Testing Shifts in Financial Models with Conditional Heteroskedasticity: An Empirical Distribution Function Approach*

Shinn-Juh Lin
School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia
Shinn-Juh.Lin@uts.edu.au

Jian Yang
Department of Economics
University of Western Ontario
London, Ontario
Canada, N6A 5C2
yang@julian.uwo.ca

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Abstract

This paper proposes a class of test procedures for a structural change with an unknown change point. In particular, we consider a general financial time series model with conditional heteroskedasticity. The test statistics are constructed via the empirical distribution approach and are aiming for detecting a change that may occur beyond the second moment. We derive the asymptotic null distributions of the test statistics and tabulate the critical values. Studies of the local power show that our test statistics have non-trivial local power. Finite sample performances of the proposed tests are studied via Monte Carlo methods. The test procedures are applied to test change point in the S&P 500 daily index.

Key Words: Change Point; Empirical Distribution Function; Sequential Empirical Process; Weak Convergence; Two-Parameter Brownian Bridge.
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1. Introduction

This paper considers tests for parameter instability of a general financial time series with conditional heteroskedasticity. It is important to develop such tests at least for the following two reasons. First, conditional variance of a financial time series measures market risk and plays very important roles in pricing derivative securities. Given that structural changes caused by various economic shocks occur all the time, it is therefore necessary to detect any structural shift in the conditional variance in order to correctly forecast volatility and price derivative securities. For example, models with conditional heteroskedasticity such as the autoregressive conditional heteroskedasticity (ARCH) models by Engle (1982), and the generalized autoregressive conditional heteroskedasticity (GARCH) by Bollerslev (1986), are very popular and heavily used in examining financial time series. Nevertheless, it is believed that parameter shift in the intercept term of the conditional variance equation is very likely to be blamed for biasing the estimation of a GARCH model toward an integrated GARCH (IGARCH) model (Diebold: 1986 and Lamoureux and Lastrapes: 1990b). Second, even if there is no shift in other than the conditional mean, correctly specifying the conditional variance will improve the performances of the tests on instability of the conditional mean.

Earlier studies on the change point test, or the parameter constancy test, include Chow (1960) and Quandt (1960). More recent works are Andrews (1993), Bai (1996), Delgado and Hidalgo (1996), and the references therein. Many tests in this literature focus on testing changes in the conditional mean assuming homoskedastic errors. For example, Andrews (1993) proposes alternative Sup-F types of testing procedures for the change point in parameter values characterizing the mean equation. Delgado and Hidalgo (1996) consider a test for changes of the regression function in a nonparametric setting. However, tests for instability of a financial time series with conditional heteroskedasticity has not received much attention. A few exceptions are, among others, Chu (1995), Perron (1998), and Knight, Li and Yang (1999).[1] Among these, tests for parameter instability in a simple GARCH framework have been considered by Chu (1995), who proposes Lagrange multiplier (LM) type of Sup-F tests for the constancy of variance parameters against the alternative of one-time shift in the GARCH models. These LM test statistics are constructed by estimating the model via the Gaussian Quasi-Maximum Likelihood (QML) estimation and making a comparison between the pre-shift and post-shift average score functions implied by the Gaussian likelihood. Chu's LM test statistics are mainly designed for detecting parameter shift manifested in the conditional variance. Nevertheless, as Hansen (1994) argues, higher order moments, such as the skewness and the kurtosis of the error distribution, affect the efficiency of estimation, the predictive performance of the model, and the validity of derivative security pricing, and should not be ignored. Hence, we find it important to develop tests that are capable of detecting structural changes manifested in any directions characterizing the underlying dynamics of the time series. In addition, as shown by the Monte Carlo results in Chu (1995), his LM statistics suffer from mis-specification of the error distribution simply because the score functions under Gaussian assumption have been used to construct the test statistics.

The proposed test statistics in this paper are constructed in the spirit of Bai (1991) and Bai (1996), where he examines the empirical distribution function of the residual processes to construct tests of parameter constancy in a linear regression model assuming i.i.d. or ARMA error terms. The idea is that whatever feature of the time series dynamics that is not captured by the possibly mis-specified model is going

[1]Perron (1998) applies the approach of Delgado and Hidalgo (1996) to detect jumps in the conditional variance in a nonparametric setting. Knight, Li and Yang (1999) propose test to detect joint jumps in the drift term and the diffusion term in a continuous-time term structure framework.

to be reflected in the empirical residual process. As a result, these tests are capable of detecting changes in the parameters characterizing the mean and the conditional variance as well as changes in higher moments of the underlying process, whereas conventional tests such as the Sup-F test may not be suitable for this purpose.

In this paper, we consider a general class of financial time series models with conditional heteroskedasticity. We provide these models and construct our test statistics based on the empirical residual processes of these models in Section 2. Section 3 studies the asymptotic null distributions of the test statistics and show that the tests have non-trivial local power. We also tabulate the critical values of the proposed test statistics for different sample sizes, which can be readily used by practitioners. Monte Carlo studies are conducted in Section 4 to examine the finite sample performances of the proposed tests, which suggest that the test procedures work reasonably well. The testing procedures are then applied to detect change point in the S&P 500 index, where strong evidence is found in favor of the existence of parameter instability. Section 5 concludes the paper.

2. The Model and the Test Statistic

2.1. The Model

We consider a linear regression model with conditional heteroskedasticity. Under the null hypothesis, it takes the following form,

$$y_t = \boldsymbol{x}_t \boldsymbol{\beta} + u_t,\ \ u_t = e_t v_t(\boldsymbol{\phi}|\mathcal{F}_{t-1}), \tag{2.1}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\phi}$ are the vectors of parameters; $\boldsymbol{x}_t$ includes exogenous or predetermined variables; and $v_t^2(\boldsymbol{\phi}|\mathcal{F}_{t-1})$ is the conditional variance which depends on the past information $\mathcal{F}_{t-1}$; the innovations, $\{e_t\}_{t=1}^n$, are independent and identically distributed with unknown density f and distribution function F.

Many of the popular finance models can be grouped into this framework with a few examples described below.

(1) The Capital Asset Pricing Model (CAPM)

In the CAPM, y_t is the return of an individual stock or a portfolio, $\boldsymbol{x}_t$ measures the return of the market portfolio, and $v_t^2(\boldsymbol{\phi}|\mathcal{F}_{t-1})$ can be assumed to follow a GARCH(p,q) process to capture the time-varying and the clustering features of individual stock volatility, which is

$$v_t^2 = \phi_0 + \sum_{i=1}^{p} \phi_{1i} u_{t-i}^2 + \sum_{j=1}^{q} \phi_{2j} v_{t-j}^2. \tag{2.2}$$

We shall assume that the process v_t^2 is stationary, which requires that

$$\sum_{i=1}^{p} \phi_{1i} + \sum_{j=1}^{q} \phi_{2j} < 1.$$

(2) The Arbitrage Pricing Theory (APT)

In a simple version of the APT with two factors, a market factor - market portfolio return, and a volatility factor - the conditional standard deviation at time t constitute the explanatory variable vector $\boldsymbol{x}_t$ in the conditional mean equation. When the conditional volatility is assumed to follow a GARCH process, it is a version of the so-called GARCH in mean (GARCH-M) model (see Engle (1993), and Engle, Lilien, and Robins (1987)).

(3) GARCH models with trading variables

Trading variables contain valuable information for describing stock volatility. A simple model to incorporate the information in trading variables is to assume that

$$v_t^2 = \phi_0 + \sum_{i=1}^{p} \phi_{1i} u_{t-i}^2 + \sum_{j=1}^{q} \phi_{2j} v_{t-j}^2 + \phi_3 \cdot tv_{t-1}, \tag{2.3}$$

where tv_{t-1} is the first lag of a trading variable, which could be trading volume, number of trades, or number of price changes. Model (2.3) is similar to the model of Lamoureux and Lastrapes (1990a), where the lagged trading variable is replaced by its current value.

When estimating these time series models, it is fundamental to assume stationarity of the underlying dynamics. However, sudden jumps and structural changes have been widely recognized in the finance literature. These could occur as a result technical progress, changes in policies and regulations, market crash, oil crisis and other economic shocks. Many nonlinear models have been proposed to capture such changes in financial time series, for example the threshold autoregression model of Tong (1983) and the Markov-switching models of Hamilton (1989). Nevertheless, covariance stationarity remains as a key assumption in these type of state-dependent models. It is therefore desirable to construct tests for change point in the underlying dynamics of the observed time series before we apply these proposed models.

2.2. The Test Statistics

To construct such a test, the alternative hypothesis can be specified as the following.

$$y_t = \boldsymbol{x}_t \boldsymbol{\beta}_t + e_t^* v_t(\boldsymbol{\phi}_t), \tag{2.4}$$

where $\boldsymbol{\beta}_t$ and $\boldsymbol{\phi}_t$ may not be constant over time and/or the disturbances e_t^* may not be identically distributed. Chu (1995) represents the first attempt to test for change points in this framework with GARCH-type conditional heteroskedasticity. His LM statistics are essentially constructed by (1) estimating the model under the null hypothesis by Gaussian Quasi Maximum Likelihood estimation; (2) breaking the sample, at a given breaking point, into two subsamples and computing the weighted quadratic difference of the average gradient functions under Gaussian likelihood; and (3) taking supermium over all possible breaking point. However, the tests constructed this way are not able to detect changes beyond the second moment. Furthermore, the small sample performance of the resulting statistics is sensitive to the mis-specification of the error distribution function as evidenced by Chu's Monte Carlo results.

To tackle these problems, the empirical distribution function approach of Bai

(1996) is a viable alternative. The intuition is that since such test statistics are built upon the empirical residual processes, they are robust to mis-specification of the error distribution. Furthermore, any higher order moment changes not captured by the conditional mean and the conditional variance equation would remain in the fitted residuals, and hence would be captured by the test statistics. We now describe our test statistics, which resembles the non-weighted test of Bai (1996). Let $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\phi}}$ be the quasi-maximum likelihood estimators of $\boldsymbol{\beta}$ and $\boldsymbol{\phi}$ respectively and the standardized fitted residuals

$$\hat{e}_t = \frac{y_t - \boldsymbol{x}_t\hat{\boldsymbol{\beta}}}{v_t(\hat{\boldsymbol{\phi}})}.$$

For a fixed k, define the empirical marginal distribution function (e.d.f.) based on the first k observations as

$$\hat{F}_k(z) = \frac{1}{k}\sum_{t=1}^{k} I(\hat{e}_t \le z),$$

and the e.d.f. based on the last $n-k$ observations as

$$\hat{F}^*_{n-k}(z) = \frac{1}{n-k}\sum_{t=k+1}^{n} I(\hat{e}_t \le z),$$

where $I(\cdot)$ is an indicator function. Further define the weighted difference of empirical distribution functions from the two subsamples as

$$T_n\left(\frac{k}{n}, z\right) = \frac{k}{n}\left(1-\frac{k}{n}\right)\sqrt{n}(\hat{F}_k(z) - \hat{F}^*_{n-k}(z)). \tag{2.5}$$

Based on $T_n(\cdot,\cdot)$, we can construct the following three alternative test statistics. First of all, the Sup-type test statistic, M_n, is defined as

$$M_n = \max_k \sup_z |T_n(k/n, z)|, \tag{2.6}$$

where the maximum is taken over $1 \le k \le n$ and the supermium with respect to z is taken over the entire real line. Secondly, we can construct two mean-type test statistics, $ABSA_n$ and SQA_n, where

$$ABSA_n = \frac{1}{n^2}\sum_k\sum_j \left|T_n\left(\frac{k}{n}, \hat{e}_j\right)\right|, \text{ and} \tag{2.7}$$

$$SQA_n = \frac{1}{n^2} \sum_k \sum_j \left[T_n \left(\frac{k}{n}, \hat{e}_j \right) \right]^2 . \tag{2.8}$$

Remark 1: The M_n test statistics is essentially the maximum value of weighted Kolmogorov-Smirnov statistics for all possible sample splits. This test is considered by Bai (1993) for testing changes in an ARMA process and Bai (1996) for testing changes in a regression model with a trend regressor.

