MULTIPERSON UTILITY

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Abstract: We approach the problem of preference aggregation by endowing both individuals and coalitions with partially-ordered or incomplete cardinal preferences. Consistency across preferences for coalitions comes in the form of the Extended Pareto Rule: if two disjoint coalitions A and B prefer x to y, then so does the coalition $A \cup B$. The Extended Pareto Rule has important consequences for the social aggregation of individual preferences. Restricting attention to the case of complete individual preferences, and assuming complete preferences for some pairs of agents (interpersonal comparisons of utility units), we discover that the Extended Pareto Rule imposes a "no arbitrage" condition in the terms of utility comparison between agents. Furthermore, if all the individuals and pairs have complete preferences and certain non-degeneracy conditions are met, then we witness the emergence of a complete preference ordering for coalitions of all sizes. The corresponding utilities are a weighted sum of individual utilities, with the $n - 1$ independent weights obtained from the preferences of $n - 1$ pairs forming a spanning tree in the group.

Keywords: Preference aggregation, Incomplete preferences, Extended Pareto Rule.

1 Introduction

Our approach to the problem of preference aggregation begins by endowing coalitions with cardinal preference orderings that may fail to be complete, i.e., some pairs of outcomes may be regarded as incomparable. We then consider a natural extension of the Pareto Rule, called the Extended Pareto Rule (EPR): if two disjoint coalitions A and B prefer x to y, then $A \cup B$ prefers x to y.

[1] For helpful comments, we are grateful to the participants of the IX International Game Theory Conference at Stony Brook, New York (July 1998); the UCLA workshop in Economic Theory (March 1999); and the Southern California Operations Research/Management Science Annual Meeting (June 1999).

Though it appears innocuous, EPR has important consequences for the social aggregation of individual preferences. Shapley and Shubik (1974), hereafter SS,[2] introduced EPR and claimed (p. 65):

Claim 1 "with the Pareto Principle thus strengthened, we can often weaken some of the other hypotheses [regarding completeness of the group preference] and still obtain the existence of a social utility function. For example, we can [assume complete group preferences] only for two-member groups. ... With the aid of the Extended Pareto Rule, we can then derive utility functions for all other subsets of N, including N itself."

SS (loc. cit.) illustrated this aggregation procedure with an example wherein a social preference for a group of three individuals $\succsim_{\overline{123}}$ is derived from the pair preferences $\succsim_{\overline{12}}$ and $\succsim_{\overline{23}}$. In this example, both the individual and social preferences are expressed by means of ordinal utilities. However, "even stronger conclusions can sometimes be drawn when we are working with conditions that lead to cardinal utility." (SS, p. 68).

In this paper we formalize and prove Claim 1 utilizing a cardinal framework. Still, as indicated in the three-individual example of SS, a similar results appears to hold in an ordinal setting. The justification for using cardinal utility for individuals is argued on several grounds in SS. Beside the usual interpretation that stems from choices over lotteries, where utilities can be interpreted as probabilities, SS emphasizes the ability of a cardinal utility scale to represent strength or intensity of preference (see Shapley 1974).[3] We find the notion of strength of preference very natural for interpreting statements about interpersonal comparisons of utility units (DeMeyer and Plott 1971 and Saposnik 1975). Cardinality of the group preference then gives us the possibility of aggregating and averaging individual intensities of preference.

Cardinal group preferences were first proposed by Fleming (1952) and Harsanyi (1955) on the normative ground that the Independence Axiom (the Sure Thing Prin-

[2] SS first appeared as a Rand report; it was subsequently published as chapters 4 and 5 of Shubik (1982).

[3] Under natural assumptions, the cardinal utility function representing intensities of preference coincides with the cardinal utility function representing choices over lotteries. See Sarin (1982) for a treatment of this point in the context of Subjective Expected Utility.

ciple) is desirable for coalitions as well as individuals.[4] Including this axiom to the group preference, and relaxing the Completeness Axiom, we obtain an elegant representation theorem (Aumann 1962 and Shapley[5]): in the same way that a complete (cardinal) preference is represented by a ray of utility functions (the positive multiples of some utility function u), an incomplete preference $\succsim$ can be represented by a convex cone U of utility functions, so that x is preferred to y if and only if $u(x) \geq u(y)$ for all u in U.

Section 2 summarizes the theory of incomplete preferences. In Section 3 we utilize utility cones to characterize EPR.[6] We progressively explore the implications of this characterization when individuals are assumed to have complete preferences. For example, if the coalition $\overline{12}$ has a complete pair preference $\succsim_{\overline{12}}$, then the utility representing $\succsim_{\overline{12}}$ takes the additive form $u_{\overline{12}} = (u_{\overline{1}} + \delta_{1;2} u_{\overline{2}})/(1 + \delta_{1;2})$. The weight $\delta_{1;2}$ effectively establishes terms of interpersonal comparison of utility units (utility differences), that we call the utility comparison rate between 1 and 2. Interestingly, with three (or more) individuals we find that EPR implies a "no arbitrage" condition in the utility comparison rates; if the pairs $\overline{12}$ and $\overline{23}$ have complete preferences, with utility comparison rates $\delta_{1;2}$ and $\delta_{2;3}$, then a complete preference for $\overline{13}$ must have utility comparison rate $\delta_{1;3} \equiv \delta_{1;2}\delta_{2;3}$. Surprisingly, under these conditions, $\succsim_{\overline{123}}$ is necessarily complete and has utility $u_{\overline{123}} = (u_{\overline{1}} + \delta_{1;2} u_{\overline{2}} + \delta_{1;3} u_{\overline{3}})/(1 + \delta_{1;2} + \delta_{1;3})$. Because no condition other than EPR was imposed on the originally incomplete preference $\succsim_{\overline{123}}$, we witness the emergence of a complete ordering for the triple $\overline{123}$ based on complete preferences for the individuals and two pairs.

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[4] Under cardinality of group preferences and the Pareto Rule, Harsanyi (1955) showed that the utility representing the group preference is a weighted sum of individual utilities. This result motivated Diamond's (1967) critique: if $u_{\overline{12}} = u_{\overline{1}} + u_{\overline{2}}$, then the pair is indifferent between (a) a lottery that gives each agent an expected utility of 1/2, and (b) a sure prospect giving $u_{\overline{1}} = 1$ and $u_{\overline{2}} = 0$. This indifference is, for Diamond, unacceptable. As shown in Sen (1970 p. 393-394), Diamond's argument depends crucially on the individuals utility level (and thus the "origin") being comparable. Because we only express comparisons of utility units, and not utility levels, Diamond's objection does not apply here.

[5] Shapley's original paper remains unpublished, but can be consulted in the preliminary section of Baucells and Shapley (1998). In contrast with Aumann's paper, Shapley's encompasses infinite-dimensional prospect spaces.

[6] In short, EPR holds if and only if the utility cone $U_{A \cup B}$ is contained in the convex hull of U_A and U_B, for all disjoint coalitions A and B.

A cardinal framework facilitates the geometrical representation of the previous result. Because each individual utility u_i is a point in a vector space, EPR implies that u_{12} is a point on the line segment $\overline{u_1 u_2}$. Similarly, given u_{12} and u_{23}, EPR finds u_{123} as the unique intersection of $\overline{u_1 u_{23}}$ and $\overline{u_{12} u_3}$. With four of more agents, it is interesting that Desargues's Theorem — a geometrical result attributed to the 17th century French mathematician Girard Desargues[7] — can be used to show how a preference $\succsim_S$ is consistent with EPR as applied to different partitions $\{A; B\}$ of S. The geometric representation in a plane also highlights the need for certain linear independence conditions to avoid degeneracy.

Section 4 presents the two main results, which are the extension of the previous findings to n agents. They are both stated under minimal requirements regarding linear independence. In essence, if we are given complete preferences for some pairs forming a spanning tree, then EPR implies the emergence of a complete preference for some coalitions, including N, the set of agents (Theorem 9); and if all the pairs have complete preferences, then all the coalitions have complete preferences (Theorem 12). Any such complete preference is represented by a weighted sum of the individual utilities, i.e., $u_S = \sum_{i \in S} \lambda_i u_i$ for all $S \subseteq N$. The $n - 1$ independent weights are obtained from the preferences of any $n - 1$ pairs forming a spanning tree in the group.

Section 5 discusses some of our assumptions. We point out at the relation between "stability" of the group preference and a very mild assumption called Minimal Consensus: there are two prospects $x; y$ such that all agents strictly prefer x to y. We also introduce a certain weakening of EPR in the subsection "Masters and Servants." The final subsection suggests some extensions.

Our theory naturally leads to an interpretation of preference aggregation as a process that begins with complete orderings for individuals. If we recognize the ability of pairs to establish terms of utility comparisons, then EPR dictates consistency conditions that build complete orderings, from small coalitions to large coalitions. Consequently, we discover that once the problem of welfare comparisons is resolved at a pair level, then it is resolved for the group at large.

[7] As a historical note, Girard Desargues (1593-1662) was Blaise Pascal's mentor. Desargues made important contributions to projective geometry. The theorem bearing his name was published in 1648 (see Field and Gray 1997).

In the social choice literature, EPR has been recently used in Dhillon (1998) and Dhillon and Mertens (1999). Our formulation here differs from the traditional one in the social choice literature in that we do not assume the existence of a social welfare function (a complete group preference).[8] Instead, by just assuming a partial ordering, we derive a complete ordering, thus giving an axiomatic basis for the existence of a social welfare function.

This paper focuses on finding conditions that lead to complete preferences for the group and the rest of coalitions. Using the same formulation, Baucells and Shapley (1999) considers pair agreements that exhibit some degree of incompleteness. After characterizing a measure of incompleteness satisfying cardinal invariance and additivity, we explore how the incompleteness in the pair preferences restricts the incompleteness of the group preference.

2 Review of the Theory of Incomplete Preferences

The underlying domain of prospects over which the preferences are given will be denoted by M, sometimes called a mixture space (see Hausner 1951). For simplicity, it will be assumed here that M is a finite dimensional, closed, convex subset of R^m. We further stipulate M to have dimension m, so that M contains interior points. The location of the origin with respect to M is arbitrary. For example, the simplex $M = \{(x_1, \ldots, x_m) \in R^m : \sum_{k=1}^m x_k \le 1; x_k \ge 0\}$ could represent the set of probability mixtures over $m + 1$ "pure" prospects $k \in \{0, 1, \ldots, m\}$. The pure prospect $k = 0$ occupies the origin of R^m and has probability $1 - \sum_{k=1}^m x_k$, whereas the other pure prospects take the unit vectors of R^m and have probability x_k, $k \in \{1, \ldots, n\}$. Probability mixtures of two prospects x and y are identified with the prospect $\alpha x + (1 - \alpha)y$, for suitable $\alpha \in [0, 1]$.

Here are the four axioms for an incomplete preference[9] relation $\succsim$ —they are asserted for all $x, y, z \in M$ and all $\alpha \in [0, 1]$:

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[8] A social welfare function may not exist, as shown in Arrow (1963). This non-existence result is extended by Kalai and Schmeidler (1977) to the context of cardinal preferences. For a discussion of Arrow's non-existence result, see SS pp. 69-79.

[9] Here "incomplete" is short for "possibly incomplete"; if we wish to exclude the complete case we shall say "not complete."

(P1) Reflexivity: $x \succsim x$.

(P2) Transitivity: If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

(P3) Independence: For all $\alpha \neq 0$, $x \succsim y$ if and only if $\alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z$.

(P4) Continuity: The set $\{\alpha : x \succsim \alpha y + (1-\alpha)z\}$ is closed.

