

Solving and Simulating RBC Models with Mathematica

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Abstract

This notebook illustrates how to solve and simulate a Real Business Cycle (RBC) model using Mathematica. It starts from finding the first order conditions, then derives the steady-state, log-linearizes and solves the RBC model. After solving the model, the notebook first computes the impulse response functions and then simulates the model. It ends by calculating some second moments statistics of the RBC model. When we want to solve a RBC model, soon or later, we will need to use a software. This notebook demonstrates how to do this using Mathematica without having the need to use paper and pencil to derive the equilibrium conditions and log-linearize the model first, and then use a numerical software to perform the simulations. In this notebook, all these steps are done with Mathematica and fully explained in detail.

Keywords: real business cycle; steady-state; log-linearization; simulation; impulse response functions; second moments statistics; dynamic stochastic general equilibrium models

JEL Classification: C6, E2, O3

*All comments are welcome. All Errors and inaccuracies are mine.

1. Introduction

Real Business Cycle (RBC) models have become the basic workhorse of Dynamic Stochastic General Equilibrium Models (DSGE). DSGE models themselves are now being used both by researchers and professionals, including central bankers, to study and forecast short term economic fluctuations. The hoped contribution of the notebook is the illustration of how to solve and simulate RBC models using Mathematica. Before starting, it is important to inform the reader that this notebook is written with Mathematica 8.0 in a Windows 7 machine. Thus the notebook may or may not work well in future versions of Mathematica.

2. Defining the Model

We load the PlotLegends package. This package allows us to make legends for graphs further below.

```
<< PlotLegends`
```

We present a modified version of the *Hansen (1985)* RBC model. The social planner's maximization problem can be written as:

$$\text{Max } \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\theta} - 1}{1-\theta} - \psi \frac{l_t^{1+\gamma}}{1+\gamma} \right) \right]$$

Subject to :

$$c_t + iv_t = Y_t$$

$$Y_t = z_t k_t^\alpha l_t^{1-\alpha}$$

$$k_{t+1} = (1 - \delta) k_t + iv_t$$

$$\text{Log}[z_{t+1}] = (1 - \rho) \text{Log}[z_s] + \rho \text{Log}[z_t] + \epsilon_{t+1}$$

k_0 and z_0 are given

Where :

$\mathbf{E}_0[\cdot]$ is the expectations operator

$\beta \in (0, 1)$ is the discount factor

c_t is the consumption

$\frac{1}{\theta}$ measures the constant intertemporal elasticity of substitution in consumption. We have $\theta > 0$

γ represents the inverse of the Frisch intertemporal

elasticity of substitution in labour supply. We have $\gamma \geq 0$

l_t is labor

ψ indicates the disutility weight

given to labor in the utility function. We have $\psi > 0$

iv_t is investment. We denote investment by iv_t since

I is a reserved word in *Mathematica*

y_t is output

k_t is capital at period t and k_{t+1} is capital at $t + 1$

$\delta \in (0, 1)$ is the capital depreciation rate

z_t is the exogenous technology shock. z_s is the steady - state level of z_t

$\rho \in [0, 1]$ is the autocorrelation parameter of technology

ϵ_{t+1} the innovations to technology

$$\epsilon_t \stackrel{iid}{\sim} N(0, \sigma^2)$$

t indicates the time.

The model presented here results to the *Hansen (1985)* model when θ and γ tend towards 1 and 0 respectively. Let's illustrate this:

First we take the limit of the utility function when θ goes towards 1

$$\text{ssrbcmo1} = \text{Limit}\left[\frac{c_t^{1-\theta} - 1}{1-\theta} - \psi \frac{l_t^{1+\gamma}}{1+\gamma}, \theta \rightarrow 1\right]$$

$$\frac{(1+\gamma) \text{Log}[c_t] - \psi l_t^{1+\gamma}}{1+\gamma}$$

Then we take the limit of the result when γ approaches 0.

$$\text{ssrbcmo2} = \text{Limit}[\text{ssrbcmo1}, \gamma \rightarrow 0]$$

$$\text{Log}[c_t] - \psi l_t$$

This last expression is the utility function used in the *Hansen (1985)* RBC model. The only difference is that l_t represents there leisure instead of labor supply. Also the minus sign becomes a positive sign since we are dealing there with leisure.

Next we illustrate how the budget constraint is obtained. We first enter the equation of motion of capital stock.

$$\text{ssrbcmo3} = k_{t+1} == (1 - \delta) k_t + i v_t$$

$$k_{1+t} == i v_t + (1 - \delta) k_t$$

We enter the identity stating that consumption plus investment is equal to output.

$$\text{ssrbcmo4} = c_t + i v_t == y_t$$

$$c_t + i v_t == y_t$$

We solve for investment in the previous identity.

$$\text{ssrbcmo5} = \text{Solve}[\text{ssrbcmo4}, i v_t][[1, 1]]$$

$$i v_t \rightarrow -c_t + y_t$$

We substitute investment in the equation of motion of capital.

$$\text{ssrbcmo6} = \text{ssrbcmo3} /. \text{ssrbcmo5}$$

$$k_{1+t} == -c_t + (1 - \delta) k_t + y_t$$

We replace output by its value. Hence the budget constraint.

$$\text{ssrbcmo7} = \text{ssrbcmo6} /. y_t \rightarrow z_t k_t^\alpha l_t^{1-\alpha}$$

$$k_{1+t} == -c_t + (1 - \delta) k_t + k_t^\alpha l_t^{1-\alpha} z_t$$

2.1. The Bellman Equation

The Bellman equation is

$$v[k_t, z_t] = \text{Max}_{(c_t, k_{t+1}, l_t)} \left\{ \frac{c_t^{1-\theta} - 1}{1-\theta} - \psi \frac{l_t^{1+\gamma}}{1+\gamma} + \beta \text{EX}_t[v[k_{t+1}, z_{t+1}]] \right\}$$

Subject to :

The economy resource constraint : $k_{1+t} = -c_t + (1 - \delta) k_t + 1_t^{1-\alpha} k_t^\alpha z_t$

And technology : $\text{Log}[z_{t+1}] = (1 - \rho) \text{Log}[zs] + \rho \text{Log}[z_t] + \epsilon_{t+1}$

We denote the expectations operator as

$\text{EX}_t[\cdot]$ since **E** is a reserved word in *Mathematica*.

We enter the left hand side of the Bellman equation

`ssrbcm08 = v[kt, zt]`

`v[kt, zt]`

We enter the right hand side of the Bellman equation without the Max operator.

$$\text{ssrbcm09} = \frac{c_t^{1-\theta} - 1}{1 - \theta} - \psi \frac{1_t^{1+\gamma}}{1 + \gamma} + \beta \text{EX}_t[v[k_{t+1}, z_{t+1}]]$$

$$\frac{-1 + c_t^{1-\theta}}{1 - \theta} - \psi \frac{1_t^{1+\gamma}}{1 + \gamma} + \beta \text{EX}_t[v[k_{1+t}, z_{1+t}]]$$

We enter the technology equation

`ssrbcm10 = Log[zt+1] == (1 - ρ) Log[zs] + ρ Log[zt] + εt+1`

`Log[z1+t] == (1 - ρ) Log[zs] + ρ Log[zt] + ε1+t`

We enter the right hand side of the Bellman equation without the expectations operator.

$$\text{ssrbcm11} = \frac{c_t^{1-\theta} - 1}{1 - \theta} - \psi \frac{1_t^{1+\gamma}}{1 + \gamma} + \beta v[k_{t+1}, z_{t+1}]$$

$$\frac{-1 + c_t^{1-\theta}}{1 - \theta} - \psi \frac{1_t^{1+\gamma}}{1 + \gamma} + \beta v[k_{1+t}, z_{1+t}]$$

We isolate consumption in the resource constraint.

`ssrbcm12 = ssrbcm07[[1]] - k1+t == ssrbcm07[[2]] - k1+t`

$$0 = -c_t + (1 - \delta) k_t - k_{1+t} + k_t^\alpha 1_t^{1-\alpha} z_t$$

`ssrbcm13 = ssrbcm12[[1]] + ct == ssrbcm12[[2]] + ct`

$$c_t = (1 - \delta) k_t - k_{1+t} + k_t^\alpha 1_t^{1-\alpha} z_t$$

We substitute consumption by its value in the right hand side of the Bellman Equation.

`ssrbcm14 = ssrbcm11 /. ssrbcm13[[1]] -> ssrbcm13[[2]]`

$$-\frac{\psi 1_t^{1+\gamma}}{1 + \gamma} + \frac{-1 + \left((1 - \delta) k_t - k_{1+t} + k_t^\alpha 1_t^{1-\alpha} z_t \right)^{1-\theta}}{1 - \theta} + \beta v[k_{1+t}, z_{1+t}]$$

We take the derivative of this last expression with respect to k_{t+1} and set it equal to 0.

`ssrbcm15 = D[ssrbcm14, kt+1] == 0`

$$-\left((1 - \delta) k_t - k_{1+t} + k_t^\alpha 1_t^{1-\alpha} z_t \right)^{-\theta} + \beta v^{(1,0)}[k_{1+t}, z_{1+t}] = 0$$

We take the derivative of the right hand side of the Bellman Equation with respect to l_t and set it equal to 0.

`ssrbcm16 = D[ssrbcm14, lt] == 0`

$$-\psi 1_t^\gamma + (1 - \alpha) k_t^\alpha 1_t^{-\alpha} z_t \left((1 - \delta) k_t - k_{1+t} + k_t^\alpha 1_t^{1-\alpha} z_t \right)^{-\theta} = 0$$

The Benveniste-Scheinkman envelope theorem condition is given by

$$\text{ssrbcmod17} = \text{D}[\text{ssrbcmod8}, \mathbf{k}_t] == \text{D}[\text{ssrbcmod14}, \mathbf{k}_t]$$

$$\mathbf{v}^{(1,0)}[\mathbf{k}_t, \mathbf{z}_t] == (1 - \delta + \alpha k_t^{-1+\alpha} l_t^{1-\alpha} z_t) \left((1 - \delta) k_t - k_{1+t} + k_t^\alpha l_t^{1-\alpha} z_t \right)^{-\theta}$$

We forward t by one period in the preceding equation.

$$\text{ssrbcmod18} = \text{ssrbcmod17} /. \mathbf{t} \rightarrow \mathbf{t} + 1$$

$$\mathbf{v}^{(1,0)}[\mathbf{k}_{1+t}, \mathbf{z}_{1+t}] == (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) \left((1 - \delta) k_{1+t} - k_{2+t} + k_{1+t}^\alpha l_{1+t}^{1-\alpha} z_{1+t} \right)^{-\theta}$$

We take expectations into account in equation ssrbcmod15

$$\text{ssrbcmod19} = \text{ssrbcmod15} /. \mathbf{v}^{(1,0)}[\mathbf{k}_{1+t}, \mathbf{z}_{1+t}] \rightarrow \text{EX}_t[\mathbf{v}^{(1,0)}[\mathbf{k}_{1+t}, \mathbf{z}_{1+t}]]$$

$$- \left((1 - \delta) k_t - k_{1+t} + k_t^\alpha l_t^{1-\alpha} z_t \right)^{-\theta} + \beta \text{EX}_t[\mathbf{v}^{(1,0)}[\mathbf{k}_{1+t}, \mathbf{z}_{1+t}]] == 0$$

We substitute $\mathbf{v}^{(1,0)}[\mathbf{k}_{1+t}, \mathbf{z}_{1+t}]$ by its value in the previous equation.

$$\text{ssrbcmod20} = \text{ssrbcmod19} /. \text{ssrbcmod18}[[1]] \rightarrow \text{ssrbcmod18}[[2]]$$

$$- \left((1 - \delta) k_t - k_{1+t} + k_t^\alpha l_t^{1-\alpha} z_t \right)^{-\theta} + \beta \text{EX}_t \left[(1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) \left((1 - \delta) k_{1+t} - k_{2+t} + k_{1+t}^\alpha l_{1+t}^{1-\alpha} z_{1+t} \right)^{-\theta} \right] == 0$$

Remember that consumption is given by

$$\text{ssrbcmod21} = \text{ssrbcmod13}$$

$$c_t == (1 - \delta) k_t - k_{1+t} + k_t^\alpha l_t^{1-\alpha} z_t$$

We forward t by one period in the preceding equation.

$$\text{ssrbcmod22} = \text{ssrbcmod21} /. \mathbf{t} \rightarrow \mathbf{t} + 1$$

$$c_{1+t} == (1 - \delta) k_{1+t} - k_{2+t} + k_{1+t}^\alpha l_{1+t}^{1-\alpha} z_{1+t}$$

We replace the right hand sides of ssrbcmod21 and ssrbcmod22 by c_t and c_{t+1} respectively in equation ssrbcmod20.

$$\text{ssrbcmod23} =$$

$$\text{ssrbcmod20} /. \{\text{ssrbcmod21}[[2]] \rightarrow \text{ssrbcmod21}[[1]], \text{ssrbcmod22}[[2]] \rightarrow \text{ssrbcmod22}[[1]]\}$$

$$- c_t^{-\theta} + \beta \text{EX}_t \left[c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) \right] == 0$$

This last equation is known as the Euler equation. It is also called Lucas asset pricing equation. We rewrite it without the expectations operator. But we keep in mind that there is always an expectation operator for this equation.

$$\text{ssrbcmod24} = \text{ssrbcmod23} /. \text{EX}_t \left[c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) \right] \rightarrow c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t})$$

$$- c_t^{-\theta} + \beta c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) == 0$$

Now, we replace c_t by its value in equation ssrbcmod16

$$\text{ssrbcmod25} = \text{ssrbcmod16} /. \text{ssrbcmod21}[[2]] \rightarrow \text{ssrbcmod21}[[1]]$$

$$- \psi l_t^\gamma + (1 - \alpha) c_t^{-\theta} k_t^\alpha l_t^{-\alpha} z_t == 0$$

We solve for c_t in the previous equation

$$\text{ssrbcmod26} = \text{Solve}[\text{ssrbcmod25}, c_t]$$

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{ \left\{ c_t \rightarrow \left(\frac{l_t^{-\gamma} (k_t^\alpha l_t^{-\alpha} z_t - \alpha k_t^\alpha l_t^{-\alpha} z_t)}{\psi} \right)^{\frac{1}{\theta}} \right\} \right\}$$

We transform this expression to an equation for convenience

`ssrbcm27 = ssrbcm26[[1, 1, 1]] == ssrbcm26[[1, 1, 2]]`

$$c_t = \left(\frac{l_t^{-\gamma} (k_t^\alpha l_t^{1-\alpha} z_t - \alpha k_t^\alpha l_t^{1-\alpha} z_t)}{\psi} \right)^{\frac{1}{\theta}}$$

This last equation represents the labor-leisure choice equation. We expand the powers in this equation to get:

`ssrbcm28 = PowerExpand[ssrbcm27]`

$$c_t = \psi^{-1/\theta} l_t^{-\frac{\gamma}{\theta}} (k_t^\alpha l_t^{1-\alpha} z_t - \alpha k_t^\alpha l_t^{1-\alpha} z_t)^{\frac{1}{\theta}}$$

2.2. The Steady-State Equilibrium

Now we derive the steady-state equilibrium for the main variables. We start with the Euler equation.

`ssrbcm29 = ssrbcm23`

$$-c_t^{-\theta} + \beta \text{EX}_t [c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{1+\alpha} l_{1+t}^{1-\alpha} z_{1+t})] = 0$$

We define steady-state variables by adding an s to the existing variables. For example the steady-state consumption is noted as cs for consumption star.

`ssrbcm30 = ssrbcm29 /. {c_t -> cs, c_{t+1} -> cs, k_{t+1} -> ks, l_{t+1} -> ls, z_{t+1} -> zs}`

$$-cs^{-\theta} + \beta \text{EX}_t [cs^{-\theta} (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta)] = 0$$

By the definition of steady-steady, all variables are constant. Hence their expectations are themselves. Thus we have.

