Introduction

The present paper attempts to study the dynamic properties of residential and commercial property prices. A dynamic general equilibrium model is built and certain testable hypotheses concerning the second moment of property prices are derived. To test these hypotheses, a multi-city panel dataset is used. It is demonstrated that, by and large, the theoretical predictions are consistent with the empirical findings.

The existing studies of the housing market fall into several strands. The first strand treats aggregate output and income as exogenous, and studies how housing demand and supply, and the equilibrium price react to different types of shock. The models in most of these studies are essentially static. In some, even the price, or the rent, is taken as given in order to focus on tenure decision. footnote Therefore, these studies cannot address the evolution and interdependent dynamics of the prices of different property type.

The second strand focuses on the option aspect of properties. footnote This is an increasingly important area as the market for mortgage-based securities in the US, among others, becomes more and more developed. Studies here take the dynamic aspect of housing price seriously and some of the analyses are exceedingly technical. They contribute significantly to our understanding of mortgage-financing under the possibility of default and early termination of contract. The drawback is that they mainly focus on individual property buyer's problem and are silent to the macro aspects of the housing market.

The present paper follows yet another approach, one which studies the housing market using the dynamic general equilibrium approach – as adopted in, for instance, Greenwood and Hercowitz (1991), Baxter (1996), and Gort, Greenwood and Rupert (1998). This approach enables us to endogenize and study the interdependent dynamics of property prices. Several casual observations provide justification for this approach. In reality, a construction company that builds commercial property typically could build residential property as well. There is substitutability between the two types of output. Also, an expanding construction industry inevitably absorbs resources from the rest of the economy and thereby increases the marginal cost of other sectors. Therefore, it is natural to conjecture that the prices of commercial and residential properties are simultaneously determined. This paper takes a preliminary step along this direction. In our model, the price of both commercial and residential property is endogenously determined. Moreover, our model can generate closed form solutions from which several testable hypotheses concerning the short-run commercial and residential property prices are obtained. In particular, we show that under certain conditions: (1) the volatility of commercial property is higher than that of residential property, (2) each of lagged, contemporary, and forward commercial property prices is positively correlated with the residential property price, and (3) the contemporaneous covariance between the two property prices is larger than the lagged covariance.

Intuitively, property prices should also be positively correlated with real output growth and other aggregate variables. Consider the situation where business capital and commercial property are complements in producing final goods in the economy. A temporary increase in general productivity will increase the demand for both business capital and commercial property. This might lead to an increase in commercial property price. Furthermore, if it is foreseen that there will be an increase in the amount of business capital stock in the following period, which means that the marginal product of commercial property will increase, there will be a further increase in the demand for commercial property. The interaction does not stop here. At the equilibrium, there is a trade-off between (nondurable) consumption, and the accumulation of business capital, commercial property, and residential property. An unexpected increase in productivity will drive up the demand for both consumption goods and residential capital at the given price. To clear the markets, the property prices would need to adjust. These interactions should be reflected in the correlation between real output growth and property prices. In this paper, we address formally this output growth vs. property price correlation by allowing endogenous accumulation of factors. Some testable hypothesis will be derived, which are then tested against real data.

We are not pioneers in the study of output growth and property prices correlation. For instance, Greenwood and Hercowitz (1991) attempt to explain the cyclicity of residential investment and business investment. However, their model assumes perfect irreversibility between residential capital and consumption goods, hence the relative price of residential capital is always unity. While this formulation is a clever abstraction for studies of business cycles, it precludes an investigation of property price dynamics in a general equilibrium context. They do not have commercial property in their model either. In a recent study, Gort, Greenwood and Rupert (1998) study technological progress embedded in "structures" (such as roads). However, they restrict their attention to the balanced growth path and they do not incorporate residential property into their model. A distinguishing feature of our present paper is that it puts economy output, commercial and residential property prices together in a unifying framework. footnote

This paper may also contribute to the multi-sector general equilibrium literature, which is typically very involved computationally but not very insightful analytically. While the current model is very simple, it nevertheless provides some closed form solutions and serves as a useful starting point for future investigations. This is somewhat important as there is a recent tendency in the business cycles literature to disaggregate the one good paradigm into a paradigm with several sectors. A natural outcome of this paradigm is the emergence of relative price dynamics. This paper demonstrates that under some assumptions, it is feasible to obtain closed form analytical solution for relative prices (as a function of shocks) even in a dynamic general equilibrium context. It would complement more general models which only deliver numerical results.

The organization of this paper is as follows. The theoretical model is presented in Section ref: model . Section ref: empirical tests describes our dataset and presents the empirical results. Section ref: conc concludes. Technical derivations are relegated to the appendix.

Model

Our model is similar to that of Lucas (1978), and Greenwood and Hercowitz (1991) and so the present description will be brief. In our model, time is discrete and the horizon is infinite. The population is constant and is normalized to unity. There are four goods: a non-storable consumption good, residential property, commercial property, and business capital; with the latter three goods being durable. footnote It is commonly recognized that the depreciation rate of business capital (e.g. machinery) is higher than that of property (e.g. structures). To highlight this difference, it is assumed that business capital completely depreciates after (goods) production, while residential property and commercial property only partially depreciate. footnote

Our analysis will focus on the representative agent of the economy. At time t, $t = 0, 1, 2, \ldots$, the agent maximizes life-time utility

$$\sum_{s=t}^{\infty} \beta^s u(C_s, H_s + H_s^r)$$

which is a discounted sum of the periodic utility $u(C_s, H_s + H_s^r)$, footnote where β, $0 < \beta < 1$, is the discount factor, C_s is the amount of consumption in period s, and H_s is the stock of residential property owned by the representative agent in period s, H_s^r is the stock of residential property rented from the market by the representative agent in period s, $s = t,\ t+1,\ t+2, \ldots$ Essentially, it is assumed that rented and owned residential properties are perfect substitutes in terms of service rendered. We follow the baseline model of Greenwood and Hercowitz (1991) in assuming that the preference is log-separable, $u(C_s, H_s) = (\ln C_s + \omega \ln(H_s + H_s^r)),\ H_s + H_s^r > 0,$ where $\omega > 0$ is a preference parameter governing the substitution between consumption and residential property. This is admittedly a strong assumption, yet it enables us to obtain closed form solutions and sharp predictions, as is made clear in the following. footnote