If $x \succsim y$ but not $y \succsim x$ we say that x is strictly preferred to y and write $x \succ y$; if both $x \succsim y$ and $y \succsim x$ we say that x and y are indifferent and write $x \sim y$; and if neither $x \succsim y$ nor $y \succsim x$ we say that x and y are incomparable and write $x \| y$. If M has no incomparable pairs then $\succsim$ is said to be complete; this can be expressed axiomatically by replacing (P1) with

(P1′) Completeness: Either $x \succsim y$ or $y \succsim x$.

Axioms $P1 - P4$ imply that $D(y) = \{x \in M : x \succsim y\}$, the preference set of $y \in M$, is a closed, convex cone in M with vertex y.[10] Moreover, if y and w are two interior points, then $D(y)$ and $D(w)$ contain the same "directions of preference."[11] Thus, given an interior point $y^\delta \in M$, we can define the preference cone D of $\succsim$ as the closed cone in R^m with vertex 0 that extends the directions of preference of $D(y^\delta)$ to R^m. Formally,

$$D \equiv \lambda[D(y^\delta) - y^\delta] = \{\lambda(x - y^\delta) \in R^m : x \in D(y^\delta);\ \lambda > 0\}.$$

Proposition 2 Let $\succsim$ be an incomplete preference relation defined on M, and D its preference cone. Then for all $x; y \in M$,

$$x \succsim y \iff x - y \in D \tag{1}$$

Conversely, let D be any closed convex cone in R^m with vertex 0, and let M be any convex subset of R^m. Then the relation $\succsim$ that (1) defines is an incomplete preference relation on M.

Proof. See the preliminary section of Baucells and Shapley (1998) for details. ■

Let us denote by M^* the set of all real-valued functions on M that are both linear and homogeneous. Then, M^* coincides with R^m, the space of linear homogeneous

[10] $K \subseteq V$ is a cone in $V \subseteq R^m$ with vertex v if and only if $\lambda(x - v) \in K$ for all $x \in K$ and all $\lambda > 0$ such that $\lambda(x - v) \in V$. If $V = R^m$, then K is a cone with vertex 0 if and only if K is closed under positive scalar multiplication.

[11] Consider $x; y; z; w \in M$ such that $x - y = \lambda(z - w)$ for some $\lambda > 0$, i.e., the half lines $\overrightarrow{yx}$ and $\overrightarrow{wz}$ are parallel. Then, $x \in D(y)$ if and only if $z \in D(w)$.

functions on R^m.[12] Let D be a preference cone and D^* its polar cone in M^* defined by $D^* = \{u \in R^m : u(x) \geq 0 \text{ for all } x \in D\}$. Similarly, the "polar" of D^* may be defined by $D^{**} = \{x \in R^m : u(x) \geq 0 \text{ for all } u \in D^*\}$ (see Figure 1).

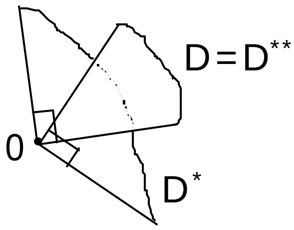

Figure 1: D^* is the polar cone of D.

The weak inequalities in these definitions ensure that D^* and D^{**} are both closed, convex cones in R^m with vertex 0. Moreover, the two polar mappings are mutual inverses, $D = D^{**}$, so that D^* can be used unambiguously to represent the incomplete preference $\succsim$ associated with D. Further, any set $U \subseteq R^m$ such that $U^* = D$ represents $\succsim$ as well. Specifically, we say that U is the utility cone of $\succsim$ if $U = D^* \backslash \{0\}$ whenever $D^* \neq \{0\}$, and $U = \{0\}$ otherwise.

Theorem 3 Let $\succsim$ be an incomplete preference relation defined on $M \subseteq R^m$. There exists a non-empty subset $U \subseteq M^* = R^m$ such that for all $x, y \in M$,

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } u \in U. \tag{2}$$

Conversely, given any set $U \subseteq R^m$, the relation defined by (2) is an incomplete preference relation with preference cone U^*.

Proof. See the preliminary section of Baucells and Shapley (1998) for details. We note that when M is in the infinite-dimensional, one needs some technical qualifications to ensure that $D = D^{**}$. ■

It is useful to describe several incomplete preferences associated with certain special types of preference and utility cones. If $\succsim$ is a non-trivial complete preference,

[12] Thus, $u(x)$ becomes the inner product of the vector $u = (u^1, \ldots, u^m)$, and the prospect $x = (x_1, \ldots, x_m)$. Because each x_k is the probability associated with pure prospect k, $u(x)$ is the "expected utility" of x. Also, observe that the normalization $u(0) = 0$ is done for mathematical convenience: the location of 0 with respect to M is arbitrary.

then D is a half-space, and U is the normal ray contained in D. A trivial preference (all pairs are regarded as indifferent) corresponds to $D = \mathbb{R}^m$ and $U = \{0\}$. Thus, if $\succsim$ is a complete preference, then any element of U is a non-negative multiple of any other element of U: we say that $\succsim$ is complete and has utility u, for some $u \in U$.

More generally, $\succsim$ contains indifference relations if and only if D contains lines. On the contrary, if D is a pointed cone (i.e., contains no complete lines, like the negative orthant), then $\succsim$ is a "strict" partial ordering, having pairs of incomparable elements but not pairs of indifferent elements.[13] More "exotic" incomplete preferences arise if D is a subspace of $\mathbb{R}^m$. Then U is the subspace orthogonal to D and $\succsim$ has no strict preferences: M decomposes into a collection of mutually incomparable indifference classes.[14]

3 Incomplete Coalition Preferences and the Extended Pareto Rule

Given a set $N = \{1, \ldots, n\}$ of individuals, we fix M as the common prospect space. For example, the set of pure prospects in M could correspond to a list of $m + 1$ public projects, or a set of $m + 1$ feasible allocations of (indivisible) goods. Endow each coalition $S \subseteq N$ with an incomplete preference $\succsim_S$ on M, and let D_S and U_S be its corresponding preference and utility cone. The empty coalition is assigned the trivial preference. Confusion does not arise if we use $\bar{i}$, $\overline{ij}$,... to indicate coalitions $\{i\}$, $\{i, j\}$,... We restrict attention to the case where the agents have non-trivial, complete preferences $\succsim_{\bar{i}}$ with utilities $0 \neq u_{\bar{i}} \in \mathbb{R}^m$, for all $i \in N$: the utility cone $U_{\bar{i}}$ associated with $\succsim_{\bar{i}}$ is the ray of positive multiples of $u_{\bar{i}}$. Let $Sp(u_{\bar{1}}, u_{\bar{2}}, \ldots)$ denote the vector subspace spanned by some collection of utilities, and $d(u_{\bar{1}}, u_{\bar{2}}, \ldots)$ its dimension.

One might want to identify the coalition preference $\succsim_S$ with the Paretian prefer-

[13] If D is a full-dimensional pointed cone, then U is also a full-dimensional pointed cone. For example, if D is the negative orthant, then U is also the negative orthant.

[14] An extreme example is given by $D = \{0\}$ and $U = \mathbb{R}^m$: all pairs of distinct prospects are regarded as incomparable. The opposite extreme, $D = \mathbb{R}^m$ and $U = \{0\}$, corresponds to a trivial preference.

ence $\succsim^p_S$ given by the unanimity rule: for all $x, y \in M$,

$$x \succsim^p_S y \iff x \succsim_i y \text{ for all } i \in S. \tag{3}$$

$\succsim^p_S$ is an incomplete preference in the sense of Axioms P1 $-$ P4, and has utility cone $U^p_S = Co(\cup_{i \in S} U_i)$, where $Co(\cdot)$ indicates the convex hull.[15] Unless all the members of S share identical preferences, the Paretian preference will contain incomparable pairs.[16] In general, for completeness of group preference to arise we need the ability of certain coalitions to establish comparisons beyond the ones given in (3). Thus, we just retain the weak Pareto Rule given by the $\Leftarrow$ implication in (3). Our setting, which treats individuals and coalitions alike, allows for an important strengthening of the Pareto rule.

A collection of preferences $\succsim_S$, $S \subseteq N$, satisfies the Extended Pareto Rule (EPR) if for all disjoint coalitions A and B, and for all $x, y \in M$,

$$x \succsim_A y;\ x \succsim_B y \implies x \succsim_{A \cup B} y; \text{ and} \tag{4}$$

$$x \succ_A y;\ x \succsim_B y \implies x \succ_{A \cup B} y. \tag{5}$$

EPR is equivalent to the seemingly more general rule in which the corresponding version of (4) and (5) holds for any partition P of a coalition S. For example, if $\{A, B, C\}$ is a partition of S and $x \succ_A y$, $x \succsim_B y$, and $x \succsim_C y$, then imposing (5) to A and B produces $x \succ_{A \cup B} y$; and imposing (5) to $A \cup B$ and C yields $x \succ_S y$. In particular, EPR implies the usual Pareto Rule.

Let $Co_{A;B}$ denote the convex hull of $U_A \cup U_B$, and $Co^{ri}_{A;B}$ denote the set of relatively internal points in $Co_{A;B}$.[17] EPR can be characterized in terms of utility cones as

[15] Using Paretian preferences in the context of multi-criteria decision making, Yu (1974) describes Pareto undominated outcomes associated with certain polyhedral utility cones, i.e., cones defined by a finite number of extreme rays.

[16] For an "exotic" example of a Paretian preference, let $\succsim_1$ and $\succsim_2$ be opposite preferences, i.e., $u_1 = -u_2 \neq 0$. Then $U_{12} = Co(U_1; U_2)$ is a subspace (a line), and D^p_{12} is the orthogonal subspace (a hyperplane): $\succsim^p_{12}$ declares $x \sim^p_{12} y$ whenever both $x \sim_1 y$ and $x \sim_2 y$, and regards all the other pairs in M as incomparable.

[17] Formally, $Co_{A;B} = \{(1-\alpha)u_A + \alpha u_B \in \mathbb{R}^m : u_A \in U_A; u_B \in U_B;\ 0 \leq \alpha \leq 1\}$; and $u^* \in Co_{A;B}$ is relatively internal to $Co_{A;B}$ if for all $u \in Co_{A;B}$ there is a u' such that $u^* = (1-\alpha)u + \alpha u'$ for some $0 < \alpha < 1$. In finite-dimensional spaces, a point is relatively internal to a convex set if and only if it is relatively interior (in the usual topology).

follows.

Proposition 4 The Extended Pareto Rule holds if and only if for any two disjoint coalitions A and B,

$$U_{A\cup B} \subseteq Co_{A;B}; \text{ and} \tag{6}$$

$$U_{A\cup B} \cap Co^{ri}_{A;B} \neq \emptyset. \tag{7}$$

Proof. By the definition of preference cone, (4) is equivalent to $D_A \cap D_B \subseteq D_{A\cup B}$; and by the properties of polar cones (see Rockafellar 1967, p. 149-151), $D_A \cap D_B \subseteq D_{A\cup B} \iff D^*_{A\cup B} \subseteq (D_A \cap D_B)^* = (D^*_A \cup D^*_B)^{**}$. Because $(\cdot)^{**}$ is the closure of the set of positive multiples of convex combinations of a given set, and $(D^*_A \cup D^*_B)$ is already a closed cone, we have that $(D^*_A \cup D^*_B)^{**} = Co(D^*_A \cup D^*_B)$, whence (4) is equivalent to $D^*_{A\cup B} \subseteq Co(D^*_A \cup D^*_B)$. A moment's reflection reveals that this last inclusion, together with the condition

$$\text{If } U_{A\cup B} = \{0\}; \text{ then } 0 \in Co_{A;B}; \tag{8}$$

is equivalent to (6). But (5) implies (8): if $0 \notin Co_{A;B}$, then $Co_{A;B}$ is a pointed cone, i.e., both $\succsim_A$ and $\succsim_B$ contain strict preferences, so that $\succsim_{A\cup B}$ also contains strict preferences and $U_{A\cup B} \neq \{0\}$. Consequently, [(4),(5) $\Rightarrow$ (6)], and [(6) $\Rightarrow$ (4)].

