\mathcal{C}'(\hat{x}^k - \theta_L) + \mathcal{C}'(\hat{x}^k - \theta_R)$ and $\frac{d^2\tilde{V}_L^k}{d(\hat{x}^k)^2} = -\mathcal{C}''(\hat{x}^k - \theta_L) + \mathcal{C}''(\hat{x}^k - \theta_R)$.
Therefore, in all cases, $\frac{d^2\tilde{V}_L^k}{d(\hat{x}^k)^2} = -\mathcal{C}''_L + \mathcal{C}''_R$, so we have:

$$\begin{aligned}
\frac{d^2\tilde{V}_L^k}{d(\hat{x}^k)^2} &= -\mathcal{C}''_L + \mathcal{C}''_R \\
&= -\frac{c''_L C''_L}{c''_L + C''_L} + \frac{c''_R C''_R}{c''_R + C''_R} \\
&= \frac{-c''_L C''_L\,(c''_R + C''_R) + c''_R C''_R\,(c''_L + C''_L)}{(c''_L + C''_L)\,(c''_R + C''_R)} \\
&= \frac{C''_L C''_R\,(c''_R - c''_L) + c''_L c''_R\,(C''_R - C''_L)}{(c''_L + C''_L)\,(c''_R + C''_R)}
\end{aligned}$$

Thus, if $c''_R \geq c''_L$ and $C''_R \geq C''_L$ then $\frac{d^2\tilde{V}_L^k}{d(\hat{x}^k)^2} \geq 0$. Since $Q_L = Q_R$, if L wins, then $\hat{x}^k - \theta_L < \theta_R - \hat{x}^k$. Thus, if $c''' > 0$ and $C''' > 0$ then we have $c''_R \geq c''_L$ and $C''_R \geq C''_L$.$\square$

Proof of Lemma 4. We will focus on the case of $i = L$. When $i = R$, we can apply the same procedure. Let's start with Lemma 4-1. Consider the case where $x^k \pm \bar{y} \in (\theta_L, \theta_R)$. There are two subcases: **(Case 1)** is the case where L wins with certainty ($\tilde{V}_L^k(x^k + \bar{y}, \theta_L, \theta_R) \geq 0$), and **(Case 2)** is the one where L may lose depending on the realization of y ($\tilde{V}_L^k(x^k + \bar{y}, \theta_L, \theta_R) < 0$).

(Case 1): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. Thus,

$$\frac{dE\tilde{V}_L^k}{dx^k} = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy < 0$$

where $\tilde{V}_L^{k\prime} = \frac{d\tilde{V}_L^k}{dx^k} = \frac{d\tilde{V}_L^k}{d\hat{x}^k}$.

(Case 2): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$, where $\tilde{V}_L^k(\bar{x}, \theta_L, \theta_R) = 0$. That is, if $x^k + y > \bar{x}$, then party L loses. (Note that $\bar{x}$ is solely determined by the value of $Q_L - Q_R$: $\frac{d\bar{x}}{d(Q_L - Q_R)} > 0$. If $Q_L = Q_R$, then $\bar{x} = 0$ holds, since $\theta_L = -\theta_R$.) Differentiating this with respect to x^k, we have

$$\begin{aligned} \frac{dE\tilde{V}_L^k}{dx^k} &= \tilde{V}_L^k(0, \theta_L, \theta_R) g(-x^k) + \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \\ &= \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy < 0 \end{aligned}$$

Thus, we have completed the proof of Lemma 4-1.

For Lemma 4-2, we classify four cases:

(Case a: $x^k - \bar{y} \geq \theta_L$ and $x^k + \bar{y} \leq \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. Thus,

$$\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy$$

(**Case b: $x^k - \bar{y} \geq \theta_L$ and $x^k + \bar{y} > \bar{x}$):** In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{-x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. That is, if $x^k + y > \bar{x} = 0$, then party L loses. Differentiating this with respect to x^k, we have

$$\begin{aligned} \frac{dE\tilde{V}_L^k}{dx^k} &= -\tilde{V}_L^k(0, \theta_L, \theta_R) + \int_{-\bar{y}}^{-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \\ &= \int_{-\bar{y}}^{-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \end{aligned}$$

Thus, the second-order derivative is

$$\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} = -\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) + \int_{-\bar{y}}^{-x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy$$

From Lemma 3-2, we know $\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) < 0$ and $\tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) > 0$. Thus, $E\tilde{V}_L^k$ is convex.

