All probabilities are equal, but some probabilities are more equal than others

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Abstract

A common procedure for selecting people is to have them draw balls from an urn in turn. Modern and ancient stories suggest that such lotteries may be viewed by the individuals as "unfair." We compare this procedure with several alternatives. They all give individuals equal chance of being selected, but have different structures. We analyze these procedures as multistage lotteries. In line with previous literature, our analysis is based on the observation that multistage lotteries are not considered indifferent to their probabilistic one-stage representations. We use a non-expected utility model and show that individuals have preferences over the different procedures.

Key words: Fair lotteries, non-expected utility, multi-stage lotteries

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1 Introduction

Thirty French hostages in a German prison in occupied France need to select three of them to be executed by their captives in retribution to the killing of three Germans by the Resistance. They tear down an old letter into thirty pieces, draw a cross on three of them, and alphabetically one after the other

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draw them out of a shoe (Graham Greene, *The Tenth Man*). Nine Greek heroes want to duel Hector. Old Nestor suggests a lottery. They mark their lots and cast them in a helmet. Nestor shakes the helmet, and out falls the lot of Ajax (*Iliad* VII, 171–182). Moses needs to select seventy out of seventy-two elders to help him in leading the Israelites in the desert. He intents to mark seventy slips "elder," to put them with two empty slips into an urn, and to ask each of the candidates, in their turn, to draw a lot. To abate a possible claim of unfairness by the elders, he actually marks seventy-two slips "elder" and put them, together with two blank slips into the urn (*Talmud Yerushalmi*, Sanhedrin 1:7).[1] Five crackers are put in a bowl full of bran. In four of them there is a check for 2 million SF, one contains a bomb, strong enough to kill the person who pulls the cracker. Mrs. Montgomery and Mr. Belmont reach the bowl together. "Mrs. Montgomery... crying 'Ladies first', knocked off the lid and plunged her hand into the bran. Perhaps she had calculated that the odds would never be as favorable again. Belmont had probably been thinking along the same lines, for he protested, 'We should have drawn for turns'." (Graham Greene, *Dr. Fischer of Geneva, or the Bomb Party*).

Each of these procedures needs to select some members of a given group, either for a good or for a bad outcome. In all cases a random tool is used which gives all candidates the same probability of selection. But are these procedures all the same? Clearly, the candidates have preferences over the different mechanisms. Are they wrong?

Statistically — for sure. But this doesn't mean that candidates may not have preferences over the way these probabilities are created. Most of these procedures require more than one stage of randomization, and following the empirical and theoretical literature, we show that the different structures lead to non-indifference between them. Moreover, we show that the intuitive preferences (as reflected by the above and other examples) are derived from the same conditions leading to some violations of expected utility theory.[2]

We discuss three procedures. 1. Pre-ordered draws, where $n-1$ balls of

[1] The Talmud is aware of the fact that more than seventy elders may be selected, but shrugs it off, saying that by a miracle it didn't happen. The probability of this "miracle" is more than 94%, so it is more a story of bold leadership.

[2] For example, violations like the common ratio effect, similar to the form $(1M, 1) \succ (5M, 0.8;\, 0, 0.2)$ while $(5M, 0.04;\, 0, 0.96) \succ (1M, 0.05;\, 0, 0.95)$. See Allais [1], Macrimmon and Larsson [10], Starmer [15] and further references there.

one color and one ball of another color are put into a box and in their turn, subjects pick a random ball with no replacement. The selected person is the one who picked the odd ball (similarly to the prison example). 2. Names lottery, where the n names are put in a box and one (or $n-1$) are randomly selected (the *Iliad* example). 3. A winning ball is added to the pre-ordered procedure, and if too many people are selected, the procedure starts over again (a variant of Moses and the elders). We compare these procedures and show how their desirability changes with the size of the group and the identity of each participant.

2 Preliminaries

Much of our analysis depends on the evaluation of multi-stage lotteries. There are basically two ways in which such lotteries can be transformed into simple, one-stage lotteries. The first one is using the reduction axiom, according to which each possible outcome is listed with its compound probability, obtained by multiplication of the probabilities along the path to that outcomes. Alternatively, each simple lottery is replaced by its certainty equivalent and when applied recursively, the value of the multi-stage lottery is computed. Expected utility is the only theory under which decision makers are indifferent between these two simplifications. Experiments tend to support the second approach (see Halevy [8], Starmer [15], and references in these two papers), and as we are interested in people's feelings, we follow this approach.