[(4),(5) $\Rightarrow$ (7)] Suppose that (7) fails so that $U_{A\cup B} \subseteq Co_{A;B} \setminus Co^{ri}_{A;B}$. We enlarge $Co_{A;B}$ and define the full dimensional cone $K_{A;B} = Co_{A;B} \times Sp(Co_{A;B})^{\perp}$, which is the Cartesian product of $Co_{A;B}$ with the subspace orthonormal to $Sp(Co_{A;B})$. Of course, $Co^{ri}_{A;B}$ is contained in the interior of $K_{A;B}$. Because $U_{A\cup B}$ is convex and contained in the boundary of $Co_{A;B}$, it has dimension strictly less than that of $Co_{A;B}$ and $K_{A;B}$: there is a hyperplane H containing $U_{A\cup B}$ and supporting $K_{A;B}$. Full dimensionality ensures that H does not intersect the interior of $K_{A;B}$. Thus, H supports $Co_{A;B}$ and does not intersect $Co^{ri}_{A;B}$. If $x - y$ is some normal vector of H, then $u(x - y) = 0$ for all $u \in U_{A\cup B}$, and $u(x - y) \geq 0$ for all $u \in Co_{A;B}$. Because $Co_{A;B}$ is non-empty, $Co^{ri}_{A;B}$ is non-empty: there is a $u \in Co_{A;B}$ such that $u(x - y) > 0$, and we can find one such u in either U_A or U_B, say U_A. Thus, $x \succ_A y$ and $x \succ_B y$, but $x \sim_{A\cup B} y$, a contradiction of (5).

[(6),(7) $\Rightarrow$ (5)] If for some $x; y \in M$, $x \succ_A y$ and $x \succsim_B y$, then $u(x - y) \geq 0$ for all $u \in Co_{A;B}$, and $u_A(x - y) > 0$ for some $u_A \in U_A$. Let $u^* \in U_{A\cup B} \cap Co^{ri}_{A;B}$ so that for some $u' \in Co_{A;B}$ and $\alpha \in (0;1)$, $u^* = (1-\alpha)u_A + \alpha u'$. By (7) $u(x-y) \geq 0$ for all $u \in U_{A\cup B}$, and $u^*(x-y) > 0$: $x \succ_{A\cup B} y$. ■

If both the individuals and the group have complete preferences (utility rays), Proposition 4 immediately implies that the utility of the group is a weighted sum of individual utilities, which is Harsanyi's (1955) Theorem.

For all $S \subseteq N$, let $U^*_S \equiv \bigcap_{\{A \overset{\cdot}{\cup} B = S\}} Co_{A;B}$, where the use of $\overset{\cdot}{\cup}$ assumes that A and B are disjoint. For EPR to hold we need $U_S \subseteq U^*_S$, and in particular $U^*_S \neq \emptyset$. Because there are $2^{|S|-1}-1$ ways to partition S into non-empty coalitions,[18] it seems than the verification of EPR is a difficult task. Fortunately, the existence of preferences satisfying EPR is not difficult to verify: the fact that Paretian preferences have utility cones $U^p_S = Co(U_i; i \in S)$ confirms (6) and (7).

One would be interested in discovering the kind of complete preferences that may be consistent with EPR. For example, if $(\lambda_1; \ldots; \lambda_n) > 0$ are positive weights, then the collection of complete preferences given by $u_S = \sum_{i\in S} \lambda_i u_{\bar{i}}$, for all $S \subseteq N$, satisfies EPR (see proof of Theorem 12 below). In fact, we anticipate that the reverse implication holds as a corollary of Theorem 12: any collection of complete preferences satisfying EPR is characterized by weights $(\lambda_1; \ldots; \lambda_n) > 0$ such that $u_S = \sum_{i\in S} \lambda_i u_{\bar{i}}$, for all $S \subseteq N$.

3.1 Two Agents and Utility Comparison Rates

The use of Proposition 4 in this paper is confined to the case of complete preferences. The following Corollary comes to no surprise considering (7).

Corollary 5 Let A and B be two disjoint coalitions, and for $S \in \{A; B; A \cup B\}$, assume that $\succsim_S$ is complete and has utility u_S. Then EPR holds if and only if for some $0 < \alpha < 1$ and $\lambda > 0$, $u_{A\cup B} = \lambda[(1-\alpha)u_A + \alpha u_B]$.

A complete preference for a pair $\overline{ij}$ is called a bilateral agreement. By setting $\lambda = 1$ in Corollary 5 we have that a bilateral agreement $\succsim_{\overline{ij}}$ has utility $u_{\overline{12}} = (1-\alpha_{1;2})u_{\bar{1}} + \alpha_{1;2}u_{\bar{2}}$, for some $0 < \alpha_{1;2} < 1$. Letting $\delta_{1;2} \equiv \alpha_{1;2}/(1-\alpha_{1;2})$ we shall prefer

[18] There are $2^{|S|}-2$ ways of choosing a non-empty proper coalition $A \subset S$ but the partition $\{A; S\backslash A\}$ is the same as $\{S\backslash A; A\}$.

to write

$$\bar{u}_{12} = (\bar{u}_1 + \delta_{1;2}\bar{u}_2)/(1 + \delta_{1;2}). \tag{9}$$

If $d(\bar{u}_1; \bar{u}_2) = 2$, i.e., $\bar{u}_1$ and $\bar{u}_2$ are linearly independent, then $\delta_{1;2}$ is unique and has the natural interpretation of a "utility comparison rate" between 1 and 2. To elicit $\delta_{1;2}$ we find prospects x and y such that $x \succ_i y$, for $i \in \{1; 2\}$. Suppose that individuals 1 (me) and 2 (you) are able to meaningfully say: "To realize prospect x in place of y is θ times more valuable for you than it is for me." For the units of utility to reflect this comparison we need to re-scale the utilities using $\lambda_1; \lambda_2 > 0$ so that $\lambda_1[\bar{u}_1(x) - \bar{u}_1(y)]$ is expressed in the same units as $\lambda_2\theta[\bar{u}_2(x) - \bar{u}_2(y)]$. Expression (9) renders $\delta_{1;2}$ as the relative scaling λ_2/λ_1, or

$$\delta_{1;2} = \theta[\bar{u}_1(x) - \bar{u}_1(y)]/[\bar{u}_2(x) - \bar{u}_2(y)].$$

The linearity of $\bar{u}_{12}$ implies that the same $\delta_{1;2}$ would be found if two other prospects were used to elicit the terms of utility comparison.[19] This has normative appeal if $\bar{u}_i(x) - \bar{u}_i(y)$ measures i's intensity of preference. Because 1 and 2 have established comparison of intensities of preference, the same terms of comparison should arise regardless of the prospects used in the elicitation. The order of agents is important in finding $\delta_{1;2}$: if we switch the agents (1 is you and 2 is me), then (9) produces $\bar{u}_{12} = (\bar{u}_2 + \frac{1}{\delta_{1;2}}\bar{u}_1)/(1 + \frac{1}{\delta_{1;2}})$ and $\delta_{2;1} = 1/\delta_{1;2}$.

Henceforth, we use $\delta_{i;j}$ ($\delta_{j;i} = 1/\delta_{i;j}$) to denote the utility comparison rate associated with a bilateral agreement between agents i and j (j and i). The choice of $\delta_{i;j}$ is exogenous in our model. The selection of $\delta_{i;j} = 1$ should not be associated with a "fair" or symmetric pair agreement. If we re-scale the individual utilities so that $\hat{\bar{u}}_i = \lambda_i\bar{u}_i$ and $\hat{\bar{u}}_j = \lambda_j\bar{u}_j$, for some $\lambda_i; \lambda_j > 0$, a given bilateral agreement having $\delta_{i;j}$ when using $\bar{u}_i$ and $\bar{u}_j$ now exhibits $\hat{\delta}_{i;j} = \delta_{i;j}\lambda_i/\lambda_j$: the magnitude of $\delta_{i;j}$ is not cardinal invariant. This fact does not preclude the agents from resorting to some normative model to find $\delta_{i;j}$.[20]

[19] Thus, if for prospects z and w, $\delta_{1;2}[\bar{u}_2(z) - \bar{u}_2(w)]/[\bar{u}_1(z) - \bar{u}_1(w)] = \theta' > 0$, then agents should express: "To realize prospect z in place of w is θ' times more important for you (2) than it is for me (1)."

[20] For example, if the pairs use Relative Utilitarianism (Dhillon and Mertens 1999) to determine a "fair" agreement, then the set M fixes $\delta_{i;j}$: the utility difference between the most preferred prospect

If one relaxes the assumption that $\succsim_{\overline{ij}}$ is complete, then we find that an incomplete pair agreement is characterized by an interval $[\delta^{\ell}_{i;j}; \delta^{h}_{i;j}]$ of utility comparison rates (see Baucells and Shapley 1999).

3.2 Three Agents and "No Arbitrage" in Utility Comparison Rates

Proposition 4 will allow us to visualize the restrictions that EPR imposes on preferences when $n = 3$. Assume $m = 3$, $d(u_{\overline{1}}; u_{\overline{2}}; u_{\overline{3}}) = 3$, and let W be the affine plane in R^3 containing the points $u_{\overline{1}}$, $u_{\overline{2}}$, and $u_{\overline{3}}$. These points can be pictured as the intersection of W with the rays $U_{\overline{i}}$. For $i \neq j$, let $\succ_{\overline{ij}}$ be some incomplete preference: $U_{\overline{ij}} \subseteq Co(U_{\overline{i}} \cup U_{\overline{j}})$ in (6) implies that the intersection of the utility cone $U_{\overline{ij}}$ with W is a closed line segment contained in $u_{\overline{i}}u_{\overline{j}}$, the line segment between $u_{\overline{i}}$ and $u_{\overline{j}}$. In Figure 2 we abuse notation and use U_S to indicate such intersections.

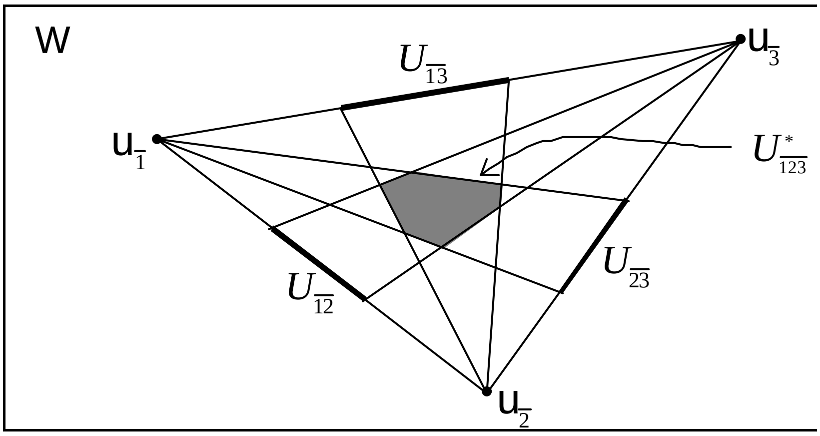

Figure 2: Geometrical illustration of EPR.