(Case c: $x^k - \bar{y} < \theta_L$ and $x^k + \bar{y} \leq \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{\bar{y}} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. Differentiating this with respect to x^k, we obtain

$$\frac{dE\tilde{V}_L^k}{dx^k} = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{\bar{y}} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy$$

The second-order derivative is

$$\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{\bar{y}} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy$$

From Lemma 3-2, we know $\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) < 0$ and $\tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) > 0$. Thus, $E\tilde{V}_L^k$ is convex.

(Case d: $x^k - \bar{y} < \theta_L$ and $x^k + \bar{y} > \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{-x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. Differentiating this with respect to x^k, we obtain

$$\begin{aligned}\frac{dE\tilde{V}_L^k}{dx^k} &= \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \\ &- \tilde{V}_L^k(0, \theta_L, \theta_R) g(-x^k) \\ &= \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy\end{aligned}$$

The second-order derivative is

$$\begin{aligned}\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} &= \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy + \int_{\theta_L - x^k}^{-x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \\ &- \tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) g(-x^k)\end{aligned}$$

From Lemma 3-2, we know $\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) < 0$ and $\tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) > 0$ when $c''(d) > 0$ and $C''(d) > 0$. Thus, $E\tilde{V}_L^k$ is convex. We have completed the proof of Lemma 4-2.

For Lemma 4-3, first observe that,

$$E\tilde{V}_i\left(\pi\right) \equiv \int_{-\bar{y}}^{\bar{y}} \sum_{k=1}^{K} max\Big\{\tilde{V}_i^k(x^k(\pi)+y,\theta_i,\theta_j),0\Big\}g(y)dy$$

For any $k \neq k'$, $\frac{\partial^2 E\tilde{V}_i}{\partial x^k \partial x^{k'}} = 0$. The Hessian matrix of $E\tilde{V}_L$ has zeros on non-diagonal parts and negative terms on the diagonal due to Lemma 4-2. Therefore, the Hessian matrix is negative semidefinite and $E\tilde{V}$ are convex function in $(x^k)_{k=1}^K$. □

Proof of Lemma 5. Note that $F(x_L^k) = \frac{k}{2K}$. Thus, to achieve x_L^k as the median voter of the kth district, we need to use all voters to the left of x_L^k. This is true for all $k = 1,...,K$. Thus, x_L is the leftmost median voter allocation in lexicographic order. We can prove the statement for x_R by a symmetric argument.□

Proof of Proposition 4. Suppose that a district structure π is not "order-and-partition." We will show that π is not an optimal partisan gerrymandering policy when the third derivatives of cost functions are positive. Since π is not order-and-partition, there exist districts k and h such that there exist $\ell \in D^k$ and $\tilde{\ell} \in D^h$ with $\mu_\ell > \mu_{\tilde{\ell}}$, in which one of the following holds: **(Case-I)** $k, h \leq K'$ and $x^k(\pi) \leq x^h(\pi)$, or **(Case-II)** $k \leq K'$ and $h > K'$. Let π' be a district structure that is generated by swapping localities ℓ and $\tilde{\ell}$ of π. Let's start with (Case-I). In this case, $x^k(\pi') < x^k(\pi) \leq x^h(\pi) < x^h(\pi')$ holds. Thus, by Lemma 4-3, $E\tilde{V}_L(\pi) < E\tilde{V}_L(\pi')$. (Case-II) is simpler. In this case, it is obvious that $x^k(\pi') < x^k(\pi)$ holds. Thus, by Lemma 4-1, $E\tilde{V}_L(\pi) < E\tilde{V}_L(\pi')$ holds. We conclude that the optimal partisan gerrymandering policy is order-and-partition.□