Decision makers may use any functional for the evaluation of simple lotteries. Some models even permit using different evaluations at different stages (see e.g. Klibanoff, Marinacci, and Mukerji [9]. See also Dillenberger and Segal [7]). Given a two-stage lottery $L = (X_1, q_1; \ldots; X_\ell, q_\ell)$ where $X_1, \ldots, X_\ell$ are simple lotteries, the lottery L is transformed by the decision maker to the simple lottery $(\mathrm{CE}(X_1), q_1; \ldots; \mathrm{CE}(X_\ell), q_\ell)$, where $\mathrm{CE}(X)$ is the certainty equivalent of X. For concrete results, we use the same rank dependent (RD) model (Quiggin [11]) at all stages (see Segal [13, 14]). According to this model the value of the lottery $X = (x_1, p_1; \ldots; x_m, p_m)$ where $x_1 \leq \ldots \leq x_m$, is

$$\mathrm{RD}(X) = \sum_{j=1}^{m-1} u(x_j) \left[f\left(\sum_{l=j}^{m} p_l \right) - f\left(\sum_{l=j+1}^{m} p_l \right) \right] + u(x_m) f(p_m) \qquad (1)$$

where f is continuous and strictly increasing, $u(0)=0, f(0)=0$, and $f(1)=1$. The value of the lottery L is

$$\mathrm{RD}\left(u^{-1}(\mathrm{RD}(X_1)), q_1; \ldots; u^{-1}(\mathrm{RD}(X_\ell)), q_\ell\right) \tag{2}$$

Let $g(p)=1-f(1-p)$. Observe that $g(0)=0, g(1)=1$, and g is concave iff f is convex. Furthermore we can rewrite eq. (1) as

$$u(x_1)g(p_1)+\sum_{j=2}^{m} u(x_j)\left[g\left(\sum_{l=1}^{j} p_l\right)-g\left(\sum_{l=1}^{j-1} p_l\right)\right] \tag{3}$$

We assume throughout that all individuals in society have the same preferences (and will therefore use the same functions u, f, and g to all), that the utility from being selected for a good outcome is 1, and the utility from being selected for a bad outcome is 0. Denote by $v^i(\Gamma)$ the value of procedure Γ to individual i using eq. (2) with either eq. (1) or (3).

The RD model represents risk aversion (in the sense of rejection of mean preserving spreads) iff f is convex (and g concave, see Chew, Karni, and Safra [5]). The elasticity of a function $h(p)$ is given by $\eta^h(p)=\frac{ph'(p)}{h(p)}$. Increasing elasticity of f is linked to the common ratio effect (Segal [12]) and to the recursive model of ambiguity (Segal [13]). We use below the following lemma from [13]:

Lemma 1. If the elasticity of f is increasing, then $f(p)f(q)\leq f(pq)$. If the elasticity of g is decreasing, then $g(p)g(q)\geq g(pq)$.[3]

Comment: Below, we analyze changes in individuals' welfare due to changes in the basic parameters, for example the number n of individuals from whom we select. These are discrete variables, but we consider them as continuous so that we differentiate with respect to them. The results are of course meaningful only for the integer values of these variables.

3 Pre-Ordered Draws

Many of the procedures discussed in the introduction assume the following procedure. Two types of balls (or slips) are put in an urn, say green and

[3] The statement of Lemma 4.1 in [13] is wrong (it suggests an iff result). The proof there only proves Lemma 1 above. Observe that this lemma does not require f to be convex or g to be concave.

red. People are pre-ordered, and then one after the other they draw from the urn. Those who drew green are selected for a good outcome receiving utility 1, and those who drew red are not selected and receive utility 0. Denote by $P(n,k)$ the case in which there are k green balls and $n-k$ red balls. We will focus on the specific cases $k=1$ and $k=n-1$.

Consider individual i in procedure $P(n,1)$. With probability $\frac{i-1}{n}$, someone already drew the green ball before his turn. In this case, he is not selected and gets 0. With the remaining probability, $\frac{n-i+1}{n}$, the green ball has not yet been drawn before his turn. Then he is facing a lottery in which with probability $\frac{1}{n-i+1}$ he draws the green ball, is selected, and gets 1. With the remaining probability, $\frac{n-i}{n-i+1}$, he draws a red ball, is not selected, and gets 0. Hence, individual i is facing the two-stage lottery

$$\left(0, \tfrac{i-1}{n}; \left(0, \tfrac{n-i}{n-i+1}; 1, \tfrac{1}{n-i+1}\right), \tfrac{n-i+1}{n}\right)$$

Using eqs. (1) and (2), the value of this lottery is

$$v^i(P(n,1)) = f\left(\tfrac{n-i+1}{n}\right) f\left(\tfrac{1}{n-i+1}\right) \tag{4}$$

Selecting $n-1$ out of n individuals for a good outcome using the pre-ordered procedure is the same as selecting 1 out of n individuals for a bad outcome. Similarly to the previous analysis, individual i is then facing the two-stage lottery

$$\left(1, \tfrac{i-1}{n}; \left(1, \tfrac{n-i}{n-i+1}; 0, \tfrac{1}{n-i+1}\right), \tfrac{n-i+1}{n}\right)$$

Using eqs. (2) and (3), the value of this lottery is

$$\begin{aligned} v^i(P(n,n-1)) &= \left[1-g\left(\tfrac{n-i+1}{n}\right)\right] + g\left(\tfrac{n-i+1}{n}\right)\left[1-g\left(\tfrac{1}{n-i+1}\right)\right] \\ &= 1-g\left(\tfrac{n-i+1}{n}\right) g\left(\tfrac{1}{n-i+1}\right) \end{aligned} \tag{5}$$

Procedures $P(n,1)$ and $P(n,n-1)$ for person i are depicted in figure 1.

