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SADDLEPOINT DYNAMICS WITH AN ENDOGENOUS ROOT OF CONVERGENCE.

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This paper reconsiders local stability in the saddlepoint sense. Market time and the root of convergence are determined endogenously using partial differential equations; consequently, there is not need of the resort to any deus-ex-machina dynamics to justify an initial jump in one of the economic variables. It is shown that the regions of stability are wider than those currently admitted and that, in some cases, there is a justification for the theoretical ambiguity regarding which variable is supposed to jump. Two examples (sluggish adjustment of salaries and exchange rate dynamics) are used to illustrate the methodology.

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1.Introduction.

In their celebrated paper, Sargent and Wallace (1973) introduced saddlepoint stability as a new way of studying convergence to long-run steady states.

Their method, successfully used by our profession since then on, consists mainly in finding (one) negative and (one) positive characteristic roots (in the continuous case, 2X2 system), and in eliminating one of the coefficients in the solution of the linear approximation. This guarantees that there is convergence after an initial jump to the stable arm of the saddle.

However, the jump variable is always chosen exogenously and the initial jump is a discontinuity, a movement not explained by the basic dynamical equations.

To justify the selection of the jump variable, economists add sounding but ad hoc assumptions. For example, when Turnovsky (1996) is referring to the well-known overshooting model, he says "...There is no uniformly correct treatment, and the appropriate formulation is dictated by the specific context and by one´s view of the world" (p.159). Also, he quotes (p.75) Calvo´s observation that "... the Sargent and Wallace solution ...does not specify when the jump takes place" (see Calvo (1977)).

It is the aim of this paper to modify the outlined procedure by determining endogenously the root of convergence, in a way consistent with the main dynamical system and, at the same time, to solve the problem of absence of continuity of the initial change.

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To do this, time will be unfolded in two reference variables. Therefore, we will have a time variable for each "market" or equation. Actual time will be determined within the model according to which market is "on" and which is "off".

It will be shown that, in general, equilibria are more stable than previously thought, because it is supposed that agents use more information than they are allowed to in current discussions of the process of adjustment. For saddlepoints specifically, it is also proved that there exist regions of "potential" stability, which give a higher chance of reaching the steady state than the "thin" manifolds of convergence accepted up to now. This does not mean that there is no room for economic policy. Though it is not necessary to be on the stable arm of the saddle, monetary or fiscal policy can help the system to be in the regions of potential stability.

Unfortunately, some indeterminacy will remain, since it is possible that the same system could converge to the long run position using any of the adjustment rules. However, it gives economic sense to the idea that the initial condition matters regarding to which market can be used as a reference for correcting prices. We can think the other way round: when the rule of adjustment has to be defined, two are too many; in fact, when we compute the characteristic roots it is with the aim of overdetermining the system, for what is important is the relation between say x(t) and y(t) and not with the abstract and ad hoc parameter t. But on the other hand, this indeterminacy justifies the theoretical ambiguity in choosing which variable is supposed to jump.

The solution involves the use of the method of separation of variables for elementary partial differential equations.

Next section will be devoted to presenting the main equations in a general case. Section 3 will address the solution mechanism with two time variables and then, in Section 4, an interpretation of the methodology is discussed. In Section 5 two well-known examples are analyzed under the new methodology. In the concluding section, possible extensions and applications are considered.

2.The general case.

Assume that the initial system of dynamic equations is

$$dx/dt = F(x,y),$$

$$dy/dt = G(x,y),$$

where x(t) and y(t) are the economic variables (for example, the price level and nominal wages respectively), and F and G are the ordinary adjustment rules, which are supposed to fulfill the regular differentiability conditions.

Both variables are being modified following the time variable, which is the one to be considered as historic time here. What happens when one of the variables, e.g. x(t) "jumps" to the stable arm is that at the initial instant is that it goes from x(0), the initial condition, to x*(0), its level consistent with the long run convergence, while the other one is constant at y(0).