Goods production technology is such that the total amount of production Y_t in period t depends on the stock of commercial property owned by the representative agent $F_t,$ the stock of commercial property rented from the market $F_t^r,$ and the stock of business capital owned by the representative agent K_t. For simplicity, it is assumed that the business capital owned by the agent and that rented from the market are perfect substitutes. And the same applies to commercial property,

$$Y_t = A_t(K_t)^{1-\alpha}(F_t + F_t^r)^{\alpha}, \qquad \#$$

$0 < \alpha < 1,\ t = 1,\ 2, \ldots$ As emphasized in Gort, Greenwood and Rupert (1998), commercial property (F_t here) plays an important role in goods production. To incorporate this idea, the aggregate production function is assumed to exhibit constant returns to scale in business capital and commercial property. footnote The amount of output, however, depends not on the amount of business capital and commercial property, but also on productivity $A_t,$ which fluctuates over time. In this paper, we assume that the fluctuation of productivity is the only exogenous (random) shock on the economy. For simplicity, the series of "productivity" $\{A_t\}_{t=0}^{\infty}$ is assumed to be bounded and positive, $0 < A_t < M < \infty,$ some constant $M,\ \forall t,$ and is identically and independently distributed (i.i.d.). Both the mean and the variance of this random productivity are bounded, footnote

$$0 < E(A_t),\ Var(A_t) < \infty,\ \forall t.$$

To close the model, it is necessary to describe the evolution of the different kinds of capital stock. Recall that business capital completely depreciates after one period. In the case of commercial and residential property, depreciation is partial and the accumulation of each is deterministic,

$$F_{t+1} = B_F(I_t^F)^{1-\varepsilon}(F_t + F_t^s)^{\varepsilon}, \qquad \#$$

and

$$H_{t+1} = B_H(I_t^H)^{1-\theta}(H_t + H_t^s)^{\theta}, \qquad \#$$

and $B_H,\ B_F$ are constants, $0 < B_H, B_F < \infty$. Equations(ref: pd f prod) says that the existing stock of commercial property, whether it is already owned by the agent, i.e. F_t, or purchased from the market, i.e. F_t^s, are complement to investment I_t^F in the formation process of the new stock of commercial property F_{t+1}. Similar reasoning applies to Equations(ref: pd h prod) for residential property. Notice that the rental market is a spot market, and hence the amount of residential or commercial property rented from the market, H_t^r and F_t^r, do not enter the equations (ref: pd f prod) and (ref: pd h prod). The specific form of the laws of motion for commercial buildings and residential property is adapted from Hercowitz and Sampson (1991). In this formulation, $1 - \varepsilon,$ where $0 < \varepsilon < 1,$ can be interpreted as the "depreciation rate" of existing commercial buildings, and $1 - \theta,$ where $0 < \theta < 1,$ can be interpreted as the "depreciation rate" of existing residential property. Besides building the properties, the representative agent can buy/sell commercial property F_t^s and residential property H_t^s at unit prices $P_{F,t}$ and $P_{H,t}$ respectively. They can also rent commercial property F_t^r and residential H_t^r from the market at rental rate $R_{F,t}$ and $R_{H,t}$ respectively. (Notice that business capital K_t depreciates completely after one period and hence the rental market of business capital can be assumed away without loss of generality). We assume that they first produce output and make payments for commercial property and residential property afterwards.

As in Greenwood and Hercowitz (1991), the dynamic optimization problem is summarized in a Bellman equation as

$$W(H_t, F_t, K_t) = \max_{C_t, K_{t+1}, H_{t+1}, F_{t+1}, I_t^H, I_t^F, H_t^s, F_t^s, F_t^r, H_t^r} \ln C_t + \omega \ln(H_s + H_s^r) + \beta E_t W(H_{t+1}, F_{t+1}, K_{t+1})$$

s.t.

$$\begin{aligned} & C_t + K_{t+1} + I_t^H + I_t^F + R_{F,t} F_t^r + R_{H,t} H_t^r \\ & \le A_t (K_t)^{1-\alpha} (F_t + F_t^r)^{\alpha} - P_{F,t} F_t^s - P_{H,t} H_t^s, \end{aligned} \qquad \#$$

and also (ref: pd f prod) and (ref: pd h prod). Let $\lambda_{1,t}$, $\lambda_{2,t}$ and $\lambda_{3,t}$ represent the multipliers of the constraints (ref: pd s budget), (ref: pd f prod) and (ref: pd h prod) respectively. It is easy to show that the first order conditions of our dynamic programming problem are

$$\lambda_{1t} = 1/C_t, \qquad \#$$

$$P_{H,t} \lambda_{1t} = \lambda_{3t} B_H \theta (I_t^H / (H_t + H_t^s))^{1-\theta}, \qquad \#$$

$$P_{F,t} \lambda_{1t} = \lambda_{2t} B_F \varepsilon (I_t^F / (F_t + F_t^s))^{1-\varepsilon}, \qquad \#$$

$$R_{F,t} = A_t \alpha (K_t / (F_t + F_t^r))^{1-\alpha}, \qquad \#$$

$$R_{H,t} = (1/\lambda_{1t})(\omega / (H_t + H_t^r)), \qquad \#$$

$$\lambda_{1t} = \lambda_{3t} B_H (1-\theta)((H_t + H_t^s)/I_t^H)^{\theta}, \qquad \#$$

$$\lambda_{1t} = \lambda_{2t} B_F (1-\varepsilon)((F_t + F_t^s)/I_t^F)^{\varepsilon}, \qquad \#$$

and

$$\begin{aligned} \lambda_{1t} &= (1-\alpha)\beta E_t[\lambda_{1,t+1} A_{t+1} (K_{t+1})^{-\alpha} (F_{t+1} + F_{t+1}^r)^{\alpha}] \\ &= (1-\alpha)\beta E_t\left[\lambda_{1,t+1}\left(\frac{Y_{t+1}}{K_{t+1}}\right)\right], \end{aligned} \qquad \#$$

$$\begin{aligned} \lambda_{2t} &= \beta E_t\{\lambda_{1,t+1} \alpha A_{t+1} (K_{t+1})^{1-\alpha} (F_{t+1} + F_{t+1}^r)^{\alpha-1} \\ &\quad + \lambda_{2,t+1} \varepsilon B_F (I_{t+1}^F / (F_{t+1} + F_{t+1}^s))^{1-\varepsilon}\} \\ &= \beta E_t\left\{\lambda_{1,t+1} \alpha \left(\frac{Y_{t+1}}{F_{t+1} + F_{t+1}^r}\right) + \lambda_{2,t+1} \varepsilon \left(\frac{F_{t+2}}{F_{t+1} + F_{t+1}^s}\right)\right\}, \end{aligned} \qquad \#$$

$$\begin{aligned} \lambda_{3t} &= \beta E_t\left\{\frac{\omega}{H_{t+1}} + \lambda_{3,t+1} \theta B_H \left(\frac{I_{t+1}^H}{H_{t+1} + H_{t+1}^s}\right)^{1-\varepsilon}\right\} \\ &= \beta E_t\left\{\frac{\omega}{H_{t+1}} + \lambda_{3,t+1} \theta \left(\frac{H_{t+2}}{H_{t+1} + H_{t+1}^s}\right)\right\}. \end{aligned} \qquad \#$$

To complete the model, it is necessary to impose equilibrium conditions. Note that this model differs from the standard real business cycle models in two ways. First, there is no explicit treatment of the firm's problem in this model. Second, closed form solutions can be obtained. We therefore skip the detailed characterization of the equilibrium and go directly to finding the solution.

Since the economy is populated by a large number of identical agents, in equilibrium, the net sale of different types of properties among agents must be zero,

$$H_t^s = F_t^s = 0, \ \forall t. \qquad \#$$

Similarly, the net trade in the rental market should also be zero,

$$H_t^r = F_t^r = 0, \ \forall t. \qquad \#$$

Given the above, we can now define the stationary equilibrium of this model.

