Endogenous Alliances in Survival Contests*

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Abstract

Esteban and Sákovics (2003) showed in their three-person game that an alliance never appears in a possibly multi-stage contest game for an indivisible prize when allies' efforts are perfectly substitutable. In this paper, we introduce allies' effort complementarity in alliances by using a CES effort aggregator function. We consider an open-membership alliance formation game followed by two contests: one played by alliances, and one within the winning alliance. We show that if allies' efforts are too substitutable or too complementary, there is no meaningful alliance in equilibrium. However, if allies' efforts are moderately complementary to each other, then competition between two alliances is a subgame perfect equilibrium, which Pareto-dominates the equilibrium in a no-alliance single-stage contest. We also show that if forming more than two alliances is supported in equilibrium, then it Pareto-dominates two-alliance equilibrium. Nevertheless, the parameter space for such an allocation to be supported as an equilibrium shrinks when the number of alliances increases.

1 Introduction

In their influential paper, Esteban and Sákovics (2003) consider a three-person strategic alliance formation in a Tullock contest model in which players compete for an indivisible prize, and demonstrate that an alliance involves strategic disadvantages (see also Konrad 2009). There are two main disadvantages to forming an alliance: First, if an alliance is formed, there will be an additional contest that dissipates

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the members' rents, even if the alliance wins the first race. Because of this *rent-dissipation* effect, the members of the alliance have lower valuations for winning in the first race, reducing their efforts and the winning probability. Second, even without the rent-dissipation problem, e.g., if the winning prize is shared equally, there are still *free-riding* incentives for the alliance members to reduce efforts and, consequently, the winning probability. As a result, they conclude that it is hard to materialize strategic alliances in a Tullock contest model.[1] Konrad (2009) points out that these disincentives are not specific to Tullock contest models—they also appear in first price all-pay auctions.[2]

However, in the real world, forming alliances in competition is ubiquitous—for example, in research and development activities, and nations in conflicts. In this paper, we provide a simple solution for this alliance paradox by using complementarity in efforts in a general but symmetric N-person game.[3] To analyze complementarity, we introduce a simple and tractable CES effort aggregator function to translate alliance members' individual efforts into the alliance's joint effort. We assume that each individual member's marginal effort cost is constant in order to limit the benefits of forming an alliance to effort complementarity only.[4] With strong complementarity in efforts, a larger alliance has the effort advantage relative to a smaller one. Although there are aforementioned disincentives, it makes sense to form an alliance as long as the benefits from complementarity exceed the costs. The complementarity parameter in the CES aggregator provides a simple measure of the strength of incentive to form alliances as its value increases from 0 to 1.[5]

[1]Konrad (2004) considers an asymmetric all-pay auction game with exogenously determined hierarchical tournament structure, and shows that the highest valuation player may not have a chance to become the final winner depending on the hierarchical structure.

[2]On the other hand, Wärneryd (1998) shows that forming alliances and competing in a multi-stage competition reduce wasteful competition and increase total welfare. However, this resource saving effect is difficult to realize due to the disadvantageous effect on alliances when members' individual efforts are perfectly substitutable.

[3]There are at least a few other ways to resolve this alliance paradox (Konrad and Leininger 2007 and Konrad and Kovenock 2008: see the literature review).

[4]In general, forming an alliance can reach higher total efforts with less individual cost when cost functions are convex. For example, Esteban and Sákovics (2003) assume quadratic individual effort functions but still got their alliance paradox.

[5]Complementarity in efforts within a group in Esteban and Ray (2011) is more subtle. They analyze the conflict between two ethnic groups by assuming that players have heterogeneous financial and human opportunity costs, and they can contribute financially to a conflict or they can directly participate as activists. This generates some sort of complementarity: more activism requires more of the two inputs, time and money. They find that within-group inequality leads to more activism. This is because alliance members are specialized: poor individuals with cheap available activist time and rich individuals with money they can contribute at a lower marginal utility

We are not the first to present this idea. Following Cornes (1993) and Cornes and Hartley (2007) in the literature of private provision of public goods, Kolmar and Rommeswinkel (2013) and Choi, Chowdhury, and Kim (2016) have already demonstrated the presence of such incentives in alliance formation (see next section). This paper goes one step further. Since players' payoffs are related to the whole alliance structure, it is important to know how other players react to the alliance structure and whether or not the alliance structure could be stable. Therefore, we need to see players' and alliances' strategic interactions, and what happens in equilibrium: in particular, we ask whether or not there exists an equilibrium alliance structure.

We set up a simple alliance (coalition) formation game with multiple stages. In stage 1, players form alliances. In stage 2, alliances compete with each other, and in stage 3, the winning alliance members compete with each other for the indivisible prize. The solution concept is the standard subgame perfect Nash equilibrium. Two things should be noted. First, we model the alliance formation process as an "open-membership" game (Yi 1997) in which players can freely choose their alliance without being excluded.[6] This setup can be motivated by examples of geographical concentrations of specialized retails stores such as car dealers (auto rows). In big cities in the United States, car dealers tend to collocate to form auto rows, despite that they must compete with each other, and that they can choose to stand alone in a different location. Consumers are attracted by auto rows since they can find a wide variety of cars at competitive prices, and stand-alone dealers have a hard time surviving.[7] The prosperity of an auto row depends on the number of retail stores and each store's efforts.[8] Car dealers choose their locations freely, knowing that big auto rows attract many customers, but that the dealers there must face fierce competition with neighboring dealers.[9] Second, given the way we set up the multi-stage game, a singleton-only alliance structure and a grand alliance structure are practically identical, since the former does not have the third stage competition, and the latter does

cost. We can interpret these results that an increase in complementarity within groups intensifies group competition.

[6]In a companion paper, Konishi and Pan (2020), we consider a sequential alliance formation game à la Bloch (1996), and compare the resulting alliance structures (see Conclusion section).

[7]See Konishi (2005) for a mechanism of the emergence of concentration of retail stores.

[8]Note that a Tullock contest success function is identical to consumers' logit demand function in a discrete choice model.

[9]Another possible example is competing technologies that have network externalities: A classic instance is the videotape format war between VHS by JVC and Betamax by Sony in the late 1970s. Japanese electric appliance companies chose one of these two technologies (JVC, Panasonic, and RCA for the former, and Sony, Toshiba, and Sanyo for the latter), but VHS won the market against Betamax. The market competititon took place among the winning technology adopters.

not have the second stage competition. The outcome of these two alliance structures coincides with that of a grand standard Tullock contest. Thus, our focus will be finding subgame perfect equilibria with non-trivial alliance structures.

We first analyze the third stage game, which is just a Tullock contest within the winning alliance from stage 2. The rate of rent dissipation increases with the size of alliance increases (Proposition 1). Substituting this as the winning payoff of stage 2, we analyze equilibrium payoffs and strategies in this stage (Theorem 1). Using these building blocks, we analyze values of CES parameter σ when *nontrivial alliance structures* emerge in equilibrium. We show that when the complementarity parameter in CES function is small, there are spin-off incentives for alliance members, while when the complementarity parameter is large, players want to join a bigger alliance and end up with a trivial grand alliance. Therefore, there is no nontrivial equilibrium structure in those ranges (Proposition 2). In order to show the existence of a nontrivial equilibrium alliance structure, we focus on the two-alliance case and provide sufficient conditions for the existence and uniqueness of this type of equilibrium (Theorem 2). We also show that equilibrium alliance structures involve two *similar-sized* alliances.

Moreover, we show that such a similar-sized two-alliance equilibrium allocation always Pareto-dominates the Tullock contest allocation (Theorem 3). That is, nontrivial alliances are not only an equilibrium phenomenon but also provide benefits to their members. Focusing on alliance structures with symmetric (i.e., equally sized) alliances, we also analyze equilibria with more than two alliances. It is shown that symmetric allocations with more alliances achieve higher payoffs, while it is harder to satisfy the equilibrium conditions when the number of alliances increases. We also illustrate how these findings are affected by the level of rent-dissipation in stage 3 by assuming a large population in the contest.

The rest of the paper is organized as follows. In the next subsection, we review the relevant literature. Section 2 introduces the model, and Sections 3 investigates subgames in stages 3 and 2. Employing simple examples, Section 4 illustrates how nontrivial alliance structure is stable when the complementary parameter is moderate. Section 5 presents the results on equilibrium alliance structures. Section 6 considers large population contests and analyzes how the level of rent dissipation affects the stability of alliances. Section 7 concludes, commenting on other alliance formation games.

1.1 Literature Review

There have been attempts to resolve the alliance paradox in Esteban and Sákovics (2003). Konrad and Leininger (2011) consider a dynamic all-pay auction game with possible side payments and endogenous timing of effort in which a group of players (alliance members) fight against a threat from an external enemy player. They show that there is a subgame perfect equilibrium, in which the alliance members exert efficient efforts against the enemy, followed by peaceful side payments from a leader of the alliance to the members in the equilibrium path. In this setup, the free-riding problem and redistributive conflicts are avoided by potential wasteful internal fighting. Konrad and Kovenock (2009) introduce budget constraints for efforts (resources) for each contest in a three-person all-pay auction game, and show that there can be a beneficial alliance for two players with tighter budgets. Konrad (2012) considers an all-pay-auction game in which each player's budget constraint is private information, considering forming an alliance as a tool of information sharing. Assuming a common willingness-to-pay, Konrad (2012) finds that merging alliances is weakly Pareto-improving, and the grand alliance emerges as equilibrium. An asymmetric three-player alliance formation game by Skaperdas (1998) may appear to be the closest to our model, in the sense that he considers complementarity in members' efforts. He shows that alliance formation is beneficial if and only if the effort aggregator function exhibits increasing returns to scale in the members' efforts, but he assumes that players' effort levels are exogenously fixed.[10] In a general symmetric n-player game, Garfinkel (2004) adopts a farsighted solution concept (in the spirit of farsighted stability in Chwe 1994), i.e., a player spins off from an alliance structure only when the eventual outcome after such a move is more preferable than the original alliance structure. With her solution concept, she shows that with a large number of players there are stable alliance structures with similar alliance sizes. In contrast, in our paper, we use the standard subgame perfect Nash equilibrium as the solution concept of our alliance formation game, and derive a stable alliance structure with similar sizes.

There are papers that use a CES aggregator function to capture effort complementarity. In the public good context, Cornes (1993) introduces complementarity in the famous voluntary public good contribution game in Bergstrom, Blume, and Varian (1986). Cornes and Hartley (2007) examine this problem extensively. In contest games, Kolmer and Rommeswinkel (2013) consider a group contest played by exoge-

[10] Tan and Wang (2010) also analyze an asymmetric model with exogenously fixed efforts. In their framework, they show that equilibrium alliance structure has only two alliances with balanced power in a three- or four-player game. Herbst et al. (2015) experimentally study a three-player alliance formation game when the winning alliance members share the prize equally.

nously formed groups using a CES effort-aggregator function when group-members *have heterogeneous abilities*. Assuming that the winning prize is enjoyed by all members of a winning team as a public good, they analyze how effort complementarity affects members' efforts. They find that the complementarity parameter has no effect on equilibrium efforts if groups are homogeneous. If groups are heterogeneous, then the divergence of efforts among group-members and, somewhat surprisingly, the winning probability decreases as the complementarity of efforts goes up, contradicting common intuitions that complementarity of efforts solves the free-riding problem. In contrast, Choi, Chowdhury, and Kim (2016) consider an indivisible private good award à la Esteban and Sákovics (2003) in an exogenous two-group model with two members each, who are heterogeneous in within-group powers. They find that the weaker player may get a higher payoff under effort complementarity. Crutzen and Sahuguet (2018) and Crutzen et al. (2020) compare political party competition with multiple party candidates under different voting systems using CES aggregator functions.

There is literature on contests among exogenously formed groups, concerning how group size and group sharing rules affect incentives to exert efforts (the prize is divisible). In his pioneering work, Olson (1965) argues that due to a free-riding problem in sharing private benefits from the prize with the members of a group, larger groups are less effective at collective effort making than smaller groups. This is the so-called "group-size paradox." Assuming individual efforts are contractable, Nitzan (1991) considers a two-part reward system that combines an egalitarian and a relative-effort-sharing system, and analyzes how the combination affects members' incentives for players in large and small groups. Lee (1995) and Ueda (2002) endogeneize group sharing rules in this class. Esteban and Ray (2001) allow for allocating the prize among the members into public and private benefits (a mixed prize), and show that the group-size paradox disappears even if private benefits are allocated in an egalitarian manner, as long as each member's marginal cost of effort increases at a sufficient speed (a sufficient condition for this is that their cost functions are quadratic). Nitzan and Ueda (2011) show that if private benefits can be allocated by an endogenously chosen relative-effort-sharing rule, then the group–size paradox disappears entirely in their class of effort functions, and larger groups tend to have more egalitarian rules.