The linear approximation of the system can be written as

$$dx/dt = ax + by,$$

$$dy/dt = cx + dy.$$

This model can be written as a second order differential equation:

$$x_{tt} = (a + d)\ x_t + (ad - bc).$$

The solution to this is given by:

$$x(t) = A_1 \exp(\lambda_1 t) + A_2 \exp(\lambda_2 t) + B,$$

with $\lambda_1\lambda_2 = ad - bc < 0$, if the stationary point is a saddle; let us define $\lambda_1 > 0$ and $\lambda_2 < 0$. Notice that B will be zero if the steady state is moved to the origin.The solution for y(t) is obtained as

$$x_t - a\ x = by.$$

Therefore, taking into account the need to converge to the steady state, the following equations determine the numerical values of the parameters:

$$x(0) = A_2 + B,$$
$$x^* = B,$$
$$y^*(0) = (\lambda_2 - a)\ A_2/b - aB/b,$$
$$y^* = - a\ x^*/b.$$

To get these equations, in the solution we put $A_1 = 0$, just to assure convergence. The normal way of solving this system is to determine endogenously A_2, B and y*(0), when x(0) and, of course, y* are fixed; alternatively, x(0) can be considered the jump variable, and in that case x*(0) substitutes for y*(0). This methodology, so usual nowadays, hinges on an implicit discontinuity of one of the variables of the system at the initial instant. If it were not so, how could it be possible to go from y_o to y*(0)?.

The main aim of this paper is trying to understand how that discontinuity can be represented and also to consider an alternative way of fulfilling those conditions by making λ_2 endogenous. This will allow us to see that: 1) the procedure habitually used is a particular case of the model to be studied here, 2) there are not clear reasons to have a jump at the initial instant, 3) the probabilities of having stability of the stationary state are higher than in the traditional methodology.

To show these, time t will be decomposed in two reference variables, one for each market or equation.

There is no need of time in the system. After all, time can be regarded as passing when some important event or change occurs and it affects the workings of the economic system. But if the postulated equations are comprehensive of the economic relations, an exogenous time variable is only a device used to make easier the solution.

3.Unfolding time.

First of all, let us consider an initial point (x_o,y_o) and try to determine λ_2 so that the stationary state is locally stable. Let L be the new unknown, and since $A_2 = x(0)-x^*$, we have

$$(x_o - x^*)(L - a) + by^* = by_o,$$

so that, for the sake of simplicity, taking $x^* = y^* = 0$, the endogenous root has to be

$$L = a + by_o/x_o.$$

It is interesting to take into account that when the condition $y^* = y_o$ is imposed we need $L = a$. Thus, we will have stability if $a + by_o/x_o < 0$, and instability if it is positive. Notice that this condition can be written as

$$(a\, x_o+by_o)/x_o = x_t(0)/x_o.$$

This implies that convergence is possible –by adjusting λ_2- is some regions of the space of states: when sgn $(x_o) \neq$ sgn $(x_t(0))$. Two well-known examples are presented below to illustrate the general determination of the regions of stability.

Of course, it is possible to conduct the same analysis with y(t); in that case the necessary and sufficient condition for the existence of a root that guarantees convergence will become:

$$(c\, x_o + dy_o)/y_o = y_t(0)/y_o < 0.$$

As it can be seen, the regions of potential stability are bounded by perpendicular rays on x^* and y^* and by the isoclines $dx/dt = 0$ and $dy/dt = 0$. But it is interesting to notice that the process will always be stable in a neighborhood of the stable saddle path. That is because on that path, by definition,

$$L = \lambda_2,$$

and (x_o,y_o) belongs to the stable path for it is one of the expressions of the corresponding characteristic vector. On the other hand, in a neighborhood of the unstable saddle path it is not possible to find a negative root that will help to reach the steady state.

However, there is a likely probability of contradiction since using one of the endogenous roots does not assure that the other will also be fulfilled unless we are already on the stable arm. It is to overcome this difficulty that the role of time has to be reconsidered.