3.1 Does the order matter?

Mrs. Montgomery and Mr. Belmont apparently believe that being the first to pick a cracker is best. The prisoners follow a reverse alphabetical order, starting with Z. Mr. Chavel, the one to offer it, doesn't seem to be indifferent

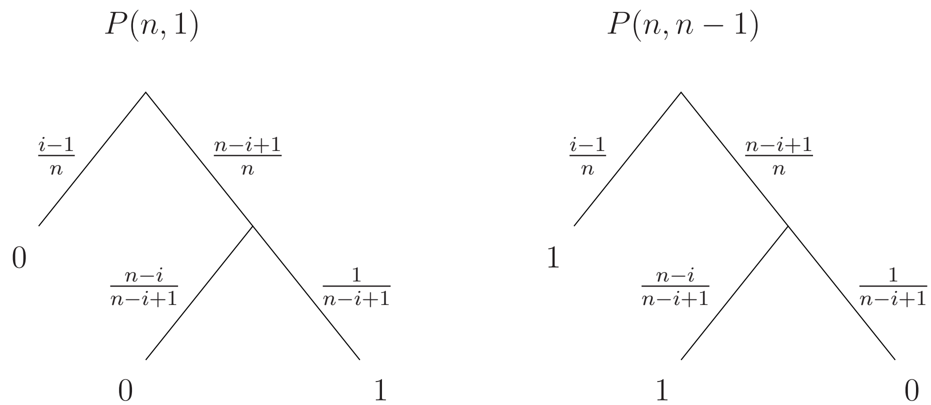

Figure 1: Procedures $P(n, 1)$ and $P(n, n-1)$ for person i

to the order in which slips are picked. Many commentators on the Talmudic story of Moses and the elders point out that the last person to pick a slip is not in the same position as everyone else and some argue that being first is best. Do any of these make sense? After all, the probability of picking a winning ball clearly does not depend on one's position in the queue.

But the position in the queue changes the structure of the two-stage lottery in which each person participates. Using recursive evaluation of such lotteries shows that all these lotteries are not the same. It turns out that under the same condition that implies the common ratio effect, people will prefer to be the first to draw.

Claim 1. Let $i^* = n + 1 - \sqrt{n}$. If the elasticity of f is increasing, then $v^i(P(n, 1))$ is decreasing in i until i^* and increasing thereafter. If the elasticity of g is decreasing, then $v^i(P(n, n-1))$ is decreasing in i until i^* and increasing thereafter.[4]

Proof: We prove the claim for f. The derivative of eq. (4) with respect to i is

$$-\frac{1}{n} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right) + \frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right) \qquad (6)$$

[4] The elasticity if the marginal divided by the average. It is easy to verify that $\eta^f(0) = 1$. Since f is convex, its elasticity cannot be decreasing throughout. Likewise, the elasticity of the concave function g cannot be increasing throughout.

Suppose $\frac{1}{n-i+1} \leq \frac{n-i+1}{n}$. Clearly $\frac{1}{n-i+1} \leq \frac{n-i+1}{n}$ if and only if $i \leq i^*$. Eq. (6) is less than or equal to zero if and only if

$$\frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right) \leq \frac{1}{n} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right)$$

$$\iff \frac{\frac{1}{n-i+1} f'\left(\frac{1}{n-i+1}\right)}{f\left(\frac{1}{n-i+1}\right)} \leq \frac{\frac{n-i+1}{n} f'\left(\frac{n-i+1}{n}\right)}{f\left(\frac{n-i+1}{n}\right)} \tag{7}$$

This inequality holds for $i \leq i^*$ since $\frac{1}{n-i+1} \leq \frac{n-i+1}{n}$ and the elasticity of f is increasing.

On the other hand, suppose $\frac{1}{n-i+1} > \frac{n-i+1}{n}$ which implies that $i > i^*$. Eq. (6) is greater than or equal to zero if and only if the inverse of eq. (7) holds, which holds for $i \geq i^*$ since $\frac{1}{n-i+1} \geq \frac{n-i+1}{n}$ and the elasticity of f is increasing.

The proof for g is similar. ■

The conditions of claim 1 are nonempty. Let $f(p) = \frac{e^p - 1}{e-1}$. Then $g(p) = \frac{e - e^{1-p}}{e-1}$. It is straightforward to verify that η^f is increasing if and only if $e^p \geq 1 + p$ and η^g is decreasing if and only if $1 - p \leq e^{-p}$. Both inequalities hold for all p by Bernoulli's inequality [4, p. 9].