Definition *For a given sequence of productivity shocks $\{A_t\}_{t=0}^{\infty}$, a stationary equilibrium is a sequence of quantity variables $\{C_t, K_{t+1}, H_{t+1}, F_{t+1}, I_t^H, I_t^F\}_{t=0}^{\infty}$, and a sequence of price variables $\{R_{F,t}, R_{H,t}, P_{F,t}, P_{H,t}\}_{t=0}^{\infty}$, such that the representative agent maximizes his expected life-time utility, subject to the constraints (ref: pd s budget), (ref: pd f prod) and (ref: pd h prod), and market clearing conditions (ref: mkt clear) and (ref: mkt clear 2).*

We now solve the equilibrium explicitly. The equilibrium quantities are solved first, followed by prices. In the appendix, it is shown that the equilibrium quantities can be characterized by the following equations,

Proposition *If the following conditions are satisfied,*

$$\beta(\alpha + \varepsilon - \alpha\varepsilon) < 1, \qquad \#$$

$$\Phi^H * \Phi_2 < 1, \qquad \#$$

some constant Φ^H, Φ_2, then the evolution of the equilibrium quantities can be summarized by the following equations,

$$C_t = \Phi Y_t,$$
$$K_{t+1} = \Gamma^K Y_t,$$
$$I_t^H = \Gamma^H Y_t,$$
$$I_t^F = \Gamma^F Y_t,$$

for some positive constant Φ, Γ^i, such that $0 < \Phi$, $\Gamma^i < 1$, $i = K,H,F$, and (ref: pd f prod), (ref: pd h prod), and (ref: prod fn), for any given initial conditions, A_0, K_0, F_0, H_0.

The conditions (ref: useless cond 1) and (ref: useless cond 2) are technical. Intuitively, they serve to ensure that the return of investing in any of the three types of capital (business capital K_t, residential property H_t and commercial property F_t) would not be too low or too high at the equilibrium, and hence guarantee a positive fraction (not exceeding one) of output to be devoted to each type of capital. To solve the equilibrium quantities *in each period* as functions of exogenous variables, it would be convenient to rewrite in log form, i.e., we write $\phi = \ln \Phi$, $\gamma^k = \ln \Gamma^K$, $c_t = \ln C_t$, $i_t^f = \ln I_t^F$, $y_t = \ln Y_t$, etc. The economy is hence represented by the following *linear* equations:

$$y_t = a_t + (1-\alpha)k_t + \alpha f_t, \qquad \#$$

$$f_{t+1} = b_f + (1-\varepsilon)i_t^f + \varepsilon f_t, \qquad \#$$

$$h_{t+1} = b_h + (1-\theta)i_t^h + \theta h_t, \qquad \#$$

$$c_t = \phi + y_t, \quad \#$$

$$k_{t+1} = \gamma^k + y_t, \quad \#$$

$$i_t^f = \gamma^f + y_t, \quad \#$$

$$i_t^h = \gamma^h + y_t, \quad \#$$

given the initial conditions $a_0,\ k_0,\ f_0,\ h_0$.

It is easy to see that (log) non-durable consumption c_t and (log) residential housing h_t are determined by (log) output y_t and (log) commercial housing f_t, given the initial conditions. This is clearly a recursive system, with the subsystems (ref: linear prod), (ref: linear f'), (ref: linear k') and (ref: linear if) determining the outcome of the large system. In fact, (ref: linear prod), (ref: linear f'), (ref: linear k') and (ref: linear if) can be combined and further simplified as

$$y_{t+1} - \alpha f_{t+1} = (1-\alpha)\gamma^k + (1-\alpha)y_t + a_{t+1}, \quad \#$$

$$f_{t+1} = b_f' + (1-\varepsilon)y_t + \varepsilon f_t, \quad \#$$

where $b_f' = b_f + (1-\varepsilon)\gamma^f$. Following Sargent (1979) and Lütkepohl (1993), footnote it is shown in the appendix that the equation system (ref: y-alpha f) and (ref: f' f) can be solved, giving rise to the simple representation:

$$\triangle y_{t+1} = b_\triangle + (1-\varepsilon L)(1-\varepsilon_\alpha L)^{-1}a_{t+1}, \quad \#$$

and

$$\triangle f_{t+1} = b_\triangle + (1-\varepsilon)L(1-\varepsilon_\alpha L)^{-1}a_{t+1}, \quad \#$$

where

$$b_\triangle = (1-\varepsilon_\alpha)^{-1}\{(1-\alpha)((1-\varepsilon)\gamma^k) + \alpha b_f'\},$$

$\triangle X_t \equiv X_t - X_{t-1},$ for any variable X_t. L is the lagged operator, $L^n X_t = X_{t-n},\ n = 1,2,3,...,$ and $Lc = c,$ for any constant c.

With (ref: change of y) and (ref: change of f) in place, the evolution of residential capital stock can be easily traced. By (ref: linear h') and (ref: linear ih), it is clear that

$$h_{t+1} = (1-\theta)^{-1}b_h' + (1-\theta)(1-\theta L)^{-1}y_t, \quad \#$$

or

$$\triangle h_{t+1} = (1-\theta)(1-\theta L)^{-1}\triangle y_t,$$

where

$$b_h' = (1-\theta)^{-1}b_h.$$

which, when combined with (ref: change of y), can be re-written as

$$\triangle h_{t+1} = b_\triangle + (1-\theta)(1-\theta L)^{-1}(1-\varepsilon L)(1-\varepsilon_\alpha L)^{-1}a_t. \quad \#$$

Equipped with (ref: change of y), (ref: change of f) and (ref: change of h), we can compute the relative prices of the two types of property. In fact, we need to solve both prices and rents of both types of property. The following result, which relates rents and prices, is useful:

Lemma *Each property rent is proportional to the corresponding property price.*

$$R_{F,t} = B^F P_{F,t}, \quad \#$$

$$R_{H,t} = B^H P_{H,t}, \quad \#$$

for some constants $B^F,\ B^H$.

This result is consistent with the empirical finding that housing prices and rents exhibit the same trend. footnote By taking log of (ref: rent price f) and (ref: rent price h), we get

$$r_{ft} = b^f + p_{ft},\ r_{ht} = b^h + p_{ht},$$

for some constant $b^f,\ b^h$. Since constants have no impact on variance and covariance terms, all the conditions and results for property prices apply directly to property rents. Therefore, without loss of generality, we can focus on the cross-relationships of prices. First, notice that (ref: pd foc h0) and (ref: pd foc ih) can be combined to yield

$$P_{Ht} = \left(\frac{\theta}{1-\theta}\right)\left(\frac{I_t^H}{H_t}\right) = \left(\frac{\theta\Gamma^H\Phi}{1-\theta}\right)\left(\frac{Y_t}{H_t}\right). \quad \#$$

Similarly, (ref: pd foc f0) and (ref: pd foc if) can be combined to yield

$$P_{Ft} = \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{I_t^F}{F_t}\right) = \left(\frac{\varepsilon\Gamma^F\Phi}{1-\varepsilon}\right)\left(\frac{Y_t}{F_t}\right). \quad \#$$

As for the quantity variables, we take the log of all price variables, for example, we write $p_{ht} = \ln P_{Ht}$. (ref: pd ph eqn) and (ref: pd pf eqn) can then be written as

$$p_{ht} = d_h + y_t - h_t, \quad \#$$

and

$$p_{ft} = d_f + y_t - f_t, \quad \#$$

where $d_h,\ d_f$ are constants,

$$d_h \equiv \ln\left(\frac{\theta\Gamma^H\Phi}{1-\theta}\right),\ d_f \equiv \ln\left(\frac{\varepsilon\Gamma^F\Phi}{1-\varepsilon}\right).$$

It is clear that whether p_{ht} and p_{ft} have finite variance depends on the terms $(y_t - h_t)$, $(y_t - f_t)$. The following lemmas provide useful characterization of these terms. Proofs can be found in the appendix.