To solve that contradiction let us now define two new time variables, one for each economic variable:

$$\partial x/\partial\theta = a\ x + b\ y,$$
$$\partial y/\partial\tau = c\ x + d\ y.$$

The first step is finding a solution; partial differentiation of x_θ with respect to τ gives a second-order partial differential equation:

$$x_{\theta\tau} = a\, x_\tau + d\, x_\theta + (bc-ad)x.$$

By separation of variables, we can look for a solution to this equation of the form:

$$x = h(\theta)g(\tau),$$

such that

$$(h'/h)(g'/g) = (g'/g)\ a + (h'/h)\ d + (bc - ad).$$

Let´s define $\alpha = h'/h$, and hence it will be necessary that

$$g'/g = (\alpha d + (bc - ad))/(\alpha - a).$$

The solution to the partial differential equation is

$$x(\theta,\tau) = A \exp(\alpha\theta) \exp((\alpha d+(bc-ad))/(\alpha-a))\tau) + B.$$

For this equation it is the value of α that is endogenous to meet the necessary initial and boundary conditions. That is to say, it has to be chosen to have stability; a good reference for the methodology, very common in physics, is Farlow (1993).

Notice that for every α it is possible to find g'/g, that allows consistency with the initial system, but that it is irrelevant if $d\tau \equiv 0$ and hence

$$dy(\theta,\tau) = (\partial y/\partial\theta)\ d\theta;$$

under this assumption, without loss of generality we can assume $\tau \equiv 0$. With this slight modification, the solution becomes

$$x(\theta,0) = A \exp(L\theta)+ B,$$

and the endogenous root has to be

$$L = a + by_o/x_o.$$

Additionally we must impose the condition $a + by_o/x_o < 0$ to prevent instability[2]. Again, this can be written as

$$(a\, x_o+by_o)/x_o = x_\theta(\theta,0)/x_o.$$

Hence, we have that the equation for, say, $y(\theta,\tau)$ is not relevant for determining the dynamics, but the variable is being modified in the right sense through θ.

Notice that: 1) if we write the system in terms of deviations, $x(\theta,0)/y(\theta,0)$ will be constant, as it is along the arms of the saddle, 2) that the stable arm always belongs to the region of potential stability (taking $L = \lambda_2$). If we keep working both time variables we will face again the potential contradiction, which is solved normally by allowing one variable to jump and by working with the overdetermined system since that moment on.

It is still possible to obtain the original presentation when

$$\theta = \tau = t/2.$$

Moreover, the characteristic roots of the system in t satisfy

$$\alpha = (\alpha d+(bc-ad))/(\alpha-a).$$

In more general terms, we could have taken also

$$\theta = \varepsilon\, t,\ \tau = \rho\, t,$$

so as to still maintain our links to real time t, but in fact t has not been defined at all. Is it a time variable representing the change in any exogenous system?. Is it time of the system itself?. In the case of this paper, t has still to be defined following the change in some endogenous variable.

Notice that in this case:

$$dx(\theta,\tau) = (x_\theta\, \varepsilon + x_\tau\, \rho)\, dt,$$

and we get again the classical representation by taking $x_\theta = x_\tau$, and $\varepsilon = \rho = ½$. On the other hand, if $\rho = 0$ then we have a case in which time passes only for the market for x.

4.Some remarks on the interpretation of the methodology.

When we want to compute the stability of the equilibrium in a system of two variables, and one of them is going to be held constant, the system converges –if it does- to a point on the isocline of the variable that is moving. This means that the familiar jump will not lead us to the stable arm of the saddle. Let us take $a < 0$, and $y^*(0) = y_o$, with $y_o \neq - a\, x^*/b$. The system will not be moving towards $x^*(0)$, but instead to a point on $x_t=0$.

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[2] This does not imply that any equilibria will have always a region of stability, in particular if it requires that e y_o/x_o to be out of the neighborhood where the linearization is valid.