Based on the line of research above, Baik and Lee (1997, 2001) endogenize the alliance formation in Nitzan's (1991) game with endogenous group sharing rules, and analyze two- and multiple-alliance cases, respectively. They use open-membership games to describe alliance formation. Bloch et al. (2006) generalize the model substantially to analyze the stability of the grand alliance in different alliance formation games, including a sequential coalition formation game in Bloch (1996), Okada (1996), and Ray and Vohra (2001). Sánchez-Páges (2007a) explores different types of stability concepts, including sequential coalition formation games in alliance formation in contests where efforts are perfect substitutes. Sánchez-Páges (2007b) considers various stability concepts in a model where players allocate endowment into productive and exploitative activities. These papers assume the award is divisible, and alliance members can write a binding contract of sharing rule in the case of the alliance's winning. In our paper, we do not allow for any side payment, and players cannot credibly commit to any intra-alliance distribution rule as in Esteban and Sákovics (2003). We only focus on the benefits of forming a larger group through complementarity of effort and analyze the endogenous formation of alliances in Tullock contests.

2 The Model

There are N players who seek to get an indivisible prize (say, to be the head of an organization). There is no side payment allowed. The set of players is also denoted by $N = \{1, ..., N\}$, and they can form alliances exclusively for the purpose of being the final winner. Each player $i \in N$ can make an effort to enhance the popularity of her alliance and that of herself. We assume that each player has an identical linear cost function $C(e_i) = e_i$ for all $e_i \geq 0$.

Starting from the inter-alliance contest, we introduce potential benefits for players who belong to an alliance—complementarity in aggregating efforts by all alliance members. That is, if player i belongs to alliance j with $N_j \subset N$ as the set of members, and these members make efforts $(e_{hj})_{h \in N_j}$, then the aggregated effort of alliance j, E_j, is described by a CES aggregator function

$$E_j = \left(\sum_{h \in N_j} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}, \tag{1}$$

where $\sigma \in (0, 1)$ is a parameter that describes the degree of complementarity: if $\sigma = 0$, it is a linear aggregator function as in Esteban and Sákovics (2003), and if $\sigma = 1$, it is a Cobb-Douglas function. Thus, as σ goes up, the complementarity of members' efforts increases.[11]

[11]Although a CES function is well-defined and concave for $\sigma \geq 1$, for those σs, an iso-quant curve of a positive aggregate effort does not touch any axis. Thus, when $\sigma \geq 1$, the departure of even a single member will make the aggregate team effort zero. This indispensability of all alliance

Candidate i in alliance j decides how much effort e_{ij} to contribute to her alliance j. The winning probabilities of an alliance is a Tullock-style contest. That is, an alliance j's "winning probability" given its members' efforts is

$$p_j = \frac{E_j}{\sum_{k \in J} E_k}. \tag{2}$$

An indivisible prize is valued as $V > 0$, which is common and normalized to 1 for all players. Since the prize is indivisible, one player in the winning alliance in the second stage must be selected as the final winner in the third-stage contest.

After a winning alliance j is determined, we assume that the final winner is determined by a Tullock contest within the winning alliance members N_j. Denoting the second-stage effort as $\hat{e}_i$, the winning probability of player $i \in N_j$ is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in N_j} \hat{e}_h}. \tag{3}$$

Before the inter- and intra-alliance contests, players form alliances using a specified process. Formally, a partition of the set of players N, $\pi = \{N_1, ..., N_J\}$ is an alliance structure, where each alliance j consists of a set of players N_j, where $\cup_{j \in J} N_j = N$ and $N_{j'} \cap N_j = \emptyset$ for any $j, j' \in J$ with $j \neq j'$. The process considered here is an *open-membership* game (Yi 1997) where players are allowed to (i) freely move from alliance to alliance or (ii) spin-off as a singleton if they want to. Following the literature, we model this process as a "location choice" problem. Suppose there is a finite number of locations. In the first stage, all players simultaneously announce their own location, and those players announcing the same location form an alliance. In equilibrium, no player wants to change their location announcement, i.e., every player prefers her own alliance, given all other players' announcement, foreseeing future inter- and intra-alliance contests.

Since we assume that players are ex-ante homogeneous, we also call $\{n_1, ..., n_J\}$ an alliance structure with $n_j = |N_j|$ for all $j = 1, ..., J$. We consider a dynamic contest game with endogenous alliances: it starts with players forming alliances, then the alliances compete for an indivisible prize in the first contest, and lastly the players in the winning alliance compete with each other to determine the final winner in the second contest. Our dynamic contest game with endogenous alliances has three stages:

members is inconsistent with the free-mobility assumption we adopt. Therefore, we concentrate on the case $0 < \sigma < 1$.

Stage 1. All players $i \in N$ choose one of locations $z_i \in Z$ simultaneously, where the number of locations is at least as many as the number of players $|Z| \geq N$. Players choosing the same integer form an alliance: $N(z) \equiv \{i \in N : z_i = z\}$ for all $z \in \mathbb{Z}$, and a collection of nonempty alliances is an alliance structure $\pi = \{N_j\}_{j=1}^{J}$.

Stage 2. All players $i \in N$ choose effort $e_i \in \mathbb{R}_+$ simultaneously, knowing the aggregated effort of her alliance is (1). The inter-alliance contest is a Tullock contest with winning probabilities equal to (2).

Stage 3. All members of the winning alliance N_j choose effort $\hat{e}_i \in \mathbb{R}_+$ simultaneously. The ultimate winner is selected in a simple Tullock contest with winning probabilities equal to (3).

We use standard subgame perfect Nash equilibrium as the solution of this dynamic game. We consider equilibria in pure strategies only. We will analyze this game by backward induction.

3 Contest Equilibrium in Stage 3 and 2

3.1 Stage 3: Final Contest within the Winning Alliance

In the third stage, all members in the winning alliance N_j in the second stage engage in a Tullock contest by exerting effort $\hat{e}_i \geq 0$. Thus, player i's winning probability is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in N_j} \hat{e}_h}.$$

For any player i in the winning group j, the expected payoff in stage 3 is

$$\tilde{V}_i = \frac{\hat{e}_i}{\hat{e}_i + \sum_{h \neq i} \hat{e}_h} - \hat{e}_i.$$

The first-order condition implies that

$$\frac{1 - p_i}{\hat{e}_i + \sum_{h \neq i} \hat{e}_h} - 1 = 0 \Rightarrow \frac{1}{\hat{e}_i} p_i(1 - p_i) - 1 = 0.$$

Since players are homogeneous, $p_i(1 - p_i) = \frac{n_j - 1}{n_j^2}$ is the same for all i in the winning group j in equilibrium. Then, we have the following proposition.

Proposition 1. *Suppose that the winning alliance of the first stage has size n_j. Then, the third-stage equilibrium strategy and payoff are*

$$\hat{e}_i = \frac{n_j - 1}{n_j^2} \text{ and } \tilde{V}^j = \frac{1}{n_j}\left(1 - \frac{n_j - 1}{n_j}\right) = \frac{1}{n_j^2}.$$

3.2 Stage 2: Contest between Alliances

Consider an inter-alliance contest problem. From Proposition 1, we know that for a given size of alliance n_j the payoff of intra-alliance contest is determined by $\tilde{V}_j = \frac{1}{n_j^2}$. Thus, the second stage maximization problem of a player ij in alliance j is to maximize the payoff

$$\begin{aligned} V_{ij} &= \frac{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + \sum_{j' \neq j} E_{j'}} \tilde{V}_j - e_{ij} \\ &= \frac{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(e_{ij}^{1-\sigma} + \sum_{h \neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + \sum_{j' \neq j} E_{j'}} \frac{1}{n_j^2} - e_{ij}. \end{aligned}$$

The first-order condition with respect to e_{ij} (if an interior solution) is

$$\frac{\left(\sum_{j'} E_{j'} - E_j\right)}{\left(\sum_{j'} E_{j'}\right)^2} e_{ij}^{-\sigma} E_j^{\sigma} \frac{1}{n_j^2} - 1 = 0.$$

This condition implies that all players in an alliance must exert the same amount of effort in equilibrium, and we can write $e_j = e_{ij}$ for all $ij \in N_j$. Therefore, the relevant information for an alliance N_j is summarized in the number of its members, n_j, and the aggregated effort can be written as $E_j = \left(n_j e_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = n_j^{\frac{1}{1-\sigma}} e_j$. Substituting this back into the above condition, we have

$$\frac{\left(\sum_{j' \neq j} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)}{\left(\sum_{j'} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)^2} n_j^{\frac{\sigma}{1-\sigma}} \frac{1}{n_j^2} - 1 = 0,$$

or

$$\frac{\left(\sum_{j'\neq j} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)}{\left(\sum_{j'} n_{j'}^{\frac{1}{1-\sigma}} e_{j'}\right)^2} - n_j^{\frac{2-3\sigma}{1-\sigma}} = 0,$$

for all $j = 1, ..., J$. This is a set of conditions that characterize the first-stage equilibrium if all coalitions exert positive efforts. Using the share function approach from Cornes and Hartley (2005), we convert our second-stage J-alliance team competition to an artificial J-person Tullock contest. We can prove the existence and uniqueness of equilibrium in the second stage under any π.

3.3 Artificial Tullock Contest Game and Share Function

To apply a method called the "share function" approach that is systematically analyzed in Cornes and Hartley (2005), we rewrite the second-stage competition as a Tullock contest with heterogeneous marginal costs.[12] Formally, let $w_j = n_j^{\frac{2-3\sigma}{1-\sigma}}$ (marginal cost) and $x_j = n_j^{\frac{1}{1-\sigma}} e_j$ (effort) for each $j = 1, ..., J$. *An artificial Tullock contest game* $(J, (w_j)_{j=1}^J)$ *corresponding to our second-stage game* is a J-person game in which each player j exerts effort x_j with constant marginal cost $w_j > 0$. Her winning probability is specified by $\pi_j = \frac{x_j}{\sum_{j'=1}^J x_{j'}}$, and her payoff is

$$u_j = \frac{x_j}{\sum_{j'=1}^J x_{j'}} - w_j x_j.$$

The payoff function is strictly concave in x_j, and the first-order condition is

$$\frac{\left(\sum_{j'\neq j} x_{j'}\right)}{\left(\sum_{j'} x_{j'}\right)^2} - w_j = 0,$$

for $j = 1, ..., J$. This set of equations are the (*interior*) first-order conditions for the artificial game that is identical to the set of first-order conditions for the original game. Thus, in order to analyze the properties of the equilibrium in the original game, it suffices to analyze the properties of the corresponding artificial game. To do that, we follow the share function approach in Cornes and Hartley (2005).

[12]Esteban and Ray (2001) and Ueda (2002) used the same method in their papers.

Let $X_{-j} = \sum_{j' \neq j} x_{j'}$. Then, $x_j > 0$ is a unique best response to X_{-j} if and only if

$$x_j^2 + 2X_{-j}x_j + X_{-j}^2 - \frac{X_{-j}}{w_j} = 0.$$

Noting that some players may have too high a marginal cost for an interior solution, player j's best response to X_{-j} is

$$\beta_j(X_{-j}) = \max\left\{-X_{-j} + \sqrt{\frac{X_{-j}}{w_j}}, 0\right\}.$$

We define player j's *replacement function* following Cornes and Hartley (2005): *a replacement function* $r_j(X)$ is a function of total effort $X = \sum_{j'} x_{j'}$ such that $r_j(X)$ is the best response to $X - r_j(X)$: i.e., $r_j(X) = \beta_j(X - r_j(X))$. Thus we obtain

$$r_j(X) = \max\left\{X - w_j X^2, 0\right\}.$$

Let group j's *share function* be $s_j(X) = \frac{1}{X} r_j(X)$:

$$s_j(X) = \max\left\{1 - w_j X, 0\right\}.$$

Note that $s_j(X)$ is a decreasing function in X. Let $s(X) = \sum_{j'} s_{j'}(X)$. This is a decreasing function as well. Order players by $w_1 \leq w_2 \leq .... \leq w_J$. The share function $s(X)$ is a piece-wise linear function with kinks at $\hat{X}^{n_j} = \frac{1}{w_j}$ for each $j = 1, ..., J$. Figure 1 depicts share functions for $j = 1, ..., J$ and $s(X)$. The equilibrium for the artificial contest is a total effort, X^*, for which every group's optimal share sums up to 1. Clearly, there exists a *unique equilibrium* X^* defined by $\sum_{j'} s_{j'}(X^*) = 1$. Moreover, at the equilibrium X^*, $s_j(X^*)$ is also the winning probability of player j. As is easily seen from Figure 1, if $\hat{X}^{n_j} = \frac{1}{w_j} < X^*$, then $s_j(X^*) = 0$ must hold, which means only those groups with smaller marginal costs are *active, i.e., exert positive efforts.* The following lemma summarizes the result of this artificial Tullock game.