By applying EPR to all the partitions of $\overline{123}$ we obtain

$$U_{\overline{123}} \subseteq U^*_{\overline{123}} \equiv Co(U_{\overline{1}} \cup U_{\overline{23}}) \cap Co(U_{\overline{2}} \cup U_{\overline{13}}) \cap Co(U_{\overline{3}} \cup U_{\overline{12}}): \qquad (10)$$

---

and least preferred alternative should be the same for every individual, or

$$\delta_{i;j} = [\max_{x \in M} u_{\overline{i}}(x) - \min_{x \in M} u_{\overline{i}}(x)]/[\max_{x \in M} u_{\overline{j}}(x) - \min_{x \in M} u_{\overline{j}}(x)]:$$

If the pair preferences coincide with the Paretian preference, then $U^{p}_{\overline{ij}}$ \ W = $u_{\overline{i}}u_{\overline{j}}$ and (10) does not restrict $U_{\overline{123}}$. However, if the pair preferences are more complete, then $U_{\overline{ij}}$ \ W is strictly contained in $u_{\overline{i}}u_{\overline{j}}$ and (10) begins to be very e¤ective in restricting $U^{¤}_{\overline{123}}$, and hence $\&_{\overline{123}}$.[21] In particular, if $U^{¤}_{\overline{123}}$ \ W were a point, then $\&_{\overline{123}}$ would necessarily be complete. But notice that this is the case if $\&_{\overline{12}}$ and $\&_{\overline{23}}$ are complete with utilities $u_{\overline{12}}$ and $u_{\overline{23}}$, i.e., (10) singles out a point $u_{\overline{123}}$ as the intersection of $u_{\overline{1}}u_{\overline{23}}$ and $u_{\overline{12}}u_{\overline{3}}$: a complete preference $\&_{\overline{123}}$ with utility $u_{\overline{123}}$ emerges from two bilateral agreements. Figure 3 below illustrates this fact, and it also reveals that $U_{\overline{13}}$ has to include the intersection $u_{\overline{13}}$ of the line segment $u_{\overline{1}}u_{\overline{3}}$ and the line $u_{\overline{2}}u_{\overline{123}}$; otherwise $U_{\overline{123}}$ µ Co($U_{\overline{2}}$ [$U_{\overline{13}}$) fails. Thus, if $\&_{\overline{13}}$ is complete, then there is a unique utility candidate, namely $u_{\overline{13}}$.[22]

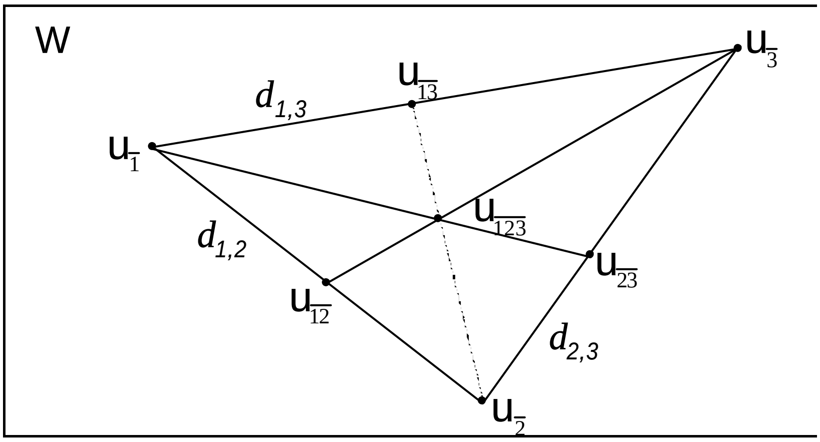

Figure 3: $\%_{\overline{123}}$ is complete and $\pm_{1;3} = \pm_{1;2}\pm_{2;3}$.

The utilities $u_{\overline{123}}$ and $u_{\overline{13}}$ that stem from this geometrical construction have an interesting interpretation as a "no arbitrage" condition in the utility comparison rates. Let $\pm_{1;2}$ and $\pm_{2;3}$ be the utility comparison rates of the bilateral agreements $\%_{\overline{12}}$ and $\%_{\overline{23}}$, respectively. If receiving prospect x in place of y is $\pm_{1;2}$ times more important for 2 than it is for 1, and $\pm_{2;3}$ times more important for 3 than it is for 2,

[21] Note that certain precautions are needed: no preference $\%_{\overline{123}}$ is consistent with EPR if $U^{¤}_{\overline{123}}$ is empty. Clearly, in Figure 2 one can choose $U_{\overline{ij}}$ so that $U^{¤}_{\overline{123}}$ = ;.

[22] Figure 3 also illustrates that EPR needs to be imposed on disjoint coalitions: $u_{\overline{123}}$ does not lie in the line segment $u_{\overline{13}}u_{\overline{23}}$. This point was brought to our attention by Bill Zame.

then it should be $\delta_{1;2}\delta_{2;3}$ times more important for 3 than it is for 1. Similar to "no arbitrage" in currency exchange rates, the natural utility comparison rate between 1 and 3 is $\delta_{1;3} = \delta_{1;2}\delta_{2;3}$. If $\delta_{1;3} < \delta_{1;2}\delta_{2;3}$, then agent 3 will prefer to communicate with 1 via 2; on the contrary, if $\delta_{1;3} > \delta_{1;2}\delta_{2;3}$, then agent 3 will prefer to communicate with 1 directly. When equality holds, any communication channel between any two individuals is acceptable.

We present a formal treatment to the previous discussion that uses a condition weaker than $d(u_{\overline{1}}; u_{\overline{2}}; u_{\overline{3}}) = 3$.

Lemma 6 Assume that EPR holds, and consider bilateral agreements $\&_{\overline{12}}$ and $\&_{\overline{23}}$ with utility comparison rates $\delta_{1;2}$ and $\delta_{2;3}$ that determine $u_{\overline{12}}$ and $u_{\overline{23}}$. Let $\delta_{1;3} \equiv \delta_{1;2}\delta_{2;3}$, $u_{\overline{123}} \equiv (u_{\overline{1}} + \delta_{1;2}u_{\overline{2}} + \delta_{1;3}u_{\overline{3}})/(1 + \delta_{1;2} + \delta_{1;3})$, and $u_{\overline{13}} \equiv (u_{\overline{1}} + \delta_{1;3}u_{\overline{3}})/(1 + \delta_{1;3})$.

a: If $u_{\overline{1}} \notin Sp(u_{\overline{3}}; u_{\overline{12}})$ and $u_{\overline{3}} \notin Sp(u_{\overline{1}}; u_{\overline{23}})$, then $\&_{\overline{123}}$ is complete and has utility $u_{\overline{123}}$.

b: If $\&_{\overline{123}}$ is complete with utility $u_{\overline{123}}$, and $u_{\overline{2}} \notin Sp(u_{\overline{1}}; u_{\overline{3}})$, then $u_{\overline{13}} \in U_{13}$, i.e., if $\&_{\overline{13}}$ is complete, then it has utility $u_{\overline{13}}$.

Proof. (a) By (6), for any $0 \neq u^* \in U_{\overline{123}}$ there are $\alpha; \beta \in [0; 1]$ and $\lambda_\alpha; \lambda_\beta \geq 0$ such that

$$\lambda_\alpha[(1 - \alpha)u_{\overline{1}} + \alpha u_{\overline{23}}] = u^* = \lambda_\beta[(1 - \beta)u_{\overline{12}} + \beta u_{\overline{3}}]: \qquad (11)$$

From the definitions of $u_{\overline{12}}$ and $u_{\overline{23}}$, $u_{\overline{12}} = \{u_{\overline{1}} + \delta_{1;2}[u_{\overline{23}} + \delta_{2;3}(u_{\overline{23}} - u_{\overline{3}})]\}/(1 + \delta_{1;2})$. Substituting this expression in the right-hand side of (11) produces an expression involving only $u_{\overline{1}}$, $u_{\overline{23}}$, $u_{\overline{3}}$. Because $u_{\overline{3}} \notin Sp(u_{\overline{1}}; u_{\overline{23}})$ (in particular $u_3 \neq 0$), we equate the coefficients of $u_{\overline{3}}$ in the modified expression (11) to conclude that $\beta = \delta_{1;3}/(1 + \delta_{1;2} + \delta_{1;3})$. Replacing β and $u_{\overline{12}} = (u_{\overline{1}} + \delta_{1;2}u_{\overline{2}})/(1 + \delta_{1;2})$ in the right-hand side of (11) yields $u^* = \lambda_\beta u_{\overline{123}}$. Thus, either $\&_{\overline{123}}$ is trivial or it has utility $u_{\overline{123}}$, but (7) excludes triviality and (a) follows.

(b) Let $\%_{\overline{13}}$ be complete with utility u^{**}. By Corollary 5, there is some $\alpha' \in (0; 1)$ and $\lambda'_\alpha > 0$ such that $u_{\overline{123}} = \lambda'_\alpha[(1 - \alpha')u_{\overline{2}} + \alpha' u^{**}]$; similarly, some $\beta' \in (0; 1)$ and $\lambda'_\beta > 0$ such that $u^{**} = \lambda'_\beta[(1 - \beta')u_{\overline{1}} + \beta' u_{\overline{3}}]$. Thus,

$$\frac{u_{\overline{1}} + \delta_{1;2}u_{\overline{2}} + \delta_{1;3}u_{\overline{3}}}{1 + \delta_{1;2} + \delta_{1;3}} = u_{\overline{123}} = \lambda'_\alpha[(1 - \alpha')u_{\overline{2}} + \alpha'\lambda'_\beta[(1 - \beta')u_{\overline{1}} + \beta' u_{\overline{3}}]]: \qquad (12)$$

Because $u_{\bar{2}} \in Sp(u_{\bar{1}}, u_{\bar{3}})$, either $\lambda'_{\alpha}(1 - \alpha') = \delta_{1;2}/(1 + \delta_{1;2} + \delta_{1;3})$ or $\beta'/(1 - \beta') = \delta_{1;3}$. In the first case, (12) implies $\alpha' u^{**} = u_{\overline{13}}$, and in the second case, $u^{**} = \lambda'_{\beta}[(1 - \beta')u_{\bar{1}} + \beta' u_{\bar{3}}] = \lambda'_{\beta} u_{\overline{13}}$. Because $\lambda'_{\alpha}, \lambda'_{\beta} > 0$, if $\succsim_{\overline{13}}$ is complete, then it has utility $u_{\overline{13}}$. Upon reflection, this is equivalent to $u_{\overline{13}} \in U_{\overline{13}}$. ■

To generalize the findings of Lemma 6 to more than three agents entails establishing at least one communication channel between each pair of agents. Moreover, the "no arbitrage" condition indicates that a chain of bilateral agreements that "cycles" (starts and finishes in the same agent) contains redundancies. If we view the agents as the nodes of a graph and the bilateral agreements as the edges, then these two conditions express that the bilateral agreements form a connected and acyclic graph, i.e., a spanning tree.

3.3 Four Agents and Desargues's Theorem

The case of four players permits the geometrical illustration of the proposed construction and reveals one unexpected difficulty. In the example of Figure 3, consider a fourth agent, which for illustration purposes has $u_{\bar{4}} \in W$ and $d(u_{\bar{2}}, u_{\bar{3}}, u_{\bar{4}}) = 3$ (see Figure 4). Let $T = \{\overline{12}, \overline{23}, \overline{34}\}$ be the spanning tree and add the bilateral agreement $\succsim_{\overline{34}}$. The application of Lemma 6a using $u_{\bar{2}}, u_{\bar{3}}, u_{\bar{4}}, \delta_{2;3}$, and $\delta_{3;4}$ produces a complete preference $\succsim_{\overline{234}}$, with $u_{\overline{234}}$ as the intersection of $u_{\bar{2}}u_{\overline{34}}$ and $u_{\overline{23}}u_{\bar{4}}$. Because we have complete preferences for $\succsim_{\overline{123}}$, we obtain a complete preference $\succsim_{\overline{1234}}$ with $u_{\overline{1234}}$ given by the intersection of $u_{\overline{123}}u_{\bar{4}}$ and $u_{\bar{1}}u_{\overline{234}}$. However, there is a third segment available, namely $u_{\overline{12}}u_{\overline{34}}$. Moreover, the two applications of Lemma 6b yield $u_{\overline{13}} \in U_{\overline{13}}$ and $u_{\overline{24}} \in U_{\overline{24}}$: if $\succsim_{\overline{13}}$ and $\succsim_{\overline{24}}$ were complete, then the segment $u_{\overline{13}}u_{\overline{24}}$ would also be available. It is impossible to have consistent and complete preferences unless these four segments are concurrent, i.e., they have a common point of intersection. This difficulty can be addressed in geometric terms by means of Desargues's theorem.

