`triprobit` and the *GHK* simulator: a short note

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1 The trivariate probit

Consider three binary variables y_1, y_2 and y_3, the trivariate probit model supposes that:

$$y_1 = \begin{cases} 1 & \text{if } X\beta + \varepsilon_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$y_2 = \begin{cases} 1 & \text{if } Z\gamma + \varepsilon_2 > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$
$$y_3 = \begin{cases} 1 & \text{if } W\theta + \varepsilon_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

with

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} \rightarrow N(0, \Sigma) \quad (2)$$

For identification reasons, the variances of the epsilons must equal 1.

Evaluation of the likelihood function requires the computation of trivariate normal integrals. For example, the probability of observing $(y_1 = 0, y_2 = 0, y_3 = 0)$ is:

$$\Pr[y_1 = 0, y_2 = 0, y_3 = 0] = \int_{-\infty}^{-X\beta} \int_{-\infty}^{-Z\gamma} \int_{-\infty}^{-W\theta} \phi_3(\varepsilon_1, \varepsilon_2, \varepsilon_3, \rho_{12}\rho_{13}\rho_{23}) \, \mathrm{d}\varepsilon_3 \mathrm{d}\varepsilon_2 \mathrm{d}\varepsilon_1 \quad (3)$$

where $\phi_3(.)$ is the trivariate normal p.d.f., and ρ_{ij} is the correlation coefficient between ε_i and ε_j.

While Stata provides commands to compute univariate and bivariate normal CDF (`norm()` and `binorm()`), no command is available for the trivariate case (as a matter of fact, numerical approximations perform poorly in computing high order integrals).

The `triprobit` command uses the *GHK* (Geweke-Hajivassiliou-Keane) smooth recursive simulator to approximate these integrals

2 The *GHK* simulator

Let us illustrate the *GHK* simulator in the trivariate case (generalization to higher orders is straightforward)

We wish to evaluate

$$\Pr(\varepsilon_1 < b_1, \varepsilon_2 < b_2, \varepsilon_3 < b_3) \quad (4)$$

where $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are normal random variables with covariance structure given in (2)

Equation (4) can be rewritten as a product of conditional probabilities:

$$\Pr(\varepsilon_1 < b_1) \Pr(\varepsilon_2 < b_2 | \varepsilon_1 < b_1) \Pr(\varepsilon_3 < b_3 | \varepsilon_1 < b_1, \varepsilon_2 < b_2) \quad (5)$$

Let L be the lower triangular Cholesky decomposition of Σ, such that: $LL' = \Sigma$:

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I am very much indebted to Wolfgang Schwerdt who has provided numerous advices

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

We get:

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{6}$$

where the ν_i are *independent* standard normal random variables.

By (6), we get:

$$\begin{aligned} \varepsilon_1 &= l_{11}\nu_1 \\ \varepsilon_2 &= l_{21}\nu_1 + l_{22}\nu_2 \\ \varepsilon_3 &= l_{31}\nu_1 + l_{32}\nu_2 + l_{33}\nu_3 \end{aligned}$$

Thus:

$$\Pr(\varepsilon_1 < b_1) = \Pr(\nu_1 < b_1 / l_{11}) \tag{7}$$

and

$$\Pr(\varepsilon_2 < b_2 | \varepsilon_1 < b_1) = \Pr(\nu_2 < (b_2 - l_{21}\nu_1)/l_{22} \,|\nu_1 < b_1/l_{11}) \tag{8}$$

and

$$\begin{gathered} \Pr(\varepsilon_3 < b_3 | \varepsilon_1 < b_1, \varepsilon_2 < b_2) = \\ \Pr(\nu_3 < (b_3 - l_{31}\nu_1 - l_{32}\nu_2)/l_{33} \,|\nu_1 < b_1/l_{11}, \nu_2 < (b_2 - l_{21}\nu_1)/l_{22}) \end{gathered} \tag{9}$$

Since (ν_1, ν_2, ν_3) are independent random variables, equation (4) can be expressed as a product of univariate CDF, but conditional on unobservables (the ν).

Suppose now that we draw a random variable ν_1^* from a truncated standard normal density with upper truncation point of b_1/l_{11}, and another one, ν_2^*, from a standard normal density with upper truncation point of $(b_2 - l_{21}\nu_1^*)/l_{22}$. These two random variables respect the conditioning events of equations (8) and (9).