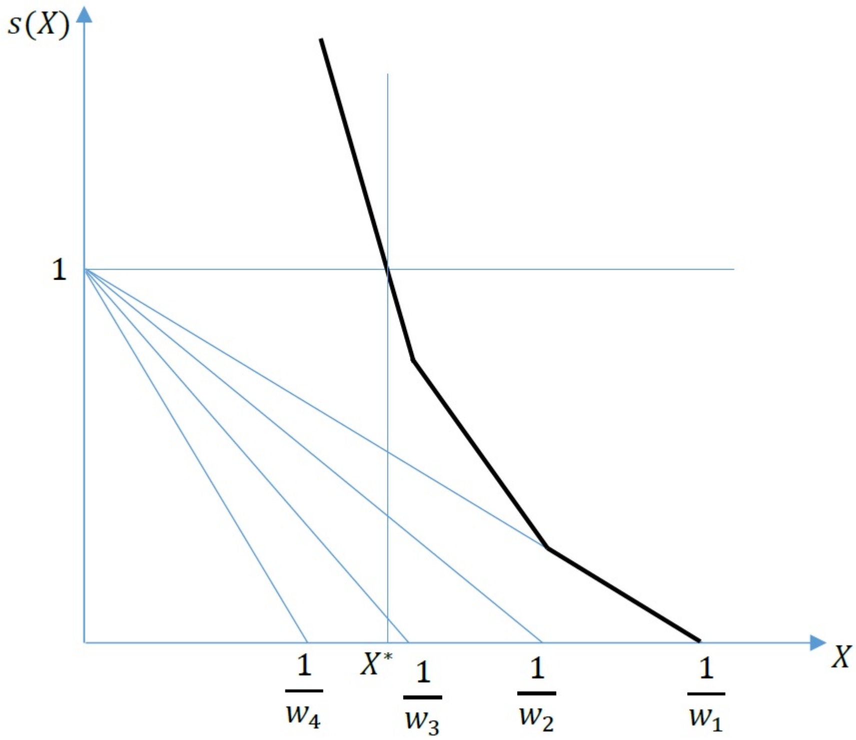

Figure 1: *$s_j(X)$ and $s(X)$ when $J = 4$. For the alliance with large $w_j = n_j^{\frac{2-3\sigma}{1-\sigma}}$, the equilibrium effort is 0, i.e., it is inactive.*

Lemma 1. [Cornes and Hartley, 2005] *An artificial Tullock game has a unique equilibrium X^* at $\sum_j s_j(X^*) = 1$. Moreover, there exists j^* such that, for each $j = 1, ..., j^*$, $x_j = X^* - w_j (X^*)^2$, and for each $j = j^* + 1, ..., J$, $\hat{X}^{n_j} \leq X^*$ (or $\sum_{j'} s_{j'}(\hat{X}^{n_j}) \geq 1$) and $x_j = 0$ hold.*

3.4 Equilibrium in Stage 2

Lemma 1 proves the existence and uniqueness of the equilibrium for the artificial contest and, therefore, for the original game's second stage contest. Given an alliance structure, we obtain the following explicit solutions by considering a special case of Kolmar and Rommeswinkel (2013). All proofs are collected in Appendix A.

Theorem 1. *There exists a unique equilibrium in the second-stage game for any partition of players $\pi = \{n_1, ..., n_j\}$ characterized by the share function $s(X^*) = 1$.*

There is $j^* \in \{1, ..., J\}$ such that $p_j^* = s_j(X^*) > 0$ (active alliance) for all $j \leq j^*$ ($\hat{X}_j > X^*$), while $p_j^* = s_j(X^*) = 0$ (inactive alliance) for all $j > j^*$ ($\hat{X}_j \leq X^*$). Then, the members of alliance $j = 1, ..., J$ obtain payoff

$$u_j = \begin{cases} \frac{1}{n_j^2}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(j^*-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}.$$

Moreover, the equilibrium total efforts are

$$X^* = \frac{j^*-1}{\sum_{j'}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}},$$

and

$$(j^*-1)n_j^{\frac{2-3\sigma}{1-\sigma}} < \sum_{j'}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}$$

holds for all $j = 1, ..., j^*$.

4 Stage 1: Alliance Formation—An Example

Before proceeding to the equilibrium analysis of the first-stage game of this dynamic contest game, we need to clarify the implications of no alliance (alliances are all singletons) and the grand alliance. If each player forms a singleton alliance, $\pi^0 = \{1, ..., 1\}$, and from Theorem 1, the resulting payoff of π^0 is $u^0 = \frac{1}{N^2}$. If players form the grand alliance, then the game will directly proceed to Stage 3, which is just a regular Tullock contest. Thus, having the grand alliance and having no alliance are both "trivial" coalition structures. We denote a single grand alliance structure and its resulting payoff by $\pi^N = \{N\}$ and $u^N = \frac{1}{N^2}$, respectively. To answer the alliance paradox, our analysis is focused on the incentives for players forming stable alliance structures other than π^0 and π^N due to the complementarity we introduced.

In this section, we consider a case of four players. If they do not form an alliance, everybody gets $u^0 = \frac{1}{16}$. Since there are only four identical players, we only need to consider the following coalition structures: (i) $\pi^0 = \{1, 1, 1, 1\}$, (ii) $\pi^1 = \{2, 1, 1\}$, (iii) $\pi^2 = \{2, 2\}$, (iv) $\pi^3 = \{3, 1\}$, and (v) $\pi^N = \{4\}$. Let us denote the payoff of a player in a size n alliance in partition π by $u(n, \pi)$. Since the key parameter in a CES aggregator function is $\sigma \in [0, 1)$, and the complementarity of team efforts increases as σ increases, we consider three values of σ in order: $\sigma = \frac{1}{2}$ (weak complementarity), $\frac{3}{4}$ (moderate complementarity), and $\frac{4}{5}$ (strong complementarity). We investigate how alliance structure is affected by the complementarity of team efforts.

4.1 Weak Complementarity $\sigma = \frac{1}{2}$

In this case, we have $\frac{2-3\sigma}{1-\sigma} = 1$ and $\frac{1-2\sigma}{1-\sigma} = 0$. Using Theorem 1, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{3}{32}$	$\frac{1}{48}$	$\frac{9}{16}$	$\frac{1}{16}$

Note that under π^1 and π^3, smaller alliances perform better than larger ones.

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: This is a Nash equilibrium.
2. $\pi^1 = \{2,1,1\}$: There is a unilateral spin-off from the size 2 alliance, resulting in π^0.
3. $\pi^2 = \{2,2\}$: There is a unilateral spin-off from one of the size 2 alliances, resulting in π^1.
4. $\pi^3 = \{3,1\}$: There is a unilateral spin-off from the size 3 alliance, resulting in π^1.
5. $\pi^N = \{4\}$: There is a unilateral spin-off from the grand alliance, resulting in π^3.

Thus, when $\sigma = \frac{1}{2}$, the complementarity of team efforts is too weak to form a nontrivial alliance.

4.2 Medium Complementarity $\sigma = \frac{3}{4}$

In this case, we have $\frac{2-3\sigma}{1-\sigma} = -1$ and $\frac{1-2\sigma}{1-\sigma} = -2$. Using Theorem 1, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	$\frac{3}{25}$	$\frac{1}{50}$	$\frac{3}{32}$	$\frac{11}{144}$	$\frac{1}{16}$	$\frac{1}{16}$

Note that under π^1 and π^3, larger alliances perform better than smaller ones.

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: Two singletons merge to form an alliance, resulting in π^1.
2. $\pi^1 = \{2,1,1\}$: Two singletons merge to form an alliance, resulting in π^2.
3. $\pi^2 = \{2,2\}$: This is a Nash equilibrium.
4. $\pi^3 = \{3,1\}$: One of the size 3 alliance members moves to merge with a singleton, resulting in π^2.
5. $\pi^N = \{4\}$: This is a Nash equilibrium.

This case allows for two Nash equilibria: a trivial grand alliance equilibrium, and an equally sized two-alliance equilibrium. One important observation is that π^2 Pareto-dominates π^N.

4.3 Strong Complementarity $\sigma = \frac{4}{5}$

In this case, we have $\frac{2-3\sigma}{1-\sigma} = 1$ and $\frac{1-2\sigma}{1-\sigma} = 0$. Using Theorem 1, we know the following:

	$u(1,\pi^0)$	$u(2,\pi^1)$	$u(1,\pi^1)$	$u(2,\pi^2)$	$u(3,\pi^3)$	$u(1,\pi^3)$	$u(4,\pi^N)$
payoff	$\frac{1}{16}$	$\frac{14}{81}$	$\frac{1}{162}$	$\frac{3}{32}$	$\frac{29}{300}$	$\frac{1}{100}$	$\frac{1}{16}$

We analyze which partition can be a Nash equilibrium of stage 1:

1. $\pi^0 = \{1,1,1,1\}$: Two singletons merge to form an alliance, resulting in π^1.
2. $\pi^1 = \{2,1,1\}$: Two singletons merge to form an alliance, resulting in π^2.
3. $\pi^2 = \{2,2\}$: One of the size 2 alliance members moves to the other alliance, resulting in π^3.
4. $\pi^3 = \{3,1\}$: A singleton merges into the size 3 alliance, resulting in π^N.
5. $\pi^N = \{4\}$: This is a Nash equilibrium.

Thus, when $\sigma = \frac{4}{5}$, the trivial grand alliance is the unique Nash equilibrium.

4.4 Observations

The above examples show that when σ is small, there is no gravity to sustain an alliance, since the effort complementarity is not sufficient enough to compensate Olson's inefficiency of alliances.[13] In this case, players prefer standing alone and competing with other single players and/or alliances. In contrast, if σ is large, a larger alliance is always relatively more attractive than a smaller alliance, resulting in the grand alliance. When σ is in the middle range, nontrivial alliances can appear and Pareto-dominate trivial allocation. For nontrivial equilibria, the complementarity is strong enough to make a singleton player unprofitable. At the same time, it is not strong enough that players prefer a smaller group to avoid severe competition in the final stage. These two forces jointly ensure stability. We will show that this is not a coincidence.

5 Two Competing Alliances

We start with the case where the number of (active) alliances is two. We first show necessary conditions for a nontrivial alliance structure to exist. The intuition is straightforward—for a two-alliance structure to be stable, σ cannot be too low or too high. If σ is too high, joining a larger group is always beneficial. If σ is too low, players have incentives to spin off. Formally stated, we have:[14]

Proposition 2. *No two-alliance structure can be stable if* $\sigma \leq \frac{2}{3}$ *or* $\frac{4}{5} \leq \sigma$.

Next, we consider a set of sufficient conditions for the *existence* and *uniqueness* of a two-alliance equilibrium. We argue that when the value of the complementarity parameter is moderate and some *No Spin-off* conditions are satisfied, there is

[13] Skaperdas (1998) shows that forming an alliance is beneficial if and only if the effort aggregator function exhibits increasing returns to scale. However, in his model, players' efforts are exogenously fixed.

[14] From now on, we implicitly assume that there are no inactive alliances in equilibrium, which is in fact consistent with equilibrium behavior. First, note that the players in an inactive alliances have 0 payoffs. Suppose that $\sigma > \frac{2}{3}$. In this case, inactive alliances must have the smallest sizes, if any. Therefore, players in an inactive alliance have incentives to join the largest alliance. This is because, by Theorem 1, the largest alliance remains active after the new entrance and yields a positive payoff. On the other hand, when $\sigma < \frac{2}{3}$, the inactive alliances are the largest. In this case, the players in those inactive alliances have incentive to spin-off since, by Theorem 1, they can enjoy a positive winning probability and payoff. Finally, when $\sigma = \frac{2}{3}$, team sizes are irrelevant, and all teams are active.

a unique two-alliance equilibrium in which the maximal difference in sizes is one. Denoting $t = \frac{3\sigma-2}{1-\sigma}$, we have the following result.

Theorem 2. *There is a* $\bar{\sigma} \in (\frac{3}{4}, \frac{4}{5})$ *such that for all* $\sigma \in (\frac{2}{3}, \bar{\sigma})$,

(i) *when* N *is an even number,* $(\frac{N}{2}, \frac{N}{2}) = (n, n)$ *is the unique two-alliance equilibrium if* $\frac{4n^3}{2n-1}\left[\frac{\max\left\{(n-1)^t+n^t-(n-1)^t n^t, 0\right\}}{(n-1)^t n^t+(n-1)^t+n^t}\right]^2 \leq 1$, *and*

(ii) *when* N *is an odd number,* $(\frac{N-1}{2}, \frac{N+1}{2}) = (n, n+1)$ *is the unique two-alliance equilibrium if*
$\frac{\left((n+1)^t+n^t\right)^2}{n^{t-2}\left((n+1)^t+n^t-n^{-1}(n+1)^t\right)}\left[\frac{\max\left\{(n+1)^t+(n-1)^t-(n-1)^t(n+1)^t, 0\right\}}{(n+1)^t+(n-1)^t+(n-1)^t(n+1)^t}\right]^2 \leq 1$, *and*
$\frac{\left((n+1)^t+n^t\right)^2\left(\max\left\{2-n^t, 0\right\}\right)^2}{(n+1)^{t-2}\left[(n+1)^t+n^t-(n+1)^{-1}n^t\right](n^t+2)^2} \leq 1.$

This theorem confirms the intuition in the examples in Section 4. The two-alliance equilibrium exists, and it is the only two-alliance when complementarity is moderate. Note that the parameter space in Theorem 2 is a proper subset of the one in Proposition 2. However, one might question what the shapes of No Spin-off conditions (i) and (ii) are. We depict those cases in Figure 2. Not surprisingly, those conditions are somewhat stricter than $\sigma > \frac{2}{3}$. However, as we will show in Section 6, those conditions are asymptotically close to $\sigma > \frac{2}{3}$ as N becomes larger and larger.