Theorem 7 (Desargues 1648) Let p_i and q_i, for $i = 1, 2, 3$ be two sets of independent points in a vector space satisfying $p_i \neq q_i$ $(i = 1, 2, 3)$. Then, the segments $p_i q_i$, $i = 1, 2, 3$ are concurrent if and only if the three points $s_{ij} = p_i p_j \cap q_i q_j$, $1 \le i < j \le 3$ are collinear.

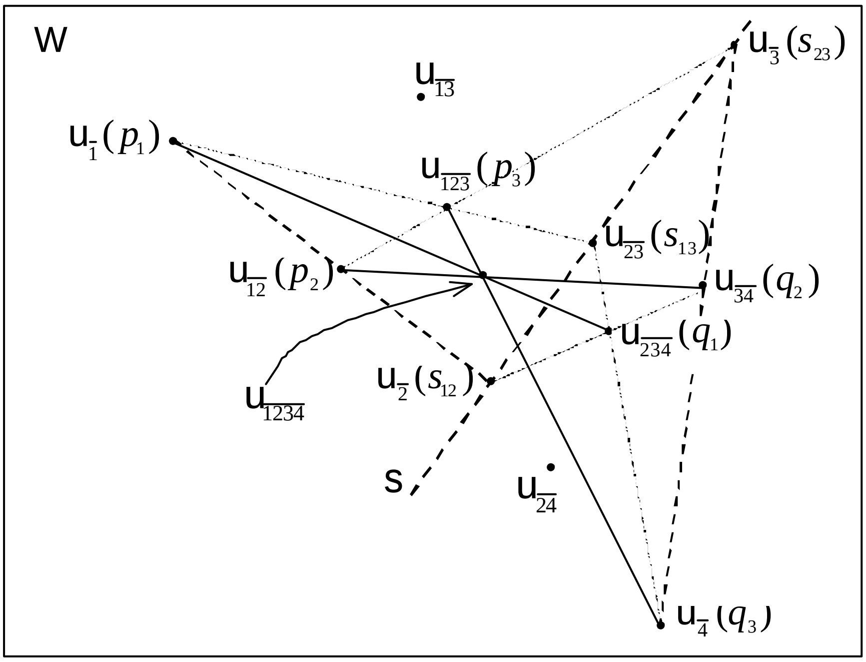

Figure 4: The Desargues's theorem.

Figure 4 illustrates Desargues's theorem as applied to

$$\left.\begin{array}{lll} p_1 = u_{\bar{1}} & p_2 = u_{\overline{12}} & p_3 = u_{\overline{123}} \\ q_1 = u_{\overline{234}} & q_2 = u_{\overline{34}} & q_3 = u_{\bar{4}} \end{array}\right\} \Rightarrow \quad s_{12} = u_{\bar{2}} \quad s_{13} = u_{\overline{23}} \quad s_{23} = u_{\bar{3}}$$

By EPR, $s_{13} \in s_{12}s_{23}$ so that the line segments $u_{\bar{1}}u_{\overline{234}}$, $u_{\overline{12}}u_{\overline{34}}$, and $u_{\overline{123}}u_{\bar{4}}$ are concurrent: $u_{\overline{1234}}$ is well defined. To see that $u_{\overline{1234}} \in u_{\overline{13}}u_{\overline{24}}$, declare $p_2' = u_{\overline{13}}$ and $q_2' = u_{\overline{24}}$, and maintain the other four points. The desired conclusion follows from $s_{13}' = u_{\overline{23}} \in u_{\bar{3}}u_{\bar{2}} = s_{12}'s_{23}'$.

Consider the following generalization of Lemma 6.

Lemma 8 For some collection of weights $(\lambda_1; \ldots; \lambda_n) > 0$, consider the utility functions $u_T = (\sum_{i \in T} \lambda_i u_{\bar{i}}) / (\sum_{i \in T} \lambda_i)$, for $T \subseteq N$. Let $A; B; C$ be disjoint coalitions and $S = A \cup B \cup C$. The following are consequences of the EPR.

a: Suppose $u_A \notin Sp(u_C; u_{A \cup B})$ and $u_C \notin Sp(u_A; u_{B \cup C})$. For $T \in \{A; C; A \cup C; B \cup C\}$, if $\succsim_T$ is complete and has utility u_T, then $\succsim_S$ is complete and has utility u_S.

b: Suppose $u_B \notin Sp(u_A; u_C)$. For $T \in \{A; B; C; S\}$, if $\succsim_T$ is complete and has utility u_T, then $u_{A \cup C} \in U_{A \cup C}$, i.e., if $\succsim_{A \cup C}$ is complete, then it has utility $u_{A \cup C}$.

Proof. To see that the utility comparison rates between two disjoint coalitions A and B is $\delta_{A;B} \equiv (\sum_{i\in B} \lambda_i)/(\sum_{i\in A} \lambda_i)$, consider

$$u_{A\cup B} = \frac{\sum_{i\in A} \lambda_i u_{\bar{i}} + \sum_{i\in B} \lambda_i u_{\bar{i}}}{\sum_{i\in A\cup B} \lambda_i} = \frac{\left(\sum_{i\in A} \lambda_i\right) u_A + \left(\sum_{i\in B} \lambda_i\right) u_B}{\sum_{i\in A\cup B} \lambda_i} = \frac{u_A + \delta_{A;B} u_B}{1+\delta_{A;B}}. \tag{13}$$

The result then follows from Lemma 6 by using u_A, u_B, u_C, $\delta_{A;B}$, and $\delta_{B;C}$ in place of $u_{\bar{1}}$, $u_{\bar{2}}$, $u_{\bar{3}}$, $\delta_{1;2}$, and $\delta_{2;3}$; and checking that $(u_A + \delta_{A;B} u_B + \delta_{A;C} u_C)/(1+\delta_{A;B}+\delta_{A;C}) = (\sum_{i\in S} \lambda_i u_{\bar{i}})/(\sum_{i\in S} \lambda_i) = u_S$. ∎

Desargues's theorem follows from Lemma 8: the two intersection points $u_{\overline{123}}u_{\bar{4}} \cap u_{\bar{1}}u_{\overline{234}}$ and $u_{\overline{123}}u_{\bar{4}} \cap u_{\overline{12}}u_{\overline{34}}$ result from applying Lemma 8a to the partitions $\{\bar{1}; \overline{23}; \bar{4}\}$ and $\{\overline{12}; \bar{3}; \bar{4}\}$, respectively. Because we can utilize the weights $\lambda_1 = 1$, $\lambda_2 = \delta_{1;2}$, $\lambda_3 = \delta_{1;2}\delta_{2;3}$, and $\lambda_4 = \delta_{1;2}\delta_{2;3}\delta_{3;4}$ in both applications, the same answer obtains: $u_{\overline{1234}} = (\sum_{i=1}^4 \lambda_i u_{\bar{i}})/(\sum_{i=1}^4 \lambda_i)$. In geometry, a true proposition is obtained if we interchange the roles of lines and points, and concurrency and collinearity. Because the Desargues's theorem is self-dual, the converse automatically holds.[23]

The Desargues's theorem is by no means restricted to coplanar points and lines. To "see" why Desargues's theorem holds in higher dimensions, consider a point u in $\mathbb{R}^m$, and three lines ℓ_i, $i \in \{1;2;3\}$ concurrent to u. Let P and Q be two planes, each intersecting the three lines at points p_i and q_i, $i \in \{1;2;3\}$, respectively. Soon we realize that the points s_{ij}, $1 \le i < j \le 3$, belong to the line s of intersection of P and Q, i.e., they are collinear. This can be seen in Figure 4 by conceiving three dimensions and letting $u = u_{\overline{1234}}$.

4 The case of n Agents: A Utility Comparison System

We now proceed to introduce some definitions in order to generalize Lemma 6 to the case of n agents. An (undirected) graph is pair $(N;G)$, where G is a collection of two-member coalitions of N. If $\overline{ij} \in G$, then we say that i is adjacent to j in $(N;G)$. Agents

[23] For the use of Desargues's theorem with larger coalitions, consider three lines given by $u_{A_i}u_{S\setminus A_i}$, $i \in \{1;2;3\}$. For $1 \le i < j \le 3$, suppose that $A_i \subset A_j$, and that coalitions A_i, $S\setminus A_i$, $A_j\setminus A_i$ have complete preferences: letting $p_i = A_i$ and $q_i = S\setminus A_i$, produces $s_{ij} = A_j\setminus A_i$. $s_{13} \in s_{12}s_{23}$ follows from $A_3\setminus A_1 = (A_2\setminus A_1) \dot{\cup} (A_3\setminus A_2)$ and EPR.

i and j are connected in $(N; G)$ if there is a sequence of agents $(i = i_1; i_2; \ldots; i_k = j)$ in N such that $\overline{i_r i_{r+1}} \in G$ for every $r \in \{1; \ldots; k-1\}$. Any such sequence is called a path in $(N; G)$. We use T instead of G whenever T is a spanning tree of N. T is a spanning tree of N if and only if there is a unique path in $(N; T)$ connecting any two agents in N. It follows that T contains precisely $n - 1$ pairs.

Equipped with a spanning tree T, and the respective utility comparison rates between pairs in T, we propose the appropriate weights to determine the utilities for coalitions. We chose an arbitrary base agent, say $i = 1$, as the "root" of the tree. Define $\lambda_1 \equiv 1$, and for $j \neq 1$, if $(1 = i_1; i_2; \ldots; i_k = j)$ is the unique path between 1 and j in T, then let

$$\lambda_j \equiv \prod_{r=1}^{k} \delta_{i_{r-1}, i_r}. \tag{14}$$

A moment's reflection reveals that a different choice of base agent, say $i^* \neq 1$, would produce weights $\delta_{i^*;1} \lambda_j$, $j \in N$. Because the utility representation of $\succsim_S$ that we are seeking is $u_S \equiv \left(\sum_{i \in S} \lambda_i u_{\bar{i}}\right) / \left(\sum_{i \in S} \lambda_i\right)$, the choice of base agent is immaterial.

4.1 Complete Preferences for Connected Coalitions

In the example with 4 agent we derive complete preferences for certain coalitions with a definite property: given an spanning tree T of N, we say that S is connected in T if $T_S \equiv \{\overline{ij} \in T : i; j \in S\}$ is a spanning tree of S. Let C denote the collection of connected coalitions in T. Singleton coalitions are always connected; a pair $\overline{ij}$ is connected if and only if $\overline{ij} \in T$. More importantly, the grand coalition N is always connected.[24] An agent $i \in S$ is terminal in a connected coalition S if there is only one $j \in S$ such that $\overline{ij} \in T_S$.

We say that T is non-degenerate if $d(u_{\bar{i}}; u_{\bar{j}}; u_{\bar{k}}) = 3$ for any $\overline{ijk} \in C$;[25] and N is

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[24] The number of connected coalitions will depend on the form of T. Consider the two extreme examples: a line tree $T^{\ell} = \{\overline{i-1; i} : i = 2; \ldots; n\}$, and a star tree $T^{*} = \{\overline{1; i} : i = 2; \ldots; n\}$. In T^{ℓ} we count $n - k + 1$ connected coalitions of size k that gather a total of $n(n+1)/2$ connected coalitions, which is small with respect to 2^n, the total number of coalitions. In T^{*} there are n connected coalitions of size 1 and $\binom{n-1}{k-1}$ connected coalitions of size $k > 1$. Because $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, the total number of connected coalitions is $2^{n-1} + n - 1$; the fraction of connected coalitions tends to $1/2$ as n increases.