Equation (5) is then rewritten as:

$$\Pr(\nu_1 < b_1/l_{11}) \Pr(\nu_2 < (b_2 - l_{21}\nu_1^*)/l_{22}) \Pr(\nu_3 < (b_3 - l_{31}\nu_1^* - l_{32}\nu_2^*)/l_{33}) \tag{10}$$

The *GHK* simulator of (4) is the arithmetic mean of the probabilities given by (10) for D random draws of ν_1^* and ν_2^*:

$$\widetilde{\Pr}_{GHK} = \frac{1}{D}\sum_{d=1}^{D} \left\{ \Phi[b_1/l_{11}] \, \Phi\left[\left(b_2 - l_{21}\nu_1^{*d}\right)/l_{22}\right] \Phi\left[\left(b_3 - l_{31}\nu_1^{*d} - l_{32}\nu_2^{*d}\right)/l_{33}\right] \right\} \tag{11}$$

where ν_1^{*d} and ν_2^{*d} are the d-th draw of ν_1^* and ν_2^*, and where $\Phi(.)$ is the univariate normal CDF.

The simulated probability (11) is then plugged into the likelihood function, and standard maximisation techniques are used.

3 An example on artificial data

```
set obs 5000
local rho12=0.3
local rho13=-0.3
local rho23=0.3
drawnorm eps1 eps2 eps3 ,corr(1     , `rho12' , `rho13' \ /*
                        */ `rho12' ,   1    , `rho23' \ /*
                        */ `rho13' , `rho23' ,  1    )
```

```
drawnorm x1 x2 x3 x4 x5 x6 x7 x8 x9
gen y3=(1+x6+x7+x8+x9+eps3>0)
gen y2=(1+x4+x5+x6+eps2>0)
gen y1=(1+y2+y3+x1+x2+x3+eps1>0) /*note that y2 and y3 are endogenous*/
triprobit ( y1= y2 y3 x1 x2 x3)(y2= x4 x5 x6)(y3 = x6 x7 x8 x9)

trivariate probit, GHK simulator, 25 draws

Comparison log likelihood = -3876.3152

initial:       log likelihood = -3876.3152
<output omited>
Iteration 5:   log likelihood = -3838.0791

                                                  Number of obs   =       5000
                                                  Wald chi2(12)   =    3576.34
Log likelihood = -3838.0791                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
y1           |
          y2 |   .9232884   .0927705     9.95   0.000     .7414615    1.105115
          y3 |   .9222976   .0765911    12.04   0.000     .7721818    1.072413
          x1 |   1.065994   .0470546    22.65   0.000     .9737688    1.158219
          x2 |    .991229   .0449885    22.03   0.000     .9030532    1.079405
          x3 |   1.037427   .0453475    22.88   0.000     .9485477    1.126307
       _cons |   1.085532   .0735326    14.76   0.000     .9414105    1.229653
-------------+----------------------------------------------------------------
y2           |
          x4 |   1.000869   .0338369    29.58   0.000     .9345499    1.067188
          x5 |    .963295   .0340263    28.31   0.000     .8966047    1.029985
          x6 |   1.066905   .0352755    30.24   0.000     .9977661    1.136044
       _cons |    1.01315   .0314987    32.16   0.000     .9514141    1.074887
-------------+----------------------------------------------------------------
y3           |
          x6 |   1.023065   .0353343    28.95   0.000     .9538105    1.092319
          x7 |   1.023166   .0351069    29.14   0.000     .9543577    1.091974
          x8 |    1.03172   .0347611    29.68   0.000     .9635901    1.099851
          x9 |   1.017668   .0348807    29.18   0.000     .9493033    1.086033
       _cons |   1.015376   .0326298    31.12   0.000      .951423    1.079329
-------------+----------------------------------------------------------------
athrho12     |
       _cons |   .1457736   .0471507     3.09   0.002       .05336    .2381872
-------------+----------------------------------------------------------------
athrho13     |
       _cons |   -.278662   .0546056    -5.10   0.000     -.385687   -.1716371
-------------+----------------------------------------------------------------
athrho23     |
       _cons |   .2598698   .0348018     7.47   0.000     .1916596      .32808
------------------------------------------------------------------------------
------------------------------------------------------------------------------
rho12=  .14474975  Std. Err.=  .04616273  z=  3.1356413  Pr>|z|=  .00171479
rho13= -.27166631  Std. Err.=  .05057554  z= -5.3714955  Pr>|z|=  7.809e-08
rho23=  .25417374  Std. Err.=  .03255343  z=  7.8078952  Pr>|z|=  5.773e-15
------------------------------------------------------------------------------
LR test of rho12=rho13=rho23=0: chi2(3) =  76.472099  Prob > chi2 =  1.752e-16
```