`ssrbcm31 = ssrbcm30 /. EX_t [cs^{-\theta} (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta)] -> cs^{-\theta} (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta)`

$$-cs^{-\theta} + cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) = 0$$

Also at the steady-state, there is no uncertainty, thus $zs = 1$.

`ssrbcm32 = ssrbcm31 /. zs -> 1`

$$-cs^{-\theta} + cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) = 0$$

We divide this last equation by $cs^{-\theta}$

$$\text{ssrbcm33} = \text{Expand} \left[\frac{\text{ssrbcm32}[[1]]}{cs^{-\theta}} == \frac{\text{ssrbcm32}[[2]]}{cs^{-\theta}} \right]$$

$$-1 + \beta + ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta - \beta \delta = 0$$

We define the following ratio

$$\text{ssrbcm34} = \text{Solve} \left[\frac{ks}{ls} == kls, kls \right] [[1, 1]]$$

$$ks \rightarrow kls \, ls$$

We substitute ks by its value in ssrbcm33

`ssrbcm35 = ssrbcm33 /. ssrbcm34 // PowerExpand`

$$-1 + \beta + kls^{-1+\alpha} \alpha \beta - \beta \delta = 0$$

We solve for kls in the previous equation.

```
ssrbcm36 = Solve[ssrbcm35, kls] [[1, 1]]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$kls \rightarrow \left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}}$$

We transform this result to an equation.

```
ssrbcm37 = ssrbcm36[[1]] == ssrbcm36[[2]]
```

$$kls == \left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}}$$

Now we turn to the labor-leisure choice equation.

```
ssrbcm38 = ssrbcm28
```

$$c_t == \psi^{-1/\theta} l_t^{-\frac{\gamma}{\theta}} (k_t^\alpha l_t^{-\alpha} z_t - \alpha k_t^\alpha l_t^{-\alpha} z_t)^{\frac{1}{\theta}}$$

We define the steady-state variables for this equation.

```
ssrbcm39 = ssrbcm38 /. {c_t -> cs, l_t -> ls, k_t -> ks, z_t -> zs}
```

$$cs == ls^{-\frac{\gamma}{\theta}} (ks^\alpha ls^{-\alpha} zs - ks^\alpha ls^{-\alpha} zs \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

We set zs = 1

```
ssrbcm40 = ssrbcm39 /. zs -> 1
```

$$cs == ls^{-\frac{\gamma}{\theta}} (ks^\alpha ls^{-\alpha} - ks^\alpha ls^{-\alpha} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

We substitute ks by its value

```
ssrbcm41 = ssrbcm40 /. ssrbcm34 // PowerExpand
```

$$cs == ls^{-\frac{\gamma}{\theta}} (kls^\alpha - kls^\alpha \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

We replace kls by its value.

```
ssrbcm42 = ssrbcm41 /. ssrbcm36
```

$$cs == ls^{-\frac{\gamma}{\theta}} \left(\left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \alpha \left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

We turn to the resource constraint.

```
ssrbcm43 = ssrbcm7
```

$$k_{1+t} == -c_t + (1 - \delta) k_t + k_t^\alpha l_t^{1-\alpha} z_t$$

```
ssrbcm44 = ssrbcm43 /. {k_{t+1} -> ks, c_t -> cs, k_t -> ks, l_t -> ls, z_t -> 1}
```

$$ks == -cs + ks^\alpha ls^{1-\alpha} + ks (1 - \delta)$$

We divide by ks.

```
ssrbcm45 = Map[1/ks # &, ssrbcm44] // Expand
```

$$1 == 1 - \frac{cs}{ks} + ks^{-1+\alpha} ls^{1-\alpha} - \delta$$

We substitute ks by its value.

ssrbcm46 = ssrbcm45 /. ssrbcm34 // PowerExpand

$$1 == 1 + kls^{-1+\alpha} - \frac{cs}{kls\ ls} - \delta$$

We replace kls and cs by their values.

ssrbcm47 = ssrbcm46 /. {ssrbcm36, ssrbcm42[[1]] -> ssrbcm42[[2]]}

$$1 == 1 - \delta + \left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{-1+\alpha} -$$

$$ls^{-1-\frac{\gamma}{\theta}} \left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{-\frac{1}{-1+\alpha}} \left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \alpha \left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

We solve for ls in the previous equation.

ssrbcm48 = Solve[ssrbcm47, ls] // FullSimplify

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{ \left\{ ls \rightarrow \left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right\} \right\}$$

We change this to an equation.

ssrbcm49 = ssrbcm48[[1, 1, 1]] == ssrbcm48[[1, 1, 2]]

$$ls == \left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}}$$

We find the value of cs.

ssrbcm50 = ssrbcm42 /. ssrbcm49[[1]] -> ssrbcm49[[2]]

$$cs == \left(\left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \alpha \left(\left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\frac{1}{\theta}} \right) \psi^{-1/\theta}$$

$$\left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^{-\frac{\gamma}{\theta}}$$

ssrbcm51 = FullSimplify[ssrbcm50]

$$cs == \left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

$$\left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^{-\frac{\gamma}{\theta}}$$

We recall the value of ks.

ssrbcm52 = ssrbcm34

$k_s \rightarrow k_l s$

We write this as an equation.

ssrbcm53 = ssrbcm52[[1]] == ssrbcm52[[2]]

$k_s = k_l s$

We then find the value of k_s .

ssrbcm54 = ssrbcm53 /. {ssrbcm36, ssrbcm49[[1]] -> ssrbcm49[[2]]}

$$k_s = \left(-\frac{-1 + \beta - \beta \delta}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}}$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}}$$

ssrbcm55 = FullSimplify[ssrbcm54]

$$k_s = \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}}$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}}$$

Next we find the value of y_s .

ssrbcm56 = $y_t == z_t k_t^\alpha l_t^{1-\alpha}$

$y_t = k_t^\alpha l_t^{1-\alpha} z_t$

ssrbcm57 = ssrbcm56 /. { $y_t \rightarrow y_s$, $z_t \rightarrow z_s$, $k_t \rightarrow k_s$, $l_t \rightarrow l_s$ }

$y_s = k_s^\alpha l_s^{1-\alpha} z_s$

We replace k_s , l_s and z_s by their respective values.

ssrbcm58 = ssrbcm57 /.

{ssrbcm55[[1]] -> ssrbcm55[[2]], ssrbcm49[[1]] -> ssrbcm49[[2]], $z_s \rightarrow 1$ }

$y_s =$

$$\left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^{1-\alpha}$$

$$\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}}$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^\alpha$$

ssrbcm59 = FullSimplify[ssrbcm58]

ys =

$$\left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma + \theta}} \right)^{1 - \alpha}$$

$$\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma + \theta}} \right)^\alpha$$

We recall the expression of iv_t .

ssrbcm60 = ssrbcm5

$iv_t \rightarrow -c_t + y_t$

We transform this expression to an equation.

ssrbcm61 = ssrbcm60[[1]] == ssrbcm60[[2]]

$iv_t == -c_t + y_t$

We find the value of ivs .

ssrbcm62 = ssrbcm61 /. {iv_t -> ivs, c_t -> cs, y_t -> ys}

$ivs == -cs + ys$

We replace cs and ys by their respective values.

ssrbcm63 =

ssrbcm62 /. {ssrbcm51[[1]] -> ssrbcm51[[2]], ssrbcm59[[1]] -> ssrbcm59[[2]]}

$$ivs == - \left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha \right)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

$$\left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma + \theta}} \right)^{-\frac{\gamma}{\theta}} +$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma + \theta}} \right)^{1 - \alpha}$$

$$\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right) \left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha \right)^{-1/\theta} \right)$$

$$\left(\left(\left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \right)^\alpha - \left(\frac{1 + \beta(-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1 + \alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma + \theta}} \right)^\alpha$$

3. Log-Linearization

Log-linearization is a mean of transforming the dynamic nonlinear equations to dynamic linear equations. This makes the calculations and interpretations easier. To do the log-linearization, we will write a variable x_t as $x_s e^{xh_t}$, where x_s is the steady-state value of x_t and $xh_t = \text{Log} \left[\frac{x_t}{x_s} \right]$. This last expression is approximately the percentage deviation of x_t around the steady-state x_s . In fact:

```
ssrbcm64 = xh_t == Series[Log[x_t/x_s], {x_t, x_s, 1}] // Normal
```

$$xh_t = \frac{-x_s + x_t}{x_s}$$

The h in xh_t is put for x hat. At the steady-state, we can verify that $xh_t = \frac{-x_s + x_t}{x_s} = 0$. In fact at the steady-state $xh_t = \frac{-x_s + x_s}{x_s} = 0$. We can check that the expression $x_s e^{xh_t}$ gives the original variable x_t by calculating:

```
ssrbcm65 = x_s e^{xh_t} /. xh_t -> Log[x_t/x_s]
```

x_t

More log-linearization techniques can be found in *Uhlig (1997)* and *Zietz (2008)*. Now let's start the log-linearization of the key equilibrium equations.

3.1. The Labor-Leisure Choice Equation

Let's write a function that performs multivariate linear Taylor series approximation. The function takes three arguments. The first represents the mathematical function we want to linearize. The second is the list of variables with respect to which we want to linearize and the third argument is the list of points around which we want to take the linearization.

```
mullinearTaylor[f_, x_, a_] := Module[{fa, gradf, gradfa}, fa = f // Thread[x -> a];
  gradf = D[f, {x}]; gradfa = gradf // Thread[x -> a]; fa + gradfa.(x - a)]
```

We recall the labor-leisure choice equation.

```
ssrbcm66 = ssrbcm28
```

$$c_t = \psi^{-1/\theta} l_t^{-\frac{\gamma}{\theta}} (k_t^\alpha l_t^{-\alpha} z_t - \alpha k_t^\alpha l_t^{-\alpha} z_t)^{\frac{1}{\theta}}$$

We first replace each variable in this equation by its equivalent transformed expression.

```
ssrbcm67 = ssrbcm66 /. {c_t -> cs e^{ch_t}, l_t -> ls e^{lh_t}, k_t -> ks e^{kh_t}, z_t -> zs e^{zh_t}}
```

$$cs e^{ch_t} = (e^{lh_t} ls)^{-\frac{\gamma}{\theta}} (e^{zh_t} (e^{kh_t} ks)^\alpha (e^{lh_t} ls)^{-\alpha} zs - e^{zh_t} (e^{kh_t} ks)^\alpha (e^{lh_t} ls)^{-\alpha} zs)^\frac{1}{\theta} \psi^{-1/\theta}$$

We take the first order Taylor series approximation of the left hand side of this equation with respect to ch_t around 0.

