Speeding Up the ARDL Estimation Command:

A Case Study in Efficient Programming in Stata and Mata

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Contents

Introduction

- Long code execution times are more than a nuisance: they negatively affect the quality of research
- strategies for speeding up execution:
  - lower-level language
  - parallelization
  - writing efficient code
- Efficient coding is often the best choice.
  - Moving to lower-level languages is tedious.
  - In many settings, speed improvements are higher than through parallelization.

Introduction: Speed of Stata and Mata

- C is the reference
  - compiled to machine instructions
- Post of Bill Gould (2014) at the Stata Forum:
  - Stata (interpreted) code is 50-200 times slower than C.
  - Mata compiled byte-code 5-6 times slower than C.
    => Mata is 10-40 times faster than Stata.
  - In real-world applications, Mata is ~2 times slower than C.
    - Mata has built-in C routines based on very efficient code.

Introduction: Efficient Coding Strategies

- Using Common Sense
  - An if-condition requires at least N comparisons. Use in-conditions instead, if possible.
  - Multiplying two 100x100 matrices requires about 2*100^3 = 2,000,000 arithmetic operations.
- Using Knowledge of Your Software (Stata, of course!)
  - Examples:
    - Mata: passing of arguments to functions
    - Efficient operators and functions (e.g. Mata's colon operator and its c-conformability)
    - Read the Stata and Mata programming manuals

Introduction: Efficient Coding Strategies

Using Knowledge of Matrix Algebra

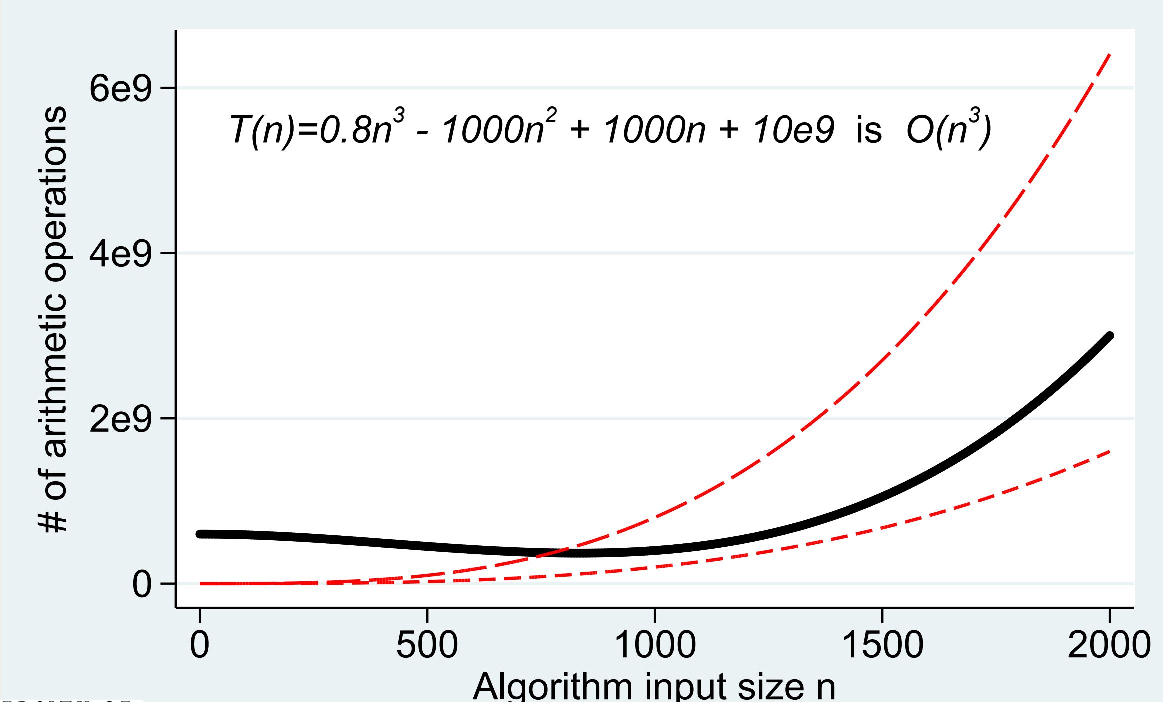
- Translating mathematical formulas one-to-one into matrix language expressions is oftentimes (very!) inefficient.
- Examples:
  - diagonal matrices (D) :
    - multiplication of a matrix by D: don't do it!
      Mata: use c-conformability of the colon operator (see **[M-2] op_colon**)
    - inverse: flip diagonal elements instead of calling a matrix solver / inverter function ($O(n)$ vs. $O(n^3)$)
  - block diagonal matrices:
    - multiplication: just multiply diagonal blocks; the latter is faster by $1/s^2$, where s is the number of diagonal blocks
    - inverse: invert individual blocks
  - order of matrix multiplication / parenthesization
    - $b = (X'X)^{-1}(X'y)$ is faster than $b = (X'X)^{-1}X'y$
      e.g. for $k = 10$, $N = 10,000$: matrix multiplications are 11 times faster!