In our case, instead, the system is supposed to take into account how $y(\theta,0)$ has to be related to $x(\theta,0)$ in order to reach the long run solution. This means that, though one of the rules of adjustment is abandoned, markets would work as if agents were fully aware of the total relative change necessary to reach the long run position; this seems to be a more "rational" approach than considering that one of the variables is temporarily constant. And, although τ is constant, $y(\theta,0)$ is being adjusted through θ.

It is interesting to explore the link between the concept just developed and "perfect stability" in the sense of Hicks (see Quirk and Saposnik (1968)). To do this, let E_i be the excess demand of good i, and assume that we have only three goods (and we leave aside the market of the numeraire). To study stability in the sense of Hicks, we need to consider the sign of dE_1/dp_1 (negative) when dp_2/dp_1 is such that

$$\partial E_2/\partial p_1 + (\partial E_2/\partial p_2)(dp_2/dp_1) \equiv 0.$$

That is, for any good, the price of which is changing, the variation of its price must be such that its market is in equilibrium. In the case of our model, it is not necessary to accept this condition on every individual point of the path from the boundary condition to the long run position, but it is required to calculate L so that this is met when the whole movement is computed. In our case what has to be computed is $d(dp_1/dt)/dp_1 = dE_1/dp_1$ (when the speed of adjustment is 1).

There is one important difference with perfect stability. In our case, it is assumed that markets are able to estimate which is the long run relation between absolute prices and therefore to compute the change in p_2 consistent with the modification in p_1. That is to say, it is not necessary that the second market be in permanent equilibrium, as it is supposed in perfect stability, but that it is in equilibrium when the market for good 1 is.

This can be written in the following way. We are defining $\emptyset$ such that

$$E_2(p_1, \emptyset\, p_1) = 0,$$

when p_1 is computed at its steady state level. And the stability condition becomes:

$$\partial\, (\, dp_1/\, dt) \,/\, \partial p_1 = [(\partial E_1/\, \partial p_1) + \emptyset\, (\partial E_1/\, \partial p_2)]\, * < 0.$$

In fact, in our presentation both "excess demands" are being reduced in absolute value at the same time, as if markets were coordinated. This can be seen computing $y_{\theta\tau}$ and using x_θ to get

$$y_{\theta\tau} = A\exp(L\theta)x_t(0)y_t(0)/x_o^2.$$

This means that "excess demand" for $y(\theta,0)$ tends to zero when θ grows.

The full picture can be summarized as follows. First of all, there is an initial computation or correction of relative prices; then a progressive movement takes the variables towards their equilibrium absolute levels. Perhaps, the initial correction gives the impression of being a sudden change in one of the variables, though it will not be a discontinuity.

Therefore, if we admit the possibility of having two time variables, we are also recovering a role for a special kind of perfect stability. Moreover, the actual time is determined within the model. After all, time passes only if something changes, and it is one of the markets (which one, is not neutral for stability) that leads the process. This is particularly important if we are referring to a closed model; if our model encompasses a whole macroeconomic system, why should we have a time variable not defined by changes in relevant (endogenous) economic variables?.

5. Two examples of stability regions.

We shall borrow a pair of examples from the text by Turnovsky (1996) to show how the regions of stability can be determined.

Example 1: Cagan model with sluggish wages. In this model the basic system is:

$$M - P = \alpha_1 Y - \alpha_2 (dP/dt),\ \alpha_1 > 0,\ \alpha_2 > 0,$$

$$Y = c + (1-m) N,\ 0 < m<1,$$

$$W - P = a - mN,$$

$$dW/dt = \gamma (N - \tilde{N}),\ \gamma > 0,$$

where Y is output, N is employment, $\tilde{N}$ is full employment , P is the price level, M is the nominal stock (constant) and W is the wage rate. The anticipated rate of inflation equals the actual, and output is variable and it is included in the demand for money. The system can be reduced to:

$$\partial W/\partial\theta = a W + bP + Z_1,$$
$$\partial P/\partial\tau = cW + dP + Z_2,$$

with $a < 0$, $b > 0$, $c < 0$, $d > 0$ and Z_1 and Z_2 are the steady state levels. However, as it was already pointed out, Turnovsky adheres to Sargent and Wallace (1973) analysis and therefore he assumes that "... the price level is free to jump instantaneously whereas wages are constrained to move continuously".