The number of spanning trees on a set of n agents, by Cayley's formula, is n^{n-2}.

[25] In Figure 4, for example, we could encompass the case where $u_{\bar{1}} = u_{\bar{4}}$.

non-degenerate if $d(u_{\bar{i}}; u_{\bar{j}}; u_{\bar{k}}) = 3$ for any $\overline{ijk} \subseteq N$. Of course, if N is non-degenerate, then so is any spanning tree of N. Note that we need $m \geq 3$ to have non-degeneracy, and that for any such m, non-degeneracy is a "generic" property.

Theorem 9 Let T be a non-degenerate spanning tree of bilateral agreements and let $(\lambda_1; \ldots; \lambda_n)$ be given as in (14). If the Extended Pareto Rule holds, then for all $S \in \mathcal{C}$, $\succsim_S$ is complete and has utility

$$u_S \equiv \left(\sum_{i\in S} \lambda_i u_i\right) \Big/ \left(\sum_{i\in S} \lambda_i\right). \qquad (15)$$

Proof. If $\mathcal{C}$ is the collection of connected coalitions in T, let $\mathcal{C}_r$ indicate the connected coalition of size r. We claim that for $r \geq 3$ and $S \in \mathcal{C}_r$, then there is a partition $\{A; B; C\}$ of S such that $\{A; C; A \cup B; B \cup C\} \subseteq \mathcal{C}$ and

$$u_A \notin Sp(u_C; u_{A\cup B}) \text{ and } u_C \notin Sp(u_A; u_{B\cup C}). \qquad (16)$$

The result easily follows from the claim: If $S \in \mathcal{C}_3$, then the partition of S given by the claim has its elements in $\mathcal{C}_1 \cup \mathcal{C}_2$. Because $\succsim_T$ is complete and has utility u_T for all $T \in \mathcal{C}_1 \cup \mathcal{C}_2$, Lemma 8a establishes this property for $\succsim_S$. Similarly, once this is established for all $T \in \mathcal{C}_\ell$, $\ell < r$, then it also holds for all $S \in \mathcal{C}_r$; the partition $\{A; B; C\}$ of S given by the claim has its members in $\mathcal{C}_\ell$, $\ell < r$, and (16) allow us to apply Lemma 8a.

We establish the claim by induction. For $r = 3$, let $\overline{ijk} \in \mathcal{C}_3$ and define the partition $\{A; B; C\} = \{\bar{i}; \bar{j}; \bar{k}\}$ of $\overline{ijk}$, so that $\{\bar{i}; \bar{k}; \overline{ij}; \overline{jk}\} \subseteq \mathcal{C}$. T non-degenerate guarantees (16).

For $r \geq 4$, assume that the claim is true for all the coalitions in $\mathcal{C}_\ell$, $\ell < r$. If degeneracies were not a problem, the proof would be as follows. If $S \in \mathcal{C}_r$ and $i \in S$ is a terminal node of S, then $S \setminus \bar{i} \in \mathcal{C}_{r-1}$. Let $\{\tilde{A}; \tilde{B}; \tilde{C}\}$ be the partition of $S\setminus\bar{i}$ given by induction, and j the unique adjacent of i in $S\setminus\bar{i}$. The partition $\{A; B; C\}$ of S is defined as follows: if $j \in \tilde{A}$, then use $\{\tilde{A} \cup \bar{i}; \tilde{B}; \tilde{C}\}$; if $j \in \tilde{B}$, then use $\{\tilde{A}; \tilde{B} \cup \bar{i}; \tilde{C}\}$; and if $j \in \tilde{C}$, then use $\{\tilde{A}; \tilde{B}; \tilde{C} \cup \bar{i}\}$. One observes that $\{A; C; A \cup B; B \cup C\} \subseteq \mathcal{C}$ in all three cases. However, Condition (16) may fail if $d(u_A; u_B; u_C) < 3$. The remedy consist of first replacing the terminal node i by a connected coalition $R \in \mathcal{C}_1 \cup \mathcal{C}_2$ such that $S\setminus R \in \mathcal{C}$ and $u_{S\setminus R} \neq 0$. If $\{\tilde{A}; \tilde{B}; \tilde{C}\}$ is the partition of $S\setminus R$ given by induction, we ensure Condition (16) by choosing which two coalitions to "glue" from $\{\tilde A; \tilde B; \tilde C; R\}$ to produce the partition $\{A; B; C\}$ of S.

To find R, let j be the node with a maximal number $t(j)$ of terminal adjacent nodes in $\mathcal{C}_r$. If $t(j) = 1$, then let i be this terminal node and define $R = \bar i$ if $u_{S\setminus\bar i} \neq 0$, and $R = \overline{ij}$ otherwise (because $\lambda_j u_{\bar j} \neq 0$, $u_{S\setminus\overline{ij}} \neq 0$). If $t(j) \geq 2$, let i and k be two terminal adjacent nodes of j and define $R = \bar i$ if $u_{S\setminus\bar i} \neq 0$, and $R = \bar k$ otherwise (if $u_{S\setminus\bar i} = 0$, then $d(u_{\bar i}; u_{\bar j}; u_{\bar k}) = 3$ implies $u_{S\setminus\bar k} \neq 0$). Thus, $R \in \mathcal{C}$, $S\setminus R \in \mathcal{C}_\ell$ for some $\ell \geq 3$ (note that when $r = 4$, a non-degenerate T guarantees that $R = \bar i$ and $S\setminus R \in \mathcal{C}_3$), and $u_{S\setminus R} \neq 0$. By induction, let $\{\tilde A; \tilde B; \tilde C\}$ be the partition of $S\setminus R$ satisfying the claim. Because of the symmetric role of $\tilde A$ and $\tilde C$, we can assume without loss of generality that either $R \cup \tilde A \in \mathcal{C}$ or $R \cup \tilde B \in \mathcal{C}$ and define the partition $\{A; B; C\}$ of S as follows:

$R \cup \tilde A \in \mathcal{C}$	A	B	C	Case
(a1)	R	$\tilde A$	$\tilde B \cup \tilde C$	if $u_{\tilde C} \in Sp(u_{\tilde A\cup R}; u_{\tilde B\cup\tilde C})$
(a2)	R	$\tilde A \cup \tilde B$	$\tilde C$	if $u_{\tilde A\cup R} \in Sp(u_{\tilde C}; u_{\tilde A\cup\tilde B\cup R})$
(a3)	$\tilde C$	$\tilde B$	$\tilde A \cup R$	otherwise.

$R \cup \tilde B \in \mathcal{C}$	A	B	C	Case
(b1)	R	$\tilde B \cup \tilde C$	$\tilde A$	if $u_{\tilde C} \in Sp(u_{\tilde A}; u_{\tilde B\cup\tilde C\cup R})$
(b2)	R	$\tilde A \cup \tilde B$	$\tilde C$	if $u_{\tilde A} \in Sp(u_{\tilde C}; u_{\tilde A\cup\tilde B\cup R})$
(b3)	$\tilde A$	$\tilde B \cup R$	$\tilde C$	otherwise.

Upon examination one confirms that $\{A; C; A \cup B; B \cup C\} \subseteq \mathcal{C}$ holds in all six cases. By construction, (16) holds in cases (a3) and (b3). We now give the details showing that (16) holds in (a1), i.e., that $u_R \notin Sp(u_{\tilde B\cup\tilde C}; u_{\tilde A\cup R})$ and $u_{\tilde B\cup\tilde C} \notin Sp(u_R; u_{S\setminus R})$. The cases (a2), (b1) and (b2) are a repetition of the same arguments. Recall that by the inductive hypotheses given by (16), both $u_{\tilde A} \notin Sp(u_{\tilde C}; u_{\tilde A\cup\tilde B})$ and $u_{\tilde C} \notin Sp(u_{\tilde A}; u_{\tilde B\cup\tilde C})$.

If (a1) applies, then $u_{\tilde C} \in Sp(u_{\tilde A\cup R}; u_{\tilde B\cup\tilde C})$ (see Figure 5), and so $u_{\tilde C} = \alpha u_{\tilde A\cup R} + \beta u_{\tilde B\cup\tilde C}$ for some α and β. That $u_{\tilde C} \notin Sp(u_{\tilde A}; u_{\tilde B\cup\tilde C})$ rules out $\alpha = 0$, and using $u_{\tilde A\cup R} = (u_R + \delta_{R;\tilde A} u_{\tilde A})/(1 + \delta_{R;\tilde A})$ as in (13) we write

$$u_R = (1 + \delta_{R;\tilde A})(u_{\tilde C} - \beta u_{\tilde B\cup\tilde C})/\alpha - \delta_{R;\tilde A} u_{\tilde A}. \quad (17)$$

Also, $u_{\tilde A} \notin Sp(u_{\tilde C}; u_{\tilde A\cup\tilde B})$ is incompatible with $u_{\tilde B\cup\tilde C} = -\gamma u_{\tilde A}$, for some $\gamma \geq 0$. Otherwise, we write $(1+\delta_{\tilde A;\tilde B\cup\tilde C})u_{S\setminus R} = u_{\tilde A}+\delta_{\tilde A;\tilde B\cup\tilde C}u_{\tilde B\cup\tilde C} = u_{\tilde A}(1 - \gamma\delta_{\tilde A;\tilde B\cup\tilde C})$. $u_{S\setminus R} \neq 0$

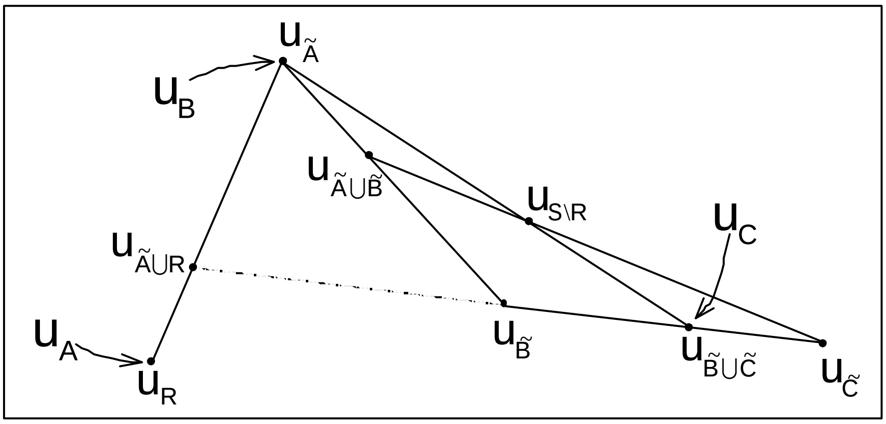

Figure 5: Identify $A = R$, $B = \tilde A$, and $C = \tilde B \cup \tilde C$ when $u_C \in Sp(u_{\tilde A\cup R}; u_{\tilde B\cup \tilde C})$.

implies $\gamma\delta_{\tilde A;\tilde B\cup\tilde C} \neq 1$ and $u_{\tilde A} \in Sp(u_{S\setminus R})$. This, coupled with $U_{S\setminus R} \subseteq Co(u_C; u_{\tilde A\cup\tilde B})$ contradicts $u_{\tilde A} \notin Sp(u_C; u_{\tilde A\cup\tilde B})$.

Now, assume that $u_R \in Sp(u_{\tilde B\cup\tilde C}; u_{\tilde A\cup R})$ so that $u_R = \alpha_1 u_{\tilde B\cup\tilde C} + \beta_1(u_R + \delta_{R;\tilde A}u_{\tilde A})$ for some $\alpha_1; \beta_1$. If $\beta_1 \neq 1$, then use (17) to eliminate u_R and obtain $u_C \notin Sp(u_{\tilde A}; u_{\tilde B\cup\tilde C})$, a contradiction. If $\beta_1 = 1$, then $u_{\tilde B\cup\tilde C} = -\delta_{R;\tilde A}u_{\tilde A}/\alpha_1$, contradicting $u_{\tilde A} \notin Sp(u_C; u_{\tilde A\cup\tilde B})$.