Appendix B: Constant Elasticity Example

In this appendix, we elaborate on the calculation involved in Example 1. Let $C(d) = a^C d^\gamma$ and $c(d) = a^c d^\gamma$, where $\gamma > 1$, $a^C > 0$, and $a^c > 0$ are parameters. In this case both party leaders and voters have the same constant elasticity γ. Thus, we have the following convenient formula. Denote $A = A(a^C, a^c) = a^C\left(\frac{\alpha}{1+\alpha}\right)^\gamma + a^c\left(\frac{1}{1+\alpha}\right)^\gamma > 0$ where $\alpha = \left(\frac{a^C}{a^c}\right)^{\frac{1}{\gamma-1}}$. Suppose that $\left(\frac{Q}{A}\right)^{\frac{1}{\gamma}} \geq 2+\bar{y}$ holds to satisfy Assumption 2. Normalizing $A = 1$, we have $\mathcal{C}(d) = Ad^\gamma = d^\gamma$. In this case, $\tilde{V}_L^k$ is concave (convex) in $\hat{x}^k$ if $\gamma \leq 2$ ($\gamma \geq 2$).

$$
\begin{aligned}
E\tilde{V}_L^k =& \int_{\theta_L - x^k}^{-x^k} \left(\mathcal{C}(\theta_R - x^k - y) - \mathcal{C}(x^k + y - \theta_L)\right) g(y)dy \\
&+ \int_{-\bar{y}}^{\theta_L - x^k} \left(\mathcal{C}(\theta_R - x^k - y) - \mathcal{C}(\theta_L - x^k - y)\right) g(y)dy
\end{aligned}
$$

$$
\begin{aligned}
\frac{dE\tilde{V}_L^k}{dx^k} =& \int_{\theta_L - x^k}^{-x^k} \left(-\mathcal{C}'(\theta_R - x^k - y) - \mathcal{C}'(x^k + y - \theta_L)\right) g(y)dy \\
&+ \int_{-\bar{y}}^{\theta_L - x^k} \left(-\mathcal{C}'(\theta_R - x^k - y) + \mathcal{C}'(\theta_L - x^k - y)\right) g(y)dy
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2E\tilde{V}_L^k}{d(x^k)^2} =& \int_{\theta_L - x^k}^{-x^k} \left(\mathcal{C}''(\theta_R - x^k - y) - \mathcal{C}''(x^k + y - \theta_L)\right) g(y)dy \\
&+ \int_{-\bar{y}}^{\theta_L - x^k} \left(\mathcal{C}''(\theta_R - x^k - y) - \mathcal{C}''(\theta_L - x^k - y)\right) g(y)dy \\
&+ (\mathcal{C}'(\theta_R) + \mathcal{C}'(-\theta_L))g(-x^k)
\end{aligned}
$$

Suppose that $\mathcal{C}(d) = d^\gamma$ ($\gamma > 1$), $\theta_L = -1$, $\theta_R = 1$ (thus $\bar{x} = 0$), and $g(y) = \frac{1}{2\bar{y}}$ if and only if $y \in [-\bar{y}, \bar{y}]$. If $\bar{y} \geq 2$ and $x^k \in [-1, 1]$ for all possible x^k, Case-d in the proof of Lemma 4 applies. In this case, we have

$$
\begin{aligned}
\frac{d^2E\tilde{V}_L^k}{d(x^k)^2} =& \frac{\gamma}{2\bar{y}} \left[-(\theta_R - x^k - y)^{\gamma-1} - (x^k + y - \theta_L)^{\gamma-1}\right]_{\theta_L - x^k}^{-x^k} \\
&+ \frac{\gamma}{2\bar{y}} \left[-(\theta_R - x^k - y)^{\gamma-1} + (\theta_L - x^k - y)^{\gamma-1}\right]_{-\bar{y}}^{\theta_L - x^k} + \frac{\gamma}{\bar{y}}\theta_R^{\gamma-1} \\
=& \frac{\gamma}{2\bar{y}}[-(\theta_R)^{\gamma-1} - (-\theta_L)^{\gamma-1} - (-(\theta_R - \theta_L)^{\gamma-1} - 0) \\
&+ (-(\theta_R - \theta_L)^{\gamma-1} + 0) - (-(\theta_R - x^k + \bar{y})^{\gamma-1} + (\theta_L - x^k + \bar{y})^{\gamma-1}) + 2\theta_R^{\gamma-1}] \\
=& \frac{\gamma}{2\bar{y}}[-2 + (1 - x^k + \bar{y})^{\gamma-1} - (-1 - x^k + \bar{y})^{\gamma-1} + 2] \\
=& \frac{\gamma}{2\bar{y}}[(1 - x^k + \bar{y})^{\gamma-1} - (-1 - x^k + \bar{y})^{\gamma-1}]
\end{aligned}
$$