```
ssrbcm68 = Series[ssrbcm67[[1]], {ch_t, 0, 1}] // Normal
```

$cs + cs ch_t$

We take the first order multivariate Taylor series approximation of the right hand side of equation `ssrbcm67` with respect to lh_t , zh_t and kh_t respectively around 0, 0 and 0.

ssrbcm69 = mulLinearTaylor[ssrbcm67[[2]], {lh_t, zh_t, kh_t}, {0, 0, 0}]

$$\begin{aligned} & \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta} + \frac{1}{\theta} \\ & \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{-1+\frac{1}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha^2) \psi^{-1/\theta} \text{kh}_t + \\ & \left(\frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{-1+\frac{1}{\theta}} (-\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha + \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha^2) \psi^{-1/\theta} - \frac{1}{\theta} \right. \\ & \left. \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \gamma \psi^{-1/\theta} \right) \text{lh}_t + \frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta} \text{zh}_t \end{aligned}$$

We equalize the two expressions ssrbcm68 and ssrbcm69 to form an equation.

ssrbcm70 = ssrbcm68 == ssrbcm69

$$\begin{aligned} \text{cs} + \text{cs} \text{ch}_t &= \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta} + \frac{1}{\theta} \\ & \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{-1+\frac{1}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha^2) \psi^{-1/\theta} \text{kh}_t + \\ & \left(\frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{-1+\frac{1}{\theta}} (-\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha + \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha^2) \psi^{-1/\theta} - \frac{1}{\theta} \right. \\ & \left. \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \gamma \psi^{-1/\theta} \right) \text{lh}_t + \frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta} \text{zh}_t \end{aligned}$$

Now, we form the steady-state equation of the labor-leisure choice equation.

ssrbcm71 = ssrbcm66 /. {c_t → cs, l_t → ls, k_t → ks, z_t → zs}

$$\text{cs} = \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta}$$

We subtract equation ssrbcm71 from equation ssrbcm70.

ssrbcm72 = ssrbcm70[[1]] - ssrbcm71[[1]] == ssrbcm70[[2]] - ssrbcm71[[2]]

$$\begin{aligned} \text{cs} \text{ch}_t &= \frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{-1+\frac{1}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha^2) \psi^{-1/\theta} \text{kh}_t + \\ & \left(\frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{-1+\frac{1}{\theta}} (-\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha + \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha^2) \psi^{-1/\theta} - \frac{1}{\theta} \right. \\ & \left. \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \gamma \psi^{-1/\theta} \right) \text{lh}_t + \frac{1}{\theta} \text{ls}^{-\frac{\gamma}{\theta}} (\text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} - \text{ks}^\alpha \text{ls}^{-\alpha} \text{zs} \alpha)^{\frac{1}{\theta}} \psi^{-1/\theta} \text{zh}_t \end{aligned}$$

We divide equation ssrbcm72 by equation ssrbcm71.

$$\text{ssrbcm73} = \frac{\text{ssrbcm72}[[1]]}{\text{ssrbcm71}[[1]]} ==$$

$$\text{Apply[Plus, Table[FullSimplify[\frac{\text{ssrbcm72}[[2, i]]}{\text{ssrbcm71}[[2]]}], \{i, 1, \text{Length}[\text{ssrbcm72}[[2]]\}]]]$$

$$\text{ch}_t = \frac{\alpha \text{kh}_t}{\theta} - \frac{(\alpha + \gamma) \text{lh}_t}{\theta} + \frac{\text{zh}_t}{\theta}$$

This last equality represents the log-linearized equation of the labor-leisure choice equation. In traditional economic notation, this equation can be rewritten as:

$$\hat{c}_t = \frac{\alpha \hat{k}_t}{\theta} - \frac{(\alpha + \gamma) \hat{l}_t}{\theta} + \frac{\hat{z}_t}{\theta}$$

Where a hat over a variable indicates percentage deviation of the variable around the steady-state.

3.2. The Euler Equation or Lucas Asset Pricing Equation

We recall the Euler equation without the expectations operator. We will put back this operator when we finish the log-linearization process.

ssrbcm74 = ssrbcm24

$$-c_t^{-\theta} + \beta c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) = 0$$

We isolate c_t .

ssrbcm75 =

$$\begin{aligned} & \text{ssrbcm74}[[1]] - \beta c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) == \text{ssrbcm74}[[2]] - \beta c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) \\ -c_t^{-\theta} & == -\beta c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t}) \end{aligned}$$

We multiply this last equality by -1.

ssrbcm76 = Map[-1 # &, ssrbcm75]

$$c_t^{-\theta} == \beta c_{1+t}^{-\theta} (1 - \delta + \alpha k_{1+t}^{-1+\alpha} l_{1+t}^{1-\alpha} z_{1+t})$$

We replace each variable in this equation by its equivalent transformed expression.

ssrbcm77 =

$$\begin{aligned} & \text{ssrbcm76} /. \{c_t \rightarrow cs e^{ch_t}, c_{t+1} \rightarrow cs e^{ch_{t+1}}, k_{t+1} \rightarrow ks e^{kh_{t+1}}, l_{t+1} \rightarrow ls e^{lh_{t+1}}, z_{t+1} \rightarrow zs e^{zh_{t+1}}\} \\ (cs e^{ch_t})^{-\theta} & == (cs e^{ch_{t+1}})^{-\theta} \beta (1 + e^{zh_{t+1}} (e^{kh_{t+1}} ks)^{-1+\alpha} (e^{lh_{t+1}} ls)^{1-\alpha} zs \alpha - \delta) \end{aligned}$$

We take the first order Taylor series approximation of the left hand side of this equation with respect to ch_t around 0.

ssrbcm78 = Series[ssrbcm77[[1]], {ch_t, 0, 1}] // Normal

$$cs^{-\theta} - cs^{-\theta} \theta ch_t$$

We take the first order multivariate Taylor series approximation of the right hand side of equation **ssrbcm77** with respect to ch_{1+t} , zh_{1+t} , kh_{1+t} and lh_{1+t} respectively around 0, 0, 0 and 0.

ssrbcm79 = mulLinearTaylor[ssrbcm77[[2]], {ch_{1+t}, zh_{1+t}, kh_{1+t}, lh_{1+t}}, {0, 0, 0, 0}]

$$\begin{aligned} cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) - cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) \theta ch_{1+t} + \\ cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs (-1 + \alpha) \alpha \beta kh_{1+t} + cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs (1 - \alpha) \alpha \beta lh_{1+t} + cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs \alpha \beta zh_{1+t} \end{aligned}$$

We equalize the two expressions **ssrbcm78** and **ssrbcm79** to form an equation.

ssrbcm80 = ssrbcm78 == ssrbcm79

$$\begin{aligned} cs^{-\theta} - cs^{-\theta} \theta ch_t & == cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) - \\ & cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) \theta ch_{1+t} + cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs (-1 + \alpha) \alpha \beta kh_{1+t} + \\ & cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs (1 - \alpha) \alpha \beta lh_{1+t} + cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs \alpha \beta zh_{1+t} \end{aligned}$$

Now, we form the steady-state equation of the Euler equation.

ssrbcm81 = ssrbcm76 /. {c_t \to cs, c_{t+1} \to cs, k_{t+1} \to ks, l_{t+1} \to ls, z_{t+1} \to zs}

$$cs^{-\theta} == cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta)$$

We subtract equation **ssrbcm81** from equation **ssrbcm80** side by side.

ssrbcm82 = ssrbcm80[[1]] - ssrbcm81[[1]] == ssrbcm80[[2]] - ssrbcm81[[2]]

$$\begin{aligned} -cs^{-\theta} \theta ch_t & == -cs^{-\theta} \beta (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) \theta ch_{1+t} + cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs (-1 + \alpha) \alpha \beta kh_{1+t} + \\ & cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs (1 - \alpha) \alpha \beta lh_{1+t} + cs^{-\theta} ks^{-1+\alpha} ls^{1-\alpha} zs \alpha \beta zh_{1+t} \end{aligned}$$

We divide both sides of the previous equation by $-cs^{-\theta}$.

$$\text{ssrbcm83} = \frac{\text{ssrbcm82}[[1]]}{-\text{cs}^{-\theta}} ==$$

$$\text{Apply}[\text{Plus}, \text{Table}[\text{FullSimplify}[\frac{\text{ssrbcm82}[[2, i]]}{-\text{cs}^{-\theta}}], \{i, 1, \text{Length}[\text{ssrbcm82}[[2]]]\}]]$$

$$\theta \text{ ch}_t = \beta (1 + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} \alpha - \delta) \theta \text{ ch}_{1+t} -$$

$$\text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} (-1 + \alpha) \alpha \beta \text{kh}_{1+t} + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} (-1 + \alpha) \alpha \beta \text{lh}_{1+t} - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} \alpha \beta \text{zh}_{1+t}$$

We divide both sides of the steady-state equation by $\text{cs}^{-\theta}$.

$$\text{ssrbcm84} = \text{Map}[\frac{1}{\text{cs}^{-\theta}} \# \&, \text{ssrbcm81}]$$

$$1 == \beta (1 + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} \alpha - \delta)$$

We rewrite this equation as:

$$\text{ssrbcm85} = \text{ssrbcm84}[[2]] == \text{ssrbcm84}[[1]]$$

$$\beta (1 + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} \alpha - \delta) == 1$$

This last equation implies that the log-linearized Euler equation is :

$$\text{ssrbcm86} = \text{ssrbcm83} / . \text{ssrbcm85}[[1]] \rightarrow \text{ssrbcm85}[[2]]$$

$$\theta \text{ ch}_t ==$$

$$\theta \text{ ch}_{1+t} - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} (-1 + \alpha) \alpha \beta \text{kh}_{1+t} + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} (-1 + \alpha) \alpha \beta \text{lh}_{1+t} - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \text{zs} \alpha \beta \text{zh}_{1+t}$$

We set $\text{zs} = 1$ in this last equation.

$$\text{ssrbcm87} = \text{ssrbcm86} / . \text{zs} \rightarrow 1$$

$$\theta \text{ ch}_t = \theta \text{ ch}_{1+t} - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha) \alpha \beta \text{kh}_{1+t} + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha) \alpha \beta \text{lh}_{1+t} - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \alpha \beta \text{zh}_{1+t}$$

This last equation represents the log-linearized Euler equation without the expectations operator. This equality implies that the log-linearized Euler equation with the expectations operator is:

$$\text{ssrbcm88} =$$

$$\text{ssrbcm87} / . \{ \text{ch}_{1+t} \rightarrow \text{EX}_t[\text{ch}_{1+t}], \text{kh}_{1+t} \rightarrow \text{EX}_t[\text{kh}_{1+t}], \text{lh}_{1+t} \rightarrow \text{EX}_t[\text{lh}_{1+t}], \text{zh}_{1+t} \rightarrow \text{EX}_t[\text{zh}_{1+t}] \}$$

$$\theta \text{ ch}_t = \theta \text{EX}_t[\text{ch}_{1+t}] - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha) \alpha \beta \text{EX}_t[\text{kh}_{1+t}] +$$

$$\text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha) \alpha \beta \text{EX}_t[\text{lh}_{1+t}] - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \alpha \beta \text{EX}_t[\text{zh}_{1+t}]$$

In traditional economic notation, this equation can be rewritten

as:

$$\theta \hat{c}_t = \theta \text{E}_t(\hat{c}_{t+1}) - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha) \alpha \beta \text{E}_t(\hat{k}_{t+1}) +$$

$$\text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha) \alpha \beta \text{E}_t(\hat{l}_{t+1}) - \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \alpha \beta \text{E}_t(\hat{z}_{t+1})$$

Where a hat over a variable indicates percentage deviation of the variable around the steady-state; $\text{E}_t(\cdot)$ is the expectations operator and an s near a variable designates the steady-state value of that variable.

3.3. The Resource Constraint

We recall the resource constraint.

$$\text{ssrbcm89} = \text{ssrbcm87}$$

$$\text{k}_{1+t} = -\text{c}_t + (1 - \delta) \text{k}_t + \text{k}_t^\alpha \text{l}_t^{1-\alpha} \text{z}_t$$

We replace each variable in this equation by its equivalent transformed expression.

$$\text{ssrbcm90} = \text{ssrbcm89} /. \{k_{t+1} \rightarrow ks e^{kh_{t+1}}, c_t \rightarrow cs e^{ch_t}, k_t \rightarrow ks e^{kh_t}, l_t \rightarrow ls e^{lh_t}, z_t \rightarrow zs e^{zh_t}\}$$

$$e^{kh_{t+1}} ks = -cs e^{ch_t} + e^{zh_t} (e^{kh_t} ks)^\alpha (e^{lh_t} ls)^{1-\alpha} zs + e^{kh_t} ks (1 - \delta)$$

We take the first order Taylor series approximation of the left hand side of this equation with respect to kh_{1+t} around 0.

$$\text{ssrbcm91} = \text{Series}[\text{ssrbcm90}[[1]], \{kh_{1+t}, 0, 1\}] // \text{Normal}$$

$$ks + ks kh_{1+t}$$

We take the first order multivariate Taylor series approximation of the right hand side of equation `ssrbcm90` with respect to ch_t , zh_t , kh_t and lh_t respectively around 0, 0, 0 and 0.

$$\text{ssrbcm92} = \text{mulLinearTaylor}[\text{ssrbcm90}[[2]], \{ch_t, zh_t, kh_t, lh_t\}, \{0, 0, 0, 0\}]$$

$$-cs + ks^\alpha ls^{1-\alpha} zs + ks (1 - \delta) - cs ch_t + (ks^\alpha ls^{1-\alpha} zs \alpha + ks (1 - \delta)) kh_t + ks^\alpha ls^{1-\alpha} zs (1 - \alpha) lh_t + ks^\alpha ls^{1-\alpha} zs zh_t$$

We equalize the two expressions `ssrbcm91` and `ssrbcm92` to form an equation.

$$\text{ssrbcm93} = \text{ssrbcm91} == \text{ssrbcm92}$$

$$ks + ks kh_{1+t} = -cs + ks^\alpha ls^{1-\alpha} zs + ks (1 - \delta) - cs ch_t + (ks^\alpha ls^{1-\alpha} zs \alpha + ks (1 - \delta)) kh_t + ks^\alpha ls^{1-\alpha} zs (1 - \alpha) lh_t + ks^\alpha ls^{1-\alpha} zs zh_t$$

Now, we form the steady-state equation of the Resource constraint.

$$\text{ssrbcm94} = \text{ssrbcm89} /. \{k_{t+1} \rightarrow ks, c_t \rightarrow cs, k_t \rightarrow ks, l_t \rightarrow ls, z_t \rightarrow zs\}$$

$$ks = -cs + ks^\alpha ls^{1-\alpha} zs + ks (1 - \delta)$$

We subtract equation `ssrbcm94` from equation `ssrbcm93` side by side.

$$\text{ssrbcm95} = \text{ssrbcm93}[[1]] - \text{ssrbcm94}[[1]] == \text{ssrbcm93}[[2]] - \text{ssrbcm94}[[2]]$$

$$ks kh_{1+t} = -cs ch_t + (ks^\alpha ls^{1-\alpha} zs \alpha + ks (1 - \delta)) kh_t + ks^\alpha ls^{1-\alpha} zs (1 - \alpha) lh_t + ks^\alpha ls^{1-\alpha} zs zh_t$$

We divide both sides of this equation by ks .

$$\text{ssrbcm96} = \frac{\text{ssrbcm95}[[1]]}{ks} == \text{Apply}[\text{Plus}, \text{Table}[\text{FullSimplify}[\frac{\text{ssrbcm95}[[2, i]]}{ks}], \{i, 1, \text{Length}[\text{ssrbcm95}[[2]]\}]]]$$

$$kh_{1+t} = -\frac{cs ch_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} zs \alpha - \delta) kh_t - ks^{-1+\alpha} ls^{1-\alpha} zs (-1 + \alpha) lh_t + ks^{-1+\alpha} ls^{1-\alpha} zs zh_t$$

We set $zs = 1$ in this last equation.

$$\text{ssrbcm97} = \text{ssrbcm96} /. zs \rightarrow 1$$

$$kh_{1+t} = -\frac{cs ch_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) kh_t - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) lh_t + ks^{-1+\alpha} ls^{1-\alpha} zh_t$$

This last equation represents the log-linearized resource constraint equation. In traditional economic notation, this equation can be rewritten as:

$$\hat{k}_{t+1} = -\frac{cs \hat{c}_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) \hat{k}_t - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \hat{l}_t + ks^{-1+\alpha} ls^{1-\alpha} \hat{z}_t$$

Where a hat over a variable indicates percentage deviation of the variable around the steady-state and an s near a variable designates the steady-state value of that variable.

3.4. The Technology Equation

We recall the technology equation.

$$\text{ssrbcm0d98} = \text{ssrbcm0d10}$$

$$\text{Log}[z_{1+t}] = (1 - \rho) \text{Log}[z_s] + \rho \text{Log}[z_t] + \epsilon_{1+t}$$

We replace each variable in this equation by its equivalent transformed expression.

$$\text{ssrbcm0d99} = \text{ssrbcm0d98} /. \{z_{1+t} \rightarrow z_s e^{zh_{t+1}}, z_t \rightarrow z_s e^{zh_t}\}$$

$$\text{Log}[e^{zh_{1+t}} z_s] = (1 - \rho) \text{Log}[z_s] + \rho \text{Log}[e^{zh_t} z_s] + \epsilon_{1+t}$$

We take the first order Taylor series approximation of the left hand side of this equation with respect to zh_{1+t} around 0.

$$\text{ssrbcm0d100} = \text{Series}[\text{ssrbcm0d99}[[1]], \{zh_{1+t}, 0, 1\}] // \text{Normal}$$

$$\text{Log}[z_s] + zh_{1+t}$$

We take the first order Taylor series approximation of the right hand side of equation `ssrbcm0d99` with respect to zh_t around 0.

$$\text{ssrbcm0d101} = \text{Series}[\text{ssrbcm0d99}[[2]], \{zh_t, 0, 1\}] // \text{Normal}$$

$$\text{Log}[z_s] + \rho zh_t + \epsilon_{1+t}$$

We equalize the two expressions `ssrbcm0d100` and `ssrbcm0d101` to form an equation.

$$\text{ssrbcm0d102} = \text{ssrbcm0d100} == \text{ssrbcm0d101}$$

$$\text{Log}[z_s] + zh_{1+t} == \text{Log}[z_s] + \rho zh_t + \epsilon_{1+t}$$

Now, we form the steady-state equation of the Technology equation.

$$\text{ssrbcm0d103} = \text{ssrbcm0d98} /. \{z_{1+t} \rightarrow z_s, z_t \rightarrow z_s, \epsilon_{1+t} \rightarrow 0\}$$

$$\text{Log}[z_s] == (1 - \rho) \text{Log}[z_s] + \rho \text{Log}[z_s]$$

We subtract equation `ssrbcm0d103` from equation `ssrbcm0d102` side by side.

$$\text{ssrbcm0d104} =$$

$$\text{ssrbcm0d102}[[1]] - \text{ssrbcm0d103}[[1]] == \text{ssrbcm0d102}[[2]] - \text{ssrbcm0d103}[[2]] // \text{Simplify}$$

$$\rho zh_t + \epsilon_{1+t} == zh_{1+t}$$

We rewrite this result as:

$$\text{ssrbcm0d105} = \text{ssrbcm0d104}[[2]] == \text{ssrbcm0d104}[[1]]$$

$$zh_{1+t} == \rho zh_t + \epsilon_{1+t}$$

This last equation represents the log-linearized technology equation. In traditional economic notation, this equation can be rewritten as:

$$\hat{Z}_{t+1} = \rho \hat{Z}_t + \epsilon_{1+t}$$

Where a hat over a variable indicates percentage deviation of the variable around the steady-state. Taking expectations of both sides of the technology equation, we get:

$$\text{ssrbcm0d106} = \text{EX}_t[zh_{1+t}] == \rho zh_t$$

$$\text{EX}_t[zh_{1+t}] == \rho zh_t$$

We rewrite this last equation without the expectations operator but we keep in mind that there is always an expectations operator with this equation written in this way.

$$\text{ssrbcm107} = \text{zh}_{1+t} == \rho \text{zh}_t$$

$$\text{zh}_{1+t} == \rho \text{zh}_t$$

3.5. The Output Equation

We recall the equation of output.

$$\text{ssrbcm108} = \text{ssrbcm56}$$

$$y_t = k_t^\alpha l_t^{1-\alpha} z_t$$

We replace each variable in this equation by its equivalent transformed expression.

$$\text{ssrbcm109} = \text{ssrbcm108} /. \{y_t \rightarrow y_s e^{y_{ht}}, k_t \rightarrow k_s e^{k_{ht}}, l_t \rightarrow l_s e^{l_{ht}}, z_t \rightarrow z_s e^{z_{ht}}\}$$

$$e^{y_{ht}} y_s = e^{z_{ht}} (e^{k_{ht}} k_s)^\alpha (e^{l_{ht}} l_s)^{1-\alpha} z_s$$

We take the first order Taylor series approximation of the left hand side of this equation with respect to y_{ht} around 0.

$$\text{ssrbcm110} = \text{Series}[\text{ssrbcm109}[[1]], \{y_{ht}, 0, 1\}] // \text{Normal}$$

$$y_s + y_s y_{ht}$$

We take the first order multivariate Taylor series approximation of the right hand side of equation `ssrbcm109` with respect to z_{ht} , k_{ht} and l_{ht} respectively around 0, 0 and 0.

$$\text{ssrbcm111} = \text{mulLinearTaylor}[\text{ssrbcm109}[[2]], \{z_{ht}, k_{ht}, l_{ht}\}, \{0, 0, 0\}]$$

$$k_s^\alpha l_s^{1-\alpha} z_s + k_s^\alpha l_s^{1-\alpha} z_s \alpha k_{ht} + k_s^\alpha l_s^{1-\alpha} z_s (1-\alpha) l_{ht} + k_s^\alpha l_s^{1-\alpha} z_s z_{ht}$$

We equalize the two expressions `ssrbcm110` and `ssrbcm111` to form an equation.

$$\text{ssrbcm112} = \text{ssrbcm110} == \text{ssrbcm111}$$

$$y_s + y_s y_{ht} = k_s^\alpha l_s^{1-\alpha} z_s + k_s^\alpha l_s^{1-\alpha} z_s \alpha k_{ht} + k_s^\alpha l_s^{1-\alpha} z_s (1-\alpha) l_{ht} + k_s^\alpha l_s^{1-\alpha} z_s z_{ht}$$

Now, we form the steady-state equation of the Output equation.

$$\text{ssrbcm113} = \text{ssrbcm108} /. \{y_t \rightarrow y_s, k_t \rightarrow k_s, l_t \rightarrow l_s, z_t \rightarrow z_s\}$$

$$y_s = k_s^\alpha l_s^{1-\alpha} z_s$$

We subtract equation `ssrbcm113` from equation `ssrbcm112` side by side.

$$\text{ssrbcm114} = \text{ssrbcm112}[[1]] - \text{ssrbcm113}[[1]] == \text{ssrbcm112}[[2]] - \text{ssrbcm113}[[2]]$$

$$y_s y_{ht} = k_s^\alpha l_s^{1-\alpha} z_s \alpha k_{ht} + k_s^\alpha l_s^{1-\alpha} z_s (1-\alpha) l_{ht} + k_s^\alpha l_s^{1-\alpha} z_s z_{ht}$$

We divide equation `ssrbcm114` by equation `ssrbcm113` side by side.

$$\text{ssrbcm115} = \frac{\text{ssrbcm114}[[1]]}{\text{ssrbcm113}[[1]]} == \frac{\text{ssrbcm114}[[2]]}{\text{ssrbcm113}[[2]]} // \text{Expand}$$

$$y_{ht} = \alpha k_{ht} + l_{ht} - \alpha l_{ht} + z_{ht}$$

We collect l_{ht} .

$$\text{ssrbcm116} = \text{Collect}[\text{ssrbcm115}, l_{ht}]$$

$$y_{ht} = \alpha k_{ht} + (1-\alpha) l_{ht} + z_{ht}$$

This last equation represents the log-linearized equation of output. In traditional economic notation, this equation can be rewritten as:

$$\hat{y}_t = \alpha \hat{k}_t + (1 - \alpha) \hat{l}_t + \hat{z}_t$$

Where a hat over a variable indicates percentage deviation of the variable around the steady-state.

3.6. The Identity Consumption plus Investment is Equal to Output

We recall the expression of Investment.