Lemma $(y_t - h_t)$ *can be written as a discounted sum of past productivity shocks,*

$$y_t - h_t = \left(\sum_{i=0}^{\infty} \eta_i a_{t-i}\right) + (constant\ terms), \quad \#$$

where

$$\varepsilon_\alpha = \varepsilon * (1-\alpha),$$

$$\eta_i = \left[\sum_{j=0}^{i}(\theta)^j(\varepsilon_\alpha)^{i-j} - \varepsilon\sum_{j=0}^{i-1}(\theta)^j(\varepsilon_\alpha)^{i-j}\right],$$

with $\eta_0 \equiv 1$.

Lemma *$(y_t - f_t)$ can be written as a discounted sum of past productivity shocks,*

$$y_t - f_t = \left(\sum_{i=0}^{\infty} (\varepsilon_\alpha)^i a_{t-i} \right). \qquad \#$$

It is clear that $(y_t - f_t)$ is stationary. footnote We assume that $(y_t - h_t)$ is too. footnote As will be made clear, the magnitude of η_i plays a key role in the results over relative volatility. It is perhaps instructive to explicitly compute the variance of (log) property prices. To do this, assumptions on $\{a_t\}$ need to be made. For expositional purposes, it is assumed that $\{a_t\}$ is an i.i.d. process. In the appendix, the case of $\{a_t\}$ being an AR(1) process is analyzed. For the present section, since $\{a_t\}$ is i.i.d. with finite first and second moments, mean and variance of p_{ht} exist. In particular, by (ref: ph log) and (ref: y h lemma),

$$var(p_{ht}) = var\left(\sum_{i=0}^{\infty} \eta_i a_{t-i} \right) = \sigma_a^2 \sum_{i=0}^{\infty} (\eta_i)^2, \qquad \#$$

where $\sigma_a^2 = var(a_t)$. footnote The existence of $var(p_{ft})$ can be shown in the same manner. By (ref: y f lemma),

$$var(p_{ft}) = var\left(\sum_{i=0}^{\infty} (\varepsilon_\alpha)^i a_{t-i} \right) = \sigma_a^2 \left(\frac{1}{1 - (\varepsilon_\alpha)^2} \right). \qquad \#$$

Combining (ref: var ph) and (ref: var pf) yields the first principal result of this paper, which concerns the relative volatility of the prices of commercial versus residential properties.

Proposition *Commercial property price is more volatile than residential housing price if the following condition is satisfied,*

$$var(p_{ft})/var(p_{ht}) > 1 \Leftrightarrow \left(\frac{1}{1 - (\varepsilon_\alpha)^2} \right) > \sum_{i=0}^{\infty} (\eta_i)^2.$$

The message of the proposition can be made more transparent by considering two limiting cases. First, consider the case where ε is equal to unity. footnote In this case, commercial property does not depreciate at all. It follows that $\eta_i = (\theta)^i$. Hence, $\sum_{i=0}^{\infty} (\eta_i)^2 = \left(1 - (\theta)^2\right)^{-1}$. On the other hand, $\varepsilon = 1$ means that $\varepsilon_\alpha = 1 - \alpha$. The proposition states that if the contribution of business capital in goods production, $(1 - \alpha)$, is larger than the contribution of existing residential property in the formation of future (i.e. next period's) residential property, θ, then commercial property price will exhibit higher volatility than residential property price. The intuition is clear. Where $\varepsilon = 1$, that the stock of commercial property is fixed. By (ref: pf log), $var(p_{ft}) = var(y_t)$. The variance of (log) output clearly depends on the variance of (log) business capital, which itself is part of output. This is why the contribution of business capital in goods production matters – because it determines the extent that current output is "affected" by previous output. On the other hand, the variance of (log) residential price will depend on the variance of (log) output relative to the stock of residential property, $var(y_t - h_t)$ by (ref: ph log). Since both business capital investment K_t, and residential property investment I_{t-1}^H are fixed fractions of output Y_{t-1}, they are perfectly correlated. Hence $var(y_t - h_t)$ would ultimately depend on the contribution of commercial property F_t to current output Y_t (captured by α), relative to the contribution of previous period's residential property H_{t-1} to the formation of current residential property H_t (captured by θ). However, the stock of commercial property is fixed in this case. Therefore, the only parameter that matters is θ.

Next, we consider the limiting case where θ is equal to zero. It corresponds to the case where residential property completely depreciates after one period and the stock of residential property adjusts very quickly to changes in market condition. In this case, $\eta_i = (\varepsilon_\alpha)^i (1 - \varepsilon)$. Hence,

$\sum_{i=0}^{\infty}(\eta_i)^2 = (1-\varepsilon)\left(1-(\varepsilon_\alpha)^2\right)^{-1}$. By the stated proposition, commercial property price has to be more volatile than its residential counterpart, $var(p_{ft}) > var(p_{ht})$. The intuition is also clear. Suppose the economy experiences a negative shock and the demand for both commercial and residential property decreases. For residential property, the existing stock would vanish in the next period. However, this is not the case for commercial property. Thus, the relative price of commercial property must adjust downward to clear the market. Similar reasoning would apply if the economy experiences a positive shock. We therefore arrive at the conclusion that the price of commercial property is more volatile. In general, both ε and θ lie strictly between zero and unity so that which property price is more volatile is not certain a priori. The discussion of the two limiting cases does illustrate the forces at work and suggest some insight in the stated proposition.

The simplicity of our model not only allows us to calculate the variance terms of property prices, it also allows us to calculate the covariances as well. The algebra is straightforward and only the crucial steps are shown.

$$\begin{aligned} cov(p_{ht},\ p_{ft}) &= cov\left(\sum_{i=0}^{\infty}\eta_i a_{t-i}, \sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t-i}\right) \\ &= \sigma_a^2 \sum_{i=0}^{\infty}(\varepsilon_\alpha)^i \eta_i. \end{aligned} \qquad \#$$

Similarly,

$$cov(p_{ht},\ p_{f,t+1}) = \sigma_a^2 \sum_{i=0}^{\infty}(\varepsilon_\alpha)^{i+1}\eta_i, \qquad \#$$

$$cov(p_{ht},\ p_{f,t-1}) = \sigma_a^2 \sum_{i=1}^{\infty}(\varepsilon_\alpha)^{i-1}\eta_i. \qquad \#$$

Note that $0 < (\varepsilon_\alpha) < 1$. Thus, it is relatively easy to ensure the positivity of covariances:

Proposition *Under some mild conditions, the prices of commercial and residential property move together, i.e.*

$$If \sum_{i=0}^{\infty}(\varepsilon_\alpha)^i \eta_i > 0,\ then\ cov(p_{ht},\ p_{ft}) > 0.$$

$$If \sum_{i=0}^{\infty}(\varepsilon_\alpha)^{i+1} \eta_i > 0,\ then\ cov(p_{ht},\ p_{f,t+1}) > 0.$$

$$If \sum_{i=0}^{\infty}(\varepsilon_\alpha)^{i-1} \eta_i > 0,\ then\ cov(p_{ht},\ p_{f,t-1}) > 0.$$

Assuming that all the covariance terms are positive, we can compare (ref: cov t t) and (ref: cov t t+1) and arrive at the following proposition,

Proposition *The covariance between p_{ht} and p_{ft} is larger than that between p_{ht} and $p_{f,t+1}$, i.e.*

$$cov(p_{ht}, p_{ft}) > cov(p_{ht}, p_{f,t+1}).$$

Corollary *The correlation between p_{ht} and p_{ft} is larger than that between p_{ht} and $p_{f,t+1}$,*

$$corr(p_{ht}, p_{ft}) > corr(p_{ht}, p_{f,t+1}).$$

[Proof] The proof is trivial.

$$corr(p_{ht}, p_{fs}) = \frac{cov(p_{ht}, p_{fs})}{\sqrt{var(p_{ht})}\sqrt{var(p_{fs})}},$$

$s = t, t+1$. *However, the variance of commercial property price is constant over time,* $var(p_{ft}) = var(p_{f,t+1})$, *and the corollary follows.*

A comparison of the relative magnitudes of $cov(p_{ht}, p_{ft})$ and $cov(p_{ht}, p_{f,t-1})$ is however non-trivial. The following proposition however provides the necessary and sufficient condition.