Since $x^k - \bar{y} < 0$, *we conclude* $(1 - x^k + \bar{y})^{\gamma-1} - (-1 - x^k + \bar{y})^{\gamma-1} > 0$ *for any* $\gamma > 1$. Thus, $E\tilde{V}_L^{k\prime\prime} > 0$ holds as long as Case 4 holds ($\bar{y} \geq 1$: there is a chance to win district k for any x^k). That is, the expected utility is convex in x^k, although $\mathcal{C}'''(d) < 0$ holds. So, even without positive third derivatives, the order-and-partition strategy is the optimal gerrymandering policy in the constant average constraint case.□

Appendix C

In the benchmark model, we assume that the losing payoff of a candidate is 0. In this section, we extend our model to a general setup à la Wittman (1983). Formally, by losing in district k, party j's leader gets utility

$$V_j^k = -\sigma C(|\beta_i^k - \theta_j|)$$

where $\sigma \in [0, 1]$ naturally (candidate cares about her own policy more than her opponent's policy). Supposing that the winning party is L, the winner's maximization problem (5) is not changed. Therefore, the optimal strategy combination is still $\hat{\beta}_L(\hat{x}^k, \theta_L)$ and $t_L^{k^*}(\bar{U}_R^k) = \bar{U}_R^k + c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|)$. From Lemma 1, the losing R chooses an equilibrium strategy combination as if he tried to maximize the median voter's payoff given its utility of no less than $-\sigma C(|\beta_L^k - \theta_R|)$. Formally, R solves the following problem

$$\max_{\beta_R^k, t_R^k} U_{\hat{x}^k}(R) = t_R^k - c(|\hat{x}^k - \beta_R^k|)$$
$$\text{subject to } t_R^k \geq 0 \text{ and } Q_R - t_R^k - C(\left|\theta_R - \beta_R^k\right|) \geq -\sigma C(|\beta_L^k - \theta_R|).$$

Notice that, since the losing payoff is negative, the losing party promises more aggressively to the median voter to maximize her chance of winning in a weakly undominated strategy. When the second constraint is binding, the problem becomes

$$\max_{\beta_R^k} Q_R - C(\left|\theta_R - \beta_R^k\right|) - c(|\hat{x}^k - \beta_R^k|) + \sigma C(|\beta_L^k - \theta_R|).$$

Thus, R again chooses $\hat{\beta}_R^k$, which is a function of $\hat{x}^k$ and θ_R minimizing $C(\left|\theta_R - \beta_R^k\right|) + c(|\hat{x}^k - \beta_R^k|)$. On the other hand, since the winning candidate L tries to maximize her payoff $Q_L - C(\left|\theta_L - \beta_L^k\right|) - t_L^k = Q_L - C(\left|\theta_L - \beta_L^k\right|) - U_{\hat{x}^k}(R) - c(|\hat{x}^k - \beta_L^k|)$ by providing the median voter exactly the same utility $U_{\hat{x}^k}(R)$ that candidate R assures, the equilibrium strategies are

$$\begin{aligned}
\beta_L^{k^*} &= \hat{\beta}_L(\hat{x}^k, \theta_L) \\
t_L^{k^*} &= Q_L - Q_R - C(\left|\theta_L - \hat{\beta}_L\right|) - c(|\hat{x}^k - \hat{\beta}_L|) + C(|\theta_R - \hat{\beta}_R|) + c(|\hat{x}^k - \beta_R|) \\
&\quad - \sigma C(|\hat{\beta}_L - \theta_R|),
\end{aligned}$$

and

$$\beta_R^{k^*} = \hat{\beta}_R(\hat{x}^k, \theta_R) \text{ and } t_R^{k^*} = Q_R - C(|\theta_R - \hat{\beta}_R|) + \sigma C(|\hat{\beta}_L - \theta_R|)$$

where $\hat{\beta}_L(\hat{x}^k, \theta_L) = \arg\min_\beta C(|\theta_L - \beta|) + c(|\hat{x}^k - \beta|)$. Since this is a simultaneous move game, candidate L does not take the externality to candidate R into account when she chooses β. Focusing on the case where $Q_L = Q_R$ and $|\theta_L| = \theta_R$, candidate L's winning payoff is