Similarly, assume that $u_{\tilde B\cup\tilde C} \in Sp(u_R; u_{S\setminus R})$, so that $u_{\tilde B\cup\tilde C} = \alpha_2 u_R + \beta_2(u_{\tilde A} + \delta_{\tilde A;\tilde B\cup\tilde C}u_{\tilde B\cup\tilde C})$ for some $\alpha_2; \beta_2$. If $\alpha_2 \neq 0$, then use (17) to eliminate u_R and obtain $u_C \notin Sp(u_{\tilde A}; u_{\tilde B\cup\tilde C})$, a contradiction. If $\alpha_2 = 0$, then $u_{\tilde B\cup\tilde C} = \beta_2 u_{\tilde A}/(1 - \beta_2\delta_{\tilde A;\tilde B\cup\tilde C})$, contradicting $u_{\tilde A} \notin Sp(u_C; u_{\tilde A\cup\tilde B})$, provided $\beta_2\delta_{\tilde A;\tilde B\cup\tilde C} \neq 1$. This same contradiction arises if $\beta_2\delta_{\tilde A;\tilde B\cup\tilde C} = 1$ and $u_{\tilde A} = 0$. ■

Degeneracies aside, the proof of Theorem 9 reveals that to establish $\succsim_S$ complete with utility u_S we just need four coalitions A, C, $A \cup B$, $B \cup C$ having this same property and forming a partition $\{A; B; C\}$ of S. Thus, a fraction of the coalitions in $\mathcal{C}$ is needed to reach the conclusion that $\succsim_N$ is complete and has utility u_N. More formally, to derive the completeness of $\succsim_N$, one needs a collection $\mathcal{E}$ of coalitions containing the singletons, the pairs in some spanning tree, and N, and possessing the following property: If $S \in \mathcal{E}$, then there is a partition $\{A; B; C\}$ of S such that $\{A; C; A \cup B; B \cup C\} \subseteq \mathcal{E}$. Of course, $\mathcal{C}$ is one such collection and an interesting question for future research is finding the minimal such collections.

4.2 Complete Preferences for all Coalitions

Theorem 9 generalizes Lemma 6a to all the connected coalitions. One expects that the generalization of Lemma 6b is to conclude $u_S \in U_S$ for $S \subseteq N$, where u_S is given as in (15). This is in fact true for the pairs and "almost" true for the rest of the coalitions.

Example 10 EPR and N non-degenerate is compatible with $u_S \notin U_S$ for some $S \in C$. Thus, we may have $\succsim_S$ complete and having utility $u'_S \neq \lambda u_S$, for all $\lambda > 0$.

Figure 6 illustrates an example where $N = \{1; 2; 3; 4\}$ is non-degenerate, $\succsim_{134}$ is complete, but has utility $u'_{134} \neq u_{134}$.

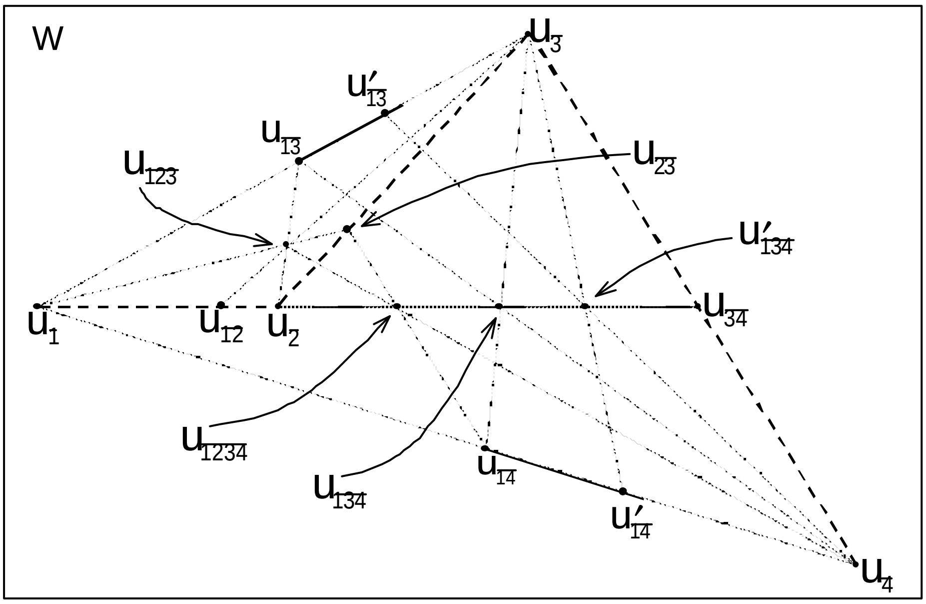

Figure 6: EPR does not necessarily imply $u_S \in U_S$.

If $d(u_1; u_2; u_{34}) = 2$, then the lines u_1u_{34} and u_2u_{1234} are parallel, and any u'_{134} chosen from the segment $u_{1234}u_{34}$ satisfies $u'_{134} \in u_1u_{34}$ and $u_{1234} \in u'_{134}u_2$. It remains to show that EPR as applied to the partitions $\{14; 3\}$ and $\{13; 4\}$ is compatible with $u'_{134} \neq u_{134}$. Indeed, if $\succsim_{13}$ and $\succsim_{14}$ are not complete, then the utility cones U_{13} and U_{14} may contain utilities u'_{13} and u'_{14} such that $u'_{134} \in u_4u'_{13}$ and $u'_{134} \in u_3u'_{14}$. ■

The phenomenon exhibited in Example 10 is very unstable. We would conclude that $u_{134} \in U_{134}$ if $d(u_1; u_2; u_{34}) = 3$; or if either $\succsim_{13}$ or $\succsim_{14}$ were complete. Fortunately, this pathology does not affect pairs.

Theorem 11 Let T be a non-degenerate spanning tree of bilateral agreements, with $(\lambda_1, \ldots, \lambda_n)$ as in (14) and u_S as in (15). If the Extended Pareto Rule holds, then for all $\overline{ik} \subseteq N$, $u_{\overline{ik}} \in U_{\overline{ik}}$: if $\succsim_{\overline{ik}}$ is complete, then it has utility $u_{\overline{ik}}$.

Proof. The result is trivial if $\overline{ik} \in T$. For $\overline{ik} \notin T$, let $(i = i_1, \ldots, i_\ell = k)$ be the path in T between i and k, and $T = \cup_{r=2}^{\ell-1} \overline{i_r} \neq \emptyset$. We claim that there is a coalition $R \subseteq T$ such that $R \in \mathcal{C}$, $u_R \notin Sp(u_{\bar{i}}, u_{\bar{k}})$, and $u_{R \cup \overline{ik}} \in U_{R \cup \overline{ik}}$: because $\succsim_R$ is complete, the result follows from letting $A = \bar{i}$, $B = R$, and $C = \bar{k}$ in Lemma 8b. To establish the claim we define R as follows. If $u_T \notin Sp(u_{\bar{i}}, u_{\bar{k}})$, then let $R = T$ (this is always the case if $\ell = 3$). On the contrary, if $u_T \in Sp(u_{\bar{i}}, u_{\bar{k}})$, then T non-degenerate implies $d(u_{\bar{i}}, u_{\overline{i_2}}, u_{\overline{i_3}}) = 3$, so that either $u_{\overline{i_2}} \notin Sp(u_{\bar{i}}, u_{\bar{k}})$ or $u_{\overline{i_3}} \notin Sp(u_{\bar{i}}, u_{\bar{k}})$, the former being always true by T non-degenerate if $\ell = 4$. Accordingly, let $R = \overline{i_2}$ or $R = \overline{i_2 i_3}$ so that $T \setminus R \neq \emptyset$, $u_T \in Sp(u_{\bar{i}}, u_{\bar{k}})$, and $u_R \notin Sp(u_{\bar{i}}, u_{\bar{k}})$. Having $(1 + \delta_{R,T\setminus R}) u_T = (u_R + \delta_{R,T\setminus R} u_{T\setminus R})$ for some $\delta_{R,T\setminus R} > 0$ implies $u_{T\setminus R} \notin Sp(u_{\bar{i}}, u_{\bar{k}})$. Because $\{R \cup \bar{i}, T\setminus R, \bar{k}, T \cup \overline{ik}\} \subseteq \mathcal{C}$ we use $A = R \cup \bar{i}$, $B = T \setminus R$, and $C = \bar{k}$ in Lemma 8b to conclude $u_{R \cup \overline{ik}} \in U_{R \cup \overline{ik}}$.

Geometrically, $u_{R \cup \overline{ik}}$ is found as the intersection of the line segment $u_{R \cup \bar{i}} u_{\bar{k}}$ and the half line $u_{T \cup \overline{ik}} u_{T \setminus R}$. Similarly, $u_{\overline{ik}}$ is found as the intersection of the line segment $u_{\bar{i}} u_{\bar{k}}$ and the half line $u_{R \cup \overline{ik}} u_R$. Because $u_{\overline{ik}} = (\lambda_i u_{\bar{i}} + \lambda_k u_{\bar{k}})/(\lambda_i + \lambda_k)$, the utility comparison rate between i and k is given by $\delta_{i,k} = \lambda_k / \lambda_i = \prod_{r=1}^{\ell} \delta_{i_{r-1}, i_r}$. ■

Theorem 12, our second main result, derives complete preferences for all coalitions by assuming complete pair preferences, thus establishing Claim 1 using minimal premises regarding linear independence.

Theorem 12 Suppose N is non-degenerate, every pair $\overline{ij}$ in N has complete preferences, and u_S is given as in (15) for some spanning tree T of N. Then the Extended Pareto Rule holds if and only if for all $S \subseteq N$, $\succsim_S$ is complete and has utility u_S.

Proof. ($\Rightarrow$) The proof is based on choosing a spanning tree that renders some desired coalition S connected, and then applying Theorem 9 to conclude that $\succsim_S$ is complete and has utility u_S. To verify that u_S does not depend on the choice of spanning tree suffices to check that if $\hat{\lambda}_i$ and λ_i are the weights computed as in (14) using spanning trees $\hat{T}$ and T, then $\hat{\lambda}_i = \lambda_i$. Clearly $\hat{\lambda}_1 = \lambda_1 = 1$, and for $i \neq 1$, let $(1 = i_1, i_2, \ldots, i_\ell = i)$ be the unique path between 1 and i in $\hat{T}$. By

Theorem 11, all pairs $\overline{i_{r-1}i_r} \in \hat{T}$ have complete preferences with utilities $u_{\overline{i_{r-1}i_r}} = (\lambda_{i_{r-1}} u_{\overline{i_{r-1}1}} + \lambda_{i_r} u_{\overline{i_r}})/(\lambda_{i_{r-1}} + \lambda_{i_r})$. Thus, $\delta_{i_{r-1};i_r} = \lambda_{i_r}/\lambda_{i_{r-1}}$ and $\hat{\lambda}_i = \prod_{r=1}^{\ell} \delta_{i_{r-1};i_r} = \prod_{r=1}^{\ell} (\lambda_{i_r}/\lambda_{i_{r-1}}) = \lambda_i$.

($\Leftarrow$) Noting that $\lambda_i > 0$ for all $i \in N$, if $x \succsim_A y$ and $x \succsim_B y$, then

$$\sum_{i \in S} \lambda_i u_i(x) \geq \sum_{i \in S} \lambda_i u_i(y),\ S \in \{A; B\} \Rightarrow u_{A \cup B}(x) \geq u_{A \cup B}(y);$$

and if $x \succ_A y$, then $u_A(x) > u_A(y)$ and $u_{A \cup B}(x) > u_{A \cup B}(y)$. ■

Theorem 12 does not place special importance to any particular spanning tree. Because all the information regarding coalition preferences is summarized in the n individual utilities and the $n - 1$ independent weights, a particular spanning tree has no relevance other than embodying this information in the form of $n - 1$ utility comparison rates. We can draw an analogy with currency exchange rates. Consider n countries, and choose country 1 as the reference. Having the currency exchange rates between country 1 and all the other countries (a particular spanning tree), allow us to compute the exchange rate between any two countries given by the path $\delta_{i;j} = \delta_{i;1}\delta_{1;j} = \delta_{1;j}/\delta_{1;i}$.