By eqs. (4) and (5), $v^1(P(n,1)) = v^n(P(n,1))$ and $v^1(P(n,n-1)) = v^n(P(n,n-1))$. That is, being the first in line or the last one to draw a ball yield the same utility. Observe that this is true from an ex-ante point of view, before the first ball is drawn. Under $P(n,1)$ person 1 is facing the lottery $(1, \frac{1}{n}; 0, 1 - \frac{1}{n})$. But person n is facing at this stage the same lottery. There is a $\frac{1}{n}$ chance that no one before him will pick up the green ball, in which case he wins for sure, and there is a $1 - \frac{1}{n}$ chance that by the time his turn arrives, the green ball will no longer be in the urn, in which case his utility is zero. Together with claim 1 we get that under both scenarios, when one of n people is selected by the pre-ordered mechanism either for the good or for the bad outcome, it is best to be either first or last one to draw. Formally:

Corollary 1. If the elasticity of f is increasing, then, for all i

$$f\left(\tfrac{1}{n}\right) = v^1(P(n,1)) = v^n(P(n,1)) \geq v^i(P(n,1))$$

If the elasticity of g is decreasing, then, for all i

$$1 - g\left(\tfrac{1}{n}\right) = v^1(P(n,n-1)) = v^n(P(n,n-1)) \geq v^i(P(n,n-1))$$

3.2 Changing n

When $n-1$ out of n people are going to be selected for a good outcome, it seems almost obvious that each of them would like the number n to be as high as possible, since the ex-ante probability of being selected, $\frac{n-1}{n}$, is increasing with n. But as we have seen in the previous section, the structure and evaluation of the multi stage lotteries faced by the n individuals may cause them to have preferences over lotteries with the same reduced probabilities. In fact, some people may even prefer a lower value of n.

Consider person #900 out of 1,000. By the time his turn arrives, it is very likely that the red ball was already picked by someone else, in which case he knows for sure that he is going to get the good outcome. However, if he is #900 out of 10,000, then it is very likely that 899 people before him picked a green ball, so he will have to participate in a lottery in which he may pick the red ball. He may therefore prefer the former case to the latter. We don't claim that this is likely to happen, only that it *may* happen. In this section we offer conditions under which decision makers behave according to expectation when people are selected for a good outcome. That is, they prefer lower n when one of n is selected and higher n when $n-1$ of n are selected. But we also show, by means of examples, that the opposite is possible as well.

We say that a function h satisfies condition $(*)$ for individual i if

$$1 - \frac{n-i+1}{n} \leq \frac{\eta^h\left(\frac{1}{n-i+1}\right)}{\eta^h\left(\frac{n-i+1}{n}\right)} \tag{8}$$

Claim 2. Suppose an individual is added at position i_0 without changing the order of the rest. Then using the $P(n+1,1)$ procedure, every individual $i \geq i_0$ in the original list is strictly worse off, and individual $i < i_0$ is worse off iff f satisfies condition $(*)$. For $P(n+1,n)$, every individual $i \geq i_0$ is strictly better off, and individual $i < i_0$ is better off iff g satisfies condition $(*)$.

Proof: We prove the claim for $P(n+1,1)$. First, observe that for all $i \geq i_0$, person i in $P(n,1)$ becomes person $i+1$ in $P(n+1,1)$. Then

$$\begin{aligned} v^{i+1}(P(n+1,1)) &= f\left(\tfrac{(n+1)-(i+1)+1}{n+1}\right) f\left(\tfrac{1}{(n+1)-(i+1)+1}\right) \\ &= f\left(\tfrac{n-i+1}{n+1}\right) f\left(\tfrac{1}{n-i+1}\right) \\ &< f\left(\tfrac{n-i+1}{n}\right) f\left(\tfrac{1}{n-i+1}\right) = v^i(P(n,1)) \end{aligned}$$

Thus, all individuals i for which $i \geq i_0$ are strictly worse off.

Second, consider $i < i_0$. Recall that the value of $P(n,1)$ to person i is given by eq. (4). The derivative of eq. (4) with respect to n is

$$\tfrac{i-1}{n^2} f'\left(\tfrac{n-i+1}{n}\right) f\left(\tfrac{1}{n-i+1}\right) - \tfrac{1}{(n-i+1)^2} f\left(\tfrac{n-i+1}{n}\right) f'\left(\tfrac{1}{n-i+1}\right)$$

which is non-positive if and only if

$$\begin{aligned} &\tfrac{i-1}{n^2} f'\left(\tfrac{n-i+1}{n}\right) f\left(\tfrac{1}{n-i+1}\right) \leq \tfrac{1}{(n-i+1)^2} f\left(\tfrac{n-i+1}{n}\right) f'\left(\tfrac{1}{n-i+1}\right) \\ \iff &\left(\tfrac{i-1}{n}\right) \times \frac{\frac{n-i+1}{n} f'\left(\frac{n-i+1}{n}\right)}{f\left(\frac{n-i+1}{n}\right)} \leq \frac{\frac{1}{n-i+1} f'\left(\frac{1}{n-i+1}\right)}{f\left(\frac{1}{n-i+1}\right)} \\ \iff &\left(1 - \tfrac{n-i+1}{n}\right) \leq \frac{\eta^f\left(\frac{1}{n-i+1}\right)}{\eta^f\left(\frac{n-i+1}{n}\right)} \end{aligned}$$