$$\mathcal{C}_R - \mathcal{C}_L - \sigma C(|\hat{\beta}_L - \theta_R|)$$

and L wins if and only if $\left|\theta_L - \hat{x}^k\right| < \left|\theta_R - \hat{x}^k\right|$. Therefore,

$$\tilde{V}_L^k = \begin{cases} \mathcal{C}_R - \mathcal{C}_L - \sigma C(|\hat{\beta}_L - \theta_R|) & \text{if } \left|\theta_L - \hat{x}^k\right| < \left|\theta_R - \hat{x}^k\right| \\ -\sigma C(|\hat{\beta}_R - \theta_L|) & \text{if } \left|\theta_L - \hat{x}^k\right| > \left|\theta_R - \hat{x}^k\right| \end{cases}$$

We argue that Lemma 4 continues to hold if σ is small enough. We focus on case (d) in the proof of Lemma 4, i.e., $x^k - \bar{y} < \theta_L$ and $x^k + \bar{y} > 0$. For other cases, the proofs are similar. First,

$$\begin{aligned} E\tilde{V}_L^k = & \int_{-\bar{y}}^{-x^k} \left[\mathcal{C}_R - \mathcal{C}_L\right] g(y)dy - \sigma \int_{-\bar{y}}^{-x^k} C(|\hat{\beta}_L - \theta_R|)g(y)dy \\ & - \sigma \int_{-x^k}^{\bar{y}} C(|\hat{\beta}_R - \theta_L|)g(y)dy \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dE\tilde{V}_L}{dx^k} = & - \left[-\mathcal{C}(\theta_R) - \mathcal{C}(-\theta_L)\right] g(-x^k) + \int_{-\bar{y}}^{-x^k} \left[-c_R'(\hat{\beta}_R - (x^k + y)) - c_L'((x^k + y) - \hat{\beta}_L)\right] g(y)dy \\ & + \sigma C(\theta_R - \hat{\beta}_L(0, \theta_L))g(-x^k) + \sigma \int_{-\bar{y}}^{-x^k} C'(\theta_R - \hat{\beta}_L)\frac{\partial \hat{\beta}_L}{\partial \hat{x}^k} g(y)dy \\ & - \sigma C(\hat{\beta}_R(0, \theta_R) - \theta_L)g(-x^k) - \sigma \int_{-x^k}^{\bar{y}} C'(\hat{\beta}_R - \theta_L)\frac{\partial \hat{\beta}_R}{\partial \hat{x}^k} g(y)dy \\ = & \int_{-\bar{y}}^{-x^k} \left[-c_R'(\hat{\beta}_R - (x^k + y)) - c_L'((x^k + y) - \hat{\beta}_L)\right] g(y)dy \\ & + \sigma \left[\int_{-\bar{y}}^{-x^k} C'(\theta_R - \hat{\beta}_L)\frac{\partial \hat{\beta}_L}{\partial \hat{x}^k} g(y)dy - \int_{-x^k}^{\bar{y}} C'(\hat{\beta}_R - \theta_L)\frac{\partial \hat{\beta}_R}{\partial \hat{x}^k} g(y)dy\right] \end{aligned}$$

This may fail to be negative for x^k around zero when σ is large, especially when $-x^k$ and $\bar{y}$ are close (the second term in the last bracket is small); i.e., L wins district k with high probability. When σ is large, the negative first line can be dominated by the first term of the second line, which is the payoff suppression effect caused by the loser's payoff in electoral competition. Therefore, Lemma 4.1 may fail to hold locally for x^k around $-\bar{y}$ when σ is large enough (the second line has higher weight, and payoff suppression effect is large).