Proposition

$$\begin{array}{rccl} & cov(p_{ht}, p_{ft}) & > & cov(p_{ht}, p_{f,t-1}) \\ \Leftrightarrow & 1 & > & \sum_{i=1}^{\infty}[1-\varepsilon_\alpha](\varepsilon_\alpha)^{i-1}\eta_i. \end{array}$$

The proof follows directly from combining (ref: cov t t) and (ref: cov t t-1) and is therefore skipped. By the constancy of the variance of the prices, the following corollary can be easily derived,

Corollary

$$\begin{array}{rccl} & corr(p_{ht}, p_{ft}) & > & corr(p_{ht}, p_{f,t+1}) \\ \Leftrightarrow & 1 & > & \sum_{i=1}^{\infty}[1-\varepsilon_\alpha](\varepsilon_\alpha)^{i-1}\eta_i. \end{array}$$

Our results here provide us with a set of hypotheses which we can test using empirical data. The next section describes the econometric procedure and the city-level dataset that we employ for exactly this purpose. However, before we move on, we should mention that the present model also generates testable implications on the covariance between the growth rate of output $\triangle y_t$ and the property prices $p_{h,t}, p_{f,t}$. These yield interesting hypotheses as well.

Proposition *If* $\left(\eta_0 + (\varepsilon_\alpha - \varepsilon)\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i\eta_{i+1}\right) > 0$, *then*

$$cov(\triangle y_t, p_{h,t}) > 0. \qquad \#$$

Proposition *If* $\left(\eta_0 + (\varepsilon_\alpha - \varepsilon)\sum_{i=1}^{\infty}(\varepsilon_\alpha)^{2i-1}\right) > 0$, *then*

$$cov(\triangle y_t, p_{f,t}) > 0. \qquad \#$$

Empirical Tests

In this section the propositions concerning the second moments of p_h, p_f and Δy are tested. The tests are based on housing price data extracted from the National Real Estate Index published by ***CB Richard Ellis National Real Estate Index***. From the fourth quarter of 1985 through the first quarter of 1998, the database provides quarterly data for 56 major U.S. cities' apartment and office rents. footnote The National Real Estate Index are supplemented by the annual data of metropolitan level per capita income from the Bureau of Economic Analysis of the US Department of Commerce. Since the data frequency of the National Real Estate Index data is different that of the Bureau of Economic Analysis data, we match only the fourth quarter National Real Estate Index data with the Bureau of Economic Analysis data. This leaves us with 496 observations. The data are deflated using the CPI (all urban consumers) from the Bureau of Labor Statistics.

The National Real Estate Index database contains data on housing price and rent. However, the high proportion of missing observations in the price series prevents us from using price data for empirical analysis. Instead, we use the rent series. The descriptive statistics of the sample are presented in Table ref: sample .

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Descriptive

Table

renewcommand

The examination of the validity of Propositions 4–6 and 12–13 could be accomplished by testing the hypotheses listed in Table ref: teststat . The 95% confidence intervals of the test statistics are computed and reported in Table ref: teststat . The confidence intervals are computed using the bootstrap methodology (see Efron and Tibshirani, 1993). The following briefly describes the construction of the confidence intervals. We first draw 1000 random samples (of the same size as the original sample with replacement) from the original sample. From each of the 1000 artificial samples we estimate a statistic q, yielding $q_b \equiv (q_1, \ldots, q_{1000})$, which constitute an empirical distribution of q. A confidence interval of α significance level, denoted by $\{c_{l_\alpha}, c_{u_\alpha}\}$, is constructed with c_{l_α} being the $(1000 \times \alpha)$th element of $\tilde{q}_b$, and c_{u_α} the $[1000 \times (1-\alpha)]$th element, where $\tilde{q}_b$ is equivalent to q_b sorted in ascending order. Bootstrapping is a computer-intensive (i.e., time-consuming) numerical method. We use this method in order to avoid imposing distributional assumptions on the test statistics. All computations are coded in GAUSS.

We accept the null hypothesis of $H_0 : q = x$ if the confidence interval of statistic q does not cover x, i.e., if $c_{u_\alpha} < x$ or $c_{l_\alpha} > x$. The results in Table ref: teststat suggests that except for hypothesis 5, all the null hypotheses are rejected. In other words, except for proposition 6, all the testable propositions are supported by the data. It is likely that the acceptance of hypothesis 5 is due to the strong autocorrelation for $p_{h,t}$.

We conclude from the empirical results that the validity of most of the testable hypotheses in the present paper are confirmed.

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Hypothesis Testing

Table

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Conclusions

There is growing attention on how different asset markets interact with the aggregate economy, and how this interaction would affect business cycles. Housing market is one of the major issue on the agenda since a significant proportion of households have real estate making up the largest share of their physical wealth. Following Greenwood and Hercowitz (1991), Baxter (1996), Gort, Greenwood and Rupert (1998), among others, this paper explores some ignored aspects of the housing market. Unlike previous works, which are mainly numerical, this paper adopts the formulations of Hercowitz and Sampson (1991) and generates some theoretical predictions concerning the stochastic features of the property prices, and their relationship with aggregate output. Thus, this paper complements the literature in the following ways. We document some stylized facts concerning the cross-dynamics and volatility of different types of property price. We formulate a dynamic general equilibrium model which generates predictions consistent with our empirical findings. Along the way, we also demonstrate a possibility of studying the dynamics of relative price in a dynamic general equilibrium context. We believe that this work contributes to enriching our understanding of the housing market and its relationship with the economy. In future, we will extend the model to allow for endogenous heterogeneity of the agents to enhance our understanding of property market transactions.