```
ssrbcm117 = ssrbcm61
```

$$iv_t = -c_t + y_t$$

We replace each variable in this equation by its equivalent transformed expression.

```
ssrbcm118 = ssrbcm117 /. {iv_t -> ivs e^{ivh_t}, c_t -> cs e^{ch_t}, y_t -> ys e^{yh_t}}
```

$$e^{ivh_t} ivs = -cs e^{ch_t} + e^{yh_t} ys$$

We take the first order Taylor series approximation of the left hand side of this equation with respect to ivh_t around 0.

```
ssrbcm119 = Series[ssrbcm118[[1]], {ivh_t, 0, 1}] // Normal
```

$$ivs + ivs ivh_t$$

We take the first order multivariate Taylor series approximation of the right hand side of equation `ssrbcm118` with respect to ch_t and yh_t respectively around 0 and 0.

```
ssrbcm120 = mulLinearTaylor[ssrbcm118[[2]], {ch_t, yh_t}, {0, 0}]
```

$$-cs + ys - cs ch_t + ys yh_t$$

We equalize the two expressions `ssrbcm119` and `ssrbcm120` to form an equation.

```
ssrbcm121 = ssrbcm119 == ssrbcm120
```

$$ivs + ivs ivh_t = -cs + ys - cs ch_t + ys yh_t$$

Now, we form the steady-state equation of the Identity.

```
ssrbcm122 = ssrbcm117 /. {iv_t -> ivs, c_t -> cs, y_t -> ys}
```

$$ivs = -cs + ys$$

We subtract equation `ssrbcm122` from equation `ssrbcm121` side by side.

```
ssrbcm123 = ssrbcm121[[1]] - ssrbcm122[[1]] == ssrbcm121[[2]] - ssrbcm122[[2]]
```

$$ivs ivh_t = -cs ch_t + ys yh_t$$

We divide both sides of this equation by `ivs`.

```
ssrbcm124 = Map[1/ivs # &, ssrbcm123] // Expand
```

$$ivh_t = -\frac{cs ch_t}{ivs} + \frac{ys yh_t}{ivs}$$

This last equation represents the log-linearized equation of the Identity Consumption plus Investment is Equal to Output. In traditional economic notation, this equation can be rewritten as:

$$\hat{i}_t = -\frac{cs \hat{c}_t}{ivs} + \frac{ys \hat{y}_t}{ivs}$$

Where a hat over a variable indicates percentage deviation of the variable around the steady-state and an s near a

variable designates the steady-state value of that variable.

4. Solving the Model

We start by defining the matrix form representation of the model.

4.1. Matrix Form Representation

Following *Hartley, Hoover and Salyer (1998)*, we write the model as a rational expectations equilibrium system of difference equation using the following equations: the labor-leisure choice equation, the Euler equation, the resource constraint and the technology equation. First, we define the vectors $u_t = (\hat{c}_t, \hat{k}_t, \hat{l}_t, \hat{z}_t)$ and $u_{t+1} = (\hat{c}_{t+1}, \hat{k}_{t+1}, \hat{l}_{t+1}, \hat{z}_{t+1})$. With this notation, the system of difference equation can be written as $A u_t = B E_t (u_{t+1})$, where A and B are matrices to be found and $E_t(\cdot)$ is the rational expectations operator. We enter the vectors u_t and u_{t+1} .

```
ssrbcm125 = u_t == {ch_t, kh_t, lh_t, zh_t}
```

```
u_t == {ch_t, kh_t, lh_t, zh_t}
```

```
ssrbcm126 = ssrbcm125 /. t -> t + 1
```

```
u_{t+1} == {ch_{t+1}, kh_{t+1}, lh_{t+1}, zh_{t+1}}
```

We remind the four log-linearized main equations of our model.

```
ssrbcm127 = ssrbcm73
```

```
ssrbcm128 = ssrbcm87
```

```
ssrbcm129 = ssrbcm97
```

```
ssrbcm130 = ssrbcm107
```

$$ch_t = \frac{\alpha kh_t}{\theta} - \frac{(\alpha + \gamma) lh_t}{\theta} + \frac{zh_t}{\theta}$$

$$\theta ch_t = \theta ch_{t+1} - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta kh_{t+1} + ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta lh_{t+1} - ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta zh_{t+1}$$

$$kh_{t+1} = -\frac{cs ch_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) kh_t - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) lh_t + ks^{-1+\alpha} ls^{1-\alpha} zh_t$$

$$zh_{t+1} = \rho zh_t$$

We cast these equations in the form of $A u_t = B E_t (u_{t+1})$ by putting all t variables on the left hand side and all t + 1 variables on the right hand side.

```
ssrbcm131 = ssrbcm127[[1]] - ssrbcm127[[2]] == 0
```

```
ssrbcm132 = ssrbcm128
```

```
ssrbcm133 = ssrbcm129[[2]] == ssrbcm129[[1]]
```

```
ssrbcm134 = ssrbcm130[[2]] == ssrbcm130[[1]]
```

$$ch_t - \frac{\alpha kh_t}{\theta} + \frac{(\alpha + \gamma) lh_t}{\theta} - \frac{zh_t}{\theta} = 0$$

$$\theta ch_t = \theta ch_{t+1} - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta kh_{t+1} + ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta lh_{t+1} - ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta zh_{t+1}$$

$$-\frac{cs ch_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) kh_t - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) lh_t + ks^{-1+\alpha} ls^{1-\alpha} zh_t = kh_{t+1}$$

$$\rho zh_t = zh_{t+1}$$

Now, we start extracting the A matrix.

```
ssrbcmod135 = CoefficientArrays[
  {ssrbcmod131, ssrbcmod132, ssrbcmod133, ssrbcmod134}, ssrbcmod125[[2]]] // Normal
{ {0, -\theta ch_{1+t} + ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta kh_{1+t} - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta lh_{1+t} + ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta zh_{1+t},
  -kh_{1+t}, -zh_{1+t}}, {{1, -\frac{\alpha}{\theta}, \frac{\alpha + \gamma}{\theta}, -\frac{1}{\theta}}, {\theta, 0, 0, 0}},
  { -\frac{cs}{ks}, 1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta, -ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha), ks^{-1+\alpha} ls^{1-\alpha} }, {0, 0, 0, \rho} } }
```

We take the A matrix.

```
ssrbcmod136 = A == ssrbcmod135[[2]]
A == { {1, -\frac{\alpha}{\theta}, \frac{\alpha + \gamma}{\theta}, -\frac{1}{\theta}}, {\theta, 0, 0, 0}},
  { -\frac{cs}{ks}, 1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta, -ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha), ks^{-1+\alpha} ls^{1-\alpha} }, {0, 0, 0, \rho} }
```

We make a copy of the previous expression.

```
ssrbcmod137 = ssrbcmod136
A == { {1, -\frac{\alpha}{\theta}, \frac{\alpha + \gamma}{\theta}, -\frac{1}{\theta}}, {\theta, 0, 0, 0}},
  { -\frac{cs}{ks}, 1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta, -ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha), ks^{-1+\alpha} ls^{1-\alpha} }, {0, 0, 0, \rho} }
```

We display this result in matrix form.

```
ssrbcmod138 = MatrixForm[ssrbcmod137] // TraditionalForm
A = \begin{pmatrix} 1 & -\frac{\alpha}{\theta} & \frac{\alpha + \gamma}{\theta} & -\frac{1}{\theta} \\ \theta & 0 & 0 & 0 \\ -\frac{cs}{ks} & ks^{\alpha-1} \alpha ls^{1-\alpha} - \delta + 1 & -ks^{\alpha-1} ls^{1-\alpha} (\alpha - 1) & ks^{\alpha-1} ls^{1-\alpha} \\ 0 & 0 & 0 & \rho \end{pmatrix}
```

Now, we start extracting the B matrix.

```
ssrbcmod139 = CoefficientArrays[
  {ssrbcmod131, ssrbcmod132, ssrbcmod133, ssrbcmod134}, ssrbcmod126[[2]]] // Normal
{ {ch_t - \frac{\alpha kh_t}{\theta} + \frac{(\alpha + \gamma) lh_t}{\theta} - \frac{zh_t}{\theta}, \theta ch_t,
  -\frac{cs ch_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) kh_t - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) lh_t + ks^{-1+\alpha} ls^{1-\alpha} zh_t, \rho zh_t},
  {0, 0, 0, 0}, {-\theta, ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta, -ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta, ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta},
  {0, -1, 0, 0}, {0, 0, 0, -1} } }
```

We extract the second term of this result.

```
ssrbcmod140 = ssrbcmod139[[2]]
{ {0, 0, 0, 0}, {-\theta, ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta, -ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) \alpha \beta, ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta},
  {0, -1, 0, 0}, {0, 0, 0, -1} }
```

We multiply this last result by -1 to obtain the B matrix.

```
ssrbcm141 = B == Map[-1 # &, ssrbcm140, {2}]
```

$$B = \left\{ \{0, 0, 0, 0\}, \left\{ \theta, -ks^{-1+\alpha} ls^{1-\alpha} (-1+\alpha) \alpha \beta, ks^{-1+\alpha} ls^{1-\alpha} (-1+\alpha) \alpha \beta, -ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta \right\}, \right. \\ \left. \{0, 1, 0, 0\}, \{0, 0, 0, 1\} \right\}$$

We make a copy of the previous expression.

```
ssrbcm142 = ssrbcm141
```

$$B = \left\{ \{0, 0, 0, 0\}, \left\{ \theta, -ks^{-1+\alpha} ls^{1-\alpha} (-1+\alpha) \alpha \beta, ks^{-1+\alpha} ls^{1-\alpha} (-1+\alpha) \alpha \beta, -ks^{-1+\alpha} ls^{1-\alpha} \alpha \beta \right\}, \right. \\ \left. \{0, 1, 0, 0\}, \{0, 0, 0, 1\} \right\}$$

We display this result in matrix form.

```
ssrbcm143 = MatrixForm[ssrbcm142] // TraditionalForm
```

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta & -ks^{\alpha-1} ls^{1-\alpha} (\alpha-1) \alpha \beta & ks^{\alpha-1} ls^{1-\alpha} (\alpha-1) \alpha \beta & -ks^{\alpha-1} ls^{1-\alpha} \alpha \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4.2. Defining the Parameters (Calibration)

In this section, we define the values of the parameters (calibration). We take these values from the literature. $\theta = 3$ is from *Barro and Sala-i-Martin (2004)*; $\gamma = 0.33$ is from *Mandelman and Zanetti (2008)*; $\beta = 0.99$, $\alpha = 0.36$, $\delta = 0.025$, $\psi = 3$, $\sigma^2 = 0.007$ (so $\sigma = 0.083666$) and $\rho = 0.95$ are from *Hartley, Hoover and Salyer (1998)*. With these values of the parameters we can find the values of the steady-state variables, all equations and, the matrices A and B.

We start by finding the numerical steady-state value of labor. We remind the equation of the steady-state value of labor.

```
ssrbcm144 = ssrbcm49
```

$$ls = \left(\left(-(-1+\alpha) \left(\left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{-1/\theta} \left(\left(\left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}}$$

Then we have:

```
ssrbcm145 = ssrbcm144 /. {alpha -> 0.36, beta -> 0.99, delta -> 0.025, theta -> 3, psi -> 3, gamma -> 0.33}
```

$$ls = 0.374007$$

We remind the steady-state value of consumption.

```
ssrbcm146 = ssrbcm51
```

$$cs = \left(-(-1+\alpha) \left(\left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{\frac{1}{\theta}} \psi^{-1/\theta} \\ \left(\left(\left(-(-1+\alpha) \left(\left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} \right)^{-1/\theta} \left(\left(\left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \right)^{\alpha} - \left(\frac{1+\beta(-1+\delta)}{\alpha\beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^{-\frac{\gamma}{\theta}}$$

We find the numerical steady-state value of consumption.