Thus, individual $i < i_0$ is worse off iff condition $(*)$ holds for individual i.

The proof for $P(n+1,n)$ is similar. ■

The LHS of eq. (8) is always less than 1. The RHS is greater than 1 whenever the elasticity at $\frac{1}{n-i+1}$ is greater than the elasticity at $\frac{n-i+1}{n}$. This observation leads to the following corollary.

Corollary 2. Let $i^* = n+1-\sqrt{n}$. If the elasticity of f is increasing, then, for $i^* \leq i < i_0$, $v^i(P(n,1))$ is increasing in n. If the elasticity of g is decreasing, then, for $i \leq i^*$, $v^i(P(n,n-1))$ is increasing in n.

Proof: We prove the corollary for f. By claim 2, condition $(*)$ must hold. Suppose $\frac{1}{n-i+1} \geq \frac{n-i+1}{n}$ which implies $i \geq i^*$. Since the elasticity of f is increasing, the RHS of eq. (8) is at least 1. Furthermore, the LHS of eq. (8) is at most 1, so condition $(*)$ is satisfied. If, on the other hand, $\frac{1}{n-i+1} < \frac{n-i+1}{n}$ which implies $i < i^*$, then condition $(*)$ is not necessarily satisfied.

The proof for g is similar. ■

Clearly condition $(*)$ for individual i is satisfied for $h(p) = p^\alpha$ (observe that the LHS of eq. (8) is less than one and the elasticity of $p^\alpha \equiv \alpha$).[5] However, for concave g with decreasing elasticity this is the only function, since eq. (8) becomes

$$1 - \tfrac{1}{n} \leq \eta^g(1)/\eta^g(\tfrac{1}{n}) \leq 1$$

As $n \to \infty$, we get $\eta^g(1) = \eta^g(0)$. The solution of the differential equation $\eta^g(p) \equiv \alpha$ together with $g(1) = 1$ implies $g = p^\alpha$ and concavity implies $\alpha \in (0,1)$. Observe however that claim 2 did not require convex f or concave g.

Condition $(*)$ does not necessarily hold for all functions f, even if they are convex. For example, let

$$f^{n,2}(p) = \begin{cases} \frac{n-1}{n^2} p & \text{if } p \leq \frac{1}{n-1} \\ \frac{(n-1)(n^2-1)}{(n-2)n^2} p + 1 - \frac{(n-1)(n^2-1)}{(n-2)n^2} & \text{if } p > \frac{1}{n-1} \end{cases}$$

The function $f^{n,2}$ is convex, it does not satisfy condition $(*)$ for individual 2, and, indeed, individual 2 prefers $P(n+1,1)$ to $P(n,1)$ when another person is added after him.

4 Names Lottery

When the nine Greek heroes draw a lottery, they mark their lots and cast them in a helmet. The person whose lot falls out is the winner and will fight Hector. But this is not the mechanism on which the thirty French prisoners are about to agree.

> "How do we draw?" Krogh asked.
>
> Chavel said, "The quickest way would be to draw marked papers out of a shoe..."

[5] Condition $(*)$ for individual i is not empty even if we require that f is convex with strictly increasing elasticity. One can prove, using numerical methods, that the function $f(p) = \frac{e^p - 1}{e-1}$ satisfies this condition for all $n \leq 10{,}000$ and for all i.

> Krogh said contemptuously, “Why the quickest way? This is the last gamble some of us will have. We may as well enjoy it.”
>
> “The only way is to draw,” the mayor said.
>
> The clerk prepared the draw, sacrificing for it one of his letters from home. He tore it into thirty pieces. On three pieces he made a cross in pencil, and then folded each piece. They shuffled the pieces on the floor and then dropped them into the shoe.

There is an important difference between the heroes and the prisoners. The former group needs to choose one person for a desired outcome, while the latter is to select a small number of its members for a terrible outcome. We show in this section that these reversed preferences for the selection mechanism are consistent with our formal analysis.

Define the “names lottery” $N(n,1)$ as follows. The n individuals' names are placed in an urn. An impartial observer draws one name from the urn and the individual whose name has been drawn is selected. Observe that all individuals are facing the same lottery $\left(1, \frac{1}{n}; 0, \frac{n-1}{n}\right)$. Using eq. (1), for every person i, the RD value of this lottery is $f(\frac{1}{n})$.