Remark 2: All of the tset statistics, M_n, $ABSA_n$, and SQA_n, in equation (2.6) - (2.8) share the same property as Chu's LM test in the sense that it requires no re-estimating the parameters for each subsample via numerical optimization, which is computational intensive and may suffer from numerical difficulties in a small subsample. On the other hand, these statistics are applicable for more general time series models with heteroskedasticity rather than a simple GARCH model as in Chu (1995).

Remark 3: Since the test statistics are constructed by using the standardized residuals from the Gaussian QML estimation rather than using the sample mean gradients of Gaussian likelihood, they are no longer sensitive to non-Gaussian errors. In addition, they are capable of detecting a change in the conditional mean, the conditional variance, as well as other higher order moments.

Remark 4: As Bai (1996) shows, assuming no change in the error distribution function, a test constructed by using an empirical distribution function approach might be more powerful than a Sup-Wald test in small samples with non-gaussian errors.

3. The Asymptotic Properties

To obtain the asymptotic properties of the proposed test statistics, we make the following assumptions.

Assumption A.1: *Under the null hypothesis, the disturbances e_t have zero mean and are equipped with a marginal distribution function, F, and a density function f,*

$f > 0$. Both $f(z)$ and $zf(z)$ are uniformly continuous on the real line. Further, there exists a finite number L such that $|zf(z)| < L$ and $|f(z)| < L$ for all z.

Assumption A.2: *The disturbances e_t are independent of all contemporaneous and past regressors.*

Assumption A.3: *Let*

$$\boldsymbol{z}_t = \left(\frac{\boldsymbol{x}_t^\top}{v_t(\boldsymbol{\phi})}, \frac{1}{v_t(\boldsymbol{\phi})} \frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} \right)^\top.$$

Under the null, $\boldsymbol{z}_t$ satisfies

$$\operatorname{plim} \frac{1}{n} \sum_{t=1}^{[ns]} \boldsymbol{z}_t \boldsymbol{z}_t^\top = sQ \text{ uniformly in } s \in [0,1],$$

where Q is a $k \times k$ nonrandom positive definite matrix.

Assumption A.4: *Under the null, $\boldsymbol{z}_t$ satisfies*

$$\operatorname{plim} \frac{1}{n} \sum_{t=1}^{[ns]} \boldsymbol{z}_t = s\overline{\boldsymbol{z}} \text{ uniformly in } s \in [0,1],$$

where $\overline{\boldsymbol{z}}$ is a $k \times 1$ constant vector.

Assumption A.5: *Under the null, there exists a $\sqrt{n}$-consistent estimator of $(\boldsymbol{\beta}^\top, \boldsymbol{\phi}^\top)^\top$.*

Assumption A.1 is a regularity assumption for empirical processes as in Bai (1996), Boldin (1989), Koul (1992), and Kreiss (1991). Assumptions A.2 allows for dynamic variables. Assumptions A.3 and A.4 ensure that the test statistics are asymptotically distribution free. Assumption A.5 is satisfied by the Gaussian Quasi-Maximum Likelihood Estimator given the regularity conditions as shown by Bollerslev and Wooldridge (1992).

3.1. The Null Distribution

Based on the assumptions we have just made, we study the asymptotic null distributions for the proposed test statistics. We will focus on the Sup-type statistic, M_n, observing that the asymptotics of the other statistics can be obtained in a similar manner. We will follow essentially Bai (1996) to first prove the weak convergence for $T_n(s,z)$ and then apply the continuous mapping theorem. We can rewrite $T_n(k/n, z)$ as follows,

$$\begin{aligned} T_n\left(\frac{k}{n}, z\right) &= n^{-1/2}\sum_{t=1}^{k} I(\hat{e}_t \le z) - \frac{k}{n} n^{-1/2}\sum_{t=1}^{n} I(\hat{e}_t \le z) \\ &= n^{-1/2}\sum_{t=1}^{k} \left[I(\hat{e}_t \le z) - F(z)\right] - \frac{k}{n} n^{-1/2}\sum_{t=1}^{n} \left[I(\hat{e}_t \le z) - F(z)\right], \end{aligned}$$

where F is assumed to be the marginal distribution function of e_t.

Let $B(s,u)$ be a Gaussian process on $[0,1]^2$ with zero mean and covariance function

$$E\{B(r,u)B(s,v)\} = (r \wedge s - rs)(u \wedge v - uv).$$

$B(s,u)$ so defined is called a two-parameter Brownian bridge on $[0,1]^2$.

Theorem 1: *Under assumptions A.1 through A.5, we have*

$$T_n\left(\frac{[n\cdot]}{n}, \cdot\right) \xrightarrow{d} B(\cdot, F(\cdot)),$$

where the notation "$\longrightarrow$" is used to denote the weak convergence in the space of $D(T)$ where $T = [0,1]^2$ under the Skorohod J_1 topology.

Given the above weak convergence result, we can simply apply the continuous mapping theorem to obtain the null distributions of the test statistics. For example, if we assume $G(\cdot)$ denote the d.f. of the r.v. $\sup_{0\le s\le 1}\sup_{0\le u\le 1}|B(s,u)|$, we then have the following corollary.

Corollary 1: *Under the assumptions of Theorem 1,*

$$\lim_{n\to\infty} P(M_n \le a) = D(a), \ \ a > 0.$$

It is observed from the above results that M_n statistic is asymptotically distribution free. Similar conclusions can be drawn for the the mean-type test statistics and

Table 3.1: Selected Quantiles of the M_n Statistics

	85.0%	90.0%	92.5%	95.0%	97.5%	99.0%	99.5%
$n = 100$	0.7000	0.7360	0.7600	0.7950	0.8500	0.9120	0.9600
$n = 200$	0.7113	0.7495	0.7750	0.8103	0.8644	0.9316	0.9786
$n = 300$	0.7178	0.7563	0.7823	0.8172	0.8730	0.9397	0.9846
$n = 600$	0.7253	0.7648	0.7898	0.8244	0.8799	0.9471	0.9941
$n = 1000$	0.7295	0.7687	0.7941	0.8283	0.8846	0.9511	0.9994

the result of Theorem 1 implies that $ABSA_n$ and SQA_n converge in distribution to $\int_0^1 \int_0^1 |B(s,t)| dsdt$ and $\int_0^1 \int_0^1 B(s,t)^2 dsdt$ respectively. Critical values of the three test statistics are tabulated in Table 3.1 - 3.3, which are computed via simulation with 200,000 replications. In each repetition, a sequence of i.i.d. uniformly distributed random variables on $[0,1]$ is generated. The process $T_n(k/n, z)(0 \leq z \leq 1)$ is constructed using this sequence. The value of M_n, $ABSA_n$ and SQA_n are then obtained. In Table 3.1, we observe that, when the sample size is equal to 200, the critical values of M_n statistics are very close to that of Bai (1996), who computes these values via simulation with the same sample size 200 and 100,000 replications. To get an overall look at the distribution of the three statistics, we plot the cumulative distribution functions (CDF) of the three statistics in Figure 1-3.[2] One interesting observation from these plots is that we find the shapes of the CDFs are not very sensitive to different sample sizes. In particular, since the mean-type statistics $ABSA_n$ and SQA_n average out most of the outliers, we can barely tell the difference among the CDFs with different sample sizes. For the same reason, their supports are much tighter than that of the M_n's CDF. As a result, for applications, we can use the quantiles in the $n = 1000$ cases for any larger sample size greater than 1000.

[2]These plots are generated from non-parametric density estimate of the simulated data, with Gaussian kernel on 200 equally spaced points and bandwidth $4 \times 1.06 \times \min(\hat{\sigma}, IQR) \times n^{-1/5}$ as suggested in Siverman (1986, pp.45-47).

Table 3.2: Selected Quantiles of the $ABSA_n$ Statistics

	85.0%	90.0%	92.5%	95.0%	97.5%	99.0%	99.5%
$n = 100$	0.1536	0.1654	0.1736	0.1847	0.2033	0.2269	0.2438
$n = 200$	0.1532	0.1647	0.1729	0.1842	0.2030	0.2269	0.2439
$n = 300$	0.1532	0.1649	0.1732	0.1846	0.2033	0.2266	0.2447
$n = 600$	0.1528	0.1645	0.1728	0.1841	0.2037	0.2274	0.2449
$n = 1000$	0.1529	0.1644	0.1726	0.1840	0.2029	0.2268	0.2446

Table 3.3: Selected Quantiles of the SQA_n Statistics

	85.0%	90.0%	92.5%	95.0%	97.5%	99.0%	99.5%
$n = 100$	0.0409	0.0473	0.0519	0.0585	0.0702	0.0864	0.0989
$n = 200$	0.0408	0.0470	0.0516	0.0583	0.0702	0.0863	0.0993
$n = 300$	0.0408	0.0472	0.0519	0.0586	0.0704	0.0868	0.0994
$n = 600$	0.0406	0.0470	0.0517	0.0585	0.0707	0.0870	0.0998
$n = 1000$	0.0407	0.0470	0.0517	0.0583	0.0702	0.0866	0.0999

3.2. Local Power Analysis

When examining the local power, we consider model (2.1) with the class of local alternatives:

$$H_1: \quad (\boldsymbol{\beta}_t^\top, \boldsymbol{\phi}_t^\top)^\top = (\boldsymbol{\beta}^\top, \boldsymbol{\phi}^\top)^\top + \Delta_1 g(t/n) n^{-1/2}, \tag{3.1}$$

where $\Delta_1 = (\Delta_{11}^\top, \Delta_{12}^\top)^\top$. When testing shift only in the mean parameters, researchers often assume that the function g is defined on $[0, 1]$ and is Riemann-Stieltjes integrable, and define the vector function

$$\lambda_g(s) = \int_0^s g(x)dx - s\int_0^1 g(x)dx. \tag{3.2}$$

Unfortunately, under this class of local alternatives, the behavior of gaussian-QML estimator is not well justified. Therefore, we consider a simple shift function g such that $g(x) = 0$ for $x \leq \tau$ and $g(x) = 1$ for $x > \tau$, where $\tau \in (0, 1)$, then $\lambda_g(s) = -(\tau \wedge s)(1 - \tau \vee s)$.