Having the weights λ_i invites us to modify the scales of the corresponding utilities u_S so as to drive all the utility comparison rates to 1. Such an additive representation is readily obtainable if we set, for all $S \subseteq N$, $\tilde{u}_S \equiv (\sum_{i \in S} \lambda_i) u_S$. It follows that for any two disjoint coalitions A and B, $\tilde{u}_{A \cup B} = \tilde{u}_A + \tilde{u}_B$, and so $\tilde{u}_S = \sum_{i \in S} \tilde{u}_i$. Recall that agent 1 was chosen as a base to compute the individual weights λ_i in (14). As a consequence, the additive representation expresses all the utilities in the units of agent 1. If we want to use the utility units of some other agent $i^* \neq 1$, it suffices to re-scale each u_S by the factor $\delta_{i^*;1} = 1/\lambda_{i^*}$.

5 Further Remarks and Extensions

5.1 Stability of Group Preference and Minimal Consensus

We observe that the conditions $u_A \notin Sp(u_C; u_{A \cup B})$ and $u_C \notin Sp(u_A; u_{B \cup C})$ are satisfied if and only if either $d(u_A; u_B; u_C) = 3$, or

$$u_A = -\gamma u_{B \cup C} \neq 0 \text{ and } u_C = -\gamma' u_{A \cup B} \neq 0, \text{ for some } \gamma; \gamma' > 0: \qquad (18)$$

Condition (16) in Theorem 9 cannot be replaced with $d(u_A;u_B;u_C) = 3$ if we are to handle the case of $u_S = 0$ for some $S \subseteq N$. We illustrate this point by means of the following example.

Example 13 Let N be non-degenerate. EPR may imply that $\succsim_S$ is trivial for some $S \subseteq N$.

Let $m = 3$, $N = \{1;2;3;4;5\}$, and individual utilities given by $u_{\overline{1}} = (0;2;0)$, $u_{\overline{2}} = (2;0;0)$, $u_{\overline{3}} = (-1;-1;1)$, $u_{\overline{4}} = (-1;-1;-1)$, and $u_{\overline{5}} = (-1;0;0)$. Let $T = \{\overline{12};\overline{23};\overline{34};\overline{45}\}$ and $\delta_{i;j} = 1$, $1 \le i < j \le 5$, so that $u_S = \left(\sum_{i\in S} u_i\right)/|S|$: $u_{\overline{1234}} = 0$ and $u_{\overline{12345}} = u_{\overline{5}}/5$. To determine $\succsim_{\overline{1234}}$, observe that any partition $\{A;B;C\}$ of $\overline{1234}$ satisfies (18), but exhibits $d(u_A;u_B;u_C) < 3$. Nevertheless, EPR implies that $\succsim_{\overline{1234}}$ is trivial. To determine $\succsim_{\overline{12345}}$ by means of a partition $\{A';B';C'\}$ we cannot count on $\overline{1234}$ as an element of $\{A';B';C';A' \cup B';B' \cup C'\}$. Thus, the only choices of partition are $\{\overline{12};\overline{3};\overline{45}\}$ or $\{\overline{1};\overline{23};\overline{45}\}$, which in the proof of Theorem 9 corresponds to choosing $R = \overline{45}$. ∎

When $u_S = 0$, the preference $\succsim_S$ is extremely unstable: had some individual utility been slightly different, say $u'_{\overline{i}} = u_{\overline{i}} + u$, for some $u \neq 0$, then the corresponding group preference would have had utility $u'_S = u$. However, the choice $u''_{\overline{i}} = u_{\overline{i}} - u$ would produce $u''_S = -u$, i.e., exactly the opposite preference. This undesired behavior is ruled out by imposing the condition of Minimal Consensus: there exists two prospects $x;y \in M$ such that for all $i \in N$, $x \succ_{\overline{i}} y$. Clearly, Minimal Consensus and (5) imply $x \succ_S y$: no coalition has a trivial preference.

5.2 Continuity under Minimal Consensus

Non-degeneracy of N is a "generic" property whenever $m \ge 3$, i.e., it holds for an open dense set in the space $\mathbb{R}^{mn}$ of individual profiles $(u_{\overline{i}})_{i\in N}$. Intuitively, if we choose n utilities at random from $\mathbb{R}^m$, then the probability that any three of them are linearly dependent is zero. This observation suggests extending our results to degenerate individual profiles by using continuity.

Let a collection $(u_S)_{S\subseteq N}$ be a group profile. If all coalitions have complete preferences, then let $(u_S)_{S\subseteq N}$ be a valid group profile. Suppose that $(u_{\overline{i}})_{i\in N}$ is a degenerate individual profile. We can use (15) to compute an invalid group profile $(u_S)_{S\subseteq N}$, i.e., some preference $\succsim_S$ may not necessarily be complete. Let $(u_{\overline{i};k})_{i\in N}$ be a sequence of non-degenerate individual profiles and $(u_{S;k})_{S\subseteq N}$ the corresponding valid group profile. The construction of the sequence $(u_{\bar{i};k})_{i\in N}$ is always possible if $m \geq 3$. By continuity of (15), if $(u_{\bar{i};k})_{i\in N} \to (u_{\bar{i}})_{i\in N}$, then $(u_{S;k})_{S\subseteq N} \to (u_S)_{S\subseteq N}$. The following Continuity Condition extends our two main results to degenerate domains:

If $\succsim_{S;k}$ is complete and $u_{S;k} \to u_S$; then $\succsim_S$ is complete and has utility u_S.

Note that this Continuity Condition is not meaningful when the invalid group profile produces $u_S = 0$ for some $S \subseteq N$. For example, if $u_S = 0$, then consider the sequence $u_{S;k} = u/k$, for some $u \neq 0$. Clearly, $u_{S;k} \to u_S$ but $\succsim_{S;k}$ is the non-trivial preference with utility u, whereas $\succsim_S$ is trivial. Thus, $\succsim_{S;k}$ does not converge to $\succsim_S$ in terms of preference.[26] Minimal Consensus is a sufficient condition that eludes this difficulty.

5.3 Agents with Trivial Preferences

Regarding the possibility of encompassing agents with trivial preferences, they can be included if the following precaution is considered: if N^* is the coalition of agents with non-trivial preferences, then choose $\mathcal{T}$ in Theorem 9 so that $N^* \in \mathcal{C}$ and $\mathcal{T}_{N^*}$ is non-degenerate. Similarly, if N^* is non-degenerate, then Theorem 12 holds.

Both claims are verified by observing that Lemma 6 holds if $u_{\bar{1}} = 0$ and $u_{\bar{2}} \neq 0$, or if $u_{\bar{1}} = u_{\bar{2}} = 0$, but fails if $u_{\bar{1}} \neq 0$, $u_{\bar{3}} \neq 0$, and $u_{\bar{2}} = 0$.

5.4 Masters and Servants

It is interesting to explore a relaxation of Condition (5) in EPR, where agents are treated in an asymmetric way. An example will be illustrative. Let $N = \{1; 2; 3; 4\}$ and $\mathcal{T} = \{\overline{12}; \overline{23}; \overline{34}\}$. We still impose the weak condition (4) in EPR to all disjoint coalitions, but now reserve the strong condition (5) to individuals as follows: if $1 \leq i < j \leq 4$, then

$$\text{for all } x; y \in M;\ x \succ_{\bar{i}} y;\ x \succsim_{\bar{j}} y \Rightarrow x \succ_{\overline{ij}} y: \tag{19}$$

[26] It is illustrative to examine the corresponding preference cones: the preference cones of $\succsim_{S;k}$ are identical to the half space with normal $u \neq 0$, but this sequence of cones does not converge to $\mathbb{R}^m$, the preference cone of the trivial preference $\succsim_S$.

(19) allows for $i < j$ to prevail over j in the bilateral agreement $\succsim_{\overline{ij}}$, i.e., to have $u_{\overline{ij}} = u_{\overline{i}}$, or $\delta_{i,j} = 0$. We may think of i as a master and j as i's servant. For example, let $\delta_{2,3} = 0$ and $\delta_{i,j} = 1$ for all other pairs in T so that 2 dominates 3. Noting that Lemma 6 encompasses $u_{\overline{23}} = u_{\overline{2}}$ whenever $u_{\overline{34}} \neq u_{\overline{4}}$, we find that EPR produces that both 1 and 2 dominate 3 and 4. Thus, if we require completeness of the pair preference as in Theorem 12, then we obtain that EPR implies the following utilities

$$
\begin{aligned}
&u_{\overline{13}} = u_{\overline{14}} = u_{\overline{134}} = u_{\overline{1}};\\
&u_{\overline{23}} = u_{\overline{24}} = u_{\overline{234}} = u_{\overline{2}}; \text{ and}\\
&u_{\overline{123}} = u_{\overline{124}} = u_{\overline{1234}} = u_{\overline{12}}.
\end{aligned}
$$

Taking 1 as base agent, Formula (14) gives $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = 0$, which produces the correct utilities for all coalitions except for $u_{\overline{34}} = (u_{\overline{3}} + u_{\overline{4}})/2 \neq 0$. This calls for the following modification in the procedure to compute a given u_S: choose a base agent in S who is undominated in S, and compute $\lambda_{i,S}$ as in (14) for all $i \in S$. Then, (14) will give

$$
u_S \equiv \left(\sum_{i \in S} \lambda_{i,S} u_i\right) \Big/ \left(\sum_{i \in S} \lambda_{i,S}\right). \tag{20}
$$

The example could be extended as follows. First we establish a relation of dominance between pairs of agents, assumed irreflexive and acyclic. Then we impose (19) to all pairs i and j such that i dominate j. The application of EPR after assuming complete preferences for all the pairs produces complete preferences for all coalitions, with utilities computed as in (20). The set of agents divides itself in hierarchical classes of masters and servants, with $\lambda_{i,S} = 0$ if S contains some agent whose class is higher than i's, and $\lambda_{i,S} > 0$ otherwise.

6 Extensions

We assumed a finite-dimensional prospect space for reasons for simplicity. The extension to infinite-dimensional spaces is quite direct, if we bear in mind that the preliminary section of Baucells and Shapley (1998) articulates the theory of incomplete preferences in such large spaces. One also imagines the extension to countably many agents once the natural definitions using limits are in place. More challenging seems the extension to uncountably many non-atomic agents, as Aumann and Shapley (1974) accomplished in the context of cooperative game theory.

In the interpretation of cardinal utility as strength of preference, we want coalitions to aggregate individual strength of preferences.[27] Thus, it is convenient to develop the theory of incomplete strength of preference. In Alt (1971) and Shapley (1974) strength of preference is defined as a binary relation over pairs of prospects that is superimposed on some ordinal preference relation over prospects. Under suitable axioms, one finds a unique cardinal representation of the originally ordinal preference that does not involve lotteries. Moreover, the strength of preference relation is represented by utility differences. With this in mind, one could introduce a binary relation over pairs, $(x;y)_1 \succsim_{12} (z;w)_2$, as the basis to express interpersonal comparisons of strength of preference. Such a relation could be incomplete, and hopefully represented by a cone of utility functions $U_{ij} \subseteq Co_{i;j}$. Thus, a cardinal framework for group utility may be obtained without involving lotteries.

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October 1999.

[27] See Elster and Roemer (1991) for a collection of articles on interpersonal comparisons of welfare.

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