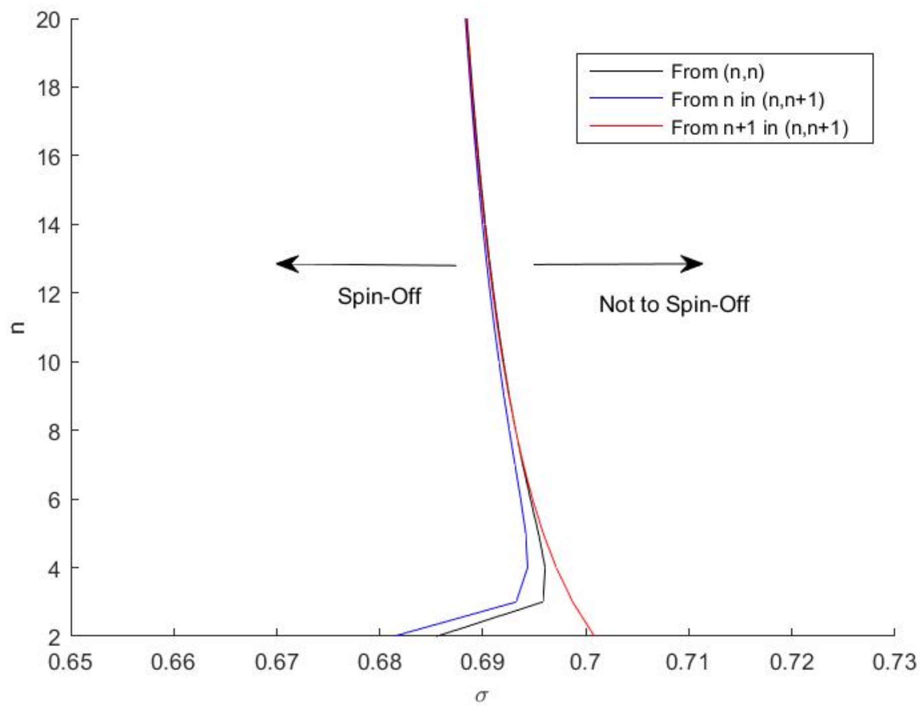

Figure 2: *No Spin-Off Conditions.*

The following theorem shows an important welfare implication of having a chance to form alliances. The emergence of alliances in subgame perfect equilibrium is not only an equilibrium phenomenon (like prisoners' dilemma games), but also a Pareto-improvement for players' welfare, because it has dynamic contests instead of a single round contest.

Theorem 3. *Every two-alliance equilibrium* $\{n_1, n_2\}$ *with* $|n_1 - n_2| \leq 1$ *Pareto-dominates a no-alliance contest outcome.*

5.1 Multi-Alliance Case

Is a symmetric alliance structure, i.e., all alliances are of the same size, stable when $J > 2$? First of all, forming multiple alliances may be welfare-improving. In fact, if the alliances are symmetric, players' welfare improves as the number of alliances increases. Formally,

Proposition 3. *Let symmetric alliance structure π_J be a structure that has $\frac{N}{J} \geq 2$ players in each alliance. If $\pi_{J'}$ and $\pi_{J''}$ with $J'' > J'$ are both equilibrium alliance structures, then $\pi_{J''}$ Pareto dominates $\pi_{J'}$.*

However, the remaining question is whether a multi-alliance structure is stable or not. The benefit from forming a larger alliance is that the new alliance has a higher winning probability in the inter-alliance contest. However, this effect is offset by a stronger intra-alliance competition in the third stage. This winning-probability-enhancement effect is stronger if each alliance only has a smaller number of members and is weaker if the number of alliances is larger. Thus, we expect that, when the number of alliances is more than two, it requires a larger membership in each alliance to be a symmetric equilibrium allocation. This intuition leads us to the following example.

Example 1. Consider the case when $J = 3$, $n = 7$ or 8, and $\sigma = \frac{3}{4}$

$$u(7, \{7, 7, 7\}) = 0.0061548 < u(8, \{8, 6, 7\}) = 0.0061581$$

$$u(8, \{8, 8, 8\}) = 0.0047743 > u(9, \{9, 7, 8\}) = 0.0047736$$

The above example shows that even when the complementarity between players is moderate, a symmetric three-alliance structure is not immune to a unilateral move if $n = 7$. But, a larger membership ($n = 8$) again guarantees stability. In fact, $\sigma = \frac{3}{4}$ is the borderline case for No Symmetry Breaking when $J = 3$, as will be seen in Corollary 1.

Proposition 2 says that there is no stable two-alliance structure if $\sigma \leq \frac{2}{3}$. However, there may be stable alliance structure with many alliances even if $\sigma \leq \frac{2}{3}$ holds. We demonstrate this in the following example, showing that a spin-off may not be profitable when σ is close to $\frac{2}{3}$ and there are many alliances.

Example 2. Suppose $\sigma = \frac{2}{3}$, π being a structure with J n-member alliances, and π' being the structure that one player spins off to form a singleton alliance from π. We can greatly simplify $u(1, \pi')$ and $u(n, \pi)$ in this case:

$$u(1, \pi') = \frac{1}{1}\left[1 - J\frac{1}{J - 1 + 1 + 1}\right]\left[1 - J\frac{1}{J - 1 + 1 + 1}\right] = \frac{1}{(J + 1)^2}$$

$$\begin{aligned} u(n, \pi) &= \frac{1}{n^2}\left[1 - (J - 1)\frac{1}{J}\right]\left[1 - (J - 1)\frac{\frac{1}{n}}{J}\right] \\ &= \frac{1}{n^2}\frac{1}{(J)^2}\left(J - \frac{J - 1}{n}\right). \end{aligned}$$

Note that $u(1, \pi') > u(n, \pi)$ holds for all $n \geq 2$ and all $J \leq 4$; i.e., there exist spin-off incentives, and π cannot be a subgame perfect equilibrium outcome. However, when $J = 5$ and $n = 2$, $u(1, \pi') = \frac{1}{36}$ and $u(2, \pi) = \frac{1}{4}\frac{1}{25}(5 - \frac{4}{2}) = \frac{3}{100} > \frac{1}{36}$, no player has incentives to spin off and form a singleton alliance. Moreover, since the size of an alliance has no effect when $\sigma = \frac{2}{3}$, the payoff of deviating from a two-player alliance and forming a three-player alliance is $\frac{1}{9}\frac{1}{5^2}\left(5 - \frac{4}{3}\right) < \frac{3}{100}$. Therefore, $\{2, 2, 2, 2, 2\}$ is in fact a stable structure. Since payoffs are continuous in σ, this example can be extended to those σs that are close to but smaller than $\frac{2}{3}$. ■

Finally, the following proposition assures that for any number of alliances $J \geq 2$, there is a spin-off incentive for every player who belongs to an alliance, if σ is small enough.

Proposition 4. *Suppose that $\sigma \leq \frac{1}{2}$. Then, from any alliance structure π with a non-singleton alliance, there is a player with an incentive to spin-off to form a singleton alliance.*

Example 1 seems to imply that players have stronger incentives to join a larger group when there are more alliances, and the parameter space for a stable symmetric alliance structure shrinks as the number of alliances increases as a result. In the following section, we analytically confirm this intuition using a heuristic approach that approximates the case with large alliances.

6 Symmetric Alliance Structure with Large Population

In the previous section, we analyzed equilibrium conditions by finding the parameter ranges that discourage forming a larger alliance and satisfy No Spin-Off conditions. In this section, we will try to interpret these conditions in the case of a large population, and thus large alliance sizes. We also generalize our analysis by allowing for different continuation games to observe the relevance of continuation payoffs on the equilibrium alliance structure. Consider the following generalization of Stage 3: After team j wins the inter-alliance competition, the winner of the subsequent inter-alliance competition gets a fraction q as a *private* reward. The remaining fraction $(1 - q)$ is the *public* reward enjoyed by all members on the winning team (Esteban and Ray 2001). Note that if $q = 1$, this corresponds to the original setup. If $q = 0$, then there is no Stage 3 competition. If $0 < q < 1$, it is the *mixed* reward case.

We will show that the generalized model above is equivalent to parameterize the expected continuation payoff for team j's victory as $V(n_j) = 1/n_j^{\delta}$.

Lemma 2. *When the fraction of private reward is* $q \in [0,1]$*, the continuation payoff is uniquely written as*

$$V(n_j) = \frac{1}{n_j^{\delta}},$$

where $\delta = -\ln\left(qn_j^{-2} + (1-q)\right) / \ln n_j$.

That is, if the continuation game is a simple Tullock contest $q = 1$ (private prize), $\delta = 2$ holds. If $\delta = 1$, this means an equal sharing of $V = 1$ without further rent dissipation, and if $1 < \delta < 2$, it can be interpreted as a case *partial rent dissipation within the winning alliance.* If $\delta = 0$ or $q = 0$, this is the public reward case. A slight modification of Theorem 1 covers all of these cases:[15]

Theorem 1'. *Suppose that, in the winning size* n_j *alliance the member's subsequent payoff is* $V(n_j) = \frac{1}{n_j^{\delta}}$*. There exists a unique equilibrium in the second stage game for any partition of players* $\pi = \{n_1, ..., n_J\}$ *characterized by the share function* $s(X^*) = 1$ *and a unique* $j^* \leq J$ *such that players in alliance* $j \leq j^*$ *obtain payoff*

$$u_j = \frac{1}{n_j^{\delta}}\left[1 - (J-1)\frac{n_j^{\delta - \frac{\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\delta - \frac{\sigma}{1-\sigma}}}\right]\left[1 - (J-1)\frac{n_j^{\delta - \frac{1}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\delta - \frac{\sigma}{1-\sigma}}}\right]$$

and alliance j's winning probability is

$$p_j = 1 - (J-1)\frac{n_j^{\delta - \frac{\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\delta - \frac{\sigma}{1-\sigma}}}.$$

First note that an increase in n_j will increase p_j if and only if $\delta - \frac{\sigma}{1-\sigma} < 0$ (or $\sigma > \frac{\delta}{1+\delta}$). It is easy to see that a smaller alliance is always better than a larger alliance, and a singleton alliance is better than any nontrivial alliance, if this condition is violated. Thus, this condition can be considered as the No Spin-Off condition when n_j is large. Second, we investigate how much u_j is affected by one player moving from alliance j' to alliance j when there are J symmetric alliances with

[15] Kolmar and Rommeswinkel (2013) call $n_j^{\frac{\sigma}{1-\sigma}}$ a *social interaction effect,* which can affect the presence of Olson's (1965) group-sized paradox. See Kolmar and Rommeswinkel (2011, 2019) for more discussions on group-sized paradox with a CES aggregator producing group efforts.

large populations. With the first-order approximation, we can show that u_j does not increase with such a move if and only if $\frac{\sigma}{1-\sigma} \leq \frac{J}{J-1}\delta$. This condition is equivalent to the one where a player does not gain by moving from one alliance to the other when the two alliances have the same size (*No Symmetry Breaking condition*), and there is no snowball effect by breaking a symmetry structure.

Proposition 5. *Suppose that the size n_j alliance members' winning payoff is $V(n_j) = \frac{1}{n^\delta}$. Then, when the* population *is large enough, an alliance structure with J symmetric alliance structure is stable if (i) $\sigma > \frac{\delta}{1+\delta}$ (No Spin-Off), and (ii) $\sigma \leq \frac{J\delta}{J-1+J\delta} = \frac{\delta}{1+\delta-\frac{1}{J}}$ (No Symmetry Breaking). Moreover, a $J+1$ symmetric alliance structure Pareto-dominates a J symmetric alliance structure.*

Our benchmark case corresponds to $\delta = 2$.

Corollary 1. *Let $\delta = 2$. Then, when the population is large enough, an alliance structure with J symmetric alliances is stable if $\frac{2}{3} < \sigma \leq \frac{2}{3-\frac{1}{J}}$.*

When $J = 2$, the limit condition is written as $\frac{2}{3} < \sigma \leq \frac{4}{5}$, and when $J = 3$, it is $\frac{2}{3} < \sigma \leq \frac{3}{4}$. These conditions correspond to the necessary condition in Proposition 2 and Example 1. This is also shown in the following graph, in which we depict the parameter space for stable symmetric two- and three-alliance structures.

