Robust income distribution analysis

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[outline]

The problem of data contamination/extreme incomes

- Income distribution analysis:
  1. summary measures of inequality (and other distributional features)
  2. dominance checks (stochastic dominance, Lorenz dominance)
- Both very sensitive to extreme incomes ('valid' outliers? contamination?)
  - unbounded influence function (Cowell & Victoria-Feser, *Econometrica* 1996, 2002)

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PSELL-3 (equivalised) household income data (waves 1-3):

	2002			2003			2004		
	Top 10 incomes								
		37,260			16,925			41,830	
		34,242			15,280			32,569	
		28,292			15,132			18,341	
		...			...			...	
		15,407			10,464			11,095	
	Summary measures								
	Raw	Trim	Wins.	Raw	Trim	Wins.	Raw	Trim	Wins.
μ	2,689	2,635	2,666	2,674	2,631	2,667	2,734	2,685	2,715
Gini	0.272	0.259	0.266	0.262	0.252	0.259	0.262	0.250	0.257
$\frac{CV^2}{2}$	0.192	0.129	0.147	0.138	0.116	0.129	0.159	0.112	0.123

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Remedial actions

1. Remove extremely high incomes, or impose a top code
   - Easy, but not efficient and dependence to trimming fractions
2. Use functional form assumptions:
   - model tails of distribution parametrically (e.g. Pareto distribution)[1]
   - model the full distribution parametrically (e.g. log-Normal, Gamma, Singh-Maddala)
   - But... classical ML estimators are themselves non-robust to extreme incomes!

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A robust alternative to classical ML

- OBRE is an M-estimator: θ solution to $\sum_{i=1}^{N} \psi(x_i, \theta) = 0$
  - (For ML: $\psi(x_i, \theta^{ML}) = s(x_i, \theta^{ML})$ is the score function)
- OBRE estimator is the solution to

$$\psi(x_i, \theta^{OB}) = (s(x_i, \theta^{OB}) - a(\theta^{OB}))W_c(x_i; \theta^{OB})$$

  where

$$W_c(x_i; \theta^{OB}) = \min\left(1; \frac{c}{G(s(x_i, \theta^{OB}), a(\theta^{OB}), A(\theta^{OB}))}\right)$$

- $W_c(x; \theta^{OB})$ imposes a bound on influence function by downweighting extreme values (values deviating from model)
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Optimal B-Robust Estimators (OBRE) (ctd.)

A robust alternative to classical ML

- $a(\theta^{OB})$ and $A(\theta^{OB})$ are such that

$$\begin{aligned} E(\psi(x,\theta^{OB})\psi(x,\theta^{OB})') &= (A(\theta^{OB})A(\theta^{OB})')^{-1} \\ E(\psi(x,\theta^{OB})) &= 0 \end{aligned}$$

The resulting estimator is the optimal (minimum variance) M-estimator with bounded influence function[2]

- If $c \rightarrow \infty$ then $\theta^{OB} = \theta^{ML}$

[2] For a thorough treatment, see Hampel et al. (1986), *Robust Statistics: The approach based on influence functions.*

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Implementation

- Given number of implicit definitions of parameters and constraints, estimation is not easy
- But relatively detailed algorithms are available (fortunately!). I follow Ronchetti & Victoria-Feser (*Canadian Journal of Statistics*, 1994).
- Estimation involves
  1. matrix operations
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  $\Longrightarrow$ Mata!

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Implementation (ctd.)

- Implementation is relatively easy with Mata (but familiarity with matrix algebra can help!)
- Builds on suite of commands by Stephen Jenkins to fit functional forms to unit record data by ML[3]
  - just replace ML engine by home-brewed OBRE engine (call a Mata function, rather than `ml model`)
- I implemented Pareto Type I distribution and 3-parameters Singh-Maddala distribution[4]
- Compatible with Nick Cox's diagnostic commands `psm` and `qsm`

[3] `ssc describe smfit`

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Practical programming issues

- Precision of numerical integration functions revealed very important
- Difficulty to set multiple tolerance and precision parameters – trade-off between speed and accuracy (still subject to changes...)
- As in ML estimation, using re-parameterization $\tilde{\theta} = \ln(\theta)$ can help convergence (in all models considered, $\theta > 0$)

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Empirical illustration

- Data from panel survey PSELL-3 (Panel ‘Living in Luxembourg’), 2003–2005
- Representative of Luxembourg residents
- Single-adult-equivalent real household income (incomes of 2002-2004)

ML vs. OBRE parameter estimates

Pareto Type I parameters

		ML	OBRE			
			$c = 200$	$c = 5$	$c = 3$	$c = 2$
Pareto Type I	2002	3.635	3.635	3.633	3.720	3.926
(upper 5%)	2003	4.075	4.075	4.060	4.007	3.911
	2004	4.306	4.306	4.383	4.425	4.498