```
ssrbcm147 = ssrbcm146 /. {alpha -> 0.36, beta -> 0.99, delta -> 0.025, theta -> 3, psi -> 3, gamma -> 0.33}
```

$$cs = 1.03014$$

We remind the steady-state value of capital.

ssrbcm148 = ssrbcm55

$$k_s = \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}}$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}}$$

We find the numerical steady-state value of capital.

ssrbcm149 = ssrbcm148 /. {α -> 0.36, β -> 0.99, δ -> 0.025, θ -> 3, ψ -> 3, γ -> 0.33}

ks = 14.2083

We remind the steady-state value of output.

ssrbcm150 = ssrbcm59

ys =

$$\left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^{1-\alpha}$$

$$\left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)$$

$$\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^\alpha$$

We find the numerical steady-state value of output.

ssrbcm151 = ssrbcm150 /. {α -> 0.36, β -> 0.99, δ -> 0.025, θ -> 3, ψ -> 3, γ -> 0.33}

ys = 1.38534

We remind the steady-state value of investment.

ssrbcm152 = ssrbcm63

$$\begin{aligned}
 \text{ivs} = & - \left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{\frac{1}{\theta}} \psi^{-1/\theta} \\
 & \left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} + \\
 & \left(\left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} 1-\alpha \\
 & \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \left(\left(-(-1 + \alpha) \left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha \right)^{-1/\theta} \right. \right. \\
 & \left. \left. \left(\left(\left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \right)^\alpha - \left(\frac{1 + \beta (-1 + \delta)}{\alpha \beta} \right)^{\frac{1}{-1+\alpha}} \delta \right) \psi^{\frac{1}{\theta}} \right)^{-\frac{\theta}{\gamma+\theta}} \right)^\alpha
 \end{aligned}$$

We find the numerical steady-state value of investment.

ssrbcm153 = ssrbcm152 /. {α -> 0.36, β -> 0.99, δ -> 0.025, θ -> 3, ψ -> 3, γ -> 0.33}

ivs = 0.355206

An alternative way to find the numerical steady-state values is to enter the values of the parameters with an equal sign at the beginning of this section and recall all the equations we want. But we avoid this method since we might still need symbolic values of the equations in the remaining of the notebook. Now we find the numerical values of the matrices A and B. We first remind the value of matrix A.

ssrbcm154 = ssrbcm136

$$\begin{aligned}
 A = & \left\{ \left\{ 1, -\frac{\alpha}{\theta}, \frac{\alpha + \gamma}{\theta}, -\frac{1}{\theta} \right\}, \{ \theta, 0, 0, 0 \}, \right. \\
 & \left. \left\{ -\frac{\text{cs}}{\text{ks}}, 1 + \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \alpha - \delta, -\text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} (-1 + \alpha), \text{ks}^{-1+\alpha} \text{ls}^{1-\alpha} \right\}, \{ 0, 0, 0, \rho \} \right\}
 \end{aligned}$$

Then we find the numerical value of matrix A.

ssrbcm155 = ssrbcm154 /. {α -> 0.36, δ -> 0.025, θ -> 3, γ -> 0.33, ρ -> 0.95, cs -> ssrbcm147[[2]], ks -> ssrbcm149[[2]], ls -> ssrbcm145[[2]]} // N

A = {{1., -0.12, 0.23, -0.333333}, {3., 0., 0., 0.},
{-0.0725028, 1.0101, 0.0624018, 0.0975028}, {0., 0., 0., 0.95}}

We make a copy of matrix A.

ssrbcm156 = ssrbcm155

A = {{1., -0.12, 0.23, -0.333333}, {3., 0., 0., 0.},
{-0.0725028, 1.0101, 0.0624018, 0.0975028}, {0., 0., 0., 0.95}}

We display this last result in matrix form.

```
ssrbcm157 = MatrixForm[ssrbcm156] // TraditionalForm
```

$$A = \begin{pmatrix} 1. & -0.12 & 0.23 & -0.333333 \\ 3. & 0. & 0. & 0. \\ -0.0725028 & 1.0101 & 0.0624018 & 0.0975028 \\ 0. & 0. & 0. & 0.95 \end{pmatrix}$$

We remind the value of matrix B.

```
ssrbcm158 = ssrbcm141
```

$$B = \{ \{0, 0, 0, 0\}, \{ \theta, -ks^{-1+\alpha} 1s^{1-\alpha} (-1+\alpha) \alpha \beta, ks^{-1+\alpha} 1s^{1-\alpha} (-1+\alpha) \alpha \beta, -ks^{-1+\alpha} 1s^{1-\alpha} \alpha \beta \}, \\ \{0, 1, 0, 0\}, \{0, 0, 0, 1\} \}$$

Then we find the numerical value of matrix B.

```
ssrbcm159 = ssrbcm158 /. 
```

```
{\alpha -> 0.36, \theta -> 3, \beta -> 0.99, ks -> ssrbcm149[[2]], 1s -> ssrbcm145[[2]]} // N
```

$$B = \{ \{0., 0., 0., 0.\}, \{3., 0.02224, -0.02224, -0.03475\}, \{0., 1., 0., 0.\}, \{0., 0., 0., 1.\} \}$$

We make a copy of matrix B.

```
ssrbcm160 = ssrbcm159
```

$$B = \{ \{0., 0., 0., 0.\}, \{3., 0.02224, -0.02224, -0.03475\}, \{0., 1., 0., 0.\}, \{0., 0., 0., 1.\} \}$$

We display this last result in matrix form.

```
ssrbcm161 = MatrixForm[ssrbcm160] // TraditionalForm
```

$$B = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 3. & 0.02224 & -0.02224 & -0.03475 \\ 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 1. \end{pmatrix}$$

To solve the model, we will follow the solution method proposed by *Hartley, Hoover and Salyer (1998)*. Remember that we wrote the model as $A u_t = B E_t (u_{t+1})$. Premultiply both sides of this system by A^{-1} . We get $u_t = A^{-1} B E_t (u_{t+1})$. Then we decompose the matrix $A^{-1} B$ as: $A^{-1} B = Q \Lambda Q^{-1}$. Where Q is a matrix which has as columns the eigenvectors of the matrix $A^{-1} B$ and Λ is a diagonal matrix which has as diagonal elements the eigenvalues of the matrix $A^{-1} B$. Replacing the matrix $A^{-1} B$ by its value, we get: $u_t = Q \Lambda Q^{-1} E_t (u_{t+1})$. We premultiply both sides of this equation by Q^{-1} . We then have:

$Q^{-1} u_t = \Lambda Q^{-1} E_t (u_{t+1})$. We rewrite this as: $Q^{-1} u_t = \Lambda E_t (Q^{-1} u_{t+1})$. We set $d_t = Q^{-1} u_t$ and $d_{t+1} = Q^{-1} u_{t+1}$. Hence we have:

$d_t = \Lambda E_t(d_{t+1})$. Since Λ is a diagonal matrix, we can rewrite this last equation as independent equations having the following form: $d_{i,t} = \lambda_i E_t(d_{i,t+1})$. Where i indicates the equations and λ_i is an eigenvalue of the matrix $A^{-1} B$. It is important to note that $i = 1..4$ here since Q^{-1} is 4 by 4 matrix in this model. Performing recursive forward substitution in the equations $d_{i,t} = \lambda_i E_t(d_{i,t+1})$ for T periods yields: $d_{i,t} = \lambda_i^T E_t(d_{i,t+T})$. This last equation implies that the stable rational expectations solution to our model is related with the eigenvalue that has an absolute value less than unity. This result comes from the fact that taking the limit of the expression $\lambda_i^T E_t(d_{i,t+T})$ as T goes toward infinity, gives 0 only if $|\lambda_i|$ is less than one. This implies that $d_{i,t} = 0$ for that particular eigenvalue. This last expression is the solution we are looking for.

Now let's find the $A^{-1} B$ matrix.

```
ssrbcm162 = Inverse[ssrbcm155[[2]]].ssrbcm159[[2]]
```

$$\{ \{1., 0.00741333, -0.00741333, -0.0115833\}, \{0.329748, 0.961531, -0.00244453, -0.193557\}, \\ \{-4.17578, 0.469437, 0.0309565, 1.47493\}, \{0., 0., 0., 1.05263\} \}$$

Then we find the eigenvalues and eigenvectors of matrix $A^{-1} B$.


```
ssrbcm163 = Eigensystem[ssrbcm162]
```

```
{ {1.05263, 1.04626, 0.946224, 0.},
  { {-0.199763, -0.840665, 0.500974, 0.0490218}, {-0.213813, -0.84617, 0.48814, 0.},
  {-0.0337368, 0.818392, 0.573669, 0.}, {0.00741313, -5.00885 × 10-18, 0.999973, 0.} } }
```

We extract the eigenvalues from the previous result.

```
ssrbcm164 = ssrbcm163[[1]]
```

```
{1.05263, 1.04626, 0.946224, 0.}
```

We form the Λ matrix.

```
ssrbcm165 =  $\Lambda$  == DiagonalMatrix[ssrbcm164]
```

```
 $\Lambda$  == { {1.05263, 0., 0., 0.}, {0., 1.04626, 0., 0.}, {0., 0., 0.946224, 0.}, {0., 0., 0., 0.} }
```

We make a copy of the previous result.

```
ssrbcm166 = ssrbcm165
```

```
 $\Lambda$  == { {1.05263, 0., 0., 0.}, {0., 1.04626, 0., 0.}, {0., 0., 0.946224, 0.}, {0., 0., 0., 0.} }
```

We display this last result in matrix form.

```
ssrbcm167 = MatrixForm[ssrbcm166] // TraditionalForm
```

$$\Lambda = \begin{pmatrix} 1.05263 & 0. & 0. & 0. \\ 0. & 1.04626 & 0. & 0. \\ 0. & 0. & 0.946224 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

This last result shows us that the third eigenvalue equal to 0.946224 has an absolute value less than one. This result implies that the stable rational expectations solution to our model is related to this eigenvalue.

Now we extract the eigenvectors from expression ssrbcm163.

```
ssrbcm168 = ssrbcm163[[2]]
```

```
{ {-0.199763, -0.840665, 0.500974, 0.0490218}, {-0.213813, -0.84617, 0.48814, 0.},
  {-0.0337368, 0.818392, 0.573669, 0.}, {0.00741313, -5.00885 × 10-18, 0.999973, 0.} }
```

We form the matrix Q^{-1} .

```
ssrbcm169 = Inverse[ssrbcm168] // Transpose
```

```
{ {0., 0., 0., 20.3991}, {-3.89544, -0.180826, 0.0288782, -19.2699},
  {-4.02766, 1.03494, 0.0298584, 1.03028}, {4.21218, -0.505461, 0.968801, -1.40406} }
```

We make a copy of the previous result.

```
ssrbcm170 = ssrbcm169
```

```
{ {0., 0., 0., 20.3991}, {-3.89544, -0.180826, 0.0288782, -19.2699},
  {-4.02766, 1.03494, 0.0298584, 1.03028}, {4.21218, -0.505461, 0.968801, -1.40406} }
```

We display Q^{-1} in matrix form.

```
ssrbcm171 = MatrixForm[ssrbcm170] // TraditionalForm
```

$$Q^{-1} = \begin{pmatrix} 0. & 0. & 0. & 20.3991 \\ -3.89544 & -0.180826 & 0.0288782 & -19.2699 \\ -4.02766 & 1.03494 & 0.0298584 & 1.03028 \\ 4.21218 & -0.505461 & 0.968801 & -1.40406 \end{pmatrix}$$

As we explained previously, a stable rational expectations solution is related to the eigenvalue less than one in absolute value which is here the third eigenvalue. This implies that the third row of the matrix Q^{-1} gives the linear

restriction we are looking for. Hence we extract the third row of matrix Q^{-1} .

```
ssrbcm172 = ssrbcm169[[3]]
{-4.02766, 1.03494, 0.0298584, 1.03028}
```

Remember that the rational expectations solution is given by the expression $d_{i,t} = 0$. Also recall that $d_t = Q^{-1} u_t$. Hence we form the equation $d_{i,t} = 0$, which is our rational expectations solution.

```
ssrbcm173 = ssrbcm172.ssrbcm125[[2]] == 0
-4.02766 ch_t + 1.03494 kh_t + 0.0298584 lh_t + 1.03028 zh_t == 0
```

We isolate ch_t in this last expression.

```
ssrbcm174 = Solve[ssrbcm173, ch_t] // Expand
{{ch_t -> 0.256959 kh_t + 0.00741333 lh_t + 0.2558 zh_t}}
```

We transform this expression to an equation. This is our rational expectations solution.

```
ssrbcm175 = ssrbcm174[[1, 1, 1]] == ssrbcm174[[1, 1, 2]]
ch_t == 0.256959 kh_t + 0.00741333 lh_t + 0.2558 zh_t
```

In addition to this last equation, the other equations describing the equilibrium conditions of our model are: the economy resource constraint, the labor-leisure choice equation, the equation of output, the identity consumption plus investment is equal to output and the technology equation. In what follows, we find the numerical values of these equations.

We recall the economy resource constraint.

```
ssrbcm176 = ssrbcm97
```

$$kh_{1+t} = -\frac{cs ch_t}{ks} + (1 + ks^{-1+\alpha} ls^{1-\alpha} \alpha - \delta) kh_t - ks^{-1+\alpha} ls^{1-\alpha} (-1 + \alpha) lh_t + ks^{-1+\alpha} ls^{1-\alpha} zh_t$$

We calculate the numerical value of the resource constraint.

```
ssrbcm177 = ssrbcm176 /. {alpha -> 0.36, delta -> 0.025,
  cs -> ssrbcm147[[2]], ks -> ssrbcm149[[2]], ls -> ssrbcm145[[2]]}
kh_{1+t} == -0.0725028 ch_t + 1.0101 kh_t + 0.0624018 lh_t + 0.0975028 zh_t
```

We recall the labor-leisure choice equation.

```
ssrbcm178 = ssrbcm73
```

$$ch_t = \frac{\alpha kh_t}{\theta} - \frac{(\alpha + \gamma) lh_t}{\theta} + \frac{zh_t}{\theta}$$

We isolate lh_t in this last equation.

```
ssrbcm179 = Solve[ssrbcm178, lh_t] // Expand
```

$$\left\{ \left\{ lh_t \rightarrow -\frac{\theta ch_t}{\alpha + \gamma} + \frac{\alpha kh_t}{\alpha + \gamma} + \frac{zh_t}{\alpha + \gamma} \right\} \right\}$$

We transform this expression to an equation.

```
ssrbcm180 = ssrbcm179[[1, 1, 1]] == ssrbcm179[[1, 1, 2]]
lh_t == -\frac{\theta ch_t}{\alpha + \gamma} + \frac{\alpha kh_t}{\alpha + \gamma} + \frac{zh_t}{\alpha + \gamma}
```

We find the numerical value of the labor-leisure choice equation.

```
ssrbcm181 = ssrbcm180 /. {θ → 3, α → 0.36, γ → 0.33}
```

$$lh_t = -4.34783 ch_t + 0.521739 kh_t + 1.44928 zh_t$$

We recall the equation of output.

```
ssrbcm182 = ssrbcm116
```

$$yh_t = \alpha kh_t + (1 - \alpha) lh_t + zh_t$$

We find the numerical value of the equation of output.

```
ssrbcm183 = ssrbcm182 /. α → 0.36
```

$$yh_t = 0.36 kh_t + 0.64 lh_t + zh_t$$

We recall the identity consumption plus investment is equal to output.

```
ssrbcm184 = ssrbcm124
```

$$ivh_t = -\frac{cs ch_t}{ivs} + \frac{ys yh_t}{ivs}$$

We find the numerical value of the identity consumption plus investment is equal to output.

```
ssrbcm185 =
```

```
ssrbcm184 /. {cs -> ssrbcm147[[2]], ivs -> ssrbcm153[[2]], ys -> ssrbcm151[[2]]}
```

$$ivh_t = -2.90011 ch_t + 3.90011 yh_t$$

We recall the technology equation.

```
ssrbcm186 = ssrbcm105
```

$$zh_{1+t} = \rho zh_t + \epsilon_{1+t}$$

We lag t by one period in this last equation.

```
ssrbcm187 = ssrbcm186 /. t → t - 1
```

$$zh_t = \rho zh_{-1+t} + \epsilon_t$$

We find the numerical value of the technology equation.

```
ssrbcm188 = ssrbcm187 /. ρ → 0.95
```

$$zh_t = 0.95 zh_{-1+t} + \epsilon_t$$

We express the variables ch_t , lh_t , yh_t , kh_{1+t} and ivh_t in function of kh_t and zh_t only.

```
ssrbcm189 = Solve[{ssrbcm175, ssrbcm177, ssrbcm181, ssrbcm183, ssrbcm185},
  {ch_t, lh_t, yh_t, kh_{1+t}, ivh_t}]
```

```
{ {ch_t → 0.252683 kh_t + 0.258221 zh_t,
  lh_t → -0.576882 kh_t + 0.326575 zh_t, yh_t → -0.00920447 kh_t + 1.20901 zh_t,
  kh_{1+t} → 0.955782 kh_t + 0.0991599 zh_t, ivh_t → 0. - 0.768707 kh_t + 3.9664 zh_t} }
```

We rearrange this last expression as:

```
ssrbcm190 = Chop[ssrbcm189] // Flatten
```

```
{ ch_t → 0.252683 kh_t + 0.258221 zh_t,
  lh_t → -0.576882 kh_t + 0.326575 zh_t, yh_t → -0.00920447 kh_t + 1.20901 zh_t,
  kh_{1+t} → 0.955782 kh_t + 0.0991599 zh_t, ivh_t → -0.768707 kh_t + 3.9664 zh_t }
```

Now we transform each expression of this preceding result to an equation. We begin with the equation of consumption.