On Figures 1a and 1b, we can see the new stability regions, that have been shaded. Both of them include the stable arm of the saddle path. It is not longer necessary to suppose that P will jump; however the probability of having stability is higher now if we look to the saddle in the mathematical sense. Notice that the region of stability when the rule of adjustment of W is used is wider than the one obtained from dP/dt. At point A, only θ should be ruling the dynamics if we want to get to Q, and along AQ, the ratio W/P will be constant[3]; first of all, we will see an adjustment of the relation of W and P, and afterwards a movement of W to its long run level at Q.

But there is an area where there is an overlapping of stability regions too; this is the area were some theoretical ambiguity is allowed. Then, a legitimate doubt is if it is not possible to keep using both θ and τ, defined as linear approximations of t, that is $\theta = \varepsilon\, t$ and $\tau = \rho\, t$, and taking $L = \lambda_2$. Perhaps that could help to solve the indeterminacy in the overlapping regions.

The idea then is trying to determine endogenously ε and ρ so that the following system of conditions is satisfied:

$$by^*(0)/x^*(0) + a = \lambda_2(\varepsilon+\rho),$$
$$cx^*(0)/y^*(0) + d = \lambda_2(\varepsilon+\rho);$$

but it can be seen that both equations can be fulfilled at the same time only if $(x^*(0),y^*(0))$ is already on the stable arm of the saddle, where

$$x/y = (ax+by)/(cx+dy).$$

Example 2: Exchange rate dynamics. The classical example of overshooting of Dornbusch (1976) is:

$$R = R^* + dE/dt,$$

$$M - P = \alpha_1 Y - \alpha_2 R, \alpha_1 > 0 , \alpha_2 R, \alpha_1 > 0, \alpha_2 > 0 ,$$

$$dP/dt = \rho \{\beta_0 + (\beta_1 - 1) Y - \beta_2 R + \beta_3 (E - P)\} ,$$

where $0 < \beta_1 < 1$, $\beta_2 > 0$, $\beta_3 > 0$, $\rho > 0$, r^* is the (exogenous) foreign nominal interest rate, R is the domestic interest rate, E is the logarithm of the current exchange rate, dE/dt is the expected percentage rate of change of the exchange variable (equal to the actual rate), M is the logarithm of the domestic money supply (exogenous), P is the logarithm of the domestic price level and Y is the logarithm of the domestic output level (fixed).

Let us define e and p as the deviations of E and P with respect to the equilibrium and put it in terms of θ and τ; the dynamical system becomes:

$$\partial e/\partial\theta = a\, e + b\, p ,$$

$$\partial p/\partial\tau = c\, e + d\, p ,$$

where $a = 0$, $b > 0$, $c > 0$ and $d > 0$.

Figures 2a and 2b show the stability regions. Not always is $e(\theta,\tau)$ the variable that can rule the dynamics; the region of potential stability of $p(\theta,\tau)$ is wider, and hence, from B, it is τ that has to lead the dynamics to get to Q in the long run. There will be an initial change in the relation e/p to $(e/p)^*$ and then a progressive movement of p towards p^*. In

[3] Computing $x_{\theta\theta}$ and using the equation for x_θ, we have $y/x = (L-a)/b$.

Figure 2a, there is not going to be a jump of the exchange rate to C, but a slow movement along BQ. On the other hand, if θ governs the dynamics, we will not see a sudden jump of the exchange rate from D to F, in Figure 2b, but a change along DQ.

6.Conclusions and further extensions.

This paper explores a change in the methodology currently used to study stability, not only in the saddlepoint sense. The techniques that are discussed are very simple to understand but their use introduces a major change in interpretation, both in the definition of time in economics, as in the hierarchy of markets to reach long run positions.