Similarly we can define the procedure $N(n, n-1)$ where the n names are placed in an urn and the $n-1$ names drawn by an impartial observer are selected. Notice that unlike $N(n,1)$, this is a multi-stage lottery. With probability $\frac{1}{n}$, each person i is selected in the first round. If not selected, with probability $\frac{1}{n-1}$ he is selected in the second round, and so on until $n-1$ names have been drawn. That is, each individual i is facing the lottery

$$\left(1, \tfrac{1}{n}; \left(1, \tfrac{1}{n-1}; \left(\ldots \left(1, \tfrac{1}{2}; 0, \tfrac{1}{2}\right) \ldots\right), \tfrac{n-2}{n-1}\right), \tfrac{n-1}{n}\right)$$

These procedures are depicted in figure 2.

Using eqs. (2) and (3), for every person i, the RD value of this lottery is

$$v(N(n, n-1)) = 1 - \prod_{i=1}^{n-1} g\left(\tfrac{n-i}{n-i+1}\right) \tag{9}$$

Nestor and the Greek heroes' preferences for the names lottery and the French prisoners' preferences for the pre-ordered procedure are consistent with the next claim:

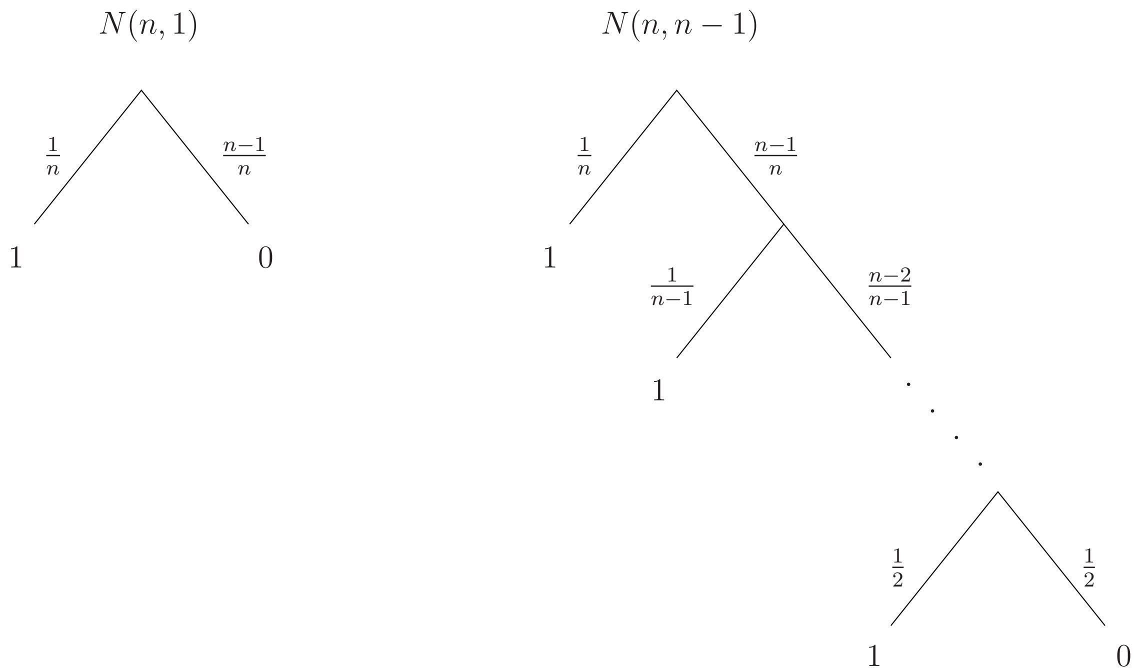

Figure 2: Procedures $N(n,1)$ and $N(n,n-1)$ for each person i

Claim 3. If the elasticity of f is increasing, then $v^i(P(n,1)) \leq v(N(n,1))$ for all i. If the elasticity of g is decreasing, then $v^i(P(n,n-1)) \geq v(N(n,n-1))$ for all i.

Proof: Since $v(N(n,1)) = f\left(\frac{1}{n}\right)$, the first claim follows immediately by corollary 1.