Asymptotic Notation

Definition

An algorithm with input size n and running time $T(n)$ is said to be $\Theta(g(n))$ ("theta of g of n") or to have an *asymptotically tight bound* $g(n)$ if there exist positive real numbers $c_1, c_2, n_0 > 0$ such that

$$c_1 g(n) \leqslant T(n) \leqslant c_2 g(n) \quad \forall n \geqslant n_0$$



Asymptotic Notation

- $O\left(g\left(n\right)\right)$("(big) oh of g of n"), as opposed to $\Theta(g(n))$, is used here to only denote an upper bound. Notation differs in the literature.
- Technically, $\Theta(g(n))$ and $O(g(n))$ are sets of functions, so we write e.g. $T\left(n\right) \in O(g(n))$.
- For matrix operations, $g\left(n\right)$ is frequently n raised to some low integer power.
  - $\Theta\left(n\right)$ is much better than $\Theta\left(n^2\right)$, which in turn is much better than $\Theta\left(n^3\right)$
  - (Square) matrix multiplication is $\Theta\left(n^3\right)$: each element of the new $n \times n$ matrix is a sum of n terms. Costly!
  - Many types of matrix inversion, e.g. the LU-decomposition, are also $\Theta\left(n^3\right)$. Costly!
  - Inner vector products are $\Theta\left(n\right)$.
- When $T\left(n\right)$ is an i-th order polynomial, the leading term asymptotically dominates: $T\left(n\right) \in O\left(n^i\right)$.
- $\Theta\left(a^n\right)$ is worse than $\Theta\left(n^a\right)$; $\Theta\left(lg\ n\right)$ is better than $\Theta\left(n\right)$

ARDL: Model Setup

- ARDL $(p, q_1, \ldots, q_k)$: autoregressive distributed lag model
- Popular, long-standing single-equation time-series model for continuous variables
- Linear model :

$$y_t = c_0 + c_1 t + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{i=0}^{q} \beta_i' \mathbf{x}_{t-i} + u_t, \;\; u_t \sim iid\left(0, \sigma^2\right)$$

- $(y_t, \mathbf{x}_t')'$ can be purely $I(0)$, purely $I(1)$, or cointegrated: can be used to test for cointegration (bounds testing procedure). (Pesaran, Shin, and Smith, 2001). => econometrics of ARDL can be complicated.
- `net install ardl , from(http://www.kripfganz.de/stata)`
- This talk: programming; for the statistics of `ardl`, see Kripfganz/Schneider (2016).

ARDL: Computational Considerations

- Despite its complex statistical properties, estimating an ARDL model is just based on OLS!
- The computational costly parts are:
  - determination of optimal lag orders (e.g. via AIC or BIC)
    - treated at length in this talk
  - simulation of test distributions for cointegration testing (PSS 2001, Narayan 2005).
    - not covered by this talk

Optimal Lag Selection: The Problem

- For $k+1$ variables (indepvars + depvar) and *maxlag* lags for each variable, run a regression and calculate an information criterion (IC) for each possible lag combination and select the model with the best IC value.

- Example: 2 variables (v1 v2) , $maxlag = 2$

```
regress v1 L(1/1).v1 L(0/0).v2
regress v1 L(1/2).v1 L(0/0).v2
regress v1 L(1/1).v1 L(0/1).v2
regress v1 L(1/2).v1 L(0/1).v2
regress v1 L(1/1).v1 L(0/2).v2
regress v1 L(1/2).v1 L(0/2).v2
```

- # of regressions to run is exponential in k: $maxlags \cdot (maxlags+1)^k$:

$k+1$	*maxlags*	# regressions
3	4	100
3	8	∼650
4	8	∼5,800
6	8	∼470,000
8	8	∼38,000,000

Lag Selection: Preliminaries

- Lag combination matrix for $k = 3$ and $maxlags = 2$: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ & \dots & \\ 2 & 2 & 2 \end{bmatrix}$
- e.g. row 3: $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$ corresponds to regressors `L.v1 L(0/0).v2 L(0/2).v3` = $\begin{bmatrix} v1_{t-1} & v2_t & v3_t & v3_{t-1} & v3_{t-2} \end{bmatrix}$
- called “lagcombs” in pseudo-code to follow