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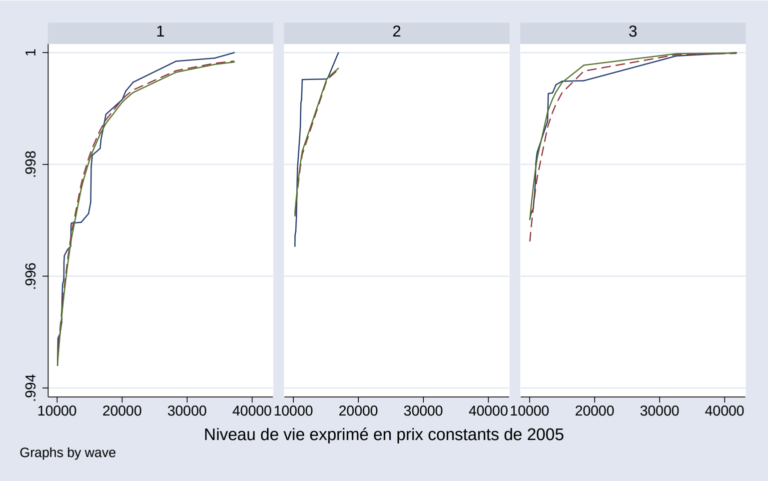
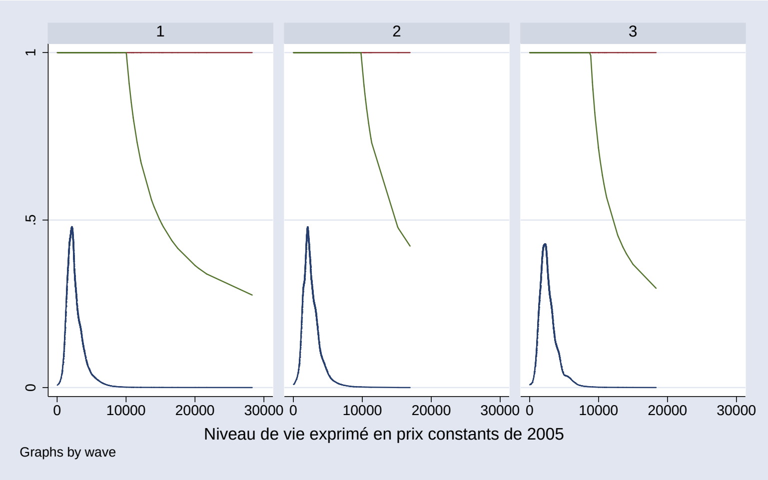
ML vs. OBRE parameter estimates

Empirical CDF and estimated Pareto Type I CDF



OBRE robustness weights

Pareto Type I distribution



ML vs. OBRE parameter estimates

Singh-Maddala parameters

		ML	OBRE			
			$c = 200$	$c = 10$	$c = 5$	$c = 4$
Singh-Maddala	2002	4.131	4.141	4.170	4.417	4.726
		2,159	2,159	2,146	2,022	1,912
		0.797	0.797	0.784	0.664	0.555
	2003	3.643	3.463	3.713	4.035	4.326
		2,477	2,477	2,428	2,214	2,060
		1.094	1.094	1.040	0.822	0.666
	2004	3.666	3.666	3.716	3.980	4.262
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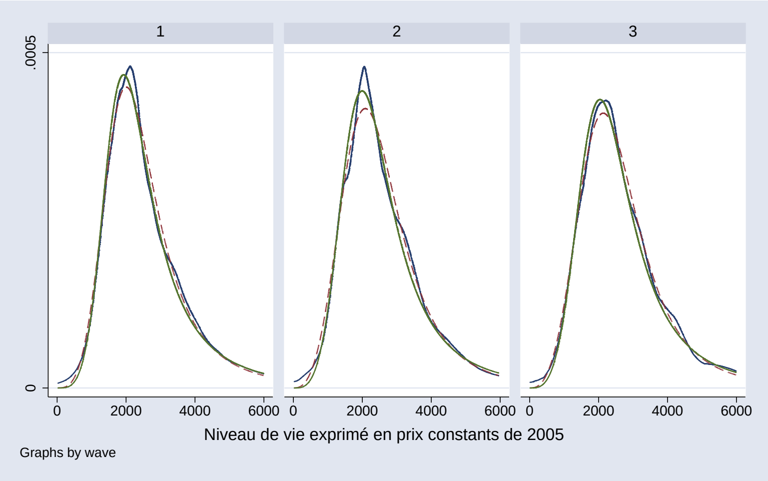
Non-parametric estimates and estimated Singh-Maddala PDFs



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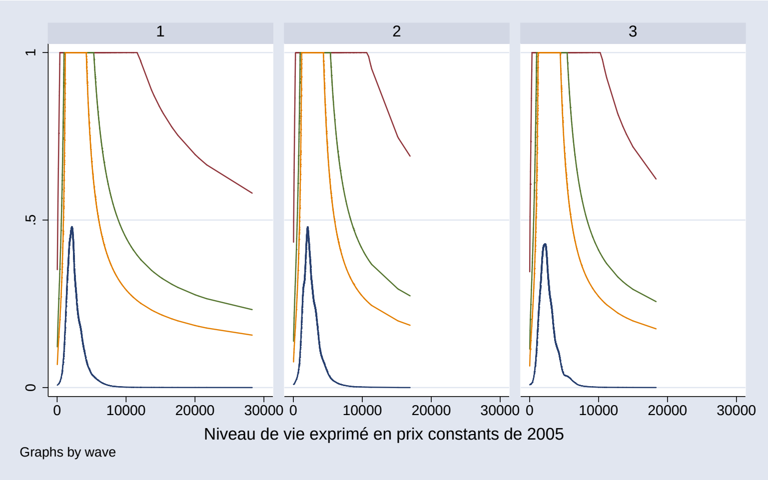

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- Mata makes estimators such as OBRE feasible within Stata
- In theory, OBRE estimators have great relevance in income distribution analysis... implementation in Stata may help putting this claim to broader practical assessment
- At present, it is a prototype (but looks ok). Minor developments still needed for
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