There are some gains and some losses stemming from these modifications. If the proposed "mechanics" are accepted, time is not longer a parameter without sense, imposed on the system for the sake of convenience; it becomes endogenous to the changes coming from the market that leads the dynamics. Moreover, the "first day" discontinuity is not necessary. Agents forecast the long run relation between the variables and accept or reject transactions in the leading market, but taking fully into account that long run ratio.

This view is more optimistic that the current one because under these conditions, markets and policy makers face a less demanding process: they have to do things well enough, but they do not have to compute exactly a point on the stable arm.

Where are the losses?. Paradoxically, the elimination of the discontinuity seems to be less elegant than coping with it; it introduces also indeterminacy when more than one market is able to lead the dynamics. That is something that has to be explored; but there is a compensation: since we have overlapping regions for stability, it helps to defend the ambiguity of the theory when defining the variable that is supposed to jump to the converging manifold. It also allows to establish better the role of each variable, showing that for some initial conditions there is no hope of getting to the long run solution.

The future research agenda includes the extension of the results to the case of several markets or variables ($n > 2$), as well as the evaluation of how local stability has to be understood in other relevant cases (e.g., node and focus).

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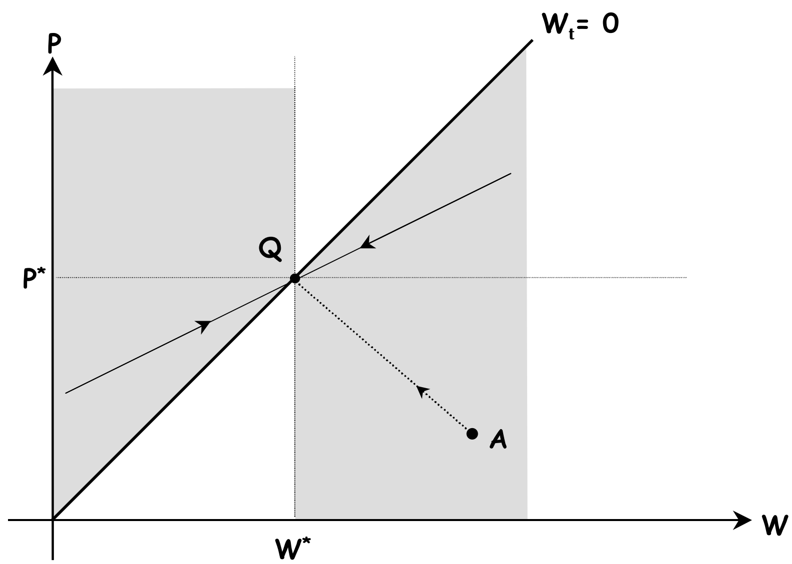