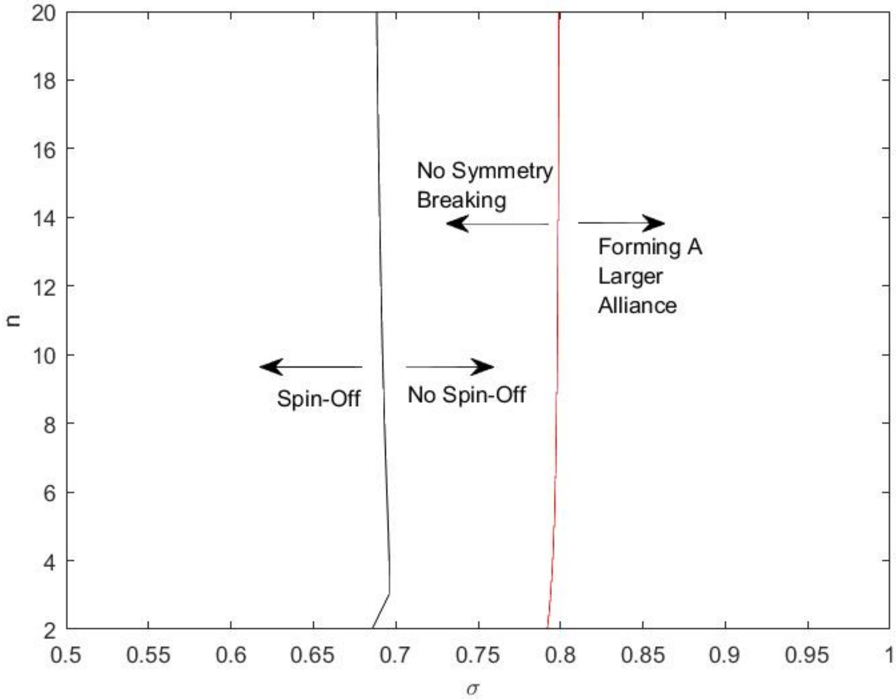

Figure 3: *The stability of a symmetric two-alliance structure.*

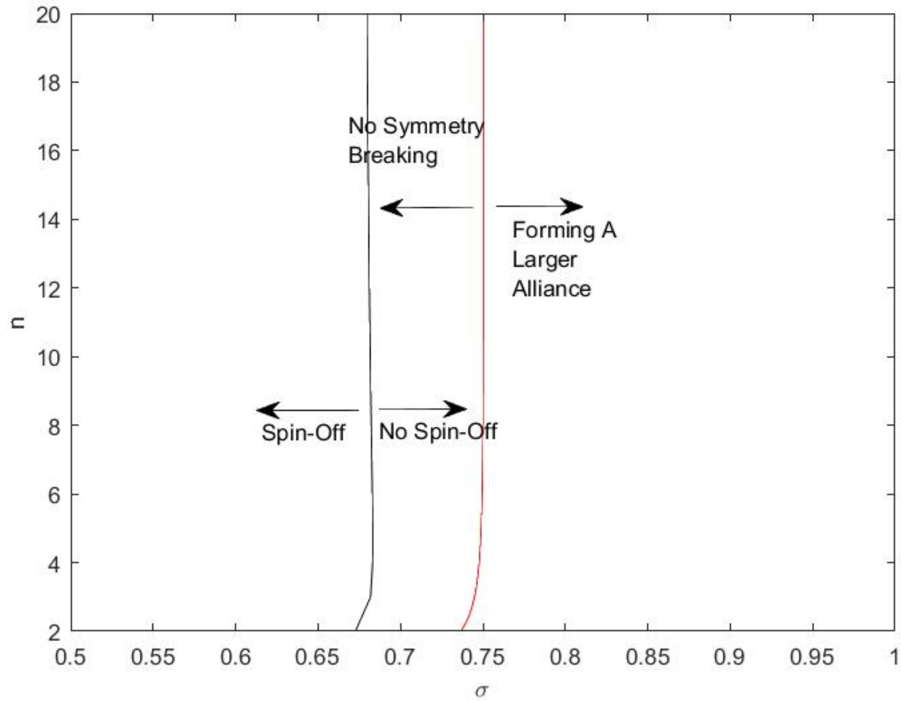

Figure 4: *The stability of a symmetric three-alliance structure.*

If $\delta = 1$ and $J = 2$, then the limit conditions (i) and (ii) in Proposition 5 become $\frac{1}{2} \leq \sigma \leq \frac{2}{3}$, so smaller values of σ achieve stable alliance structures. If the rent dissipation in stage 3 is milder than the simple Tullock contest, such as partial prize sharing ($1 < \delta < 2$) with $J = 2$, then the values of σ for stability are somewhere in between.

For each value of δ, the values of σ that support the stability of J symmetric alliance structure are $\frac{\delta}{1+\delta} < \sigma < \frac{\delta}{1+\delta-\frac{1}{J}}$. Thus, as J goes up, the parameter range of σ for stable alliance structures shrinks, although players' expected payoffs increase.

7 Concluding Remarks

In this paper, we used a CES effort aggregator function to describe incentives to form alliances by effort complementarity, and we show that there exist stable alliances in an *open-membership* two-stage alliance formation game when the effort complementarity is moderately strong. When complementarity is too strong, alliances become too attractive, and all players end up forming a grand alliance, which simply defers the noncooperative contest by one period.

There are alternative alliance formation games in the literature (see Hart and Kurz 1983). Using a noncooperative game approach, Bloch (1996), Okada (1996), and Ray and Vohra (1999) consider an interesting sequential coalition formation game. In a companion paper, Konishi and Pan (2020), we study equilibrium alliance structure by adopting their game: In this game, the alliance formation stage has multiple steps, and a player proposes an alliance at each step, and if all called upon members agree to form a group, then an alliance is formed, and multiple alliances are formed sequentially. That is, these alliances can exclude outsiders in this alternative setup. Allowing for side payments, Bloch et al. (2006) consider a sequential alliance formation game in contests, which allows alliances to limit their memberships (exclusion), and show that the grand alliance would be formed by sharing the prize peacefully. However, in our game without side payments (an indivisible prize), the grand alliance would not be formed, since this is identical to not forming an alliance. We show that there is always a subgame perfect equilibrium and that there can be at most two alliances in equilibrium, one large and one small, without any fringe players (all players belong to one of the two alliances) if the complementarity parameter σ is large enough. In this case, the large alliance is formed first, and the leftover players form the second smaller alliance, and the former alliance achieves higher payoffs than the latter.[16]

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Appendix A (Proofs)

Proof of Theorem 1. The artificial game we constructed has the same first-order conditions as the original first-stage game. This implies that j^* is uniquely defined, as in the statement of Lemma 1, only $j = 1, ..., j^*$ exert efforts in equilibrium. Since $p_j^* = 1 - \frac{\sum_{j' \neq j} x_{j'}}{\sum_{j'=1}^{j^*} x_{j'}}$, the first-order conditions can be written as

$$\frac{\left(1 - p_j^*\right)}{\left(\sum_{j'=1}^{j^*} x_{j'}\right)} - n_j^{\frac{2-3\sigma}{1-\sigma}} = 0$$

or

$$1 - p_j^* = \left(\sum_{j'=1}^{j^*} x_{j'}\right) n_j^{\frac{2-3\sigma}{1-\sigma}}.$$

Summing up the above from $j = 1$ to j^*, we have

$$j^* - 1 = \left(\sum_{j'=1}^{j^*} x_{j'}\right) \sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}.$$

Eliminating $\frac{\sum_{j'=1}^{j^*} x_{j'}}{V}$ from the first-order condition, we obtain:

$$p_j^* = 1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}.$$

Since $p_j \left(\sum_{j'} x_{j'}\right) = x_j$, we have

$$x_j = \left[1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \frac{j^* - 1}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}.$$

Notice that $x_j = n_j^{\frac{1}{1-\sigma}} e_j$, which means the equilibrium e_j in the original problem is

$$e_j = \frac{1}{n_j^{\frac{1}{1-\sigma}}} \left[1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \frac{j^* - 1}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}.$$

Therefore, the equilibrium payoff of the original problem is

$$\begin{aligned}
u_j &= p_j^* \tilde{V}_j - e_j \\
&= \left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\frac{1}{n_j^2} - \left[\frac{1}{n_j^{\frac{1}{1-\sigma}}}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\frac{j^*-1}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{1}{n_j^2} - \frac{1}{n_j^{\frac{1}{1-\sigma}}}\frac{(j^*-1)}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \frac{1}{n_j^2}\left[1-(j^*-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(j^*-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right].
\end{aligned}$$

We completed the proof.■

Proof of Proposition 2

This proposition can be proved by the following three lemmas.

Lemma A1. *For any two-alliance structure $\pi = (n_1, n_2)$ with $n_1 \geq n_2 \geq 2$, it is beneficial to move to the larger group (group 1) from the smaller group whenever $\sigma = \frac{4}{5}$.*

Proof. Suppose that the initial structure is $\pi = (n_1, n_2)$, and if a player in alliance 2 moves to 1, $\pi' = (n_1 + 1, n_2 - 1)$ realizes. Recall

$$u(n_2, \pi) = \frac{1}{n_2^2}\left(1 - \frac{\frac{1}{n_2^2}}{\frac{1}{n_1^2}+\frac{1}{n_2^2}}\right)\left(1 - \frac{\frac{1}{n_2^3}}{\frac{1}{n_1^2}+\frac{1}{n_2^2}}\right) = \left(\frac{1}{n_1^2+n_2^2}\right)\left(\frac{n_1^2+n_2^2-n_1^2 n_2^{-1}}{n_1^2+n_2^2}\right).$$

We will compare $u(n_1+1,\pi')$ with $u(n_2,\pi)$.

$$
\begin{aligned}
& u(n_1+1,\pi') - u(n_2,\pi) \\
= \; & \left(\frac{1}{(n_1+1)^2+(n_2-1)^2}\right)\left(\frac{(n_1+1)^2+(n_2-1)^2-(n_2-1)^2(n_1+1)^{-1}}{(n_1+1)^2+(n_2-1)^2}\right) \\
& -\left(\frac{1}{n_1^2+n_2^2}\right)\left(\frac{n_1^2+n_2^2-n_1^2n_2^{-1}}{n_1^2+n_2^2}\right) \\
= \; & \left(\frac{1}{(n_1+1)^2+(n_2-1)^2}\right)-\left(\frac{1}{n_1^2+n_2^2}\right)-\left(\frac{(n_2-1)^2(n_1+1)^{-1}}{\left((n_1+1)^2+(n_2-1)^2\right)^2}\right)+\left(\frac{n_1^2n_2^{-1}}{(n_1^2+n_2^2)^2}\right) \\
= \; & \frac{1}{n_2(n_1+1)\left((n_1+1)^2+(n_2-1)^2\right)^2(n_1^2+n_2^2)^2}\times \\
& \big[n_1^7-2n_1^6n_2+4n_1^5n_2^2-5n_1^4n_2^3+5n_1^3n_2^4-4n_1^2n_2^5+2n_1n_2^6-n_2^7 \\
& +5n_1^6-12n_1^5n_2+18n_1^4n_2^2-20n_1^3n_2^3+17n_1^2n_2^4-8n_1n_2^5+4n_2^6 \\
& +12n_1^5-27n_1^4n_2+28n_1^3n_2^2-26n_1^2n_2^3+16n_1n_2^4-7n_2^5 \\
& +16n_1^4-28n_1^3n_2+16n_1^2n_2^2-12n_1n_2^3+8n_2^4+12n_1^3-12n_1^2n_2-4n_2^3+4n_1^2\big].
\end{aligned}
$$

The contents of the bracket can be rewritten as follows:

$$
\begin{aligned}
[\cdot] \; = \; & (n_1-n_2)\left(n_1^2+n_2^2-n_1n_2\right)\left(n_1^2+n_2^2\right)^2 \\
& +n_1(n_1-n_2)\left(n_1(n_1-n_2)\left(5n_1^2-2n_1n_2+9n_2^2\right)+8n_2^4\right)+4n_2^6 \\
& +12n_1^5-27n_1^4n_2+28n_1^3n_2^2-26n_1^2n_2^3+16n_1n_2^4-7n_2^5+4n_2^5-4n_2^5 \\
& +16n_1^4-28n_1^3n_2+16n_1^2n_2^2-12n_1n_2^3+8n_2^4+12n_1^3-12n_1^2n_2-4n_2^3+4n_1^2 \\
= \; & (n_1-n_2)\Big\{\left(n_1^2+n_2^2-n_1n_2\right)\left(n_1^2+n_2^2\right)^2+n_1\left(n_1(n_1-n_2)\left(5n_1^2-2n_1n_2+9n_2^2\right)+8n_2^4\right)\Big\}+ \\
& +\left(10n_1n_2^2-3n_1^2n_2+12n_1^3-3n_2^3\right)(n_1-n_2)^2-4n_2^5 \\
& +4\left(4n_1^2+2n_2^2+n_1n_2\right)(n_1-n_2)^2+12n_1^3-12n_1^2n_2-4n_2^3+4n_1^2 \\
= \; & (n_1-n_2)\Big[\left(n_1^2+n_2^2-n_1n_2\right)\left(n_1^2+n_2^2\right)^2+n_1\left\{n_1(n_1-n_2)\left(5n_1^2-2n_1n_2+9n_2^2\right)+8n_2^4\right\}\Big] \\
& +(n_1-n_2)^2\left(10n_1n_2^2-3n_1^2n_2+12n_1^3-3n_2^3\right)+4(n_1-n_2)^2\left(4n_1^2+2n_2^2+n_1n_2\right) \\
& +4n_2^6-4n_2^5+12n_1^3-12n_1^2n_2-4n_2^3+4n_1^2.
\end{aligned}
$$

Since $n_1 \geq n_2$ holds, if we can show $4n_2^6-4n_2^5+12n_1^3-12n_1^2n_2-4n_2^3+4n_1^2 >$

0, $[\cdot] > 0$ holds. Rewriting this, we have

$$\begin{aligned}
& 4n_2^6 - 4n_2^5 + 12n_1^3 - 12n_1^2 n_2 - 4n_2^3 + 4n_1^2 \\
= \; & 4\left[n_2^6 - n_2^5 + 3n_1^2 (n_1 - n_2) - n_2^3 + n_2^2 - n_2^2 + n_1^2\right] \\
= \; & 4\left[n_2^6 - n_2^5 - n_2^3 + n_2^2 + 3n_1^2 (n_1 - n_2) + (n_1 - n_2)(n_1 + n_2)\right] \\
= \; & 4\left[n_2^2 (n_2 - 1)^2 \left(n_2^2 + n_2 + 1\right) + 3n_1^2 (n_1 - n_2) + (n_1 - n_2)(n_1 + n_2)\right] > 0.
\end{aligned}$$

We have completed the proof.□

Next, we argue that $u(n_1 + 1, \pi') - u(n_2, \pi) > 0$ for not only $\sigma = \frac{4}{5}$ also all $\sigma \geq \frac{4}{5}$.

