This document spells out the derivation of the recipe for logistic scores as centres of gravity of slices of a logistic distribution.

We need to evaluate the definite integral

$$\int_A^B \ln\left[P/(1-P)\right]\, dP,$$

which can be rewritten as

$$\left\{\int_A^B \ln\, P\, dP\right\} - \left\{\int_A^B \ln(1-P)\, dP\right\}.$$

The first integral in braces $\{\}$ is a standard form, namely in indefinite terms

$$P\, [\ln\, P - 1].$$

Finding that is an exercise in integration by parts. See for example Abbott (1940, p.190); Thompson (1946, p.168); Hardy (1952, p.267); Courant and Robbins (1978, p.508); Hamming (1985, p.444); Hughes-Hallett *et al.* (1994, p.362); Spivak (1994, p.363); Gullberg (1997, p.229); Banner (2007, p.397); or Strang (2017, pp.342–343). The integral is listed in reference compendia such as Abramowitz and Stegun (1965, p.69), Zeidler (2003, p.182) or Olver *et al.* (2010, p.110).

The point of all these references is not that you may have access to any of these books, but rather that you should be able to find what is needed in any good introduction to calculus or general mathematical reference.

Here and below extra constants that appear on integration are neglected, as they will disappear in the final calculations.

The second integral in braces is evidently a kindred form. Putting $1 - P = u$ so that $-1 = du/dP$ and $-dP = du$ we get

$$-u\, [\ln\, u - 1] = -(1-P)\, [\ln(1-P) - 1]$$

and putting the integrals together we then have

$$P\, [\ln\, P - 1] + (1-P)\, [\ln(1-P) - 1] = P \ln\, P - P + (1-P)\ln(1-P) + P - 1$$

$$= P \ln\, P + (1-P)\ln(1-P) - 1.$$

In the definite form, the constant -1 disappears by subtraction and we find

$$[B \ln\, B + (1-B)\ln(1-B)] \quad - \quad [A \ln\, A + (1-A)\ln(1-A)].$$

In programming the main detail is to ensure that $0 \ln 0$ is returned as 0, rather than a missing or undefined value. Plotting $P \ln P$ as a function of P is an easy way to see that as a natural limit, which can be justified more rigorously.

Abbott, P. 1940. *Teach Yourself Calculus.* London: English Universities Press.

Abramowitz, M. and I.A. Stegun (Eds) 1965. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* New York: Dover.

Banner, A. 2007. *The Calculus Lifesaver.* Princeton, NJ: Princeton University Press.

Courant, R. and H. Robbins. 1978. *What is Mathematics? An Elementary Approach to Ideas and Methods.* New York: Oxford University Press.

Gullberg, J. 1997. *Mathematics from the Birth of Numbers.* New York: W.W. Norton.

Hamming, R.W. 1985. *Methods of Mathematics Applied to Calculus, Probability, and Statistics.* Englewood Cliffs, NJ: Prentice-Hall.

Hardy, G.H. 1952. *A Course of Pure Mathematics.* London: Cambridge University Press.

Hughes-Hallett, D. and 14 others. 1994. *Calculus.* New York: John Wiley.

Olver, F.W.J., D.W. Lozier, R.F. Boiswert, and C.W. Clark (Eds). 2010. *NIST Handbook of Mathematical Functions.* New York: Cambridge University Press.

Spivak, M. 1994. *Calculus.* Cambridge: Cambridge University Press.

Strang, G. 2017. *Calculus.* Wellesley, MA: Wellesley-Cambridge Press.

Thompson, S.P. 1946. *Calculus Made Easy.* London: Macmillan.

Zeidler, E. (Ed.) 2003. *Oxford Users' Guide to Mathematics.* Oxford: Oxford University Press.

NJC 13 May 2022/6 July 2023/27 July 2026