```
ssrbcm191 = ssrbcm190[[1, 1]] == ssrbcm190[[1, 2]]
```

```
cht == 0.252683 kht + 0.258221 zht
```

The equation of labor.

```
ssrbcm192 = ssrbcm190[[2, 1]] == ssrbcm190[[2, 2]]
```

```
lht == -0.576882 kht + 0.326575 zht
```

The equation of output.

```
ssrbcm193 = ssrbcm190[[3, 1]] == ssrbcm190[[3, 2]]
```

```
yht == -0.00920447 kht + 1.20901 zht
```

The equation of next period capital (kh_{1+t}).

```
ssrbcm194 = ssrbcm190[[4, 1]] == ssrbcm190[[4, 2]]
```

```
kh1+t == 0.955782 kht + 0.0991599 zht
```

The equation of investment.

```
ssrbcm195 = ssrbcm190[[5, 1]] == ssrbcm190[[5, 2]]
```

```
ivht == -0.768707 kht + 3.9664 zht
```

Now we write functions that numerically evaluate each of these preceding equations. We start with the technology equation. We first remind the technology equation.

```
ssrbcm196 = ssrbcm188
```

```
zht == 0.95 zh-1+t + εt
```

The function that numerically evaluates the technology equation is given in the next line. This function takes three arguments: the first is the autocorrelation parameter, the second represents a list containing the numerical values of the innovations to technology and the third is the total time length we want for the technology series.

```
fzh[ρ_, e_, T_] :=
```

```
Module[{res1, zh, t, T1}, T1 = T + 1; zh = Table[0, {i, T1}]; res1 = Table[0, {i, T1}];
  For[t = 2, t ≤ T1, t++,
    zh[[t]] = ρ zh[[t - 1]] + e[[t]];
    res1[[t]] = zh[[t]];
  ];
  res1]
```

We recall the equation of next period capital stock (kh_{1+t}).

```
ssrbcm197 = ssrbcm194
```

```
kh1+t == 0.955782 kht + 0.0991599 zht
```

Now we write a function that calculates the series for the capital stock (kh_t). This function takes four arguments: the first two arguments represent the numerical coefficients of the equation of next period capital, the third argument is the total time length we want for the capital stock series and the fourth is the technology series.

```
fkh[a_, b_, T_, zh_] :=
```

```
Module[{res1, res2, t, T1, kh}, T1 = T + 1; kh = Table[0, {i, T1}]; res1 = Table[0, {i, T1}];
  For[t = 1, t ≤ (T1 - 1), t++,
    kh[[t + 1]] = a kh[[t]] + b zh[[t]];
    res1[[t + 1]] = kh[[t + 1]];
  ];
  res2 = res1; Most[res2]]
```

We remind the equation of consumption.

```
ssrbcm198 = ssrbcm191
cht == 0.252683 kht + 0.258221 zht
```

We write a function that calculates the consumption series. This function takes four arguments: the first two are respectively the coefficients of capital and technology in the equation of consumption, the third is the capital series and the fourth the technology series.

```
fch[ac_, bc_, kh_, zh_] := ac kh + bc zh
```

We remind the equation of labor.

```
ssrbcm199 = ssrbcm192
lht == -0.576882 kht + 0.326575 zht
```

We write a function that calculates the labor series. This function takes four arguments: the first two are respectively the coefficients of capital and technology in the equation of labor, the third is the capital series and the fourth the technology series.

```
flh[a1_, b1_, kh_, zh_] := a1 kh + b1 zh
```

We remind the equation of output.

```
ssrbcm200 = ssrbcm193
yht == -0.00920447 kht + 1.20901 zht
```

We write a function that calculates the output series. This function takes four arguments: the first two are respectively the coefficients of capital and technology in the equation of output, the third is the capital series and the fourth the technology series.

```
fyh[ay_, by_, kh_, zh_] := ay kh + by zh
```

We remind the equation of investment.

```
ssrbcm201 = ssrbcm195
ivht == -0.768707 kht + 3.9664 zht
```

We write a function that calculates the investment series. This function takes four arguments: the first two are respectively the coefficients of capital and technology in the equation of investment, the third is the capital series and the fourth the technology series.

```
fiyh[aiv_, biv_, kh_, zh_] := aiv kh + biv zh
```

5. Studying the Economy

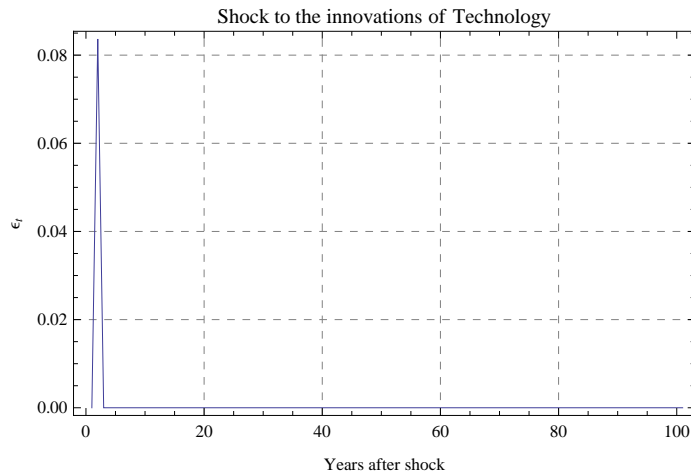
In this section we study the economy formed by our theoretical model.

5.1. Impulse Response Functions

To generate the impulse response functions, we first create the innovations to technology. We shock these innovations by one standard deviation and plot the result.

```
ep1 = Table[0, {i, 100}];
ep2 = Insert[ep1, 0.083666, 2];
```

```
ssrbcm202 = ListLinePlot[ep2, FrameLabel -> {"Years after shock", "\u03b5\u209c"},
  PlotLabel -> "Shock to the innovations of Technology",
  Frame -> True, GridLines -> Automatic, GridLinesStyle -> Dashed]
```



Now we remind the equation of technology and calculate the impulse response function of technology.

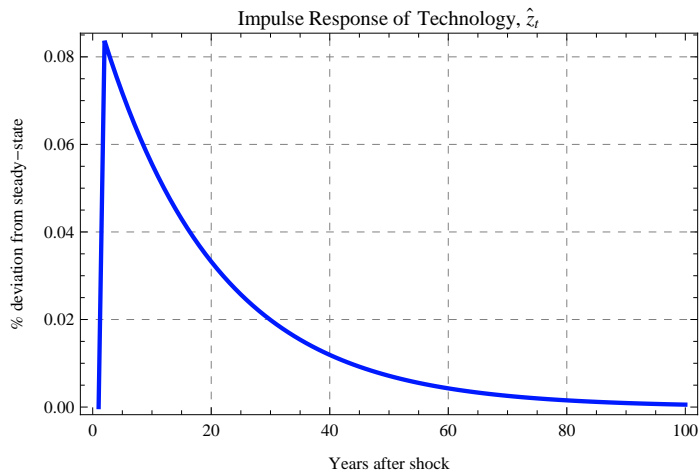
```
ssrbcm196
```

$$zh_t = 0.95 zh_{-1+t} + \epsilon_t$$

```
ssrbcm203 = fzh[0.95, ep2, 99];
```

We plot the impulse response function of technology. The graph shows that technology starts from its steady-state value of 0 and increases in response to the shock of one standard deviation, and then gradually decreases towards its steady-state value.

```
ssrbcm204 = ListLinePlot[ssrbcm203,
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response of Technology, \u0177\u209c", Frame -> True, GridLines -> Automatic,
  GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.007], Hue[0.65]}]
```



We remind the equation of next period capital stock (kh_{t+1}) and calculate the impulse response function of capital stock (kh_t).

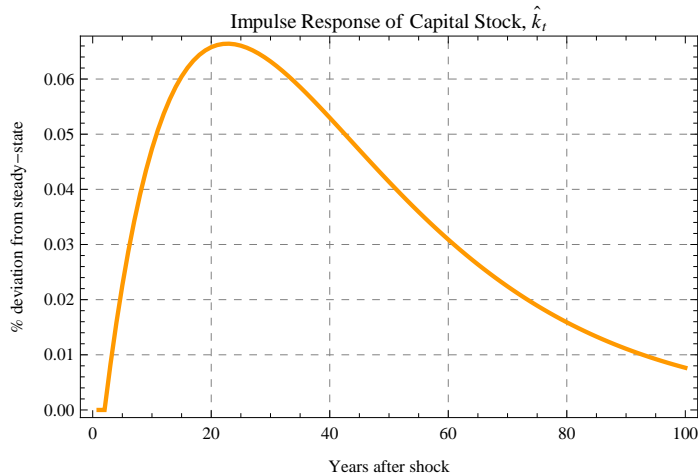
```
ssrbcm197
```

$$kh_{1+t} = 0.955782 kh_t + 0.0991599 zh_t$$

```
ssrbcm205 = fkh[ssrbcm197[[2, 1, 1]], ssrbcm197[[2, 2, 1]], 100, ssrbcm203];
```

We plot the impulse response function of capital stock (kh_t). We see that the graph of the impulse response function of capital stock is hump-shaped.

```
ssrbcm206 = ListLinePlot[ssrbcm205,
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response of Capital Stock,  $\hat{k}_t$ ", Frame -> True,
  GridLines -> Automatic, GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.007], Hue[0.1]}]
```



We remind the equation of consumption.

```
ssrbcm198
```

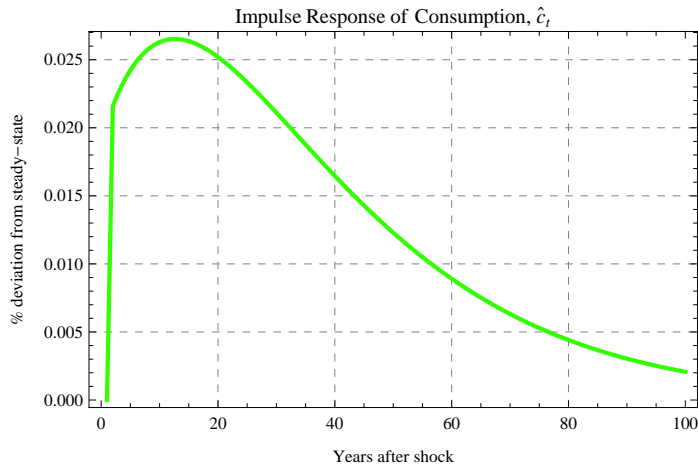
$$ch_t = 0.252683 kh_t + 0.258221 zh_t$$

We calculate the impulse response function of consumption.

```
ssrbcm207 = fch[ssrbcm198[[2, 1, 1]], ssrbcm198[[2, 2, 1]], ssrbcm205, ssrbcm203];
```

We plot the impulse response function of consumption. Like the capital stock, we see that the graph of the impulse response function of consumption is hump-shaped.

```
ssrbcm208 = ListLinePlot[ssrbcm207,
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response of Consumption,  $\hat{c}_t$ ", Frame -> True, GridLines -> Automatic,
  GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.007], Hue[0.3]}]
```



We remind the equation of labor.

```
ssrbcm192
```

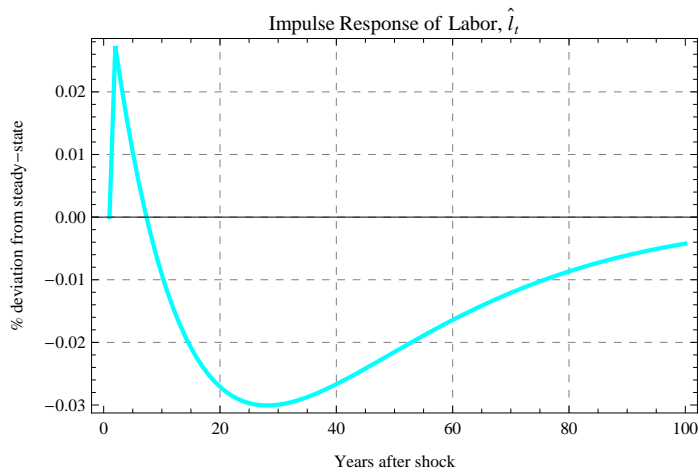
$$lh_t = -0.576882 kh_t + 0.326575 zh_t$$

We calculate the impulse response function of labor.

```
ssrbcm209 = flh[ssrbcm192[[2, 1, 1]], ssrbcm192[[2, 2, 1]], ssrbcm205, ssrbcm203];
```

We plot the impulse response function of labor. We see that labor first augments and then decreases gradually, and increases again progressively to its steady-state level.

```
ssrbcm210 = ListLinePlot[ssrbcm209,
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response of Labor,  $\hat{l}_t$ ", Frame -> True, GridLines -> Automatic,
  GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.007], Hue[0.5]}]
```



We remind the equation of output.

```
ssrbcm200
```

$$yh_t = -0.00920447 kh_t + 1.20901 zh_t$$

We calculate the impulse response function of output.

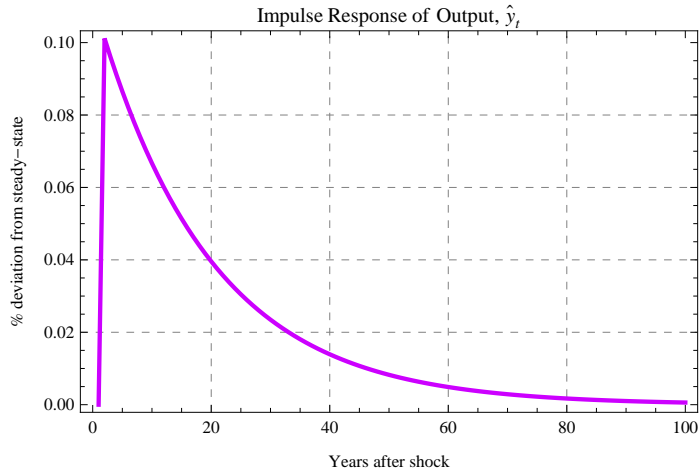
```
ssrbcm211 = fyh[ssrbcm200[[2, 1, 1]], ssrbcm200[[2, 2, 1]], ssrbcm205, ssrbcm203];
```

We plot the impulse response function of output. As technology, we observe that output increases in response to the shock and then gradually falls to its steady-state level.


```

ssrbcm212 = ListLinePlot[ssrbcm211,
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response of Output,  $\hat{y}_t$ ", Frame -> True, GridLines -> Automatic,
  GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.007], Hue[0.8]}]

```



We remind the equation of investment.

```
ssrbcm201
```

$$ivh_t = -0.768707 kh_t + 3.9664 zh_t$$

We calculate the impulse response function of investment.

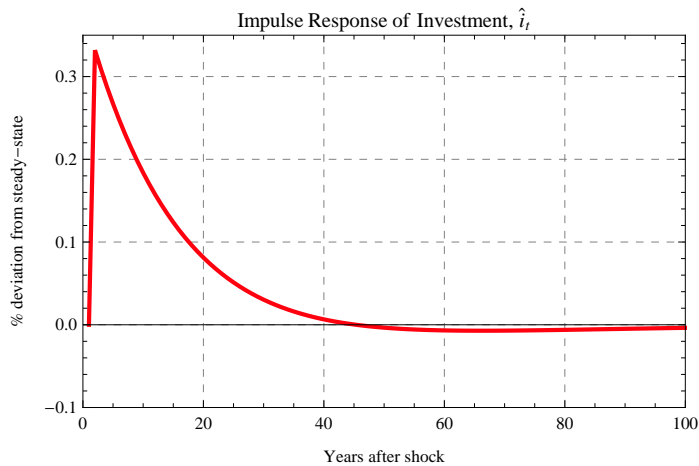
```
ssrbcm213 = fivh[ssrbcm201[[2, 1, 1]], ssrbcm201[[2, 2, 1]], ssrbcm205, ssrbcm203];
```

We plot the impulse response function of investment. As output, we see that investment rises in response to the shock and then progressively falls towards its steady-state level.