Lemma A2. *For any two-alliance structure $\pi = (n_1, n_2)$ with $n_1 \geq n_2 \geq 2$, $\frac{u(n_1+1,\pi')}{u(n_2,\pi)}$ is an increasing function in σ.*

Proof. Let $t = \frac{3\sigma-2}{1-\sigma}$. Then, we have

$$\begin{aligned}
\frac{u_1(n_1+1, n_2-1)}{u_2(n_1, n_2)} \; = \; & \left(\frac{n_1+1}{n_2}\right)^{t-2} \left[\frac{n_1^t + n_2^t}{(n_1+1)^t + (n_2-1)^t}\right]^2 \\
& \times \left[\frac{(n_1+1)^t + (n_2-1)^t - (n_2-1)^t (n_1+1)^{-1}}{n_1^t + n_2^t - n_1^t n_2^{-1}}\right].
\end{aligned}$$

Let $M = \left(\frac{n_1+1}{n_2}\right)^{t-2} \left(\frac{n_1^t+n_2^t}{(n_1+1)^t+(n_2-1)^2}\right)$, and $L = \left(\frac{n_1^t+n_2^t}{(n_1+1)^t+(n_2-1)^2}\right)\left(\frac{(n_1+1)^t+(n_2-1)^t-(n_2-1)^t(n_1+1)^{-1}}{n_1^t+n_2^t-n_1^t n_2^{-1}}\right)$. Moreover, let "$\simeq$" stand for "has the same sign as." Then,

$$\begin{aligned}
\frac{\partial M}{\partial t} \; \simeq \; & n_1^t \ln(n_1)(n_1+1)^t + n_1^t \ln(n_1)(n_2-1)^t + n_2^t \ln(n_2)(n_1+1)^t + n_2^t \ln(n_2)(n_2-1)^t \\
& - n_1^t \ln(n_1+1)(n_1+1)^t - n_1^t \ln(n_2-1)(n_2-1)^t - n_2^t \ln(n_1+1)(n_1+1)^t \\
& - n_2^t \ln(n_2-1)(n_2-1)^t + n_1^t \ln\left(\frac{n_1+1}{n_2}\right)(n_1+1)^t + n_1^t \ln\left(\frac{n_1+1}{n_2}\right)(n_2-1)^t \\
& + n_2^t \ln\left(\frac{n_1+1}{n_2}\right)(x+1)^t + n_2^t \ln\left(\frac{n_1+1}{n_2}\right)(n_2-1)^t \\
= \; & n_1^t (n_1+1)^t \ln\left(\frac{n_1}{n_2}\right) + n_1^t (n_2-1)^t \ln\left(\frac{n_1(n_1+1)}{(n_2-1)n_2}\right) + n_2^t (n_2-1)^t \ln\left(\frac{n_1+1}{n_2-1}\right) > 0.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\frac{\partial L}{\partial t} \;\simeq\;& -{n_1}^{2\,t}\,\ln(n_1+1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t+{n_1}^{2\,t}\,\ln(n_2-1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&+{n_1}^t\,{n_2}^t\,\ln(n_1)\,\left(n_1+1\right)^{2\,t}-{n_1}^t\,{n_2}^t\,\ln(n_2)\,\left(n_1+1\right)^{2\,t} \\
&+{n_1}^t\,{n_2}^t\,\ln(n_1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t-{n_1}^t\,{n_2}^t\,\ln(n_2)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&+n_1\,{n_1}^t\,{n_2}^t\,\ln(n_1)\,\left(n_1+1\right)^{2\,t}+n_1\,{n_1}^t\,{n_2}^t\,\ln(n_1)\,\left(n_2-1\right)^{2\,t} \\
&-n_1\,{n_1}^t\,{n_2}^t\,\ln(n_2)\,\left(n_1+1\right)^{2\,t}-n_1\,{n_1}^t\,{n_2}^t\,\ln(n_2)\,\left(n_2-1\right)^{2\,t} \\
&-{n_1}^t\,{n_2}^t\,\ln(n_1+1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t+{n_1}^t\,{n_2}^t\,\ln(n_2-1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&+{n_1}^{2\,t}\,n_2\,\ln(n_1+1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t+n_2\,{n_2}^{2\,t}\,\ln(n_1+1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&-{n_1}^{2\,t}\,n_2\,\ln(n_2-1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t-n_2\,{n_2}^{2\,t}\,\ln(n_2-1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&+2\,n_1\,{n_1}^t\,{n_2}^t\,\ln(n_1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t-2\,n_1\,{n_1}^t\,{n_2}^t\,\ln(n_2)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&+2\,{n_1}^t\,n_2\,{n_2}^t\,\ln(n_1+1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t-2\,{n_1}^t\,n_2\,{n_2}^t\,\ln(n_2-1)\,\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&= -n_1^{2t}\left(n_1+1\right)^t\left(n_2-1\right)^t\left[\ln\left(n_1+1\right)-\ln\left(n_2-1\right)+n_2\ln\left(n_2-1\right)-n_2\ln\left(n_1+1\right)\right] \\
&-n_1^t n_2^t\left(n_1+1\right)^{2t}\left[\ln\left(n_2\right)-\ln\left(n_1\right)-n_1\ln\left(n_1\right)+n_1\ln\left(n_2\right)\right] \\
&-n_1^t n_2^t\left(n_2-1\right)^{2t}n_1\left[\ln\left(n_2\right)-\ln\left(n_1\right)\right] \\
&-n_1^t n_2^t\left(n_1+1\right)^t\left(n_2-1\right)^t \\
&\quad\times\left[\ln\left(n_2\right)-\ln\left(n_1\right)+\ln\left(n_1+1\right)-\ln\left(n_2-1\right)+2n_1\ln\left(n_2\right)\right. \\
&\quad\left.-2n_1\ln\left(n_1\right)+2n_2\ln\left(n_2-1\right)-2n_2\ln\left(n_1+1\right)\right] \\
&-n_2 n_2^{2t}\left(n_1+1\right)^t\left(n_2-1\right)^t\left[\ln\left(n_2-1\right)-\ln\left(n_1+1\right)\right] \\
=\;& -n_1^{2t}\left(n_1+1\right)^t\left(n_2-1\right)^t\left[\left(n_2-1\right)\left(\ln\left(\frac{n_2-1}{n_1+1}\right)\right)\right] \\
&-n_1^t n_2^t\left(n_1+1\right)^{2t}\left[\left(n_1+1\right)\ln\left(\frac{n_2}{n_1}\right)\right]+n_1^{t+1}n_2^t\left(n_2-1\right)^{2t}\ln\left(\frac{n_2}{n_1}\right) \\
&-n_1^t n_2^t\left(n_1+1\right)^t\left(n_2-1\right)^t\left[\left(2n_1-1\right)\ln\left(\frac{n_2}{n_1}\right)+\left(2n_2-1\right)\ln\left(\frac{n_2-1}{n_1+1}\right)\right] \\
&-n_2 n_2^{2t}\left(n_1+1\right)^t\left(n_2-1\right)^t\ln\left(\frac{n_2-1}{n_1+1}\right) \\
>\;& 0.
\end{aligned}
$$

Therefore, $\frac{\partial L}{\partial t}>0$. Since $\frac{u(n_1+1,\pi')}{u(n_2,\pi)}=M\times L$, $\frac{u(n_1+1,\pi')}{u(n_2,\pi)}$ is an increasing function in $t=\frac{3\sigma-2}{1-\sigma}$ and σ as well. Combined with Lemma A1, we have that $\frac{u(n_1+1,\pi')}{u(n_2,\pi)}$ for all $\sigma\geq\frac{4}{5}$.□

The next lemma shows that when $\sigma\leq\frac{2}{3}$, players have incentives to spin off. Formally,

Lemma A3. *For any two-alliance structure $\pi = (n_1, n_2)$ with $n_1 \geq n_2 \geq 2$, it is beneficial to spin off from the larger group whenever $\sigma \leq \frac{2}{3}$.*

Proof. Note that the payoff in the size-n_1 group is

$$\begin{aligned} u(n_1, \pi) &= \left[\frac{1}{n_1^2}\right]\left[\frac{n_1^t}{n_1^t + n_2^t}\right]\left[\frac{n_1^t + n_2^t - n_2^t n_1^{-1}}{n_1^t + n_2^t}\right] \\ &\leq \left[\frac{1}{n_1^2}\right]\left[\frac{n_1^t}{n_1^t + n_2^t}\right] \leq \frac{1}{4}\left[\frac{n_1^t}{n_1^t + n_2^t}\right]. \end{aligned}$$

Let $\pi'' = (1, n_1 - 1, n_2)$. Then we have

$$u(1, \pi'') = \left[\frac{(n_1-1)^t + n_2^t - (n_1-1)^t n_2^t}{(n_1-1)^t + n_2^t + (n_1-1) n_2^t}\right]^2.$$

Note that

$$\begin{aligned} & 2\left[\frac{(n_1-1)^t + n_2^t - (n_1-1)^t n_2^t}{(n_1-1)^t + n_2^t + (n_1-1) n_2^t}\right] - \left[\frac{n_1^t}{n_1^t + n_2^t}\right] \\ =& \frac{{n_1}^t (n_1-1)^t + 2\,{n_2}^t (n_1-1)^t + 2\,{n_2}^{2\,t} - 2\,{n_2}^{2\,t} (n_1-1)^t + {n_1}^t {n_2}^t - 3\,{n_1}^t {n_2}^t (n_1-1)^t}{\left({n_2}^t + {n_2}^t (n_1-1)^t + (n_1-1)^t\right)\left({n_1}^t + {n_2}^t\right)} \\ >& 0. \end{aligned}$$

The last inequality holds because $(n_1 - 1)^t \leq 1$ whenever $n_1 \geq 2$ and $t \leq 0$, i.e., $\sigma \leq \frac{2}{3}$. It implies $u(1, \pi'') > u(n_1, \pi)$.□

By Lemma A1 and A2, we know that $\frac{u(n_1+1,\pi')}{u(n_2,\pi)} > 1$ at all $\sigma \geq \frac{4}{5}$. So, there are incentives to move to a larger group for all $\sigma \geq \frac{4}{5}$. By Lemma A3, we know that $\frac{u(1,\pi'')}{u(n_1,\pi)} > 1$ at all $\sigma \leq \frac{2}{3}$. Therefore, there are incentives to spin off from the larger group in any 2-alliance structure. Those two facts together imply that for a 2-alliance structure to be stable, it must be the case of $\frac{2}{3} < \sigma < \frac{4}{5}$.■

Proof of Theorem 2

To prove the theorem, we need the following lemma.