Lag Selection: Naive Approach Using `regress`

Stata/Mata-like pseudocode:

```
// note: may contain incorrect syntax,
//        fictitious commands/options/return values, etc.
lagcombs , k(3) maxlag(8)
    // defines matrix with all lag combs

scalar ic = .
forvalues i=1/numrows(lagcombs) {
    matrix lags = lagcombs(`i',1...)
    regress v1 v2 v3 , lags(lags)
    estat ic
    if r(aic)<ic {
        scalar ic = r(aic)
        matrix optimlag = lags
    }
}
```

Lag Selection: Timings

Timings in seconds (2.5GHz, single core) for N=1000:

$k+1$	*maxlags*	# regressions	regress
3	4	100	1.6
3	8	$\sim$650	12.5
4	8	$\sim$5,800	132
6	8	$\sim$470,000	$\sim$14000
8	8	$\sim$38,000,000	(13 days?)

Lag Selection: Mata I

Stata/Mata-like pseudocode:

```
// note: may contain incorrect syntax,
//        fictitious commands/options/return values, etc.
lagcombs , k(3) maxlag(8)
    // defines matrix with all lag combs

mata:
    lagcombs = st_matrix("lagcombs")
    ic = .
    for (i=1; i<=rows(lagcombs); i++) {
        y = st_data( DEPVAR )
        X = st_data( PULL DATA FOR SPECIFIC LAG COMBINATION )

        // calculate AIC
        ee = y'y - y'X*invsym(X'X)*X'y  // sum of squared residuals
        ic = T*log(2*pi()) + T*log(ee/T) + T + 2*k

        if (ic<ic_min) {
            ic_min = ic
            optimrow = i
        }
    }
end
```

Lag Selection: Mata II (no redundant calculations)

```
// note: may contain incorrect syntax, fictitious commands/options/return values, etc.
lagcombs , k(3) maxlag(8)
mata:
    y = st_data( DEPVAR )
    X = st_data( PULL ALL DATA : ALL VARIABLES, ALL LAGS)

    // calculate terms for full lag specification
    XX = X'X ; Xy = X'y ; yy = y'y

    lagcombs = st_matrix("lagcombs")
    ic = .
    for (i=1; i<=rows(lagcombs); i++) {
        // cross-products for current iteration
        idx = [ GET INDEX VECTOR FOR CURRENT LAG STRUCTURE ]
        XXi = XX[idx,idx] ; Xyi = Xy[idx]

        ee = yy - Xyi'*invsym(XXi)*Xyi  // calculate sum of squared residuals for AIC

        ic = T*log(2*pi()) + T*log(ee/T) + T + 2*k
        if (ic<ic_min) {
            ic_min = ic ;
            optimrow = i
        }
    }
end
```

Lag Selection: Mata III

- A sticky point are the many matrix inversions, which are $\Theta\left(n^3\right)$.
- We will further improve matters by using results from linear algebra.
- We will introduce and use pointer variables in the process.
- The following will put forth a somewhat complicated algorithm that affects many parts of the loop.
- In this talk, we could have focused our attention on many smaller changes for code optimization, but both things are not possible within the time window for this presentation.

Updating $(X'X)^{-1}$ Using Partitioned Matrices

- For $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, with A, $A_{11}$and A_{22} square and invertible:

$$A^{-1} = \begin{bmatrix} D & -DA_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D & -A_{22}^{-1} + -A_{22}^{-1}A_{21}DA_{12}A_{22}^{-1} \end{bmatrix}$$

  with $D = A_{11}^{-1} + A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} A_{21}A_{11}^{-1}$

- Here: Let $X_v = \begin{bmatrix} X & v \end{bmatrix}$. The cross-product matrix becomes

$$X_v'X_v = \begin{bmatrix} A_{11} = X'X & A_{12} = X'v \\ A_{21} = A_{12}' & A_{22} = v'v \end{bmatrix}$$

- Task: calculate $\left(X_v'X_v\right)^{-1}$based on the known terms of: $X'X$, $(X'X)^{-1}$, $X'v$, $v'v$
  - Slight complication: Inserting a column to X , not just appending.
  - Can be solved by permutation vectors (see **[M-1] permutation**).
- Let's call this procedure PMAC (partioned matrices / append column) to ease exposition.

Updating $(X'X)^{-1}$ Using Partitioned Matrices

- Problem: columns are sometimes deleted, not just added.
- Lag combination matrix ($maxlags = 2$ for all variables): $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ & \dots & \\ 2 & 2 & 2 \end{bmatrix}$
- e.g. moving from row 3: $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$ to row 4: $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ deletes two lags of the last regressor
- Solution: store matrices the algorithm can jump back to using *pointers*.