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appendix

Proofs

Proof of Proposition 1

To compute the equilibrium quantities, different variables will first be written as functions of non-durable consumption, and by use of goods market equilibrium condition, the non-durable consumption is computed. The first target relationship is in between the investment in residential capital and non-durable consumption. By (ref: mkt clear), (ref: pd foc h') can be written as

$$\lambda_{3,t}H_{t+1} = \beta E_t\{\omega + \theta\lambda_{3,t+1}(H_{t+2})\}$$

since H_{t+1} is a choice variable at time period t. Assuming no bubble conditions, which means that

$$\lim_{s\to\infty}\beta^{t+s}E_t(\lambda_{3,t+s}H_{t+s+1}) = 0,$$
$$\lim_{s\to\infty}\beta^{t+s}E_t(\lambda_{3,t+s}F_{t+s+1}) = 0,$$

it can be shown that

$$\lambda_{3t} = \left(\frac{\omega\beta}{H_{t+1}}\right)(1+\theta\beta+(\theta\beta)^2+\dots)$$
$$= \left(\frac{\omega\beta}{H_{t+1}}\right)\left(\frac{1}{1-\theta\beta}\right). \qquad \#$$

>From (ref: pd foc ih),

$$\frac{\lambda_{3t}}{\lambda_{1t}} = \left(\frac{1}{(1-\theta)F_H}\right)\left(\frac{I_t^H}{H_t}\right)^{\theta}.$$

Combined with (ref: pd foc c), it gives

$$\left(\frac{\omega\beta}{1-\theta\beta}\right)\left(\frac{C_t}{H_{t+1}}\right) = \left(\frac{1}{(1-\theta)F_H}\right)\left(\frac{I_t^H}{H_t}\right)^{\theta}. \qquad \#$$

However, (ref: pd h prod) can be written as

$$H_{t+1} = B_H(I_t^H)\left(\frac{H_t}{I_t^H}\right)^{\theta}.$$

Thus, (ref: c-h' ratio) will reduce to

$$I_t^H = \Gamma_C^H C_t, \qquad \#$$

where

$$\Gamma_C^H = \left(\frac{(1-\theta)\omega\beta}{1-\theta\beta}\right). \qquad \#$$

It is clear that $0 < \Gamma_C^H$. (ref: ih-c) says that the investment in building houses is proportional to the amount of non-durable consumption at any period of time.

It is possible to derive a similar relationship among other variables. By (ref: pd foc if), the left hand side of (ref: pd foc f') becomes

$$\left(\frac{1}{(1-\varepsilon)B_F}\right)\left(\frac{I_t^F}{F_t}\right)^{\varepsilon}\left(\frac{1}{C_t}\right),$$

and the right hand side becomes

$$\beta E_t\left\{\left(\frac{1}{C_{t+1}}\right)\left[\left(\frac{\alpha Y_{t+1}}{F_{t+1}}\right)+\left(\frac{\varepsilon}{(1-\varepsilon)B_F}\right)\left(\frac{I_{t+1}^F}{F_{t+1}}\right)^{\varepsilon}\left(\frac{F_{t+2}}{F_{t+1}}\right)\right]\right\}.$$

However, by (ref: pd f prod),

$$F_{s+1} = B_F(I_s^F)^{1-\varepsilon}(F_s)^{\varepsilon},\ \forall s.$$

Hence, (ref: pd foc f') can be simplified as

$$\left(\frac{1}{(1-\varepsilon)}\right)\left(\frac{I_t^F}{C_t}\right) = \beta E_t\left\{\left[\left(\frac{\alpha Y_{t+1}}{C_{t+1}}\right)+\left(\frac{\varepsilon}{(1-\varepsilon)}\right)\left(\frac{I_{t+1}^F}{C_{t+1}}\right)\right]\right\}. \qquad \#$$

Obviously, if the ratio of output relative to non-durable consumption is a constant, so will the ratio of commercial buildings investment relative to the non-durable consumption.

Conjecture *The amount of non-durable consumption is a fixed fraction of output,*

$$C_t = \Phi Y_t, \qquad \#$$

$0 < \Phi < 1,$ *and the commercial building investment is also proportional to non-durable consumption,*

$$I_t^F = \Gamma_C^F C_t, \qquad \#$$

$0 < \Gamma_C^F, \ \forall t.$

By (ref: conject c-y) and (ref: conject if-c), (ref: if-c prelim) is reduced to

$$\Gamma_C^F = \left(\frac{(1-\varepsilon)\alpha\beta}{1-\varepsilon\beta} \right) \left(\frac{1}{\Phi} \right). \qquad \#$$

Now, by (ref: ih-c) and (ref: conject if-c), (ref: pd s budget) can be re-written as

$$\begin{aligned} Y_t &= C_t + K_{t+1} + I_t^H + I_t^F \\ &= K_{t+1} + (\Gamma_C^H + \Gamma_C^F + 1)C_t \end{aligned}$$

which implies that

$$K_{t+1} = \Gamma^K Y_t, \qquad \#$$

where

$$\Gamma^K = [1 - (\Gamma_C^H + \Gamma_C^F + 1)\Phi]$$

by (ref: conject c-y). Notice that (ref: pd foc k') can be written as

$$\frac{1}{C_t} = (1-\alpha)\beta E_t \left[\left(\frac{1}{C_{t+1}} \right) \frac{Y_{t+1}}{K_{t+1}} \right],$$

or

$$\frac{1}{\Phi} = (1-\alpha)\beta E_t \left[\left(\frac{1}{\Phi} \right) \frac{1}{[\Gamma^K]} \right].$$

This implies that

$$\Gamma^K = [1 - (\Gamma_C^H + \Gamma_C^F + 1)\Phi] = (1-\alpha)\beta, \qquad \#$$

which means that

$$0 < \Gamma^K < 1,$$

and

$$\Phi = \frac{1-(1-\alpha)\beta}{(\Gamma_C^H + \Gamma_C^F + 1)}. \qquad \#$$

If $\Gamma_C^F > 0,$ it is clear that $0 < \Phi < 1$. Substitute (ref: ata 1) into (ref: if-c) gives

$$\Gamma_C^F = \frac{(1-\varepsilon)\alpha\beta * (\Gamma_C^H + 1)}{[(1-\varepsilon\beta)(1-(1-\alpha)\beta)] - (1-\varepsilon)\alpha\beta} > 0. \qquad \#$$

To verify the conjectures (ref: conject c-y) and (ref: conject if-c), it suffices to show that the shares of different kinds of investment are in fact positive and smaller than unity, $0 < I_t^H/Y_t$, $I_t^F/Y_t < 1$. These requirements lead to certain restrictions on the parameters, as it will be made clear. Note that $I_t^F/Y_t = \Phi\Gamma_C^F = \Gamma^F > 0$. By (ref: if-c),

$$\begin{aligned} \Phi\Gamma_C^F &= \left(\frac{(1-\varepsilon)\alpha\beta}{1-\varepsilon\beta}\right) < 1 \\ &\Leftrightarrow \beta(\alpha+\varepsilon-\alpha\varepsilon) < 1. \end{aligned} \quad \#$$

Thus, this paper imposes (ref: ib-y cond) to guarantee $I_t^F/Y_t < 1$. The case for I_t^F/Y_t is similar. Note that $I_t^H/Y_t = \Phi\Gamma_C^H = \Gamma^H > 0$. However, to know what restriction will lead to $\Phi\Gamma_C^H < 1$, it is necessary to write Φ as a function of parameters first. By (ref: ata 1),

$$\Phi + \Phi\Gamma_C^H = 1-(1-\alpha)\beta - \Phi\Gamma_C^F,$$

which means that

$$\Phi = \left(\frac{1}{1+\Gamma_C^H}\right)\Phi_2, \quad \#$$

where

$$\Phi_2 = \left\{[1-(1-\alpha)\beta] - \left(\frac{(1-\varepsilon)\alpha\beta}{1-\varepsilon\beta}\right)\right\}$$

Hence, by (ref: gamma h),

$$\Phi\Gamma_C^H = \left(\frac{\Gamma_C^H}{1+\Gamma_C^H}\right)\left\{[1-(1-\alpha)\beta] - \left(\frac{(1-\varepsilon)\alpha\beta}{1-\varepsilon\beta}\right)\right\}$$

where

$$\left(\frac{\Gamma_C^H}{1+\Gamma_C^H}\right) = \left(\frac{(1-\theta)\omega\beta}{(1-\theta)\omega\beta + (1-\beta)}\right) \equiv \Phi^H. \quad \#$$

In other words,

$$\Phi\Gamma_C^H < 1 \Leftrightarrow \Phi^H * \Phi_2 < 1. \quad \#$$

In sum, this section shows how we compute the share of non-durable consumption and different kinds of investment, Φ, Γ^i, $i = K, H, F$, and their formulae are given by (ref: gamma h), (ref: extra 1), (ref: T F) and (ref: extra 2).