Lemma A4. *When $J = 2$, there is $\bar{\sigma} \in (\frac{3}{4}, \frac{4}{5})$ such that for all $\sigma \in (\frac{2}{3}, \bar{\sigma})$, the following statements hold: (i) Players in the smaller alliance do not have an incentive to move to a larger alliance. (ii) When alliance sizes are equal, players do not move to create a larger alliance. (iii) Players in a larger alliance have an incentive to move to the smaller one.*

Proof of Lemma A4. Consider $\pi = (n_1, n_2)$ and $\pi' = (n_1 + 1, n_2 - 1)$ with $n_1 \geq n_2 \geq 2$. All three statements above are equivalent to

$$\frac{u(n_1 + 1, \pi')}{u(n_2, \pi)} < 1.$$

Suppose there is a $\bar{\sigma}$ such that at $\sigma = \bar{\sigma}$, $\frac{u(n_1+1,\pi')}{u(n_2,\pi)} < 1$. By Lemma A1 and A2, we know that (a) $\bar{\sigma} < \frac{4}{5}$ and (b) $\frac{u(n_1+1,\pi')}{u(n_2,\pi)} < 1$ holds for all σ with $\frac{2}{3} < \sigma < \bar{\sigma}$. It remains to show that $\bar{\sigma} > \frac{3}{4}$. For computational purposes, let $n_1 = n + d + 1$ and $n_2 = n + 1$ with $n \geq 1$. Note that $n_1 \geq n_2$ is equivalent to $d \geq 0$. Consider the case with $\sigma = \frac{3}{4}$. We have

$$\begin{aligned}
u(n_2, \pi) &= \frac{1}{(n+1)^2} \left[\frac{n+1}{(n+d+1)+(n+1)}\right] \left[1 - \frac{1}{n+1} \frac{n+d+1}{(n+d+1)+(n+1)}\right] \\
&= \frac{1}{(n+1)^2} \left[\frac{n+1}{(n+d+1)+(n+1)}\right] \left[\frac{(n+d+1)+(n+1) - \frac{n+d+1}{n+1}}{(n+d+1)+(n+1)}\right] \\
&= \frac{1}{(n+1)^3} \left[\frac{n+1}{2n+d+2}\right] \left[n + \frac{n+1}{2n+d+2}\right] \\
&> \frac{1}{(n+1)^3} \left[\frac{(n+d+1)^t}{(n+d+1)^t + (n+1)^t}\right] \left[n + \frac{(n+d+1)^t}{2(n+1)^t}\right] \\
&= \frac{1}{(n+1)^3} P_0(t) \left[n + \frac{(n+d+1)^t}{2(n+1)^t}\right], \qquad (4)
\end{aligned}$$

and similarly

$$\begin{aligned}
u(n_1 + 1, \pi') &= \frac{1}{(n+d+2)^2} \left[\frac{(n+d+2)}{(n+d+2)+n}\right] \left[1 - \frac{(n+d+2)^{t-1}}{(n+d+2)^t + n^t}\right] \\
&< \frac{1}{(n+d+2)^2} \left[\frac{n+d+2}{2n+d+2}\right] \\
&= \frac{1}{(n+d+2)^2} P_1(t). \qquad (5)
\end{aligned}$$

Therefore,

$$\frac{u(n_1+1,\pi')}{u(n_2,\pi)} < \frac{\frac{1}{(n+d+2)^2}\left[\frac{n+d+2}{2n+d+2}\right]}{\frac{1}{(n+1)^3}\left[\frac{n+1}{2n+d+2}\right]\left[n+\frac{n+1}{2n+d+2}\right]}$$
$$= \frac{(n+1)^3}{(n+d+2)^2}\left[\frac{n+d+2}{n+1}\right]\left[\frac{2n+d+2}{n(2n+d+2)+(n+1)}\right]$$
$$= \frac{(n+1)^2}{n+d+2}\left[\frac{2n+d+2}{2n^2+(d+3)n+1}\right]$$
$$= \frac{2n^3+(d+6)n^2+(2d+6)n+d+2}{2n^3+(3d+7)n^2+\left[(d+2)(d+3)+1\right]n+d+2} < 1.$$

So, (i), (ii), and (iii) hold for $\sigma = \frac{3}{4}$. As a result, there exists $\bar{\sigma} > \frac{3}{4}$ such that $\frac{u(n_1+1,\pi')}{u(n_2,\pi)} < 1$ for all $\sigma \in (\frac{2}{3}, \bar{\sigma})$, all $d \geq 0$, and all $n \geq 1$.$\square$

From Lemma A1, for any coalition structure with $|n_1 - n_2| > 1$, a player in a larger alliance moves to the smaller one. This cannot be an equilibrium alliance structure. As a result, the only alliance structure that is immune to moving incentives is the one with $|n_1 - n_2| \leq 1$.

To complete the proof, we calculate the conditions such that plays do not have incentives to spin off from the above alliance structures. Since in a stable alliance structure we have $|n_1 - n_2| \leq 1$, we have only two possibilities (i) $(n_1, n_2) = (n, n)$, and (ii) $(n_1, n_2) = (n+1, n)$. We start with (i) $(n_1, n_2) = (n, n)$: Players' payoff in this alliance structure is

$$u(n, \{n, n\}) = \frac{1}{n^2}\frac{1}{2}\left[1 - \frac{1}{2n}\right] = \frac{2n-1}{4n^3},$$

and a spun-off player's payoff is

$$u(1, \{n, n-1, 1\}) = \left[\max\left\{1 - \frac{2}{1+(n-1)^{-t}+n^{-t}}, 0\right\}\right]^2 = \left[\frac{\max\left\{(n-1)^t + n^t - (n-1)^t n^t, 0\right\}}{(n-1)^t n^t + (n-1)^t + n^t}\right]^2,$$

since $(n-1)^t + n^t < (n-1)^t n^t$ then the spun-off player becomes inactive, obtaining zero payoff. In this case, the no spin-off condition is

$$\frac{u(1, \{n, n-1, 1\})}{u(n, \{n, n\})} = \frac{4n^3}{2n-1}\left[\frac{\max\left\{(n-1)^t + n^t - (n-1)^t n^t, 0\right\}}{(n-1)^t n^t + (n-1)^t + n^t}\right]^2 \leq 1.$$

Now, case (ii). This case is more cumbersome, since a player can spin off from both alliances. We need to consider two possible spin-off subcases $\frac{u(n+1,\{n+1,n\})}{u(1,\{n,n,1\})} \geq 1$ and $\frac{u(n,\{n+1,n\})}{u(1,\{n+1,n-1,1\})} \geq 1$.

$$\begin{aligned}
u(n,\{n,n+1\}) &= \frac{1}{n^2}\left(1-\frac{\frac{1}{n^t}}{\frac{1}{n^t}+\frac{1}{(n+1)^t}}\right)\left(1-\frac{\frac{1}{n^{t+1}}}{\frac{1}{n^t}+\frac{1}{(n+1)^t}}\right) \\
&= \frac{n^{t-2}\left((n+1)^t+n^t-n^{-1}(n+1)^t\right)}{\left((n+1)^t+n^t\right)^2} \\
u(n+1,\{n,n+1\}) &= \frac{1}{(n+1)^2}\left(1-\frac{\frac{1}{(n+1)^t}}{\frac{1}{n^t}+\frac{1}{(n+1)^t}}\right)\left(1-\frac{\frac{1}{(n+1)^{t+1}}}{\frac{1}{n^t}+\frac{1}{(n+1)^t}}\right) \\
&= \frac{(n+1)^{t-2}\left[(n+1)^t+n^t-(n+1)^{-1}n^t\right]}{\left((n+1)^t+n^t\right)^2} \\
u(1,\{n,n,1\}) &= \left(\max\left\{1-\frac{2}{\frac{2}{n^t}+1},0\right\}\right)^2 = \left(\frac{\max\{2-n^t,0\}}{2+n^t}\right)^2 \\
u(1,\{n-1,n+1,1\}) &= \left(\max\left\{1-\frac{2}{\frac{1}{(n-1)^t}+\frac{1}{(n+1)^t}+1},0\right\}\right)^2 \\
&= \left[\frac{\max\left\{(n+1)^t+(n-1)^t-(n-1)^t(n+1)^t,0\right\}}{(n+1)^t+(n-1)^t+(n-1)^t(n+1)^t}\right]^2
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{u(1,\{n-1,n+1,1\})}{u(n,\{n,n+1\})} \\
=\ &\frac{\left((n+1)^t+n^t\right)^2}{n^{t-2}\left((n+1)^t+n^t-n^{-1}(n+1)^t\right)}\left[\frac{\max\left\{(n+1)^t+(n-1)^t-(n-1)^t(n+1)^t,0\right\}}{(n+1)^t+(n-1)^t+(n-1)^t(n+1)^t}\right]^2 \leq 1
\end{aligned}$$

and

$$\frac{u(1,\{n,n,1\})}{u(n+1,\{n,n+1\})} = \frac{\left((n+1)^t+n^t\right)^2\left(\max\{2-n^t,0\}\right)^2}{(n+1)^{t-2}\left[(n+1)^t+n^t-(n+1)^{-1}n^t\right](n^t+2)^2} \leq 1.$$

We have completed the proof. ■

Proof of Theorem 3. There are two cases: Case 1 with two equally sized alliances $\{n,n\}$, and Case 2 with two alliances whose sizes differ by one $\{n,n+1\}$. We start with Case 1. The payoff from $\{n, n\}$ is $\frac{1}{n^2}\frac{1}{2}\left(1-\frac{1}{2n}\right)=\frac{2n-1}{4n}$, and the one from $\{2n\}$ is $\frac{1}{(2n)^2}=\frac{1}{4n^2}$. Thus, the two-alliance equilibrium dominates no alliance equilibrium.

Case 2: Consider allocation $\pi=\{n+1, n\}$. First, the payoff from belonging to a size n alliance is

$$\begin{aligned}
u(n,\pi) &= \frac{1}{n^2}\left[1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-\frac{n^{\frac{1-2\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \frac{1}{n^2}\left[1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}-\frac{n^{\frac{1-2\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
&= \frac{1}{n^2}\left[1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(1-\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right)\right] \\
&= \frac{1}{n^2}\left[\frac{(n+1)^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{(n+1)^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right)\right].
\end{aligned}$$

Since $u(n;\pi)$ is decreasing in σ for $\sigma \geq \frac{2}{3}$, and since there is no two-alliance equilibrium for $\sigma > \frac{4}{5}$ (see Example 2), it suffices to show that $u(n;\pi)$ exceeds $\frac{1}{(2n+1)^2}$ when $\sigma=\frac{4}{5}$. Substituting $\sigma=\frac{4}{5}$ into $u(n;\pi)$, we obtain

$$\begin{aligned}
u(n,\pi) &= \frac{1}{n^2}\left[\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}+\frac{1}{(n+1)^2}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}+\frac{1}{(n+1)^2}}\right)\right] \\
&= \frac{1}{n^2}\frac{n^2}{2n^2+2n+1}\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{n^2}{2n^2+2n+1}\right)\right] \\
&= \frac{1}{2n^2+2n+1}\times\frac{(2n^3+n^2-n-1)}{n\,(2n^2+2n+1)} \\
&= \frac{(2n^3+n^2-n-1)}{n\,(2n^2+2n+1)^2}.
\end{aligned}$$

Subtracting $\frac{1}{(2n+1)^2}$ from the above, we obtain

$$\frac{(2n^3+n^2-n-1)}{n\,(2n^2+2n+1)^2}-\frac{1}{(2n+1)^2}=1\frac{4n^5+4n^4-6n^3-11n^2-6n-1}{n\,(4n^3+6n^2+4n+1)^2}.$$

Let $f_{(n;\pi)}(n) \equiv 4n^5+4n^4-6n^3-11n^2-6n-1$. Since $f_{(n;\pi)}(2)>0$ and $f'_{(n;\pi)}(n)>0$ for $n \geq 2$, we conclude $u(n;\pi)>\frac{1}{(2n+1)^2}$.

Second, we check $u(n+1;\pi)$. We have

$$\begin{aligned}
u(n+1;\pi) &= \frac{1}{(n+1)^2}\left[1-\frac{(n+1)^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-\frac{(n+1)^{\frac{1-2\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right] \\
& \frac{1}{(n+1)^2}\left[\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[\frac{n-1}{n}+\frac{1}{n}\left(\frac{n^{\frac{2-3\sigma}{1-\sigma}}}{n^{\frac{2-3\sigma}{1-\sigma}}+(n+1)^{\frac{2-3\sigma}{1-\sigma}}}\right)\right].
\end{aligned}$$

Since $u(n+1;\pi)$ is increasing in σ for $\sigma \geq \frac{2}{3}$, we check whether or not $u(n+1;\pi) > \frac{1}{(2n+1)^2}$ for the smallest relevant sigma, $\sigma = \frac{2}{3}$.