Theorem 2: *Under the assumptions A.1 through A.5 and the local alternatives* (3.1), *we have*

$$M_n \xrightarrow{d} \sup_{0\le s\le 1}\sup_{0\le u\le 1}\Big|B(s,u)+\Delta_1 f(F^{-1}(u))F^{-1}(u)\lambda_g(s)\Big|.$$

Theorem 2 implies that the test statistic M_n has a non-trivial local power against the local alternative (3.1). We now examine the behavior of the M_n statistics under a type of changes in the error distribution functions. Similar to the test developed in Bai (1996), M_n is capable of detecting shifts other than in the form of changing mean and changing variances. We will therefore consider the following equation,

$$y_t = \boldsymbol{x}_t\boldsymbol{\beta} + e_{nt}v_t(\boldsymbol{\phi}), \tag{3.3}$$

where e_{nt} has a distribution function $F_{nt},\ t=1,\cdots,n$ associated with a density function f_{nt}. The null hypothesis specifies a special case of model (3.3), where $F_{nt}=F$ for all $t\le n$.

Both fixed and local alternatives could be studied. The fixed alternative is specified as

$$H_2:\quad F_{nt}=F,\ \text{for } t\le[n\tau] \text{ and } F_{nt}=G,\ \text{for } t>[n\tau],$$

where $F\neq G$. The local alternative is

$$H_3:\quad F_{nt}=F,\ \text{for } t\le[n\tau] \text{ and } F_{nt}=(1-\Delta_2 n^{-1/2})F+\Delta_2 n^{-1/2}H,\ \text{for } t>[n\tau],$$

where $\Delta_2>0$ and $\Delta_2 n^{-1/2}<1$, and $F\neq H$.

To present the asymptotic distributions of M_n under H_2 and H_3, we introduce a Kiefer process denoted as K_F. A Kiefer process defined on $[0,1]\times R$ is a Gaussian process with mean zero and covariance function $E\{K_F(r,y)K_F(s,z)\}=(r\wedge s)[F(y\wedge z)-F(y)F(z)]$. It also satisfies $K_F(0,\cdot)=0$. Following Bai (1996), we then define

$$\bar{K}(s,z)=K_F(s\wedge\tau,z)-sK_F(\tau,z)+K_G(s-s\wedge\tau,z)-sK_G(1-\tau,z),$$

where K_G is another Kiefer process independent of K_F with $K_G(0,\cdot) = 0$. Using these notations, we will summarize the results on local power of M_n under H_2 and H_3 in the following theorem.

Theorem 3: *Assume that Assumption (A.2)-(A.5) are satisfied and (A.1) holds for distribution functions F, G, and H. Then:*

(i) Under the fixed alternative H_2,

$$M_n = \sup_{s,z} |\bar{K}(s,z) + \sqrt{n}(s \wedge \tau)(1 - s \vee \tau)(F - G)| + O_p(1),$$

where O_p is uniform in s and z.

(ii) Under the local alternative H_3,

$$M_n \xrightarrow{d} \sup_{s,z} |B(s, F(z)) + \Delta_2(s \wedge \tau)(1 - s \vee \tau)(F - H)|.$$

The Kiefer processes K_F and K_G are uniformly bounded in probability and consequently $\bar{K}$ is also uniformly bounded in probability. This together with $\sqrt{n}(s \wedge \tau)(1 - s \vee \tau)|F - G| \to \infty$ (for some s and z if $F \neq G$) implies that the test M_n is consistent under H_1. Part (ii) implies that M_n has nontrivial power in testing local shifts in the distributions of the time series.

4. Simulation Results and an Empirical Example

To evaluate the practical usefulness, we conduct Monte Carlo studies to examine the finite-sample size and power performance of the proposed tests. For each size and each power simulation, the replication number is 5,000. We then apply these test procedures to detect the change point in S&P 500 stock returns.

4.1. Monte Carlo Studies

When studying the size performance of the test statistics, we consider the following data generating processes (DGP).[3]

Gaussian GARCH(1,1) - NGARCH:

$y_t = 0.05 + u_t, \quad u_t = v_t e_t$

$v_t^2 = 0.5 + 0.3u_{t-1}^2 + 0.3v_{t-1}^2.$

Gaussian Near-Integrated GARCH(1,1) - NIGARCH:

$y_t = 0.05 + u_t,$

$v_t^2 = 0.5 + 0.1y_{t-1}^2 + 0.8v_{t-1}^2.$

GARCH(1,1) with t density - TGARCH:

$e_t \sim t$ distribution with 10 degrees of freedom, normalized to have unit variance, and

$v_t^2 = 0.5 + 0.3u_{t-1}^2 + 0.3v_{t-1}^2.$

Near-Integrated GARCH(1,1) with t density - TIGARCH:

$e_t \sim t$ distribution with 10 degrees of freedom, normalized to have unit variance, and

$v_t^2 = 0.5 + 0.1y_{t-1}^2 + 0.8v_{t-1}^2.$

GARCH(1,1) with χ^2 density - CGARCH:

$e_t \sim \chi^2$ distribution with 10 degrees of freedom, normalized to have zero mean and unit variance and

$v_t^2 = 0.5 + 0.3u_{t-1}^2 + 0.3v_{t-1}^2.$

Near-Integrated GARCH(1,1) with χ^2 density - CIGARCH:

$e_t \sim \chi^2$ distribution with 10 degrees of freedom, normalized to have zero mean and unit variance, and

$v_t^2 = 0.5 + 0.1y_{t-1}^2 + 0.8v_{t-1}^2.$

[3]The assumptions are verified in Appendix B for these types of models.

Table 4.1: Size Simulation

	M_n		$ABSA_n$		SQA_n	
	0.05	0.10	0.05	0.10	0.05	0.10
$n = 300$						
NGARCH	0.0528	0.1048	0.0540	0.1072	0.0528	0.1124
TGARCH	0.0480	0.0942	0.0448	0.0980	0.0466	0.0956
CGARCH	0.0496	0.0956	0.0482	0.0978	0.0482	0.0962
NIGARCH	0.0552	0.1108	0.0570	0.1108	0.0576	0.1066
TIGARCH	0.0448	0.0940	0.0468	0.1018	0.0460	0.0984
CIGARCH	0.0520	0.0924	0.0490	0.1038	0.0494	0.1024
$n = 600$						
NGARCH	0.0478	0.0922	0.0476	0.0934	0.0482	0.0942
TGARCH	0.0444	0.0940	0.0438	0.0886	0.0408	0.0898
CGARCH	0.0464	0.0944	0.0430	0.0892	0.0412	0.0926
NIGARCH	0.0474	0.0954	0.0472	0.0986	0.0470	0.0960
TIGARCH	0.0502	0.0964	0.0480	0.0966	0.0474	0.0970
CIGARCH	0.0500	0.0968	0.0518	0.1010	0.0504	0.0998
$n = 1000$						
NGARCH	0.0480	0.0894	0.0440	0.0902	0.0452	0.0898
TGARCH	0.0512	0.1016	0.0558	0.1008	0.0552	0.1008
CGARCH	0.0520	0.1000	0.0524	0.1034	0.0540	0.1030
NIGARCH	0.0496	0.0968	0.0528	0.1024	0.0516	0.0948
TIGARCH	0.0486	0.0966	0.0546	0.1036	0.0540	0.1014
CIGARCH	0.0482	0.1010	0.0562	0.1090	0.0556	0.1006

The results for the three types of tests are reported in Table 4.1. Overall, all the three test statistics perform very well and none of them dominates. To get a better picture of how well our test statistics perform, we present the size discrepancy plots of the test statistics in Figure 4-6. Following the suggestion in Davidson and MacKinnon (1998), we also plot the 5% Kolmogorov-Smirnov critical values in these figures. It is clear from these figures that most of the size discrepancies come from random error during the simulation phase. These strengthen our confidence in the size performance of our test statistics. In particular, we find that even when we mis-specify the error distribution as either a $t(10)$ or a $\chi^2(10)$ distribution, the test statistics remain reliable. This is in contrast to Chu's finding, where the empirical size of his LM statistics behaves poorly under mis-specification of the error distribution.

To examine the power performance of the test statistics, we introduce the alternative of a one-time shift in the parameter vector. We set the parameter values before the break as $(0.0342, 0.0108, 0.3000, 0.3000)$. For those after the break, we study the following three scenarios $(0.0532, 0.0129, 0.3000, 0.3000)$, $(0.0722, 0.0150, 0.3000, 0.3000)$, and $(0.1102, 0.0192, 0.3000, 0.3000)$.[4] These three scenarios correspond to jump size equal to 0.5, 1.0 and 2.0. We also control the location of the break point ($\tau = 0.3$, 0.5, and 0.7), and the sample size (300, 600, and 1000). The results are reported in Table 4.2-4.4, as well as the size-power plots in Figure 7-9.[5] As expected, the power increases with the jump size and the sample size, and the highest power realizes at $\tau = 0.5$. The results also suggest that the mean-type statistics, $ABSA_n$ and SQA_n, have more power than the maximal-statistic, M_n . However, the maximal-type test gives valuable information on the possible location of the break point, while the average type test does not. Therefore, it is recommanded that, if possible, maximal test

[4]These values are obtained from examining the S&P 500 data, which we use in the following empirical application, to make the simulation results more informative.

[5]The size-power plots are size-corrected. In other words, we plot the power against the true size rather than the norminal size.

Table 4.2: Power Simulation: Jump Size = 0.5

	M_n		$ABSA_n$		SQA_n	
	0.05	0.10	0.05	0.10	0.05	0.10
$n = 300$						
$\tau = 0.3$	0.0960	0.1722	0.1088	0.1850	0.1082	0.1904
$\tau = 0.5$	0.1190	0.1974	0.1352	0.2186	0.1356	0.2220
$\tau = 0.7$	0.0978	0.1748	0.1200	0.1956	0.1174	0.1944
$n = 600$						
$\tau = 0.3$	0.1578	0.2474	0.1878	0.2884	0.1878	0.2884
$\tau = 0.5$	0.2126	0.3224	0.2450	0.3520	0.2492	0.3562
$\tau = 0.7$	0.1494	0.2372	0.1734	0.2770	0.1724	0.2752
$n = 1000$						
$\tau = 0.3$	0.2442	0.3588	0.2814	0.3990	0.2828	0.3984
$\tau = 0.5$	0.3392	0.4612	0.3648	0.5010	0.3772	0.5088
$\tau = 0.7$	0.2306	0.3476	0.2774	0.3900	0.2726	0.3886

and average test should both be performed in practice.