For the second derivative, we have

$$\begin{aligned}
\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} &= \left[c'(\hat{\beta}_R(0,\theta_R)) + c'(-\hat{\beta}_L(0,\theta_L))\right] g(-x_L^k) \\
&\quad - \int_{-\bar{y}}^{-x^k} \frac{\partial}{\partial x^k}\left(c_R'(\hat{\beta}_R - (x^k+y)) + c_L'((x^k+y) - \hat{\beta}_L)\right) g(y)dy \\
&\quad - \sigma C'(\theta_R - \hat{\beta}_L(0,\theta_L))\frac{\partial \hat{\beta}_L}{\partial \hat{x}^k}(0,\theta_L) g(-x^k) \\
&\quad - \sigma \int_{-\bar{y}}^{x^k} \left(C''(\theta_R - \hat{\beta}_L)\left(\frac{\partial \hat{\beta}_L}{\partial \hat{x}^k}\right)^2 - C'(\theta_R - \hat{\beta}_L)\frac{\partial^2 \hat{\beta}_L}{\partial (\hat{x}^k)^2}\right) g(y)dy \\
&\quad - \sigma C'(\hat{\beta}_R(0,\theta_R) - \theta_L)\frac{\partial \hat{\beta}_R}{\partial \hat{x}^k}(0,\theta_R) g(-x^k) \\
&\quad - \sigma \int_{-x^k}^{\bar{y}} \left(C''(\hat{\beta}_R - \theta_L)\left(\frac{\partial \hat{\beta}_R}{\partial \hat{x}^k}\right)^2 + C'(\hat{\beta}_R - \theta_L)\frac{\partial^2 \hat{\beta}_R}{\partial (\hat{x}^k)^2}\right) g(y)dy
\end{aligned}$$

The first two terms are the same as the case where $\sigma = 0$ and are (strictly) positive. However, the signs of the third and fifth terms are negative and the signs of all other terms are undetermined (the signs of $\frac{\partial^2 \hat{\beta}_L}{\partial (\hat{x}^k)^2}$ and $\frac{\partial^2 \hat{\beta}_R}{\partial (\hat{x}^k)^2}$ are unknown), and it is impossible to determine the sign of the second derivative when σ is large. However, as long as σ is not large, Lemma 4.2 and 4.3 prevail.

Example 1 (continuation). In this case, we can write $\hat{\beta}_L(\hat{x}^k, \theta_L) = \frac{\alpha}{1+\alpha}\hat{x}^k + \frac{1}{1+\alpha}\theta_L$ and $\hat{\beta}_R(\hat{x}^k, \theta_R) = \frac{\alpha}{1+\alpha}\hat{x}^k + \frac{1}{1+\alpha}\theta_R$. Thus, we have

$$\frac{\partial^2 \hat{\beta}_L}{\partial (\hat{x}^k)^2} = \frac{\partial^2 \hat{\beta}_R}{\partial (\hat{x}^k)^2} = 0$$

and

$$\begin{aligned}
\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} &= \frac{\gamma}{2\bar{y}}\left[(1 - x^k + \bar{y})^{\gamma-1} - (-1 - x^k + \bar{y})^{\gamma-1}\right] \\
&\quad - \sigma \frac{\gamma}{2\bar{y}} \frac{\alpha}{1+\alpha}\left[\left(1 - \frac{\alpha}{1+\alpha}(x^k - \bar{y}) + \frac{1}{1+\alpha}\right)^{\gamma-1}\right. \\
&\quad \left. + \left(\frac{\alpha}{1+\alpha}(x^k + \bar{y}) + \frac{1}{1+\alpha} + 1\right)^{\gamma-1}\right]
\end{aligned}$$

From Example 1, the first term is positive. However, all other terms are negative ones multiplied by $\sigma \in [0,1]$. Thus, the expected utility is convex in x^k, when $\sigma \geq 0$ is small enough. This is especially true, if we adopt the wildly-used quadratic cost functions; i.e. $\gamma = 2$, $\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2}$ is linear, and the above equation is simplified:

$$\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} = \frac{\gamma}{\bar{y}} \left[1 - \frac{\sigma\alpha}{1+\alpha} \left\{ 1 + \frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha}\bar{y} \right\} \right]$$

Therefore, readers can easily see the parameter space where our results hold. For example, when $\bar{y} = 1.5$, $\alpha = 1$, $-\bar{y} < x^k < \bar{y}$, and $\sigma < \frac{2}{3}$, $E\tilde{V}_L^k$ is convex. This example shows that even if candidates' losing payoffs are negative, the order-and-partition strategy can be optimal for the constrained case with the constant average constraint.□

References

[1] Bernstein, M., and M. Duchin (2017): "A Formula Goes to Court: Partisan Gerrymandering and the Efficiency Gap," *Notice of the American Mathematical Society* 64(9), pp. 1020-1024.

[2] Besley, T., and I. Preston (2007): "Electoral Bias and Policy Choice: Theory and Evidence," *Quarterly Journal of Economics*, 122(4), pp. 1473-1510.