Substituting $\sigma = \frac{2}{3}$ into $u(n+1;\pi)$, we obtain:

$$\begin{aligned}
u(n+1;\pi) &= \frac{1}{(n+1)^2}\frac{1}{2}\left[\frac{n-1}{n}+\frac{1}{2n}\right] \\
&= \frac{1}{2(n+1)^2}\left[\frac{2n-1}{2n}\right].
\end{aligned}$$

Subtracting $\frac{V}{(2n+1)^2}$ from the above, we obtain

$$\begin{aligned}
& \frac{1}{2(n+1)^2}\left[\frac{2n-1}{2n}\right]-\frac{1}{(2n+1)^2} \\
=\ & \frac{(2n-1)(2n+1)^2-4(n+1)^2 n}{4(n+1)^2 n(2n+1)^2}.
\end{aligned}$$

Denoting the numerator by $f_{(n+1;\pi)}(n)$, we have

$$\begin{aligned}
f(n) &= \left(4n^2-1\right)(2n+1)-\left(4n^3+8n^2+4n\right) \\
&= 8n^3+4n^2-2n-1-4n^3-8n^2-4n \\
&= 4n^3-4n^2-6n-1.
\end{aligned}$$

Since $f_{(n+1;\pi)}(2) = 3 > 0$ and $f'_{(n+1;\pi)}(n) >$ for $n \geq 2$, we conclude that $u(n+1;\pi) > \frac{1}{(2n+1)^2}$ for all $n \geq 2$. We have completed the proof.■

Proof of Proposition 3. This can be shown by the utility in a symmetric alliance structure

$$u(\pi_J) = \frac{1}{\left(\frac{N}{J}\right)^2}\frac{1}{J}\left(1-\frac{J-1}{N}\right) = \frac{1}{N^3}J(N-J+1)$$

$$\frac{\partial u(\pi_J)}{\partial J} = \frac{1}{N^3}\left(N - 2J + 1\right).$$

Therefore, $\frac{\partial u(\pi_J)}{\partial J} > 0$ holds for all $J \leq \frac{N+1}{2}$. Also, notice that a group of N players can sustain at most $\frac{N}{2}$ alliances. Therefore, a symmetric structure with more alliances Pareto-dominates one with less. $\square$

Proof of Proposition 4. From Theorem 1, we know that the payoff of a player who is one of n_j is

$$u(n_j, \pi) = \frac{1}{n_j^2}\left[1 - (J-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1 - (J-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right].$$

Let π'_{n_j} stand for the structure after one player in alliance j spins off to form a singleton alliance. This player has a payoff equal to

$$u(1, \pi'_{n_j}) = \left[1 - J\frac{1}{\sum_{j'=1, j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j - 1)^{\frac{2-3\sigma}{1-\sigma}} + 1}\right]^2.$$

Since $\frac{1-2\sigma}{1-\sigma} \geq 0$, $n_j^{\frac{2-3\sigma}{1-\sigma}} \geq (n_j-1)^{\frac{2-3\sigma}{1-\sigma}} \geq 1^{\frac{2-3\sigma}{1-\sigma}} = 1$ and $n_j^{\frac{1-2\sigma}{1-\sigma}} \geq 1^{\frac{1-2\sigma}{1-\sigma}} = 1$ hold for all $n_j \geq 2$. Since $n_j^{\frac{2-3\sigma}{1-\sigma}}$ is a convex function for $\sigma \in [0, \frac{1}{2}]$ ($\frac{2-3\sigma}{1-\sigma} \in [1, 2]$), we have

$$\sum_{j'=1, j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j - 1)^{\frac{2-3\sigma}{1-\sigma}} + 1 \leq \sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}.$$

This implies

$$\frac{\sum_{j'=1, j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j - 1)^{\frac{2-3\sigma}{1-\sigma}} + 1}{J} < \frac{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}{J-1}.$$

Thus, we have

$$u(1, \pi'_{n_j}) > \left[1 - (J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1 - (J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right].$$

We want to show the RHS that the above inequality is not exceeded by u_j for any $\sigma \in [0, \frac{1}{2}]$. Note that $\frac{2-3\sigma}{1-\sigma} \geq 1$ and $\frac{1-2\sigma}{1-\sigma} \geq 0$ for any $\sigma \in [0, \frac{1}{2}]$. Thus, we have

$$\begin{aligned} u(n_j, \pi) &= \frac{1}{n_j^2}\left[1-(J-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(J-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right] \\ &< \frac{1}{n_j^2}\left[1-(J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1-(J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]. \end{aligned}$$

Therefore, we conclude that for any $\sigma \in [0, \frac{1}{2}]$, a player has an incentive to spin off from any alliance with $n_j \geq 2$.■

Proof of Lemma 2. By Proposition 1, the continuation payoff for a given δ is written as

$$\left[\frac{q}{n_j^2} + (1-q)\right] = \frac{1}{n_j^\delta}$$

$$\Rightarrow \ln\left[\frac{q}{n_j^2} + (1-q)\right] = -\delta \ln n_j \Rightarrow \delta = -\frac{\ln\left[\frac{q}{n_j^2} + (1-q)\right]}{\ln n_j}.$$

Since $\frac{\ln\left[\frac{q}{n_j^2}+(1-q)\right]}{\ln n_j}$ is a monotonic function in q, the corresponding δ is unique for each q.■

Proof of Proposition 5. Without loss of generality, let's consider a unilateral move from alliance n_2 to alliance n_1. Let $D = n_1^{\delta-\frac{\sigma}{1-\sigma}} + n_2^{\delta-\frac{\sigma}{1-\sigma}} + \sum_{j\geq 3}^{J} n_j^{\delta-\frac{\sigma}{1-\sigma}}$. Recall that

$$u_1 = \frac{1}{n_j^\delta}\left[1-(J-1)\frac{n_1^{\delta-\frac{\sigma}{1-\sigma}}}{D}\right]\left[1-(J-1)\frac{n_1^{\delta-\frac{1}{1-\sigma}}}{D}\right].$$

We assume that n_js are large, so how much u_1 is affected by one player's moving from alliance n_2 to alliance n_1 can be approximated by first-order approximation of changing from (n_1, n_2) to $(n_1 + \Delta, n_2 - \Delta)$. We have

$$\frac{du_1}{d\Delta} = -\delta \frac{1}{n_1^{\delta+1}} \left[1 - \frac{(J-1)n_1^{\delta - \frac{\sigma}{1-\sigma}}}{D}\right] \left[1 - \frac{(J-1)n_1^{\delta - \frac{1}{1-\sigma}}}{D}\right]$$
$$+\frac{1}{n_1^{\delta}} \left[-(J-1) \frac{\left(\delta - \frac{\sigma}{1-\sigma}\right) D n_1^{\delta - \frac{\sigma}{1-\sigma} - 1} - \left(\delta - \frac{\sigma}{1-\sigma}\right) n_1^{\delta - \frac{\sigma}{1-\sigma}} n_1^{\delta - \frac{\sigma}{1-\sigma} - 1}}{D^2}\right] \left[1 - \frac{(J-1)n_1^{\delta - \frac{1}{1-\sigma}}}{D}\right]$$
$$+\frac{1}{n_1^{\delta}} \left[1 - \frac{(J-1)n_1^{\delta - \frac{\sigma}{1-\sigma}}}{D}\right] \left[-(J-1) \frac{\left(\delta - \frac{1}{1-\sigma}\right) D n_1^{\delta - \frac{1}{1-\sigma} - 1} - \left(\delta - \frac{\sigma}{1-\sigma}\right) n_1^{\delta - \frac{1}{1-\sigma}} n_1^{\delta - \frac{\sigma}{1-\sigma} - 1}}{D^2}\right]$$
$$-\frac{1}{n_1^{\delta}} \left[(J-1) \frac{\left(\delta - \frac{\sigma}{1-\sigma}\right) n_2^{\delta - \frac{\sigma}{1-\sigma} - 1} n_1^{\delta - \frac{\sigma}{1-\sigma}}}{D^2}\right] \left[1 - \frac{(J-1)n_1^{\delta - \frac{1}{1-\sigma}}}{D}\right]$$
$$-\frac{1}{n_1^{\delta}} \left[1 - \frac{(J-1)n_1^{\delta - \frac{\sigma}{1-\sigma}}}{D}\right] \left[(J-1) \frac{\left(\delta - \frac{\sigma}{1-\sigma}\right) n_2^{\delta - \frac{\sigma}{1-\sigma} - 1} n_1^{\delta - \frac{1}{1-\sigma}}}{D^2}\right].$$

Evaluating at $n_1 = n_2 = n_j = n$ and $D = Jn^{\delta - \frac{\sigma}{1-\sigma}}$, we obtain,

$$\begin{aligned}
\frac{du_1}{d\Delta} &= -\delta \frac{1}{n^{\delta+1}} \left[1 - \frac{(J-1)n^{\delta - \frac{\sigma}{1-\sigma}}}{Jn^{\delta - \frac{\sigma}{1-\sigma}}}\right] \left[1 - \frac{(J-1)n^{\delta - \frac{1}{1-\sigma}}}{Jn^{\delta - \frac{\sigma}{1-\sigma}}}\right] \\
&\quad - \frac{1}{n^\delta} \left[\frac{(J-1)^2 \left(\delta - \frac{\sigma}{1-\sigma}\right) n^{2\delta - 2\frac{\sigma}{1-\sigma} - 1}}{\left(Jn^{\delta - \frac{\sigma}{1-\sigma}}\right)^2}\right] \left[1 - \frac{(J-1)n^{\delta - \frac{1}{1-\sigma}}}{Jn^{\delta - \frac{\sigma}{1-\sigma}}}\right] \\
&\quad - \frac{1}{n^\delta} \left[1 - \frac{(J-1)n^{\delta - \frac{\sigma}{1-\sigma}}}{Jn^{\delta - \frac{\sigma}{1-\sigma}}}\right] \left[\frac{(J-1)\, n^{2\delta - 2\frac{\sigma}{1-\sigma} - 2} \left[J\left(\delta - \frac{1}{1-\sigma}\right) - \left(\delta - \frac{\sigma}{1-\sigma}\right)\right]}{\left(Jn^{\delta - \frac{\sigma}{1-\sigma}}\right)^2}\right] \\
&\quad - \frac{1}{n^\delta} \left[(J-1) \frac{\left(\delta - \frac{\sigma}{1-\sigma}\right) n^{2\delta - 2\frac{\sigma}{1-\sigma} - 1}}{\left(Jn^{\delta - \frac{\sigma}{1-\sigma}}\right)^2}\right] \left[1 - \frac{(J-1)n^{\delta - \frac{1}{1-\sigma}}}{Jn^{\delta - \frac{\sigma}{1-\sigma}}}\right] \\
&\quad - \frac{1}{n^\delta} \left[1 - \frac{(J-1)n^{\delta - \frac{\sigma}{1-\sigma}}}{Jn^{\delta - \frac{\sigma}{1-\sigma}}}\right] \left[(J-1) \frac{\left(\delta - \frac{\sigma}{1-\sigma}\right) n^{2\delta - 2\frac{\sigma}{1-\sigma} - 2}}{\left(Jn^{\delta - \frac{\sigma}{1-\sigma}}\right)^2}\right] \\
&= \frac{1}{n^{\delta+1}} \left[-\frac{\delta}{J} - \left(\frac{J-1}{J}\right)^2 \left(\delta - \frac{\sigma}{1-\sigma}\right) - \frac{J-1}{J^2}\left(\delta - \frac{\sigma}{1-\sigma}\right)\right] \left[1 - \frac{J-1}{Jn}\right] \\
&\quad - \frac{1}{n^{\delta+2}} \frac{1}{J} \left[\frac{(J-1)\left[J\left(\delta - \frac{1}{1-\sigma}\right) - \left(\delta - \frac{\sigma}{1-\sigma}\right)\right]}{J^2} + \frac{(J-1)\left(\delta - \frac{\sigma}{1-\sigma}\right)}{J^2}\right] \\
&= \frac{1}{n^{\delta+1}} \frac{1}{J} \left[-\delta - (J-1)\left(\delta - \frac{\sigma}{1-\sigma}\right)\right] \left[1 - \frac{J-1}{Jn}\right] \\
&\quad - \frac{1}{n^{\delta+1}} \frac{(J-1)}{nJ^2} \left(\delta - \frac{1}{1-\sigma}\right).
\end{aligned}$$

Notice that, if n is large, the second term in the last equation is close to 0. Therefore, when n is large, players have no incentive to move to another alliance unilaterally if

$$\frac{du_1}{d\Delta} < 0 \iff -\delta < (J-1)\left(\delta - \frac{\sigma}{1-\sigma}\right).$$

Rearranging the last inequality yields the No Symmetry Breaking condition in Proposition 5.

Now, we turn to the No Spin-Off condition. The payoff from a symmetric alliance structure is simply written as

$$u(n) = \frac{1}{n^{\delta}} \frac{1}{J} \left[1 - \frac{1}{nJ} \right].$$

In contrast, the payoff of a player who spun off from a symmetric alliance structure is more subtle, and we need to consider two cases. We start with the case where $\delta - \frac{\sigma}{1-\sigma} < 0$. Let $\epsilon = \left| \delta - \frac{\sigma}{1-\sigma} \right|$ If a player spins off, then there are $J + 1$ alliances, but we have

$$1 + (J-1)\frac{1}{n^{\epsilon}} + \frac{1}{(n-1)^{\epsilon}} < J \cdot 1,$$

which means that the spun-off player becomes inactive, obtaining zero payoff when n is large. Thus, if $\delta - \frac{\sigma}{1-\sigma} < 0$, then no spin-off occurs. In contrast, when $\delta - \frac{\sigma}{1-\sigma} > 0$, a large n implies

$$u(1) = \left[1 - J \times \frac{1}{1 + n^{\delta - \frac{\sigma}{1-\sigma}} + ... + n^{\delta - \frac{\sigma}{1-\sigma}} + (n-1)^{\delta - \frac{\sigma}{1-\sigma}}} \right]^2 \simeq 1,$$

resulting in a spin-off. Finally, when $\delta - \frac{\sigma}{1-\sigma} = 0$, then

$$u(1) = \left[1 - J \times \frac{1}{J+1} \right]^2 = \frac{1}{(J+1)^2} > u(n) = \frac{1}{n^{\delta}} \frac{1}{J} \left[1 - \frac{1}{nJ} \right]$$

for n large enough.■