4.2. S&P 500 Daily Returns

In what follows, we apply the proposed statistics to daily returns of the S&P 500 stock index as an illustration. The sample extends from January 2, 1980 to December 29, 1995, totaling 4045 observations. The stock return dynamics is assumed to follow a GARCH(1,1) model as follows,

$$y_t = \beta_1 + u_t, \quad u_t = v_t e_t, \text{ and } v_t^2 = \phi_0 + \phi_1 u_{t-1}^2 + \phi_2 v_{t-1}^2. \tag{4.1}$$

GAUSS and its MAXLIK library are used to estimate the parameters and the estimation results of model (4.1) are reported in Table 4.5. Based on the standardized fitted residuals of the GARCH(1,1) model, we then perform our three change-point tests and report the results in Table 4.6. All three tests consistently suggest a structural change at significance level far below 0.05. Based on the M_n statistic, the "break point" is at observation 1458. To have a closer look at this change, we re-estimate the model according to the two subsamples - before and after the "break point". These

Table 4.3: Power Simulation: Jump Size = 1.0

	M_n		$ABSA_n$		SQA_n	
	0.05	0.10	0.05	0.10	0.05	0.10
$n = 300$						
$\tau = 0.3$	0.2522	0.3756	0.2946	0.4278	0.2962	0.4262
$\tau = 0.5$	0.3594	0.4954	0.3964	0.5314	0.4046	0.5332
$\tau = 0.7$	0.2362	0.3500	0.2994	0.4156	0.2982	0.4094
$n = 600$						
$\tau = 0.3$	0.4848	0.6136	0.5364	0.6650	0.5440	0.6728
$\tau = 0.5$	0.6488	0.7586	0.6876	0.7942	0.7004	0.8012
$\tau = 0.7$	0.4516	0.5808	0.5188	0.6560	0.5226	0.6550
$n = 1000$						
$\tau = 0.3$	0.7308	0.8334	0.7738	0.8590	0.7848	0.8682
$\tau = 0.5$	0.8642	0.9252	0.8882	0.9404	0.9012	0.9440
$\tau = 0.7$	0.6922	0.7974	0.7588	0.8488	0.7660	0.8516

Table 4.4: Power Simulation: Jump Size = 2.0

	M_n		$ABSA_n$		SQA_n	
	0.05	0.10	0.05	0.10	0.05	0.10
$n = 300$						
$\tau = 0.3$	0.2522	0.3756	0.2946	0.4278	0.2962	0.4262
$\tau = 0.5$	0.3594	0.4954	0.3964	0.5314	0.4046	0.5332
$\tau = 0.7$	0.2362	0.3500	0.2994	0.4156	0.2982	0.4094
$n = 600$						
$\tau = 0.3$	0.4848	0.6136	0.5364	0.6650	0.5440	0.6728
$\tau = 0.5$	0.6488	0.7586	0.6876	0.7942	0.7004	0.8012
$\tau = 0.7$	0.4516	0.5808	0.5188	0.6560	0.5226	0.6550
$n = 1000$						
$\tau = 0.3$	0.7308	0.8334	0.7738	0.8590	0.7848	0.8682
$\tau = 0.5$	0.8642	0.9252	0.8882	0.9404	0.9012	0.9440
$\tau = 0.7$	0.6922	0.7974	0.7588	0.8488	0.7660	0.8516

Table 4.5: GARCH(1,1) Estimation Results of S&P 500 Daily Returns

	Full Sample		Sample 1:1458		Sample 1459:4045	
Parameter	Estimate	p-Value	Estimate	p-Value	Estimate	p-Value
β_1	0.0596	4.7E-07	0.0342	6.0E-02	0.0722	9.7E-07
ϕ_0	0.0126	3.3E-12	0.0108	9.2E-03	0.0150	4.6E-11
ϕ_1	0.0767	0.00000	0.0389	2.9E-07	0.0900	0.00000
ϕ_1	0.9128	0.00000	0.9476	0.00000	0.8989	0.00000

results are also reported in Table 4.5. A notable thing between these two subsamples is that the sample skewness and the sample kurtosis before and after the "break point" are $(0.3141, 4.4461)$ and $(-3.6881, 81.6225)$, respectively. It is quite clear that this structural change combines the change in conditional mean, in the conditional variance, and especially in the higher order moments. Another interesting observation is that the post-break period is equipped with both a higher average return and a higher unconditional variance. To examine whether a GARCH-M model could capture this effect and avoid possible mis-specification of the null model, we perform the tests for the following model.

$$y_t = \beta_1 + \beta_2 v_t + u_t, \quad u_t = v_t e_t, \text{ and } v_t^2 = \phi_0 + \phi_1 u_{t-1}^2 + \phi_2 v_{t-1}^2. \tag{4.2}$$

The complete estimation results are not reported here. However, the t-statistic of β_2 is equal to 1.2963, which indicates that the GARCH-in-mean effect is not significant. Furthermore, we apply our three change-point tests on the fitted residuals of this model. The results are reported Table 4.6, which suggest that a GARCH-M model cannot capture the jump in the average return associated with different market volatilities. All the three test statistics strongly support parameter instability and the M_n statistic interestingly indicates the same "location" for the change point.

Table 4.6: Test statistics Based on GARCH, and GARCH-M Models

	M_n	$ABSA_n$	SQA_n
GARCH	0.9598	0.2215	0.0763
	(0.009)	(0.012)	(0.018)
GARCH-M	0.0121	0.2399	0.0942
	(0.004)	(0.006)	(0.007)

†Numbers in the parentheses are p-values.

5. Conclusions

In this paper, we propose a class of tests for parameter constancy in a general class of financial time series models with conditional heteroskedasticity. The proposed tests are constructed via sequential empirical processes. We show that these tests are capable of detecting changes not just in the conditional mean and the conditional variance, but also in other characteristics of the underlying dynamics. Simulation results suggest that the proposed tests work well. In the application, strong evidence is found in favour of a structural change in the S&P 500 index returns.

Appendix A

Proof of Theorem 1.

Define

$$K_n(s,z) = n^{-1/2} \sum_{t=1}^{[ns]} \left[I(\hat{e}_t \leq z) - F(z) \right],$$

then we have

$$T_n\left(\frac{[ns]}{n}, z\right) = K_n(s,z) - sK_n(1,z).$$

Therefore, to study T_n, it suffices to study K_n. In the ideal case that we observe the true disturbance terms e_t, we define

$$H_n(s,z) = n^{-1/2} \sum_{t=1}^{[ns]} \left[I(e_t \leq z) - F(z) \right].$$

Let $\Im = [0,1] \times \Re$ be the parameter set with metric $\rho(\{r,e\},\{s,z\}) = |s-r| + |F(z) - F(e)|$. Let $D(\Im)$ be the set of functions defined on $\Im$ that are right continuous and have left limits. We equip $D(\Im)$ with the Skorohod metric (Pollard, 1984). The weak convergence of H_n in the space $D(\Im)$ is implied by the finite dimensional convergence together with stochastic equicontinuity.

Theorem A.1: *Under Assumption (A.1) and (A.2), the process H_n is stochastically equicontinuous on $(\Im, \rho)$. That is for any $\epsilon > 0$, $\eta > 0$, there exists a $\delta > 0$ such that for large n,*

$$P\left(\sup_{[\delta]} |H_n(r,e) - H_n(s,z)| > \eta\right) < \epsilon,$$

where $[\delta] = \{(\tau_1, \tau_2);\ \tau_1 = (r,e),\ \tau_2 = (s,z),\ \rho(\tau_1,\tau_2) < \delta\}$ with $[\delta] \subset \Im \times \Im$.

The equicontinuity of H_n is proved by Bickel and Wichura (1971) and Bai (1996), which states the stochastic equicontinuity holds for (randomly) sequential process. Let $u_t = F(e_t)$, and $u = F(z)$, therefore u_t is uniformly distributed on $[0,1]$. Define

$$Y_n(s,u) = n^{-1/2} \sum_{t=1}^{[ns]} \left[I(u_t \leq u) - u \right],$$

then $H_n(s,z) = Y_n(s,F(z))$ and Y_n and H_n are equivalent in terms of stochastic equicontinuity. Thus the proof of the equicontinuity of $H_n(s,z)$ is provided by focusing on Y_n.

Corollary A.1: *Under the assumptions in Theorem A.1, the process H_n converges weakly to a Gaussian process H with zero mean and covariance*

$$E\{H(r,e)H(s,z)\} = (r \wedge s)[F(z \wedge e) - F(z)F(e)].$$

Proof: The convergence to a normal distribution follows from the central limit theorem for martingale differences. This, together with Theorem A.1, implies that H_n converges weakly to some Gaussian process H. To verify the variance function, we consider the covariance function of $Y_n = n^{-1/2}H_n$. For $r < s$ and $u = F(z) < v = F(e)$, using double expectation and the martingale property, we obtain

$$\begin{aligned}
E\{Y_n(r,u)Y_n(s,v)\} &= n^{-1}E\left[\sum_{t1=1}^{[nr]}(I(u_{t1} \le u) - u)\sum_{t2=1}^{[ns]}(I(u_{t2} \le v) - v)\right] \\
&= n^{-1}E\left[\sum_{t1=1}^{[nr]}(I(u_{t1} \le u) - u)\sum_{t2=1}^{[nr]}(I(u_{t2} \le v) - v)\right] \\
&\quad + n^{-1}E\left[\sum_{t1=1}^{[nr]}(I(u_{t1} \le u) - u)\sum_{t2=[nr]+1}^{[ns]}(I(u_{t2} \le v) - v)\right] \\
&= rE\left[(I(u_{t1} \le u) - u)(I(u_{t2} \le v) - v)\right] \\
&\to r(u - uv), \text{ when } n \to \infty.
\end{aligned}$$

Corollary A.2: *Under the assumption of Corollary A.1, the process V_n defined as*

$$V_n(s,z) = H_n(s,z) - sH_n(1,z)$$

converges weakly to a Gaussian process V with mean zero and covariance

$$E\{V(r,e)V(s,z)\} = (r \wedge s - rs)[F(e \wedge z) - F(e)F(z)].$$

Proof: The stochastic equicontinuity of V_n follows from the stochastic equicontinuity of H_n. Let

$$V_n(s,z) = H_n(s,z) - sH_n(1,z),\ r < s$$

then

$$\begin{aligned}
E\{V(r,e)V(s,z)\} &= E\Big\{(H(r,e) - rH(1,e))\,(H(s,z) - sH(1,z))\Big\} \\
&= r\,(F(z \wedge e) - F(z)F(e)) - sr\,(F(z \wedge e) - F(z)F(e)) \\
&\quad -rs\,(F(z \wedge e) - F(z)F(e)) + rs\,(F(z \wedge e) - F(z)F(e)) \\
&= (r - rs)\,(F(z \wedge e) - F(z)F(e))\,.
\end{aligned}$$

The above results are obtained when observing the true disturbance terms. We next examine the asymptotic behavior of the sequential empirical process constructed using estimation residuals. Under model (2.1), $\hat{e}_t \le z$ if and only if

$$e_t \le z\frac{v_t(\hat{\boldsymbol{\phi}})}{v_t(\boldsymbol{\phi})} + \frac{\boldsymbol{x}_t(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{v_t(\boldsymbol{\phi})},$$

thus K_n under H_0 is given by

$$K_n(s,z) = n^{-1/2}\sum_{t=1}^{[ns]}\left\{I\left(e_t \le z\frac{v_t(\hat{\boldsymbol{\phi}})}{v_t(\boldsymbol{\phi})} + \frac{\boldsymbol{x}_t(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{v_t(\boldsymbol{\phi})}\right) - F(z)\right\}.$$

Under the local alternative of (3.1), $\hat{e}_t \le z$ if and only if

$$e_t \le z\frac{v_t(\hat{\boldsymbol{\phi}})}{v_t(\boldsymbol{\phi}_t)} + \frac{\boldsymbol{x}_t(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_t)}{v_t(\boldsymbol{\phi}_t)},$$

and K_n becomes, under H_1,

$$K_n(s,z) = n^{-1/2}\sum_{t=1}^{[ns]}\left\{I\left(e_t \le z\frac{v_t(\hat{\boldsymbol{\phi}})}{v_t(\boldsymbol{\phi}_t)} + \frac{\boldsymbol{x}_t(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_t)}{v_t(\boldsymbol{\phi}_t)}\right) - F(z)\right\}.$$

Under both H_0 and H_1, K_n can be written as

$$K_n(s,z) = n^{-1/2}\sum_{t=1}^{[ns]}\Big\{I\left(e_t \le z[1 + a_t n^{-1/2}] + b_t n^{-1/2}\right) - F(z)\Big\} + o_p(1),$$

where

$$a_t = \frac{1}{v_t(\boldsymbol{\phi})} \frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} (\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}),$$

under H_0, and

$$a_t = \frac{1}{v_t(\boldsymbol{\phi})} \frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} \left(n^{1/2}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) + \Delta_{12} g\left(\frac{t}{n}\right)\right),$$

under H_1;

$$b_t = \frac{\boldsymbol{x}_t n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{v_t(\boldsymbol{\phi})},$$

under H_0, and

$$b_t = \frac{\boldsymbol{x}_t n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + \Delta_{11} g(t/n)}{v_t(\boldsymbol{\phi})} \left[1 - \frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} \Delta_{12} g\left(\frac{t}{n}\right)\right],$$

under H_1; $o_p(1)$ is uniformly in s and z.