Proof of (ref: rent price f) and (ref: rent price h)

It is very simple. First, note that (ref: pd foc h0) and (ref: pd foc ih) can be combined to yield

$$P_{Ht} = \left(\frac{\theta}{1-\theta}\right)\left(\frac{I_t^H}{H_t}\right) = \left(\frac{\theta\Gamma^H\Phi}{1-\theta}\right)\left(\frac{Y_t}{H_t}\right). \quad \#$$

Similarly, (ref: pd foc f0) and (ref: pd foc if) can be combined to yield

$$P_{Ft} = \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{I_t^F}{F_t}\right) = \left(\frac{\varepsilon\Gamma^F\Phi}{1-\varepsilon}\right)\left(\frac{Y_t}{F_t}\right). \quad \#$$

Now, (ref: rent f) can be written as

$$R_{Ft} = \alpha\left(\frac{Y_t}{F_t}\right).$$

Combining it with (ref: pd pf eqn) yields

$$R_{Ft} = \left(\frac{\alpha(1-\varepsilon)}{\varepsilon\Gamma^F\Phi}\right)P_{Ft},$$

which is (ref: rent price f). Similarly, (ref: rent f) can be combined with (ref: pd foc c) and (ref: conject c-y) and yields

$$R_{Ht} = \omega\Phi\left(\frac{Y_t}{H_t}\right).$$

Combining this with (ref: pd ph eqn) gives

$$R_{Ht} = \left(\frac{\omega(1-\theta)}{\theta\Gamma^H}\right)P_{Ht},$$

which is (ref: rent price h).

Proof of (ref: change of y) and (ref: change of f)

Following Sargent (1979), Lütkepohl (1993), the equation system (ref: y-alpha f) and (ref: f' f) can be expressed in the following matrix form,

$$M_1\overrightarrow{y_{t+1}} = N_1 + M_2(L)\overrightarrow{y_{t+1}} + \overrightarrow{a_{t+1}}, \qquad \#$$

where

$$M_1 = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix},\ N_1 = \begin{pmatrix} (1-\alpha)\gamma^k \\ b_f' \end{pmatrix},\ M_2(L) = \begin{pmatrix} (1-\alpha)L & 0 \\ (1-\varepsilon)L & \varepsilon L \end{pmatrix},$$

$$\overrightarrow{y_{t+1}} = \begin{pmatrix} y_{t+1} \\ f_{t+1} \end{pmatrix},\ \overrightarrow{a_{t+1}} = \begin{pmatrix} a_{t+1} \\ 0 \end{pmatrix}.$$

By (ref: matrix formula), it is easy to see that the solution takes a very simple form

$$\overrightarrow{y_{t+1}} = (M_1 - M_2(L))^{-1}N_1 + (M_1 - M_2(L))^{-1}\overrightarrow{a_{t+1}}, \qquad \#$$

where

$$\begin{aligned}(M_1 - M_2(L))^{-1} &= \begin{pmatrix} 1-(1-\alpha)L & -\alpha \\ -(1-\varepsilon)L & 1-\varepsilon L \end{pmatrix}^{-1} \\ &= \begin{pmatrix} m_{11}(L) & m_{12}(L) \\ m_{21}(L) & m_{22}(L) \end{pmatrix},\end{aligned}$$

such that

$$\begin{pmatrix} m_{11}(L) & m_{12}(L) \\ m_{21}(L) & m_{22}(L) \end{pmatrix}\begin{pmatrix} 1-(1-\alpha)L & -\alpha \\ -(1-\varepsilon)L & 1-\varepsilon L \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Or,

$$\begin{cases} m_{11}(L)[1-(1-\alpha)L] + m_{12}(L)[-(1-\varepsilon)L] &= 1, \\ m_{11}(L)[-\alpha] + m_{12}(L)[1-\varepsilon L] &= 0, \\ m_{21}(L)[1-(1-\alpha)L] + m_{22}(L)[-(1-\varepsilon)L] &= 0, \\ m_{21}(L)(-\alpha) + m_{22}(L)(1-\varepsilon L) &= 1. \end{cases}$$

It is easy to show that

$$\begin{aligned} m_{11}(L) &= (1-\varepsilon L)(1-L)^{-1}(1-\varepsilon_\alpha L)^{-1}, \\ m_{12}(L) &= \alpha(1-L)^{-1}(1-\varepsilon_\alpha L)^{-1}, \\ m_{21}(L) &= (1-\varepsilon)L(1-L)^{-1}(1-\varepsilon_\alpha L)^{-1}, \\ m_{22}(L) &= (1-(1-\alpha)L)(1-L)^{-1}(1-\varepsilon_\alpha L)^{-1}. \end{aligned}$$

Hence, (ref: matrix') can be written as

$$\overrightarrow{\triangle y_{t+1}} = M_3(L)N_1 + M_3(L)\overrightarrow{a_{t+1}}$$

where

$$\overrightarrow{\triangle y_{t+1}} = \begin{pmatrix} \triangle y_{t+1} \\ \triangle f_{t+1} \end{pmatrix} \equiv \begin{pmatrix} (1-L)y_{t+1} \\ (1-L)f_{t+1} \end{pmatrix} = \begin{pmatrix} y_{t+1}-y_t \\ f_{t+1}-f_t \end{pmatrix},$$

and

$$M_3(L) = \begin{pmatrix} (1-\varepsilon L)(1-\varepsilon_\alpha L)^{-1} & \alpha(1-\varepsilon_\alpha L)^{-1} \\ (1-\varepsilon)L(1-\varepsilon_\alpha L)^{-1} & (1-(1-\alpha)L)(1-\varepsilon_\alpha L)^{-1} \end{pmatrix}.$$

Proof of (ref: y h lemma)