Figure 1a

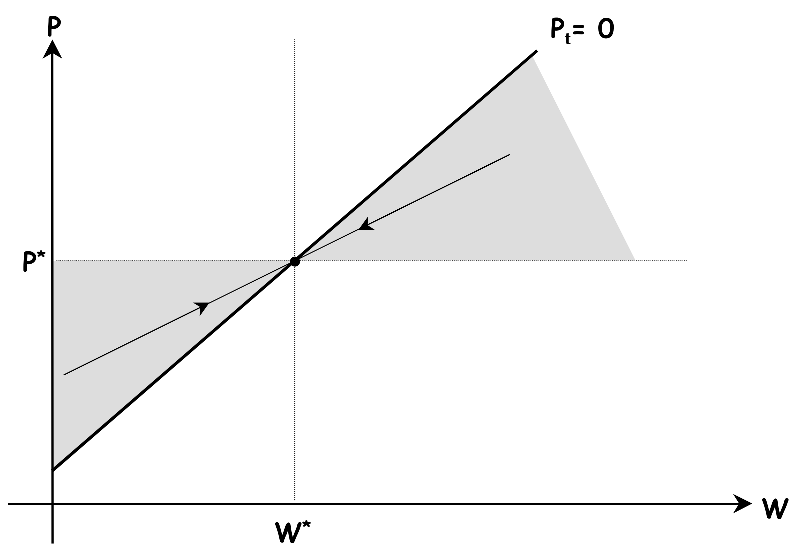

Figure 1b

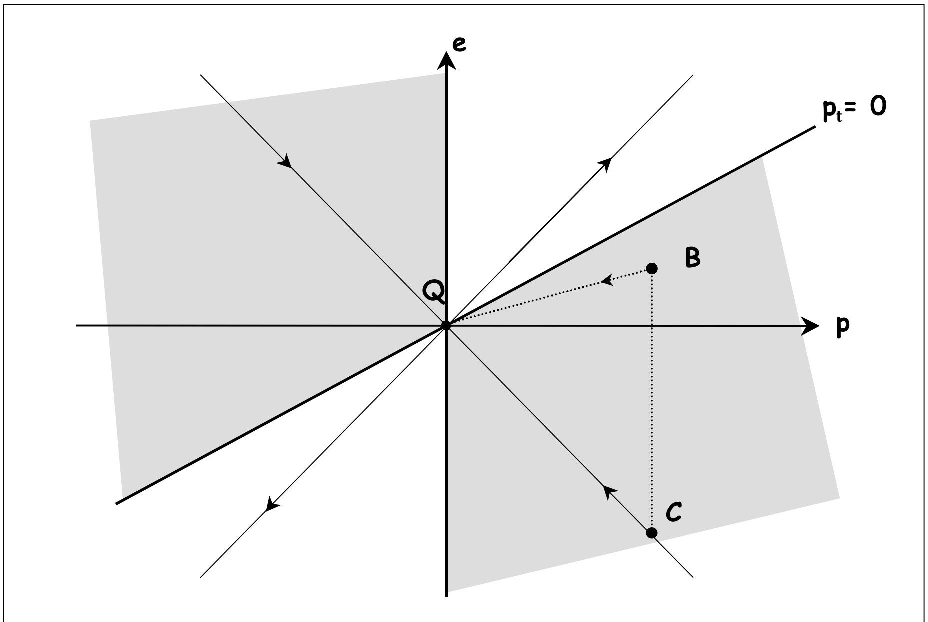

Figure 2a

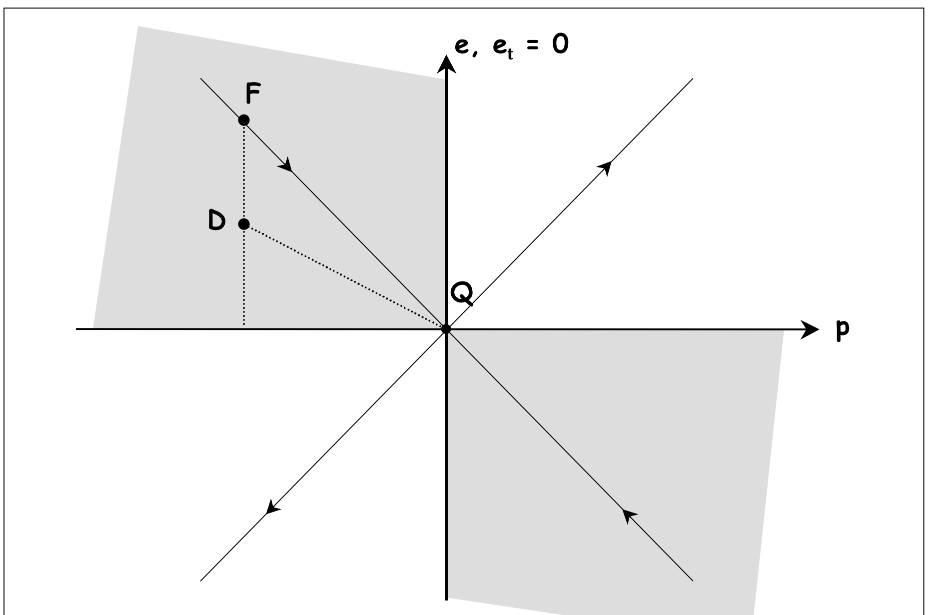

Figure 2b