```

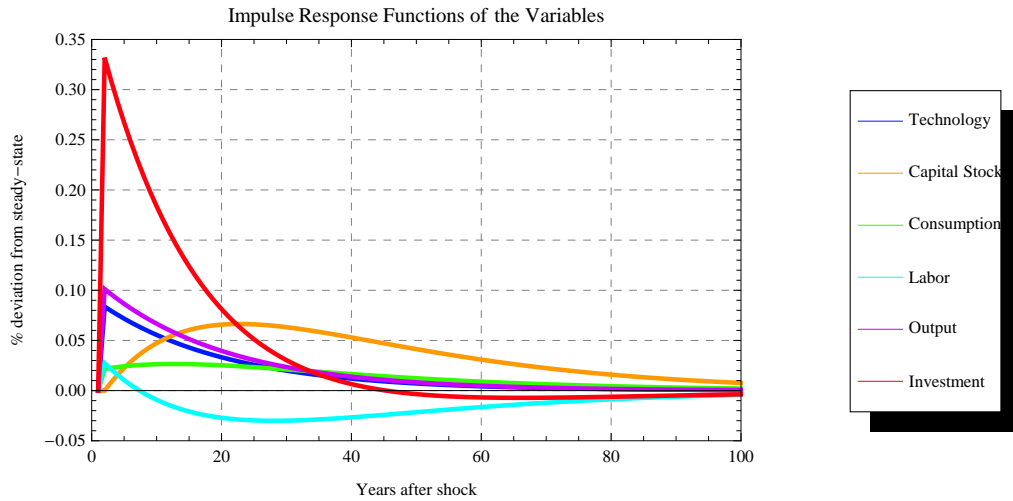
ssrbcm214 = ListLinePlot[ssrbcm213,
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response of Investment,  $\hat{i}_t$ ",
  Frame -> True, GridLines -> Automatic, GridLinesStyle -> Dashed,
  PlotStyle -> {Thickness[0.007], Hue[0.99]}, PlotRange -> {{0, 100}, {-0.1, 0.35}}]

```



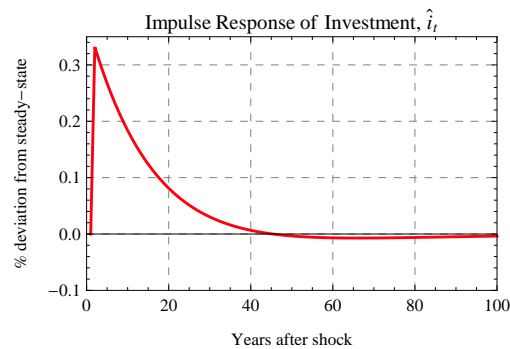
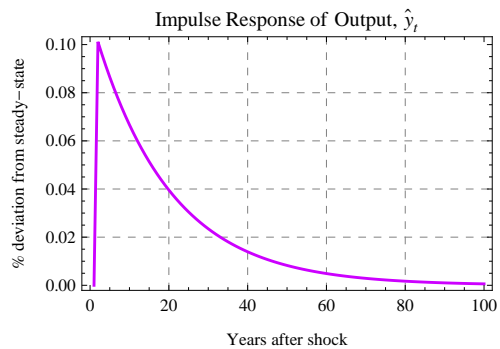
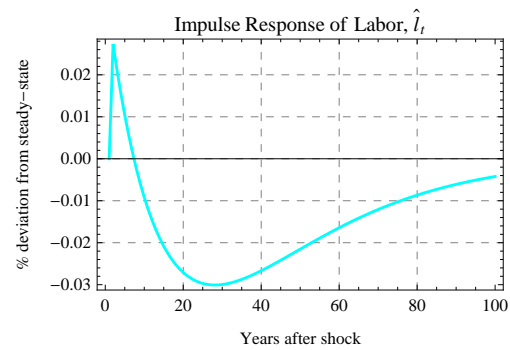
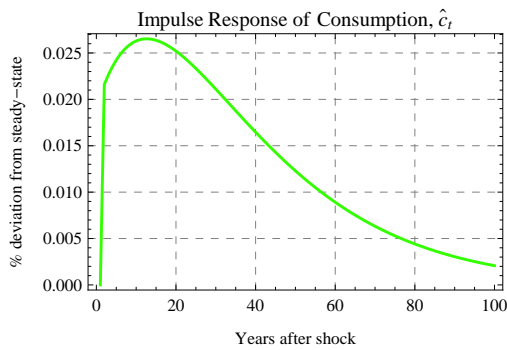
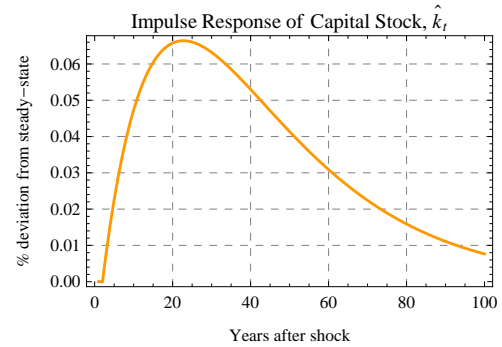
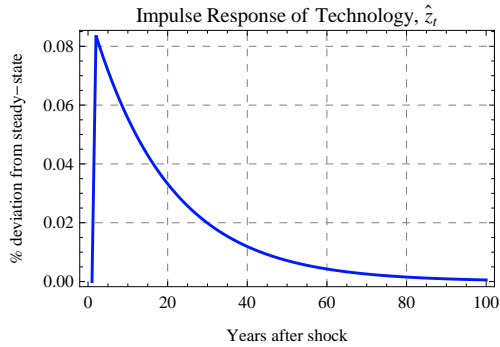
Now we combine all these preceding figures into a single graph.

```
ssrbcm215 = ListLinePlot[
  {ssrbcm203, ssrbcm205, ssrbcm207, ssrbcm209, ssrbcm211, ssrbcm213},
  FrameLabel -> {"Years after shock", "% deviation from steady-state"},
  PlotLabel -> "Impulse Response Functions of the Variables", Frame -> True,
  GridLines -> Automatic, GridLinesStyle -> Dashed, PlotLegend ->
  {"Technology", "Capital Stock", "Consumption", "Labor", "Output", "Investment"},
  LegendPosition -> {1.1, -0.4}, ImageSize -> 550,
  PlotStyle -> {{Thickness[0.007], Hue[0.65]}, {Thickness[0.007], Hue[0.1]},
  {Thickness[0.007], Hue[0.3]}, {Thickness[0.007], Hue[0.5]}, {Thickness[0.007],
  Hue[0.8]}, {Thickness[0.007], Hue[0.99]}}], PlotRange -> {{0, 100}, {-0.05, 0.35}}]
```



Now we present the graphs side by side.

```
ssrbcm216 = GraphicsGrid[{{ssrbcm204, ssrbcm206},
  {ssrbcm208, ssrbcm210}, {ssrbcm212, ssrbcm214}}, ImageSize -> 600]
```



5.2. Stochastic Simulation

We set the random seed to 2000 in order to allow the reproduction of our simulations.

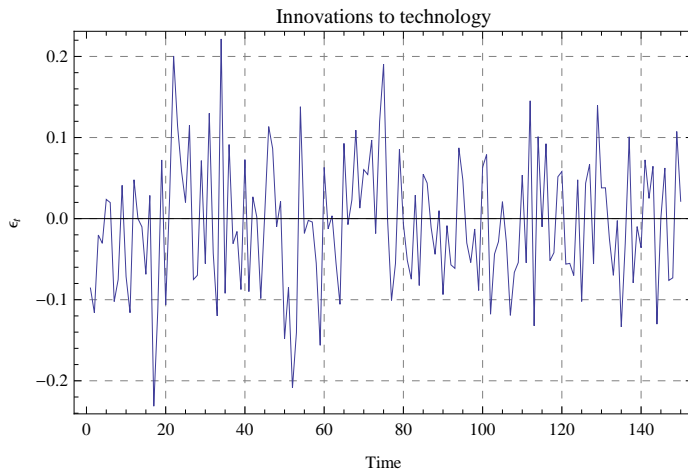
```
SeedRandom[2000]
```

We generate 5000 normal random numbers with mean 0 and standard deviation 0.083666. This series represents the innovations to technology.

```
ssrbcm217 = RandomVariate[NormalDistribution[0, 0.083666], 5000];
```

Now we plot only the last 150 values of the previous series.

```
ssrbcm218 = ListLinePlot[Take[ssrbcm217, -150],
  FrameLabel -> {"Time",  $\epsilon_t$ }, PlotLabel -> "Innovations to technology",
  Frame -> True, GridLines -> Automatic, GridLinesStyle -> Dashed]
```



Next we present the *Hodrick-Prescott (1997)* filter used to filter the data we are going to study. To find the Hodrick-Prescott (HP) filter, we minimize the following expression:

$$\text{Min}_{\{g_t\}_{t=1}^T} \sum_{t=1}^T c_t^2 + \lambda \sum_{t=3}^T [(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})]^2$$

Subject to:

$$c_t = y_t - g_t$$

Where y_t is the series we want to smooth, g_t is the trend component, c_t is the cyclical component and λ is the smoothing parameter.

Danthine and Girardin (1989) show that the solution to this optimization problem is given by:

$$g_t = (I + \lambda K' K)^{-1} y_t$$

With I an identity matrix of dimension T and K is a special matrix of dimension $(T-2) \times T$ having the following form:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}$$

Having found g_t , the cyclical component is calculated as $c_t = y_t - g_t$. Since the parameters of our model are calibrated for quarterly data, we will use the value $\lambda = 1600$. Now let us write a function that computes the HP filter. The function takes two arguments: the first is the data we want to smooth and the second is the value of the parameter λ . The function then returns the cyclical component and the trend component respectively.

```

hPfilterz[data_List, λ_?Positive] :=
Module[{totime, trend, cycle, KKMAT, IDM, MATA}, totime = Length[data]; KKMAT = SparseArray[
  {{r_, r_} -> 1., {r_, s_} /; s == r + 1 -> -2., {r_, s_} /; s == r + 2 -> 1.},
  {(totime - 2), totime}, 0.];
IDM = IdentityMatrix[totime]; MATA = IDM + λ Transpose[KKMAT].KKMAT;
trend = LinearSolve[MATA, data, Method -> "Automatic"];
cycle = data - trend; {cycle, trend}

```

We generate the simulated time series of technology.

```
ssrbcm219 = fzh[0.95, ssrbcm217, 4999];
```

We take the last 150 values of the previous series.

```
ssrbcm220 = Take[ssrbcm219, -150];
```

We apply the HP-filter to the preceding result.

```
ssrbcm221 = hPfilterz[ssrbcm220, 1600];
```

We extract the cyclical component of the technology series.

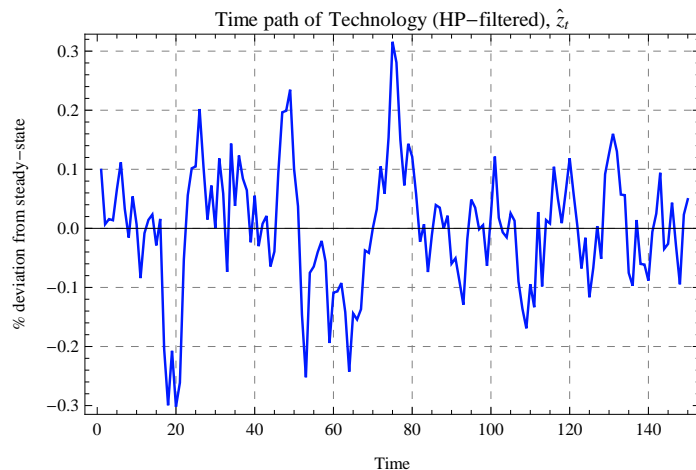
```
ssrbcm222 = ssrbcm221[[1]];
```

We plot the HP-filtered time path of technology.

```

ssrbcm223 =
ListLinePlot[ssrbcm222, FrameLabel -> {"Time", "% deviation from steady-state"},
  PlotLabel -> "Time path of Technology (HP-filtered),  $\hat{z}_t$ ", Frame -> True,
  GridLines -> Automatic, GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.004], Hue[0.65]}]

```



We generate the simulated time series of capital stock.

```
ssrbcm224 = fkh[ssrbcm219[[2, 1, 1]], ssrbcm219[[2, 2, 1]], 5000, ssrbcm219];
```

We take the last 150 values of the previous series.

```
ssrbcm225 = Take[ssrbcm224, -150];
```

We apply the HP-filter to the preceding result.

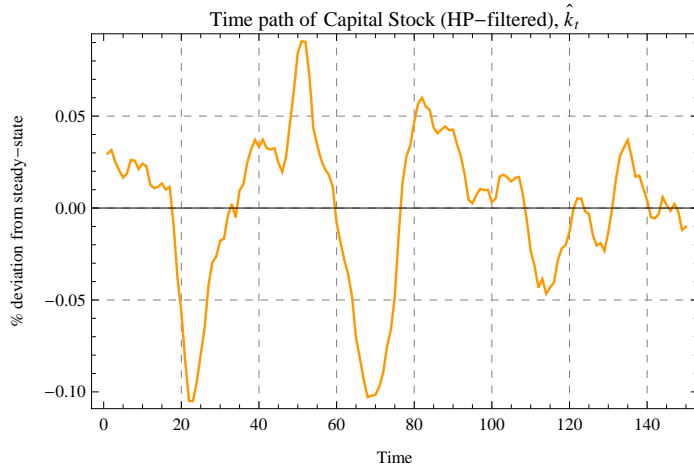
```
ssrbcm226 = hPfilterz[ssrbcm225, 1600];
```

We extract the cyclical component of the capital stock series.

```
ssrbcm227 = ssrbcm226[[1]];
```

We plot the HP-filtered time path of capital stock.

```
ssrbcm228 =
  ListLinePlot[ssrbcm227, FrameLabel -> {"Time", "% deviation from steady-state"},
  PlotLabel -> "Time path of Capital Stock (HP-filtered),  $\hat{k}_t$ ", Frame -> True,
  GridLines -> Automatic, GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.004], Hue[0.1]}]
```



We generate the simulated time series of consumption.

```
ssrbcm229 = fch[ssrbcm198[[2, 1, 1]], ssrbcm198[[2, 2, 1]], ssrbcm224, ssrbcm219];
```

We take the last 150 values of the previous series.

```
ssrbcm230 = Take[ssrbcm229, -150];
```

We apply the HP-filter to the preceding result.

```
ssrbcm231 = hpfILTERz[ssrbcm230, 1600];
```

We extract the cyclical component of the consumption series.

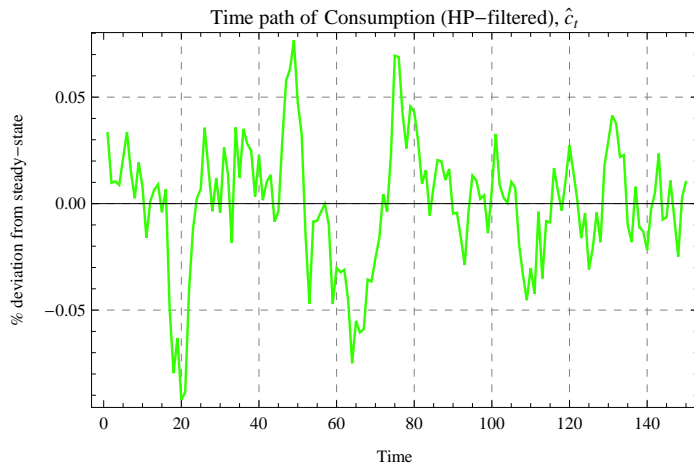
```
ssrbcm232 = ssrbcm231[[1]];
```

We plot the HP-filtered time path of consumption.