Suppose the elasticity of g is decreasing. Recall that $v(N(n,n-1)) = 1 - \prod_{j=1}^{n-1} g\left(\frac{n-j}{n-j+1}\right)$ by eq. (9) and $v(P(n,n-1)) = 1 - g\left(\frac{n-i+1}{n}\right) g\left(\frac{1}{n-i+1}\right)$

by eq. (5). Observe that, for all i,

$$\begin{aligned} v(N(n,n-1)) &= 1 - \prod_{j=1}^{n-1} g\left(\tfrac{n-j}{n-j+1}\right) \\ &= 1 - \prod_{j=1}^{i-1} g\left(\tfrac{n-j}{n-j+1}\right) \times \prod_{j=i}^{n-1} g\left(\tfrac{n-j}{n-j+1}\right) \\ &\leq 1 - g\left(\tfrac{n-i+1}{n}\right) g\left(\tfrac{1}{n-i+1}\right) = v^i(P(n,n-1)) \end{aligned}$$

where the inequality follows by lemma 1. ■

The nine heroes put their lots in a helmet and Ajax's name was drawn. They might as well have drawn eight lots for those who would *not* fight Hector. These two procedures produce the same ex ante probability for being the person to fight, but they are different in one important aspect. The one they used is a single stage lottery. The suggested alternative requires eight stages. Not surprisingly, our analysis does not consider them the same. The value of procedure N_1^{win}, which is drawing one name out of a hat of n names to win a good outcome is

$$v(N_1^{win}) = v(N(n,1)) = f\left(\tfrac{1}{n}\right) \tag{10}$$

while the value of procedure N_{n-1}^{lose}, which is drawing $n-1$ names out of a hat of n names to receive a bad outcome is

$$v(N_{n-1}^{lose}) = \prod_{j=1}^{n-1} f\left(\tfrac{n-j}{n-j+1}\right) \tag{11}$$

On the other hand, using the P procedure where individuals draw balls in a pre-ordered line, there is no difference between P_1^{win}, which is selecting the person who picks the green ball and P_{n-1}^{lose}, which is selecting the $n-1$ persons who pick a red ball, and the value of both is $f\left(\tfrac{n-i+1}{n}\right) f\left(\tfrac{1}{n-i+1}\right)$.

Claim 4. Suppose the elasticity of f is increasing. Then

$$v(N_{n-1}^{lose}) \leq v^i(P_1^{win}) = v^i(P_{n-1}^{lose}) \leq v(N_1^{win})$$

Proof: Observe that for all i

$$\begin{aligned}
v(N_{n-1}^{lose}) &= \prod_{j=1}^{n-1} f\left(\tfrac{n-j}{n-j+1}\right) \\
&= \prod_{j=1}^{i-1} f\left(\tfrac{n-j}{n-j+1}\right) \times \prod_{j=i}^{n-1} f\left(\tfrac{n-j}{n-j+1}\right) \\
&\leq f\left(\tfrac{n-i+1}{n}\right) f\left(\tfrac{1}{n-i+1}\right) = v(P(n,1)) \\
&\leq f\left(\tfrac{1}{n}\right) = v(N_1^{win})
\end{aligned}$$

where the inequalities follow by lemma 1. ■

In a similar way, we can prove the following:

Claim 5. Suppose that the elasticity of g is decreasing. Then

$$v(N_{n-1}^{win}) \leq v^i(P_{n-1}^{win}) = v^i(P_1^{lose}) \leq v(N_1^{lose})$$

5 Adding Winning Balls

Following the story of Moses and the elders (see the introduction), consider the following $W(n, n-1)$ procedure for selecting $n-1$ out of n people for a good outcome. Put n green and one red ball in an urn and let the n candidates draw balls (with no replacement) according to a pre-arranged order. If someone picks the red ball, then the other $n-1$ people are selected. If not, repeat this procedure using the same order. We show in this section that although this procedure may be attractive to some individuals, such preferences are not universal.

Person i will face a three-stage lottery. In stage 1, individuals before him draw balls and the probability that one of them has drawn the red ball, which means that person i is selected, is $\frac{i-1}{n+1}$. If not, then we move to the second stage which is person i's turn. In this stage, his probability of drawing a green ball is $\frac{n-i+1}{n-i+2}$. If he drew a red ball he gets 0 and the procedure is over. Otherwise, we move to the third stage, in which the probability of person i being selected is the probability that one of the last $n-i+1$ people drew the red ball. This probability is $\frac{n-i+1}{n-i}$. If this does not happen, then the procedure is repeated. That is, person i is facing the following multi-stage lottery:

$$W^i(n,n-1) = \left(1, \tfrac{i-1}{n+1}; \left(\left(1, \tfrac{n-i}{n-i+1}; W(n,n-1), \tfrac{1}{n-i+1}\right), \tfrac{n-i+1}{n-i+2}; 0, \tfrac{1}{n-i+2}\right), \tfrac{n-i+2}{n+1}\right)$$

We show next that there is a connection between preferences for adding balls and preferences for adding individuals as discussed in section 3.2. In the unintuitive case in which a person prefers not to add a candidate when all but one are selected for a good outcomes in the pre-ordered procedure, then he will prefer not to add an extra ball:

Claim 6. If $v^i(P(n,n-1))$ is decreasing in n when an individual is added at the end of the order, then $v^i(P(n,n-1)) \geq v^i(W(n,n-1))$.