Let $a = (a_1, a_2, \cdots, a_n)^\top$ and $b = (b_1, b_2, \cdots, b_n)^\top$, define

$$K_n(s, z, a, b) = n^{-1/2} \sum_{t=1}^{[ns]} \Big\{ I\left(e_t \le z[1 + a_t n^{-1/2}] + b_n n^{-1/2}\right) - F(z) \Big\}.$$

Given the root-n consistency of $\boldsymbol{\beta}$ and $\hat{\boldsymbol{\phi}}$, we have the following theorem.

Theorem A.2: *Under Assumption (A.1) and (A.2), we have*

$$K_n(s, z, a, b) = K_n(s, z, 0, 0) + f(z) z \left(\frac{1}{n} \sum_{t=1}^{[ns]} a_t \right) + f(z) \left(\frac{1}{n} \sum_{t=1}^{[ns]} b_t \right) + o_p(1),$$

where $o_p(1)$ is uniformly in s and z.

The proof of this theorem follows Theorem A.2 in Bai (1996).

We are now ready to prove Theorem 1: Under the null hypothesis

$$\begin{aligned} & K_n(s,z) - sK_n(1,z) \\ = \; & H_n(s,z) - sH_n(1,z) \\ & + f(z) z \left(\frac{1}{n} \sum_{t=1}^{[ns]} \frac{1}{v_t(\boldsymbol{\phi})} \frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} (\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \right) - f(z) z \left(\frac{s}{n} \sum_{t=1}^{n} \frac{1}{v_t(\boldsymbol{\phi})} \frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} (\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \right) \\ & + f(z) \left(\frac{1}{n} \sum_{t=1}^{[ns]} \frac{1}{v_t(\boldsymbol{\phi})} \boldsymbol{x}_t (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \right) - f(z) \left(\frac{s}{n} \sum_{t=1}^{n} \frac{1}{v_t(\boldsymbol{\phi})} \boldsymbol{x}_t (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \right). \end{aligned}$$

Given Assumption A.4, Theorem 1 simply follows.

Proof of Theorem 2.

Under the local alternatives (3.1), $\boldsymbol{\phi}$ is still estimable with root-n consistency. Note that a_t is dominated by

$$\frac{1}{v_t(\boldsymbol{\phi})}\frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top}\left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi} + \Delta_1 g\left(\frac{t}{n}\right) n^{-1/2}\right),$$

with the remaining term being negligible in the limit. Moreover, when

$$a_t = \frac{1}{v_t(\boldsymbol{\phi})}\frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}),$$

from the previous proof, the drift term of $K_n(s,z) - sK_n(1,z)$ is negligible. We can thus assume

$$a_t = \frac{1}{v_t(\boldsymbol{\phi})}\frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top}\left(\Delta_1 g\left(\frac{t}{n}\right) n^{-1/2}\right).$$

Following Theorem A.2, we have

$$K_n(s,z) - sK_n(s,z) = H_n(s,z) - sH_n(s,z)$$
$$+f(z)z\Big\{\frac{1}{n}\sum_{t=1}^{[ns]} g(t/n) - s\frac{1}{n}\sum_{t=1}^{n} g(t/n)\Big\} + o_p(1).$$

Given that

$$\frac{1}{n}\sum_{t=1}^{[ns]} g(t/n) = \int_0^s h(x)dx,$$

therefore,

$$f(z)z\Big\{\frac{1}{n}\sum_{t=1}^{[ns]} g(t/n) - s\frac{1}{n}\sum_{t=1}^{n} g(t/n)\Big\}$$

converges to $f(z)z\Delta_1\lambda_g(s)$, where λ_g is given by (3.2). The proof is completed.

Proof of Theorem 3.

(i) Under the fixed alternative H_2, we have

$$T_n(s,z) = n^{-\frac{1}{2}}\sum_{t=1}^{[ns]} I(e_t \le z) - \frac{[ns]}{n} n^{-\frac{1}{2}}\sum_{t=1}^{n} I(e_t \le z).$$

We consider the case $s \leq \tau$, then

$$\begin{aligned}
T_n(s,z) &= n^{-\frac{1}{2}}\sum_{t=1}^{[ns]}[I(e_t \leq z) - F(z)] - \frac{[ns]}{n}n^{-\frac{1}{2}}\sum_{t=1}^{[n\tau]}[I(e_t \leq z) - F(z)] \\
&\quad -\frac{[ns]}{n}n^{-\frac{1}{2}}\sum_{t=[n\tau]+1}^{n}[I(e_t \leq z) - G(z)] + \sqrt{n}s(1-\tau)[F(z) - G(z)] \\
&= n^{-\frac{1}{2}}\sum_{t=1}^{[ns]}[I(e_t \leq z) - F(z)] - \frac{[ns]}{n}n^{-\frac{1}{2}}\sum_{t=1}^{[n\tau]}[I(e_t \leq z) - F(z)] \\
&\quad -\frac{[ns]}{n}n^{-\frac{1}{2}}\sum_{t=[n\tau]+1}^{n}[I(e_t \leq z) - G(z)] + \sqrt{n}(s\wedge\tau)(1-s\vee\tau)[F(z) - G(z)].
\end{aligned}$$

It can be found that

$$T_n(s,z) \longrightarrow \bar{K}(s,z) + \sqrt{n}(s\wedge\tau)(1-s\vee\tau)[F(z)-G(z)] + O_p(1),$$

for any given s and z. Similar result is obtained when $s > \tau$. We then apply Theorem A.1 to obtain the first result in Theorem 2.

(ii) Under the local alternative H_3 and $s \leq \tau$, we have

$$\begin{aligned}
T_n(s,z) &= n^{-\frac{1}{2}}\sum_{t=1}^{[ns]}[I(e_t \leq z) - F(z)] - \frac{[ns]}{n}n^{-\frac{1}{2}}\sum_{t=1}^{[n\tau]}[I(e_t \leq z) - F(z)] \\
&\quad -\frac{[ns]}{n}n^{-\frac{1}{2}}\sum_{t=[n\tau]+1}^{n}[I(e_t \leq z) - F_{nt}(z)] + \Delta_2(s\wedge\tau)(1-s\vee\tau)[F(z) - H(z)].
\end{aligned}$$

It can found that

$$T_n(s,z) \xrightarrow{p} B(s,F(z)) + \Delta_2(s\wedge\tau)(1-s\vee\tau)[F(z)-H(z)],$$

for any given s and z. We then apply Theorem A.1 to obtain the second result in Theorem 3.

Appendix B

We are going to show that

$$\frac{1}{n}\frac{1}{v_t(\boldsymbol{\phi})}\frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} - \frac{s}{n}\sum_{t=1}^{n}\frac{1}{v_t(\boldsymbol{\phi})}\frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} = o_p(1), \text{uniformly}.$$

We find that

$$\frac{1}{v_t(\boldsymbol{\phi})}\frac{\partial v_t(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}^\top} = \left(\frac{1}{2v_t^2(\boldsymbol{\phi})}, \frac{u_{t-1}^2}{2v_t^2(\boldsymbol{\phi})}, \frac{v_{t-1}^2(\boldsymbol{\phi})}{2v_t^2(\boldsymbol{\phi})}\right)^\top.$$

Let consider

$$\frac{1}{n}\sum_{t=1}^{[ns]}\frac{1}{v_t^2(\boldsymbol{\phi})} - \frac{s}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})},$$

it suffers to show that

$$\frac{1}{n}\sum_{t=1}^{[ns]}\frac{1}{v_t^2(\boldsymbol{\phi})} - s\,\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})} = o_p(1), \text{uniformly}.$$

Observing that

$$\frac{1}{v_t^2(\boldsymbol{\phi})} \le \frac{1}{a} > 0,$$

$v_t^2(\boldsymbol{\phi})$ is strictly bounded from zero. For any given $\epsilon > 0$, there exists m such that

$$\frac{1}{m} < \frac{a}{4}\epsilon.$$

Define $s_i = i/m$, for $i = 1, 2, \cdots, m$, then for any s_i, there exists N_i such that for any $n \ge N_i$,

$$\text{Prob}\left\{\left|\frac{1}{[ns_i]}\sum_{t=1}^{[ns_i]}\frac{1}{v_t^2(\boldsymbol{\phi})} - \operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})}\right| > \frac{\epsilon}{2}\right\} < \eta.$$

Let $N = \max\{m, N_1, \cdots, N_m\}$, for any s, we have $s_j \le s < s_{j+1}$, we then have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{t=1}^{[ns]}\frac{1}{v_t^2(\boldsymbol{\phi})} - s\,\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})}\right| \\
\le\ &\left|\frac{1}{n}\sum_{t=1}^{[ns_j]}\frac{1}{v_t^2(\boldsymbol{\phi})} - s_j\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})}\right| + \frac{1}{n}\sum_{t=[ns_j]}^{[ns]}\frac{1}{v_t^2(\boldsymbol{\phi})} + (s - s_j)\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})} \\
\le\ &\left|\frac{1}{n}\sum_{t=1}^{[ns_j]}\frac{1}{v_t^2(\boldsymbol{\phi})} - s_j\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{v_t^2(\boldsymbol{\phi})}\right| + \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}$$

Similarly, we have the same results for $y_{t-1}^2/\ v_t^2(\boldsymbol{\phi})$ and $v_{t-1}^2(\boldsymbol{\phi})/\ v_t^2(\boldsymbol{\phi})$.

References

[1] Andrews, D. W. K. (1993): "Tests for Parametric Instability and Structural Change with Unknown Change Point," *Econometrica*, 61, 821-856.

[2] Bai, J. (1994): "Weak Convergence of Sequential Empirical Processes of ARMA Residuals," *Annals of Statistics*, 22, 2051-2061.

[3] Bai, J. (1996): "Testing for Parameter Constancy in Linear Regressions: An Empirical Distribution Approach," *Econometrica*, 64, 597-622.

[4] Bollerslev, T. (1986): "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31, 307-328.

[5] Bollerslev, T. and J. Wooldridge (1992): "Quasi-Maximum Likelihood Estimation and Inference in Dynamic Models with Time-Varying Covariances," *Econometric Reviews*, 11(2), 143-172.

[6] Bickel, P. J. and M. J. Wichura (1971): "Convergences for Multiparameter Stochastic Processes and Some Applications," *Annals of Mathematical Statistics*, 42, 1656-1670.