```

ssrbcm233 =
ListLinePlot[ssrbcm232, FrameLabel -> {"Time", "% deviation from steady-state"},
PlotLabel -> "Time path of Consumption (HP-filtered),  $\hat{c}_t$ ", Frame -> True,
GridLines -> Automatic, GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.004], Hue[0.3]}]

```



We generate the simulated time series of labor.

```
ssrbcm234 = flh[ssrbcm192[[2, 1, 1]], ssrbcm192[[2, 2, 1]], ssrbcm224, ssrbcm219];
```

We take the last 150 values of the previous series.

```
ssrbcm235 = Take[ssrbcm234, -150];
```

We apply the HP-filter to the preceding result.

```
ssrbcm236 = hPfilterz[ssrbcm235, 1600];
```

We extract the cyclical component of the labor series.

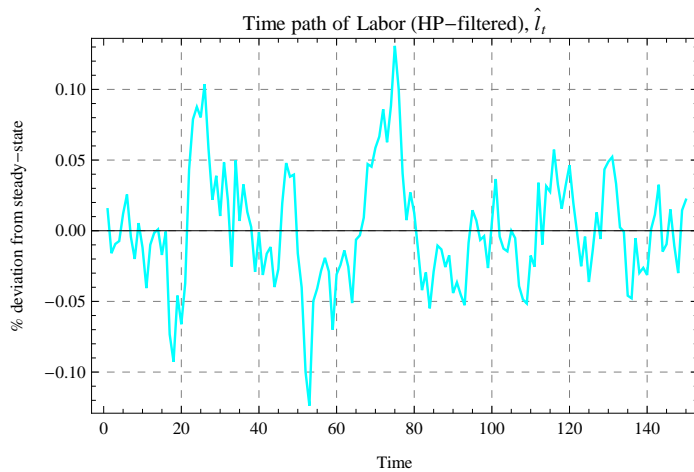
```
ssrbcm237 = ssrbcm236[[1]];
```

We plot the HP-filtered time path of labor.

```

ssrbcm238 =
ListLinePlot[ssrbcm237, FrameLabel -> {"Time", "% deviation from steady-state"},
PlotLabel -> "Time path of Labor (HP-filtered),  $\hat{l}_t$ ", Frame -> True, GridLines -> Automatic,
GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.004], Hue[0.5]}]

```



We generate the simulated time series of output.

```
ssrbcm239 = fyh[ssrbcm200[[2, 1, 1]], ssrbcm200[[2, 2, 1]], ssrbcm224, ssrbcm219];
```

We take the last 150 values of the previous series.

```
ssrbcm240 = Take[ssrbcm239, -150];
```

We apply the HP-filter to the preceding result.

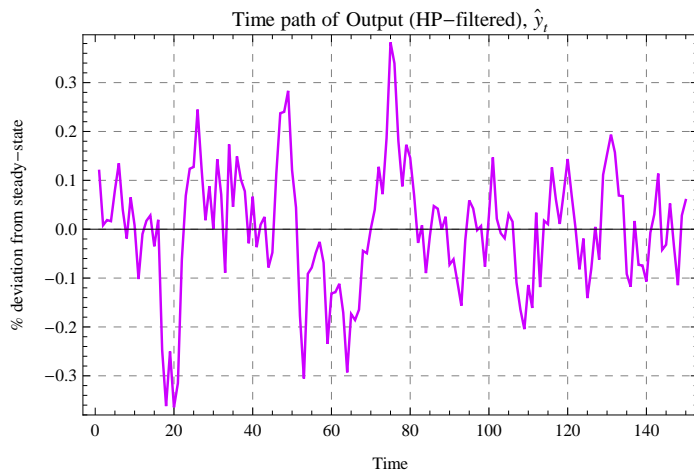
```
ssrbcm241 = hPfilterz[ssrbcm240, 1600];
```

We extract the cyclical component of the output series.

```
ssrbcm242 = ssrbcm241[[1]];
```

We plot the HP-filtered time path of output.

```
ssrbcm243 =  
ListLinePlot[ssrbcm242, FrameLabel -> {"Time", "% deviation from steady-state"},  
PlotLabel -> "Time path of Output (HP-filtered),  $\hat{y}_t$ ", Frame -> True,  
GridLines -> Automatic, GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.004], Hue[0.8]}]
```



We generate the simulated time series of investment.

```
ssrbcm244 = fivh[ssrbcm201[[2, 1, 1]], ssrbcm201[[2, 2, 1]], ssrbcm224, ssrbcm219];
```

We take the last 150 values of the previous series.

```
ssrbcm245 = Take[ssrbcm244, -150];
```

We apply the HP-filter to the preceding result.

```
ssrbcm246 = hPfilterz[ssrbcm245, 1600];
```

We extract the cyclical component of the investment series.

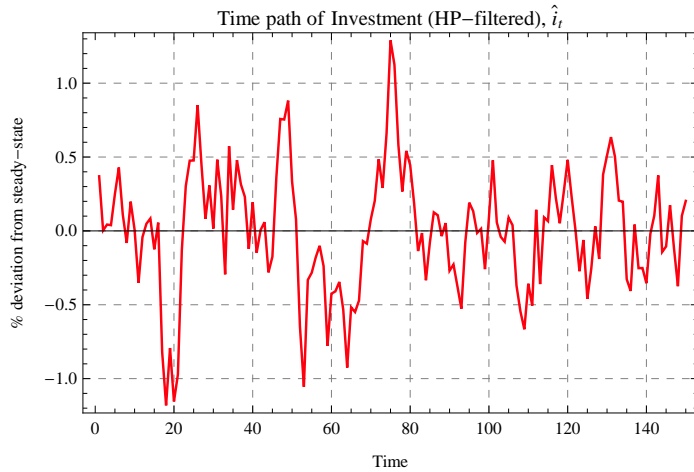
```
ssrbcm247 = ssrbcm246[[1]];
```

We plot the HP-filtered time path of investment.


```

ssrbcm248 =
ListLinePlot[ssrbcm247, FrameLabel -> {"Time", "% deviation from steady-state"},
PlotLabel -> "Time path of Investment (HP-filtered),  $\hat{i}_t$ ", Frame -> True,
GridLines -> Automatic, GridLinesStyle -> Dashed, PlotStyle -> {Thickness[0.004], Hue[0.99]}]

```

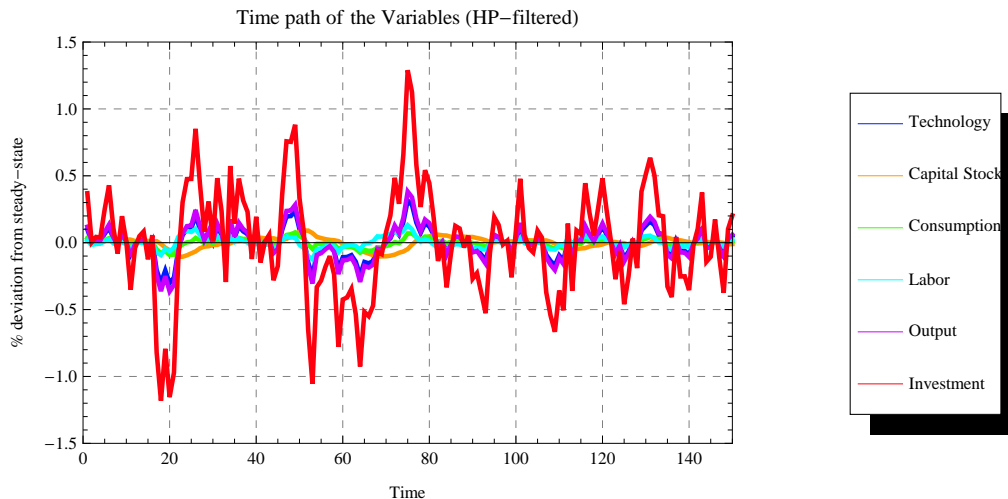


Now we combine all these preceding figures into a single graph.

```

ssrbcm249 = ListLinePlot[
{ssrbcm222, ssrbcm227, ssrbcm232, ssrbcm237, ssrbcm242, ssrbcm247},
FrameLabel -> {"Time", "% deviation from steady-state"},
PlotLabel -> "Time path of the Variables (HP-filtered)", Frame -> True,
GridLines -> Automatic, GridLinesStyle -> Dashed, PlotLegend ->
{"Technology", "Capital Stock", "Consumption", "Labor", "Output", "Investment"},
LegendPosition -> {1.1, -0.4}, ImageSize -> 550,
PlotStyle -> {{Thickness[0.007], Hue[0.65]}, {Thickness[0.007], Hue[0.1]},
{Thickness[0.007], Hue[0.3]}, {Thickness[0.007], Hue[0.5]}, {Thickness[0.007],
Hue[0.8]}, {Thickness[0.007], Hue[0.99]}}, PlotRange -> {{0, 150}, {-1.5, 1.5}}]

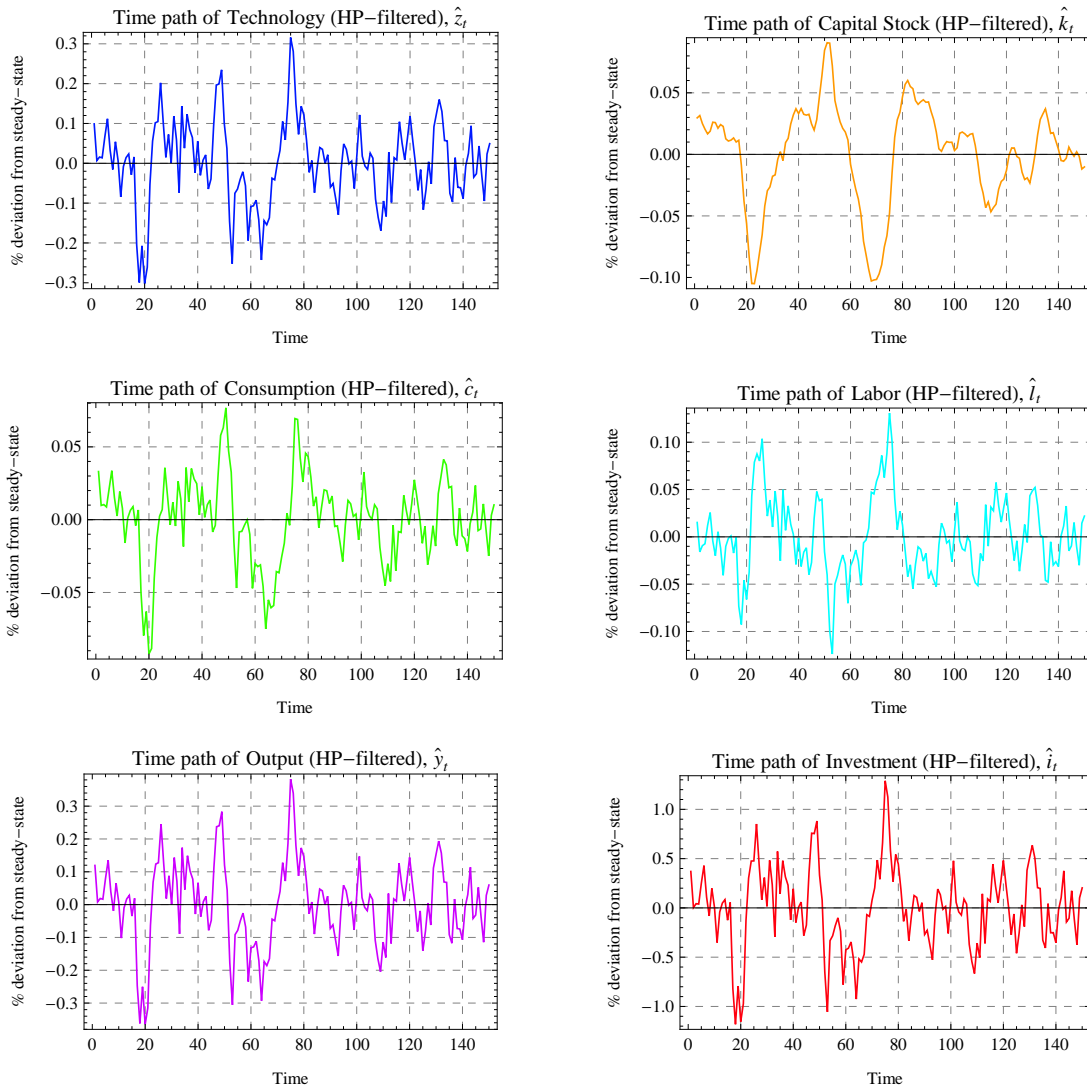
```



The graph illustrates that investment is more volatile than output while capital stock and consumption are smoother than output.

Now we present the graphs side by side.

```
ssrbcm250 = GraphicsGrid[{{ssrbcm223, ssrbcm228},
  {ssrbcm233, ssrbcm238}, {ssrbcm243, ssrbcm248}}, ImageSize -> 600]
```



5.3. Second Moments Statistics

We start by writing a function that calculates the relative standard deviation. The function takes two arguments. Both arguments are lists or time series in our case. The result of the function is the standard deviation of the first series relative to the second.

```
relativesd[mvar_List, output_List] := Module[{sdmvar, sdout},
  sdmvar = StandardDeviation[mvar]; sdout = StandardDeviation[output]; sdmvar / sdout]
```

We compute the standard deviation of each series relative output. To do this we automate the process by creating a function that takes as first argument a list of list containing the series and as second argument the output series. The function then gives as result the list of the relative standard deviations.

```
listrsdfunc[listrsd_, outrsd_] :=
Module[{res1, NN}, NN = Length[listrsd]; res1 = Table[0, {i, NN}];
Do[res1[[i]] = relativesd[listrsd[[i]], outrsd], {i, 1, NN}]; res1]
```

We enter the series for which we want to calculate the relative standard deviation with output.

```
ssrbcm251 = {ssrbcm222, ssrbcm227, ssrbcm232, ssrbcm237, ssrbcm247};
```

We calculate the relative standard deviation of technology, capital stock, consumption, labor and investment respectively with output.

```
ssrbcm252 = listrsdfunc[ssrbcm251, ssrbcm242]
{0.827272, 0.325121, 0.233494, 0.319369, 3.27558}
```

We create a function that compute the correlation of each series with output. The function takes as first argument a list of list containing the series and as second argument the output series. The result of the function is a list containing the correlation of each series with output.

```
listcorrfunc[listcorr_, outcorr_] :=
Module[{rescorr, NNC}, NNC = Length[listcorr]; rescorr = Table[0, {i, NNC}];
Do[rescorr[[i]] = Correlation[listcorr[[i]], outcorr], {i, 1, NNC}]; rescorr]
```

We compute the correlation of technology, capital stock, consumption, labor and investment respectively with output.

```
ssrbcm253 = listcorrfunc[ssrbcm251, ssrbcm242]
{0.999996, 0.0579083, 0.935252, 0.811926, 0.997319}
```

We display the relative standard deviation next to the correlation.

```
ssrbcm254 = Transpose[{ssrbcm252, ssrbcm253}]
{{0.827272, 0.999996}, {0.325121, 0.0579083},
 {0.233494, 0.935252}, {0.319369, 0.811926}, {3.27558, 0.997319}}
```

Now we display the Second Moments Statistics of the model in table form.

```
ssrbcm255 = TableForm[ssrbcm254,
TableHeadings -> {"Technology", "Capital Stock", "Consumption", "Labor", "Investment"},
{"relative volatility", "correlation"}, TableAlignments -> Center]
```

	relative volatility	correlation
Technology	0.827272	0.999996
Capital Stock	0.325121	0.0579083
Consumption	0.233494	0.935252
Labor	0.319369	0.811926
Investment	3.27558	0.997319

From this table it appears that consumption, labor and capital stock are less volatile than output. Technology is almost as volatile as output. Investment is more volatile than output. The table also illustrates that technology, investment and consumption are highly procyclic, while capital stock is less correlated with output.

6. Conclusion

This notebook has demonstrated how to solve and simulate a Real Business Cycle (RBC) model using the software Mathematica. We have also shown how to find the first order conditions, derive the steady-state and log-linearize the model. A good extension of this notebook could be:

- First, to automate some parts instead of doing it step by step. We have chosen this last method for pedagogic reasons.

- Second, to use alternatives solution methods.

7. References

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