Pointer Variables

- General and “advanced” programming concept, but the basics are easy to understand and apply.
- Each variable has a name and a type.
  - The name really is just a device to refer to a specific location in memory; every location in memory has a unique *address*.
  - Since the type of the variable is known to Mata, it knows how big of a memory range a variable name refers to, and how to interpret the value (the bits stored there).
  - Think in these terms: each variable has an address and a value.
- Pointer variables hold memory addresses of other variables. Pointer variables can point to anything: scalars, matrices, pointers, objects, functions ...

Pointer Variables

- Pointers are often assigned to using “&”; they are dereferenced using “*”.
- Read:
  - & : “the address of”
  - * : “the thing pointed to by”
- mata:

```
mata:
s = J(2,2,1)
p = &s
p // outputs something like 0xcb3cb60
*p // outputs the 2x2 matrix of ones
*p = J(2,2,-7)
s // now contains the matrix of -7s
end
```

- See **[M-2] pointers** for many more details.

Using Pointers for Updating $(X'X)^{-1}$

- Lag combination matrix : $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ & \cdots & \end{bmatrix}$
- 3-element vector of pointers $vec = \begin{bmatrix} p1 & p2 & p3 \end{bmatrix}$; each element points to a matrix. Then calculate $(X'X)^{-1}$ for...
  - ... lags (1 0 0) by ordinary matrix inversion; store using $p1$
  - ... lags (1 0 1) by PMAC using $*p1$; store using $p3$
  - ... lags (1 0 2) by PMAC using $*p3$
  - ... lags (1 1 0) by PMAC using $*p1$; store using $p2$
  - ... lags (1 1 1) by PMAC using $*p2$; store using $p3$
  - ... lags (1 1 2) by PMAC using $*p3$;
  - ... lags (1 2 0) by PMAC using $*p2$; store using $p2$ ... and so forth.

Lag Selection: Mata III (update inverses)

```
// note: may contain incorrect syntax, fictitious commands/options/return values, etc.
lagcombs , k(3) maxlag(8)
mata:
    y = st_data( DEPVAR )
    X = st_data( PULL ALL DATA : ALL VARIABLES, ALL LAGS)

    // calculate terms for full lag specification
    XX = X'X ; Xy = X'y ; yy = y'y

    lagcombs = st_matrix("lagcombs")
    ic = .
    for (i=1; i<=rows(lagcombs); i++) {
        idx = [ GET INDEX VECTOR FOR CURRENT LAG STRUCTURE ]
        XXi = XX[idx,idx] ; Xyi = Xy[idx]

        if (i==1) XXinvi = invsym(XX)
        else      XXinvi = update_XXinv( XXiold , XXinvi , ... )

        ee = yy - Xyi'*XXinvi*Xyi

        ( CALCULATE IC, SELECT IF LOWER THAN IC VALUES UP TO NOW )

        XXiold = XXi
    }
end
```

Lag Selection: Timings

Timings in seconds (2.5GHz, single core) for N=1000:

Mata 2 : no redundancies

Mata 3 : no redundancies + inverse updating

$k+1$	*maxlags*	# regressions	regress	Mata 1	Mata 2	Mata 3
3	4	100	1.6	0.36	0.11	0.14
3	8	∼650	12.5	1.33	0.09	0.13
4	8	∼5,800	132	11.8	0.31	0.27
6	8	∼470,000	∼14,000	∼1,400	53	37
8	8	∼38,000,000	(13 days?)	∼146,000	∼6,500	∼3,200

Recap

In this talk, we have discussed

- Potential strategies for improving code performance
- Basic asymptotic notation for the computing time of algorithms
- Quick look at the ARDL model
- Optimal lag selection
- Moving Stata code to Mata and optimizing the Mata code
- An advanced way of using linear algebra results to improve code performance
- Pointer variables

We have tried to illustrate that mindful code creation can be superior to the "brute force" methods of low-level programming languages and parallelization.

Thank you!

Questions? Comments?

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References

- Gould, William (2014, April 17): Using Mata Operators efficiently [Msg 8]. Message posted to https://www.statalist.org/forums/forum/general-stata-discussion/mata/993-using-mata-operators-efficiently?p=1826#post1826
- Kripfganz / Schneider (2016): ardl: Stata Module to Estimate Autoregressive Distributed Lag Models. Presentation held at the Stata Conference 2016, Chicago.
- Narayan, P.K. (2005): The Saving and Investment Nexus for China: Evidence from Cointegration Tests. Applied Economics, 37 (17), 1979-1990.
- Pesaran, M.H., Shin Y. and R.J. Smith (2001): Bounds Testing Approaches to the Analysis of Level Relationships. Journal of Applied Econometrics, 16 (3), 289-326.