[7] Billingsley, P. (1968): "Convergence of Probability Measure," Wiley.

[8] Boldin, M. V. (1989): "On Testing Hypotheses in Sliding Average Scheme by the Kolmogorov-Smirnov and ω^2 Tests," *Theory of Probability and its Applications*, 34, 699-704.

[9] Bollerslev, T. P. (1986) Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics* 31, 307-327.

[10] Chow, G. C. (1960): "Tests of Equality between Sets of Coefficients in Two Linear Regressions," *Econometrica*, 28, 591-605.

[11] Chu, C. J. (1995): "Detecting Parameter Shift in GARCH Models," *Econometric Reviews*, 14(2), 241-266.

[12] Delgado, M. A. and J. Hidalgo (1996): "Nonparametric Inference on Structural Break Points", mimeo, London School of Economics.

[13] DeLong, D. M. (1981): "Crossing Probabilities for a Square Root Boundary by a Bessel Process," *Communications in Statistics — Theory and Methods*, A10 (21), 2197-2213.

[14] Diebold, F. (1986): "Modeling the Persistence of Conditional Variances: A Comment," *Econometric Reviews*, 5, 51-56.

[15] Engle, R. F.(1982) "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation", *Econometrica* 50, 987-1007.

[16] Engle, Robert F., David M. Lilien and Russell P. Robins (1987), "Estimating Time Varying Risk Premia In The Term Structure: The Arch-M Model," *ECONOMETRICA*, 55, 391-408.

[17] Engle, R. F.(1993) "Statistical Models for Financial Volatility", *Financial Analyst Journal*, 1993, v49(1), 72-78.

[18] Hawkins, D. (1987): "A Testing for a Change Point in Parametric Model Based on a Maximal Wald-type Statistic," *Sankhya*, 49 368-376.

[19] Hamilton, J. (1989): "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle," *Econometrica*, 57, 357-384.

[20] Hansen, B. (1994): "Autoregressive Conditional Density Estimation," *International Economic Review*, 35(3), 705-730.

[21] James, B., K. L. James, and D. Siegmund (1987): "Tests for a Change-point," *Biometrika*, 74, 71-83.

[22] Knight, J., F. Li, and J. Yang (1999): "Testing Structural Change in Continuous-Time Term Structure Models," manuscript, Department of Economics, The University of Western Ontario.

[23] Koul, H. L. (1992): *Weighted Empiricals and Linear Models,* IMS Lecture Notes - Monograph Series. Hayward, CA: IMS.

[24] Kreiss, P. (1991): "Estimation of the Distribution of Noise in Stationary Processes," *Metrika*, 38, 285-297.

[25] Lamoureux, C. and W. Lastrapes (1990a): "Heteroskedasticity in Stock Return Data: Volume versus GARCH Effects," *The Journal of Finance*, 45, 221-229.

[26] Lamoureux, C. and W. Lastrapes (1990b): "Persistence in Variance, Structural Change and the GARCH Model," *Journal of Business and Economic Statistics*, 8, 225-234.

[27] Davidson, R. and J. G. MacKinnon. (1998): "Graphical Methods For Investigating The Size And Power Of Hypothesis Tests,," *Manchester School of Economics & Social Studies*, 66, 1-26.

[28] Perron, B. (1998): "Jumps in the Volatility of Financial Markets," mimeo, Department of Economics, University of Montreal.

[29] Pollard, D. (1984): *Convergence of Stochastic Processes,* New York: Springer-Verlag.

[30] Quandt, R. E. (1960): “Tests of the Hypothesis that a Linear Regression System Obeys Two Separate Regimes,” *Journal of the American Statistical Association*, 55, 324-330.

[31] Silverman, B. W. (1986): *Density Estimation for Statistics and Data Analysis,* London: Chapman and Hall.

[32] Tong, H. (1983): *Threshold Models in Nonlinear Time Series Analysis,* New York: Springer-Verlag.