By (ref: linear h' l),

$$\begin{aligned} & y_t - h_t \\ =\ & (1-(1-\theta)(1-\theta L)^{-1}L)y_t + ... \\ =\ & (1-\theta L)^{-1}((1-\theta L)-(1-\theta)L)y_t + ... \qquad\qquad \# \\ =\ & (1-\theta L)^{-1}(1-L)y_t + ... \\ =\ & (1-\theta L)^{-1}(1-\varepsilon L)(1-\varepsilon_\alpha L)^{-1}a_t + ... \text{ by (ref: change of y)}, \end{aligned}$$

with constant terms skipped. Note that

$$
\begin{aligned}
& (1-\theta L)^{-1}(1-\varepsilon L)(1-\varepsilon_\alpha L)^{-1} \\
= \ & (1-\varepsilon L)\left(\sum_{i=0}^{\infty}(\theta L)^i\right)\left(\sum_{i=0}^{\infty}(\varepsilon_\alpha L)^i\right) \\
= \ & (1-\varepsilon L)\left(\sum_{i=0}^{\infty} L^i \sum_{j=0}^{i}(\theta)^j(\varepsilon_\alpha)^{i-j}\right) \\
= \ & 1+\left\{\sum_{i=1}^{\infty} L^i\left[\sum_{j=0}^{i}(\theta)^j(\varepsilon_\alpha)^{i-j}-\varepsilon\sum_{j=0}^{i-1}(\theta)^j(\varepsilon_\alpha)^{i-j}\right]\right\} \\
= \ & \left(\sum_{i=0}^{\infty} L^i\eta_i\right)
\end{aligned}
$$

where

$$
\eta_i = \left[\sum_{j=0}^{i}(\theta)^j(\varepsilon_\alpha)^{i-j}-\varepsilon\sum_{j=0}^{i-1}(\theta)^j(\varepsilon_\alpha)^{i-j}\right]
$$

with $\eta_0 \equiv 1$. It is assumed that η_i converges to zero fast enough so that $\sum_{i=0}^{\infty}\eta_i$ is finite. footnote Thus, (ref: y-h) can be written as

$$
\begin{aligned}
& y_t - h_t \\
= \ & \left(\sum_{i=0}^{\infty} L^i\eta_i\right)a_t + ... \\
= \ & \left(\sum_{i=0}^{\infty} \eta_i a_{t-i}\right) + ...
\end{aligned}
$$

Proof of (ref: y f lemma)

By (ref: change of y) and (ref: change of f),

$$
\begin{aligned}
& (y_{t+1}-f_{t+1}) \\
= \ & (y_t-f_t)+(1-\varepsilon_\alpha L)^{-1}[(1-\varepsilon L)-(1-\varepsilon)L]a_{t+1} \\
= \ & (y_t-f_t)+(1-\varepsilon_\alpha L)^{-1}(1-L)a_{t+1} \\
= \ & (1-\varepsilon_\alpha L)^{-1}(1-L)(a_{t+1}+a_t+a_{t-1}+...) \\
= \ & (1-\varepsilon_\alpha L)^{-1}(a_{t+1}) \\
= \ & \sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t+1-i}.
\end{aligned}
$$

Proof of (ref: cov y ph) and (ref: cov y pf)

First, it is necessary to re-write equation (ref: change of y) as

$$
\triangle y_t = b_\triangle + a_t + (\varepsilon_\alpha - \varepsilon)\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t-1-i}.
$$

Hence, the covariance of change of y_t and $p_{h,t}$ is

$$\begin{aligned} & cov(\triangle y_t, p_{h,t}) \\ &= cov\left(a_t + (\varepsilon_\alpha - \varepsilon)\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t-1-i},\ \sum_{i=0}^{\infty}\eta_i a_{t-i} \right) \\ &= \sigma_a^2 * \left(\eta_0 + (\varepsilon_\alpha - \varepsilon)\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i \eta_{i+1} \right). \end{aligned}$$

Similarly, the covariance of change of y_t and $p_{f,t}$ is

$$\begin{aligned} & cov(\triangle y_t, p_{f,t}) \\ &= cov\left(a_t + (\varepsilon_\alpha - \varepsilon)\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t-1-i},\ \sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t-i} \right) \\ &= \sigma_a^2 * \left(\eta_0 + (\varepsilon_\alpha - \varepsilon)\sum_{i=1}^{\infty}(\varepsilon_\alpha)^{2i-1} \right). \end{aligned}$$

The case when $\{a_t\}$ is an AR(1)

In this section, we consider the case where $\{a_t\}$ is an AR(1), and examine how the results are affected. Formally,

$$a_t = \rho a_{t-1} + u_t, \qquad \#$$

where u_t is i.i.d., with $E(u_t) = 0,\ var(u_t) = \sigma_u^2 < \infty,\ \forall t$ and $cov(u_t, u_s) = 0,\ \forall s \neq t$. By (ref: at ar1), we have

$$a_t = \sum_{i=0}^{\infty}(\rho)^i u_{t-i}.$$

It means that

$$\begin{aligned} & var(p_{ht}) \\ &= var\left(\sum_{i=0}^{\infty}\eta_i a_{t-i} \right) \\ &= var\left(\sum_{i=0}^{\infty}\eta_i \sum_{j=0}^{\infty}(\rho)^j u_{t-i-j} \right) \\ &= var\left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j} \right) u_{t-i} \right) \\ &= \sum_{i=0}^{\infty} var\left(\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j} \right) u_{t-i} \right) \\ &= \sigma_u^2 \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j} \right)^2, \end{aligned}$$

and

$$
\begin{aligned}
& var(p_{ft}) \\
&= var\left(\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i a_{t-i}\right) \\
&= var\left(\sum_{i=0}^{\infty}(\varepsilon_\alpha)^i \sum_{j=0}^{\infty}(\rho)^j u_{t-i-j}\right) \\
&= var\left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)u_{t-i}\right) \\
&= \sum_{i=0}^{\infty} var\left(\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)u_{t-i}\right) \\
&= \sigma_u^2 \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)^2.
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
& var(p_{f,t}) > var(p_{h,t}) \\
iff\ & \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)^2 > \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j}\right)^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& cov(p_{f,t}, p_{h,t}) \\
&= cov\left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)u_{t-i},\ \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j}\right)u_{t-i}\right) \\
&= \sigma_u^2 * \sum_{i=0}^{\infty}\left\{\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j}\right)\right\}.
\end{aligned}
$$

Clearly, a sufficient but not necessary condition is that $\eta_j > 0,\ \forall j,$ then $cov(p_{ft}, p_{ht}) > 0$.

$$
\begin{aligned}
& cov(p_{f,t+1}, p_{h,t}) \\
&= cov\left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)u_{t+1-i},\ \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j}\right)u_{t-i}\right) \\
&= cov\left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i+1}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)u_{t-i},\ \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j}\right)u_{t-i}\right) \\
&= \sigma_u^2 * \sum_{i=0}^{\infty}\left\{\left(\sum_{j=0}^{i+1}(\varepsilon_\alpha)^j(\rho)^{i-j}\right)\left(\sum_{j=0}^{i}\eta_j(\rho)^{i-j}\right)\right\}.
\end{aligned}
$$

Again, if $\eta_j > 0,\ \forall j,$ then $cov(p_{f,t+1}, p_{h,t}) > 0$.

$$
\begin{aligned}
&cov(p_{f,t-1}, p_{h,t}) \\
&= cov\left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} (\varepsilon_\alpha)^j (\rho)^{i-j} \right) u_{t-1-i},\ \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \eta_j (\rho)^{i-j} \right) u_{t-i} \right) \\
&= cov\left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} (\varepsilon_\alpha)^j (\rho)^{i-j} \right) u_{t-1-i},\ \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i+1} \eta_j (\rho)^{i-j} \right) u_{t-1-i} \right) \\
&= cov\left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i-1} (\varepsilon_\alpha)^j (\rho)^{i-j} \right) u_{t-i},\ \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \eta_j (\rho)^{i-j} \right) u_{t-i} \right) \\
&= \sigma_u^2 * \sum_{i=0}^{\infty} \left\{ \left(\sum_{j=0}^{i-1} (\varepsilon_\alpha)^j (\rho)^{i-j} \right) \left(\sum_{j=0}^{i} \eta_j (\rho)^{i-j} \right) \right\}.
\end{aligned}
$$

Again, if $\eta_j > 0,\ \forall j,$ then $cov(p_{f,t-1}, p_{h,t}) > 0$.