[3] Bracco, E. (2013): "Optimal Districting with Endogenous Party Platforms," *Journal of Public Economics*, 104(C), pp. 1-13.

[4] Cain, B.E. (1985): "Assessing the Partisan Effects of Redistricting," *American Political Science Review* 79, pp. 320-333.

[5] Chambers, P.C., A.D. Miller, and J. Sobel (2017), "Flaws in the Efficiency Gap," *Journal of Law and Politics* 33(1), 1-33.

[6] Coate, S., and B. Knight (2007): "Socially Optimal Districting: A Theoretical and Empirical Exploration," *Quarterly Journal of Economics*, 122(4), pp. 1409-1471.

[7] Dixit, A., and J. Londregan (1996): "The Determinants of Success of Special Interests in Redistributive Politics," *Journal of Politics*, 58, pp. 1132-1155.

[8] Dixit, A., and J. Londregan (1998): "Ideology, Tactics, and Efficiency in Redistributive Politics," *Quarterly Journal of Economics*, 113 (2), pp. 497-529.

[9] Dekel, E., Matthew O. Jackson, and Wolinsky, A. (2008): "Vote Buying: General Elections," *Journal of Political Economy*, 116(2), pp. 351-380.

[10] Ferejohn, J. A. (1977): "On the Decline of Competition in Congressional Elections," *American Political Science Review*, 71(1), pp. 166-176.

[11] Fiorina, M. P., J. C. Pope, and S. J. Abrams (2011): *Culture War? The Myth of a Polarized America*. Boston: Pearson.

[12] Friedman, J. N., and R. T. Holden (2008): "Optimal Gerrymandering: Sometimes Pack, but Never Crack," *American Economic Review*, 98(1), pp. 113-144.

[13] Friedman, J. N., and R. T. Holden (2009): "The Rising Incumbent Reelection Rate: What's Gerrymandering Got to Do With It?" *Journal of Politics,* 71(2), pp. 593-611.

[14] Gilroux, G. (2001): "Remap's Clear Trend: Incumbent Protection," *CQ Weekly*, Nov. 3, pp. 2627-2632.

[15] Grainger, C.A. (2010): "Redistricting and Polarization: Who Draws the Lines in California?" *Journal of Law and Economics* 53(3), 545-567.

[16] Gul, F., and W. Pesendorfer (2010): "Strategic Redistricting," *American Economic Review*, 100(4), pp. 1616-1641.

[17] Krasa, S., and M. Polborn (2015): Political Competition in Legislative Elections, *working paper.*

[18] Levendusky, M. (2009): *The Partisan Sort: How Liberals Became Democrats and Conservatives Became Republicans*. Chicago: Chicago University Press.

[19] Lindbeck, A., and J. W. Weibull (1987): "Balanced-Budget Redistribution as the Outcome of Political Competition," *Public Choice*, 52 (2), pp. 273-297.

[20] McCarty, N., K. T. Poole, and H. Rosenthal (2006): *Polarized America: The Dance of Ideology and Unequal Riches.* Cambridge, MA: MIT Press.

[21] McCarty, N., K. T. Poole, and H. Rosenthal (2009): "Does Gerrymandering Cause Polarization," *American Journal of Political Science*, 53(3), pp. 666-680.

[22] Osborne, M. J. (1995): "Spatial Models of Political Competition under Plurality Rule: A Survey of Some Explanations of the Number of Candidates and the Positions They Take," *Canadian Journal of Economics*, 28(2), pp. 261-301.

[23] Owen, G., and B. N. Grofman (1988): "Optimal Partisan Gerrymandering," *Political Geography Quarterly*, 7(1), pp. 5-22.

[24] Puppe, C., and A. Tasnadi (2009): "Optimal Redistricting under Geographical Constraints: Why "Pack and Crack" Does Not Work," *Economics Letters*, 105(1), pp. 93-96.

[25] Roemer, J. E. (2001): *Political Competition: Theory and Applications.* Harvard University Press.

[26] Shotts, K. W. (2001): "The Effect of Majority-Minority Mandates on Partisan Gerrymandering," *American Journal of Political Science*, 45(1), pp. 120-135.

[27] Wittman D. (1983): "Candidate Motivation: A Synthesis of Alternative Theories," *American Political Science Review*, 77, pp.142-157.