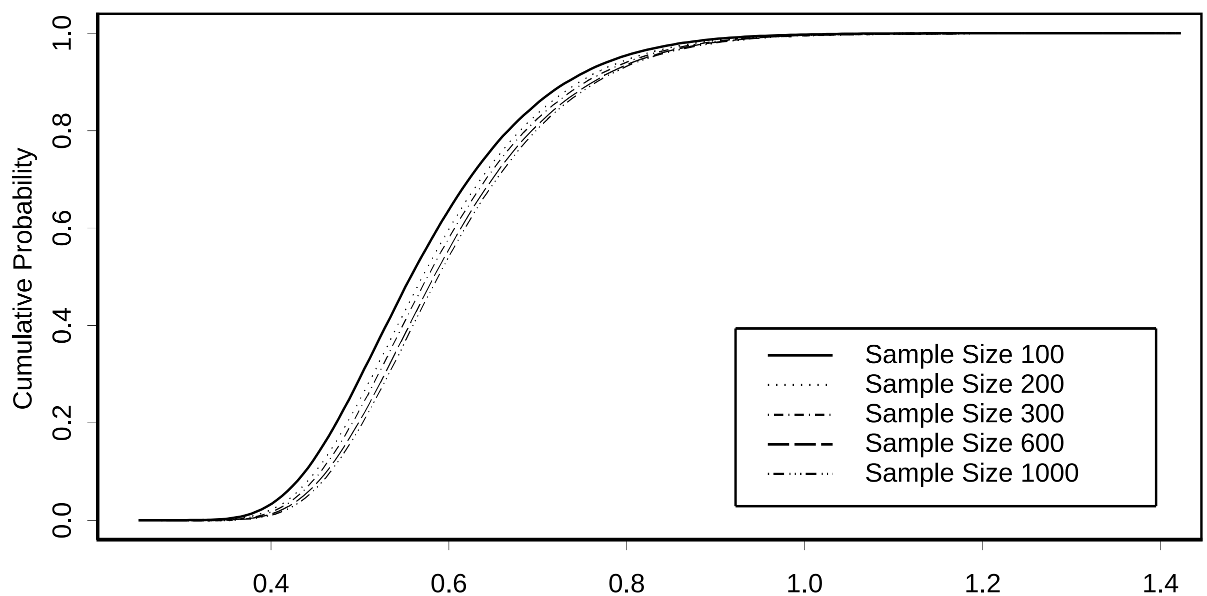


Figure 1: *CDF* of the M_n Statistics with Difference Sample Size

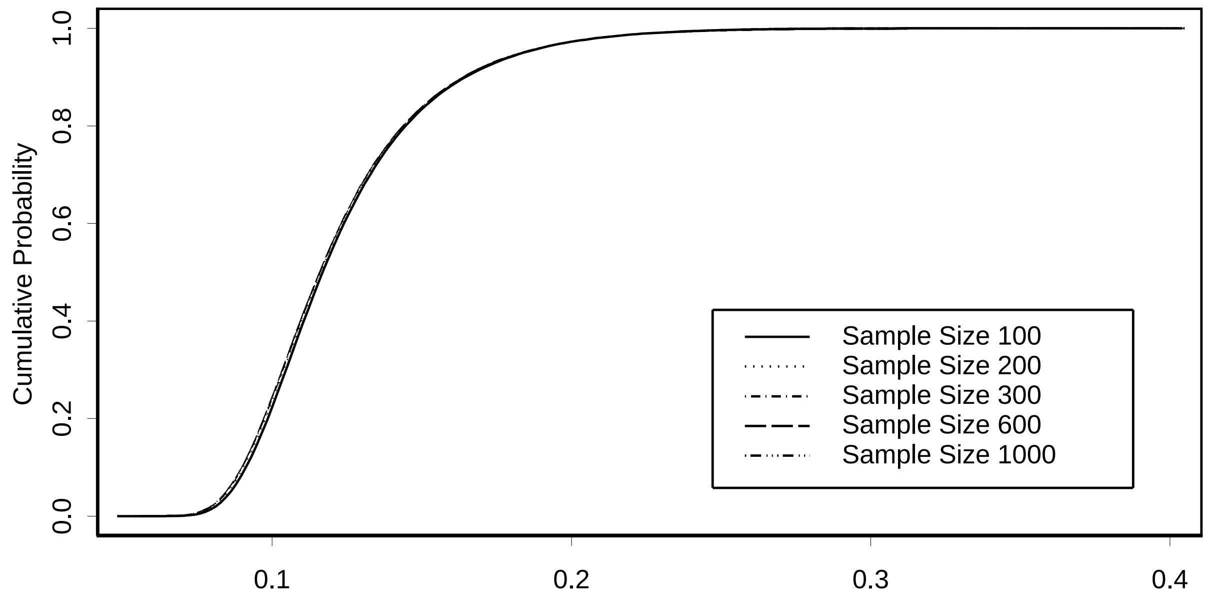


Figure 2: *CDF* of the $ABSA_n$ Statistics with Difference Sample Size

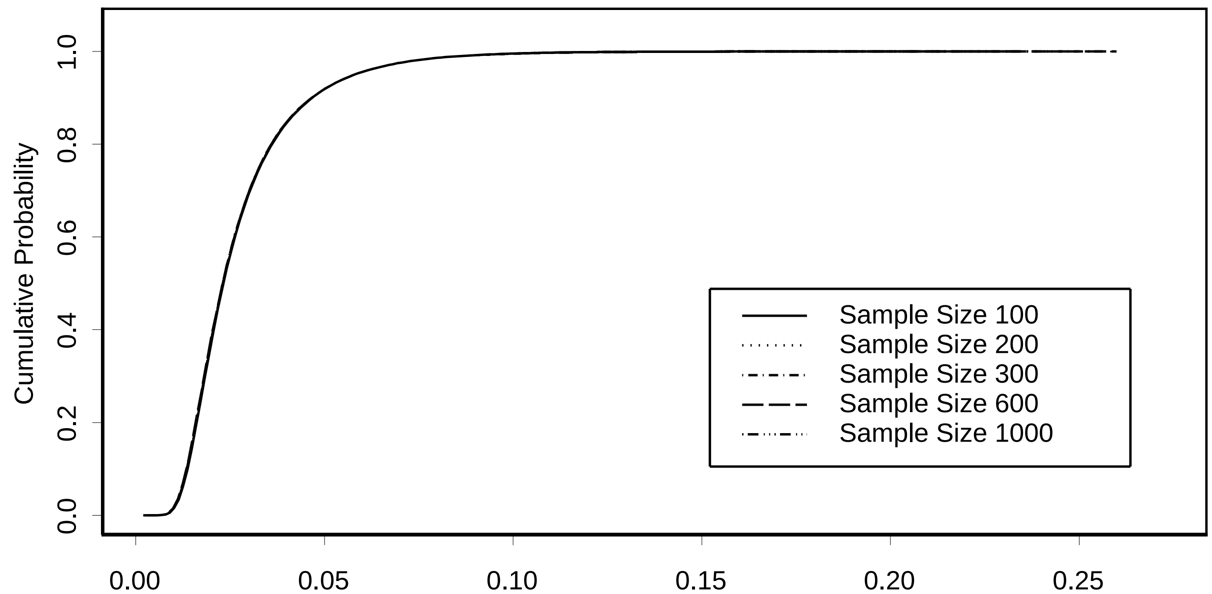


Figure 3: *CDF* of the SQA_n Statistics with Difference Sample Size

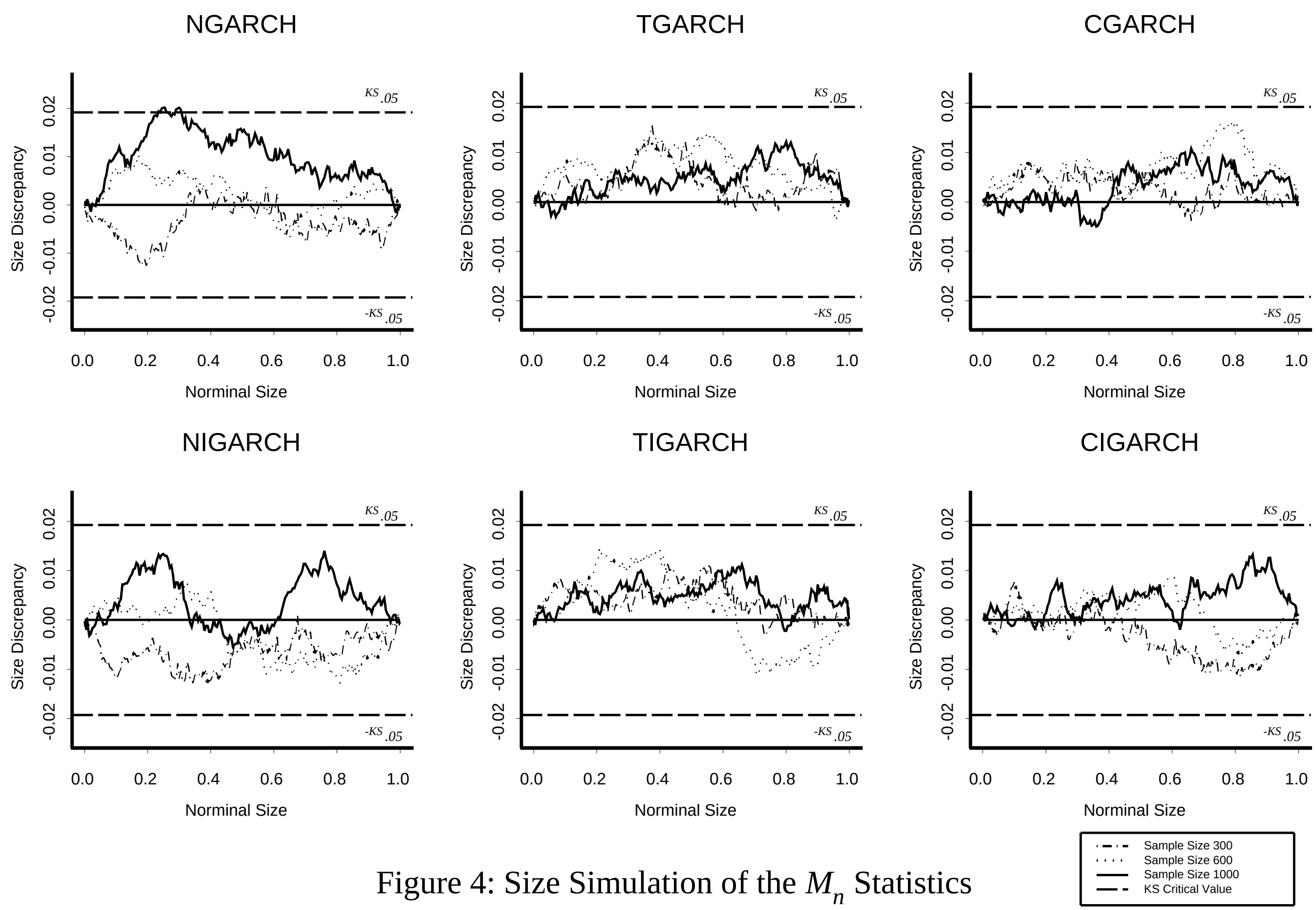


Figure 4: Size Simulation of the M_n Statistics

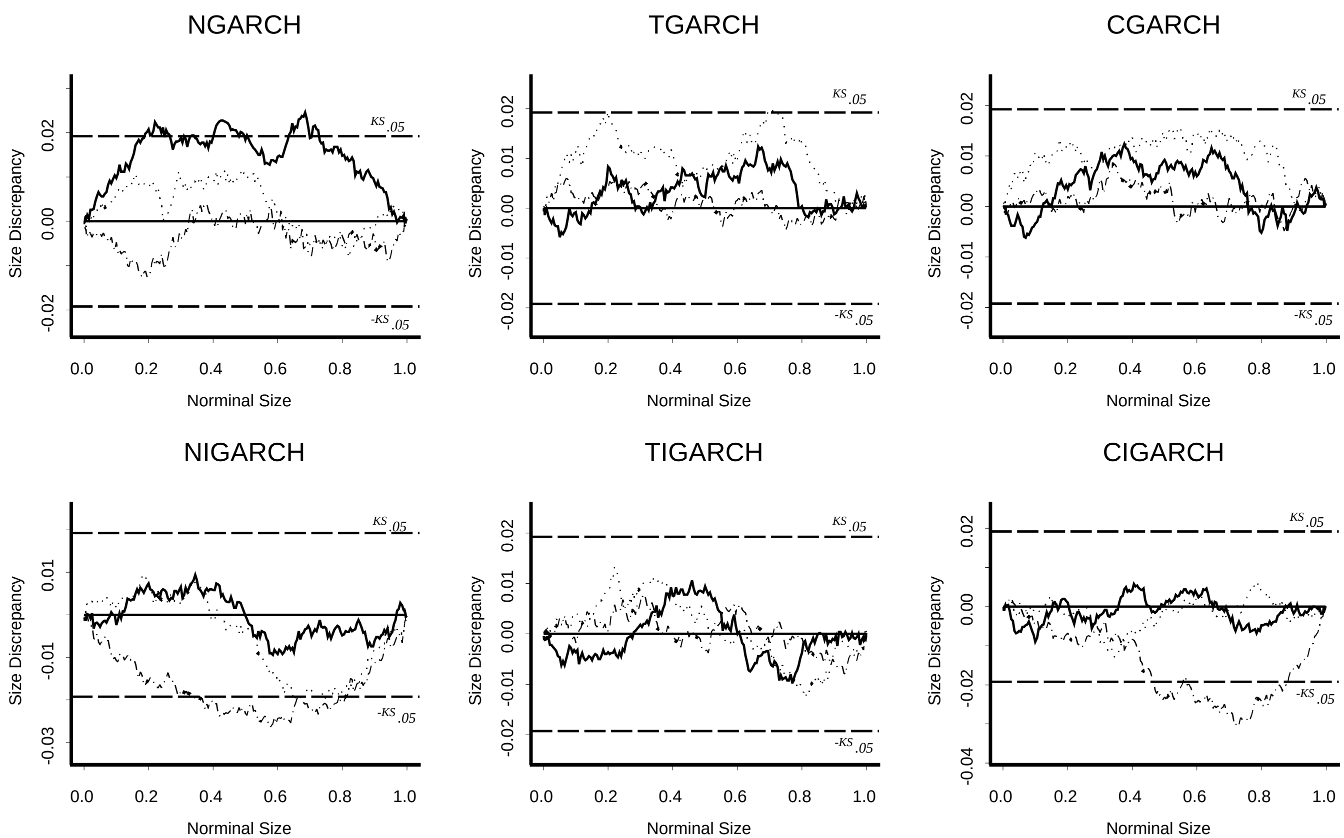


Figure 5: Size Simulation of the $ABSA_n$ Statistics

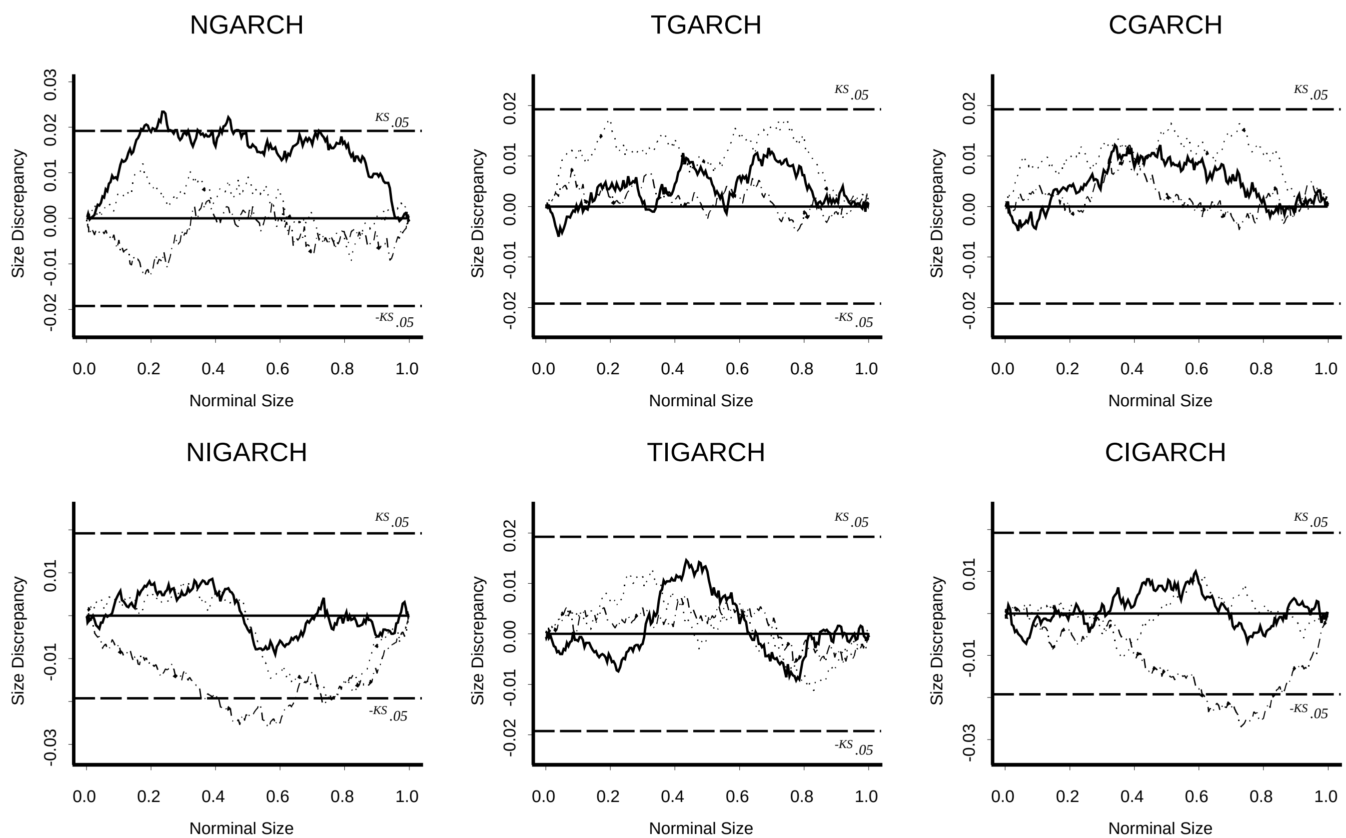
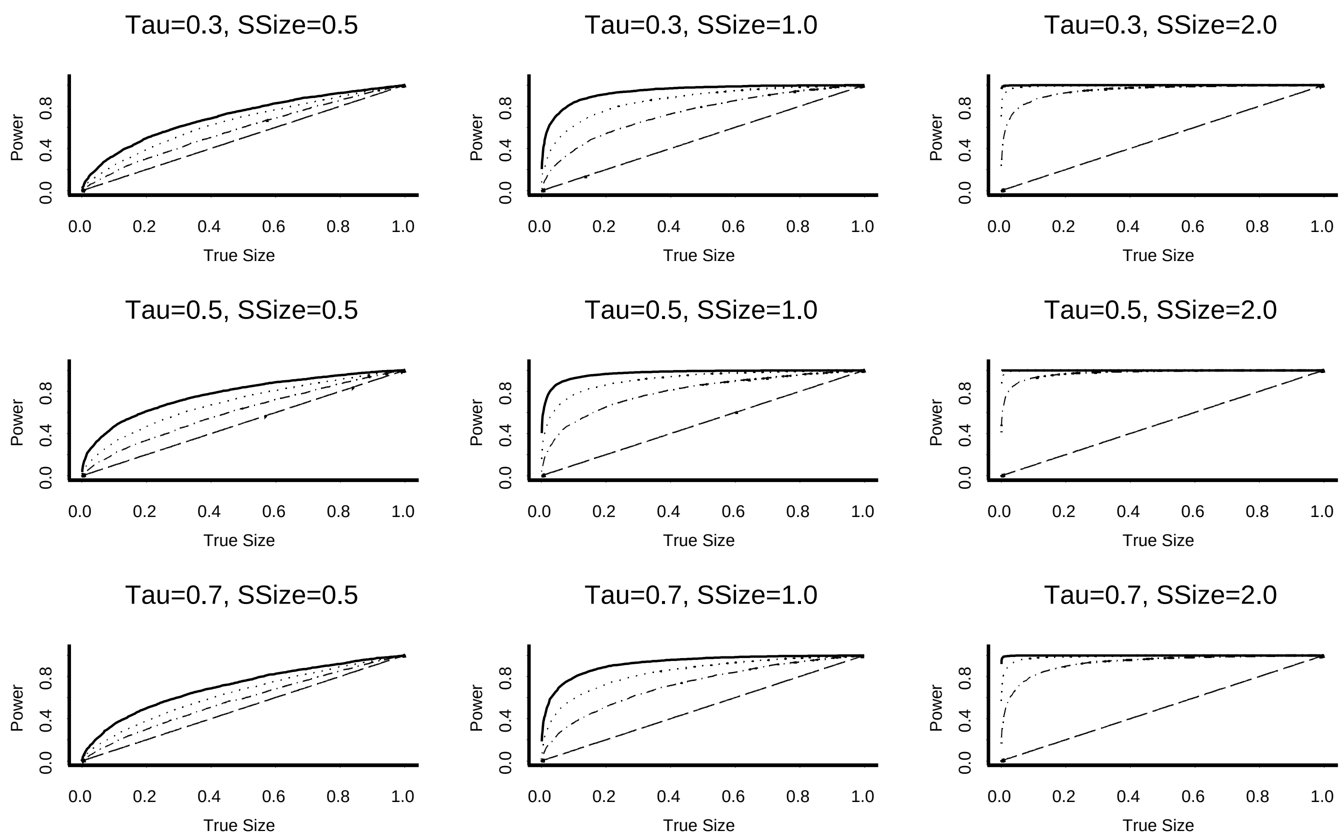