Proof: Adding an extra ball is similar to adding a person at the end of the order of the $P(n,n-1)$ procedure, only that the "behavior" of the last person will be different. Suppose this is the case and $v^i(P(n,n-1))$ is decreasing in n, so that $v^i(P(n,n-1)) \geq v^i(P(n+1,n))$. Observe that the first stages of the $P(n+1,n)$ and $W(n,n-1)$ procedures where the first $i-1$ people draw balls are equivalent (see figure 3 for a graphical comparison of the two procedures.) In the second stage, in both procedures, if person i draws a red ball, then he is not selected with equal probability. The difference between the two procedures arises when person i draws a green ball. In the P procedure, he is selected and the procedure is over for him. In the W procedure, he is selected when someone after him draws the red ball, otherwise the procedure must be repeated. Since the continuation value of the W procedure is less than 1, it follows that $v^i(P(n+1,n)) \geq v^i(W(n,n-1))$. Therefore $v^i(P(n,n-1)) \geq v^i(W(n,n-1))$. ∎

On the other hand, if $v^i(P(n,n-1))$ is increasing in n when a person is added at the end, then it is possible (though not guaranteed) that person i will like to add a winning ball to the urn, even if it means that he'll to go through the whole procedure again. For example, let $i^* = n+1-\sqrt{n}$, $s = f\left(\frac{n-i}{n-i+1}\right) \leq \frac{n-i}{n-i+1}$, $r = f\left(\frac{i-1}{n}\right) = \max\{0, s + \left(\frac{i^2-2(n+1)i+(n^2+n+1)}{n}\right)(s-1)\}$, and let f be the piecewise linear function

$$f(p) = \begin{cases} \frac{nr}{i-1}p & \text{if } p \leq \frac{i-1}{n} \\ \frac{n(1-r)}{n-i+1}p + \frac{nr-i+1}{n-i+1} & \text{if } p > \frac{i-1}{n} \end{cases}$$

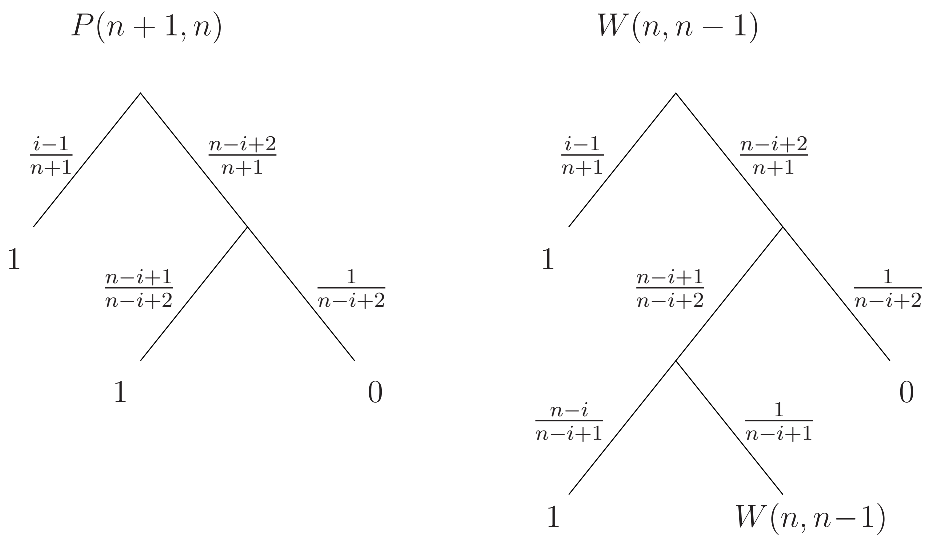

Figure 3: Procedures $P(n+1,n)$ and $W(n,n-1)$ for person i

The function f is convex, increasing in n, and is such that individual $i \le i^*$ prefers $W(n,n-1)$ to $P(n,n-1)$.

The opposite is also possible. One can prove, using numerical methods, that under the function $f(p) = \frac{e^p-1}{e-1}$, $v^i(P(n,n-1)) \ge v^i(W(n,n-1))$ while $v^i(P(n,n-1))$ is increasing in n.

6 Concluding Remarks

A fundamental requirement of fairness is equal treatment of equals. When random selection from equals is needed, this principal requires giving each candidate an equal chance (see Taurek [16] and Broome [2, 3]). But having a mechanism which is considered fair by the social planner may not be enough. Members of society, too, should consider it fair. As we show, the fact that all probabilities are equal does not imply that all individuals will consider them as such and some mechanisms will be considered to create more equal treatments than others.

The argument that such behavior is irrational and that violations of probability theory may expose decision makers to Dutch books (see de Finetti [6] and Yaari [17]) is irrelevant. There is no point in using a “fair” mechanism unless it is deemed fair by those who should bear its consequences. And if adding a green ball to the urn, or having a names lottery rather than sequential draws will make people feel that the procedure is more fair, then so be it.

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