Figure 6: Size Simulation of the SQA_n Statistics



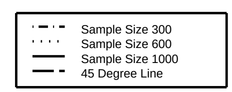


Figure 7: Power Simulation of the M_n Statistics

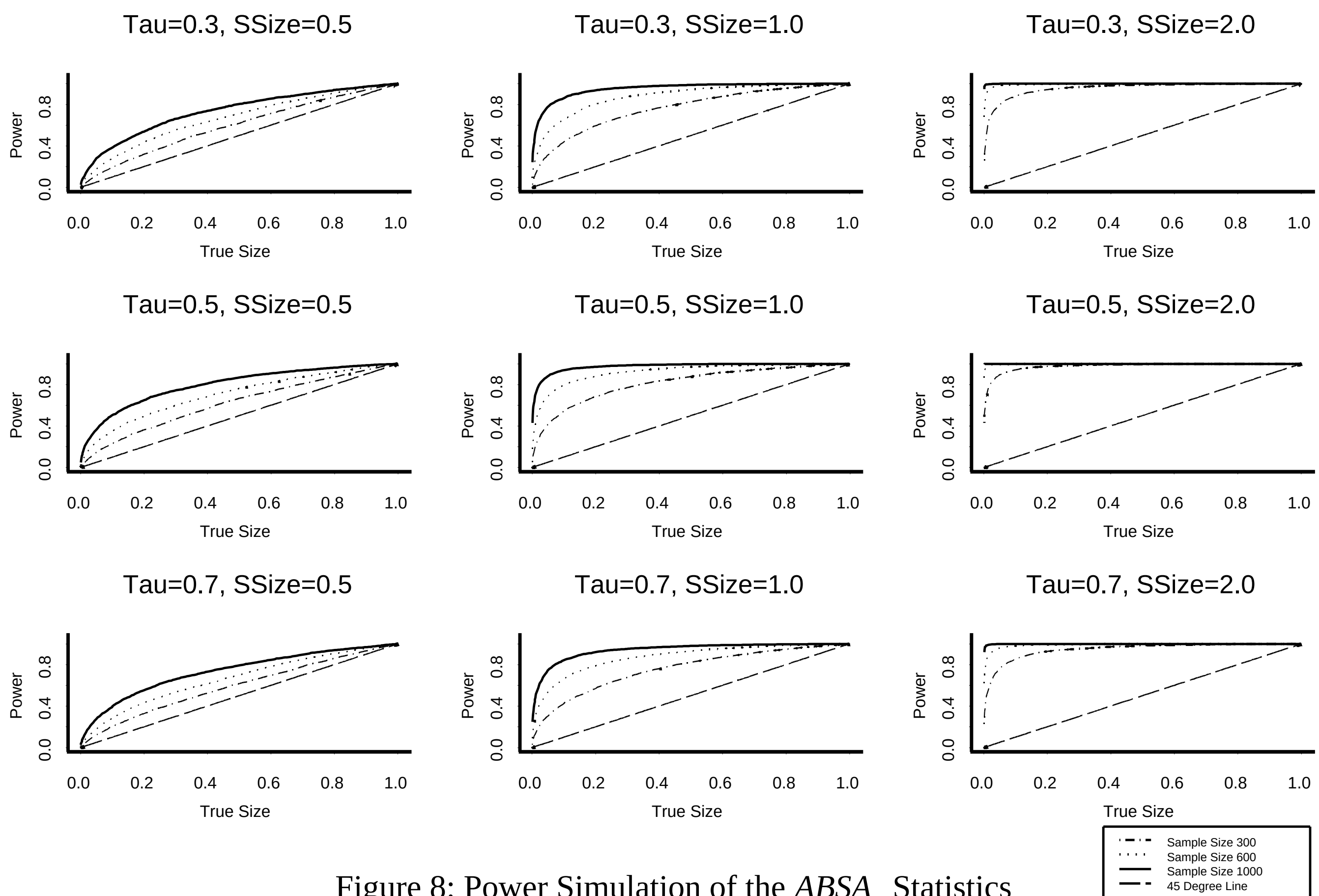


Figure 8: Power Simulation of the $ABSA_n$ Statistics

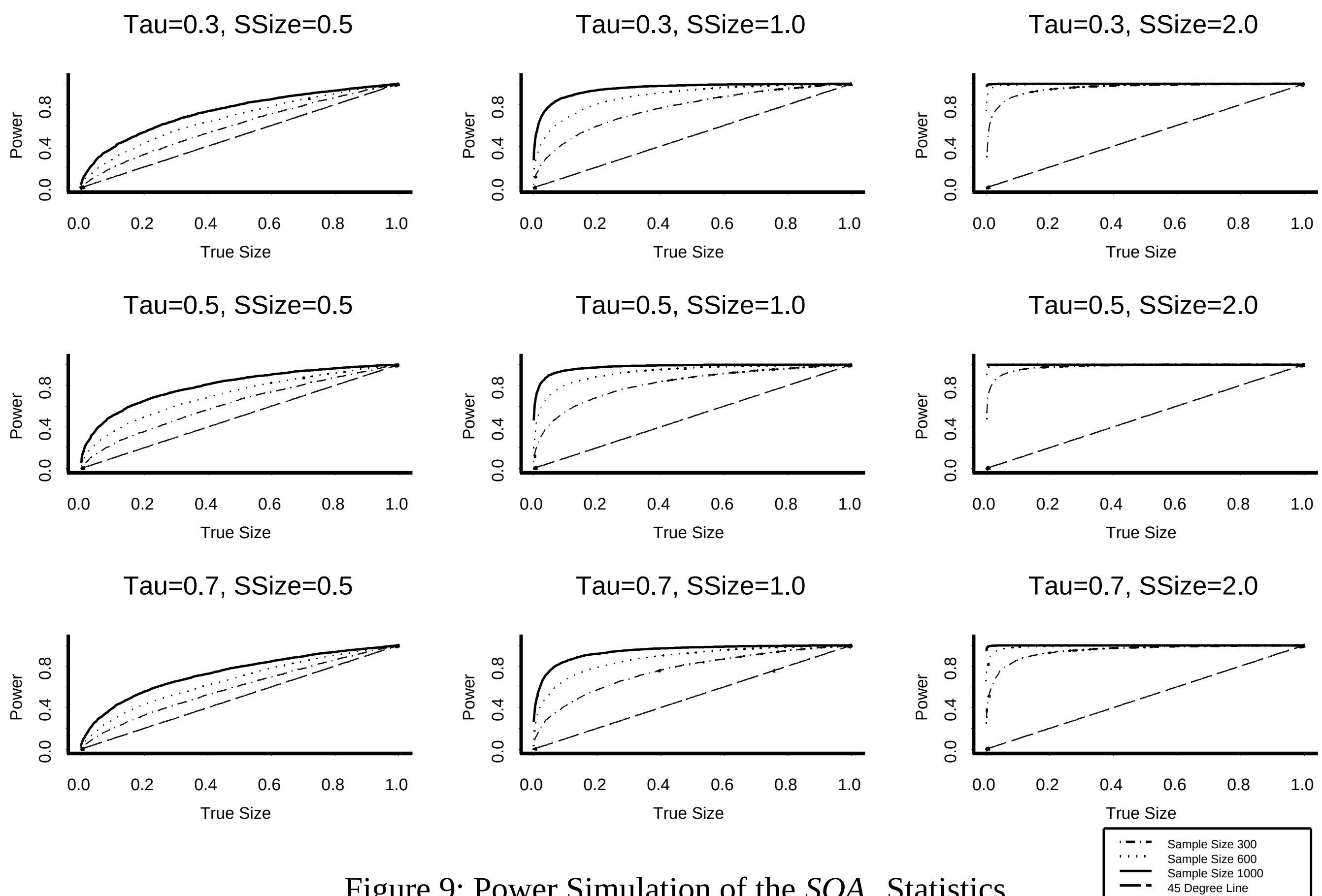


Figure 9: Power Simulation of the SQA_n Statistics