LM Tests for Functional Form and Spatial Correlation

Badi H. Baltagi and Dong Li
Texas A&M University
Department of Economics
College Station, TX 77843-4228

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Abstract

This paper derives Lagrangian Multiplier tests to jointly test for functional form and spatial error correlation. In particular, this paper tests for linear and loglinear models with no spatial error dependence against a more general Box-Cox model with spatial error correlation. Conditional LM tests and modified Rao-Score tests that guard against local misspecification are also derived. These tests are easy to implement and are illustrated using Anselin's (1988) crime data. The performance of these tests are also compared using Monte Carlo experiments.

1 Introduction

The choice of functional form is important especially when nonlinearity is suspected, see Lendent (1986) for a model of urbanization with nonlinear migration flows; Craig, Kohlhase and Papell (1991) who were concerned with the nonlinear structure of hedonic housing price regressions; Elad, Clifton and Epperson (1994) for a nonlinear hedonic model of the price of farmland in Georgia. Similar concerns over non-linearity in the spatial econometrics literature are evident in Upton and Fingleton (1985), Bailly et al. (1992), Griffith et al. (1998), and Fik and Mulligan (1998), to mention a few. The Box and Cox (1964) procedure has been used to choose among alternative functional forms, see Savin and White (1978), Seaks and Layson (1983), and Davidson and MacKinnon (1985), to mention a few. But the spatial correlation further complicates the estimation and testing of these models. Attempts at dealing with this problem vary from estimating the Box-Cox model by maximum likelihood methods ignoring the spatial correlation, see Upton and Fingleton (1985), to linearizing the Box-Cox transformation, see Bailly et al. (1992), and Griffith et al. (1998). However, linearization is an approximation that is only valid around specific values of the parameters. Misspecifying the functional form and/or ignoring the spatial dependence can result in misleading inference and raise questions about the reliability and precision of the resulting estimates.

This paper derives Lagrange Multiplier (LM) tests to jointly test for functional form and spatial dependence. To our knowledge, this is the first extension of the LM test on functional form to spatial econometrics. Testing for spatial dependence assuming a specific functional form, usually a linear regression, is studied extensively in Anselin (1988), Anselin et al. (1996), and more recently in Anselin and Bera (1998). The latter study surveys several tests for spatial dependence including Wald, LR and LM type tests. However, none of these tests jointly test for spatial dependence and functional form. The LM tests derived in this paper are computationally simple, requiring least squares regressions on linear or loglinear models. It allows the researcher to test for a linear or loglinear model with no spatial error dependence against a more general Box-Cox model with spatial error dependence. Special cases of these tests include tests for functional form given spatial dependence and tests for spatial dependence given functional form. In addition, conditional LM tests as well as Bera and Yoon (1993) modified Rao Score tests are derived. The latter guard against local misspecification.

Section 2 derives the joint, conditional and modified LM tests for the Box-Cox spatial error dependence model, while Section 3 illustrates these tests using Anselin's (1988) crime data. Section 4 compares the performance of these tests using Monte Carlo experiments, while Section 5 gives our conclusion.

2 The Model and the LM Tests

Consider the following Box-Cox model

$$y_i^{(r)} = \sum_{k=1}^{K} \beta_k \, x_{ik}^{(r)} + \sum_{s=1}^{S} \gamma_s \, z_{is} + u_i, \qquad i = 1, ..., n \tag{1}$$

where

$$x^{(r)} = \begin{cases} \frac{x^r - 1}{r} & \text{if } r \neq 0 \\ \log(x) & \text{if } r = 0 \end{cases} \tag{2}$$

is the familiar Box-Cox transformation. Both y_i and x_{ik} are subject to the Box-Cox transformation and are required to take positive values only, while the Z_i variables are not subject to the Box-Cox transformation. The Z_i's may include dummy variables and the intercept. Note that for $r = 1$, equation (1) becomes a linear model whereas for $r = 0$ it becomes a loglinear model. Following Anselin (1988) or Anselin and Bera (1998, p.248), we allow for spatial correlation in the error term

$$u = \lambda W u + v \tag{3}$$

where λ is the spatial autoregressive coefficient, W is the matrix of known spatial weights and $v \sim N(0, \sigma_v^2 I)$ is independent of u. [1]

[1] Davidson and MacKinnon (1985, p. 500) point out that the normality assumption may be untenable. In fact, except for certain values of r (including 0 and 1), $y_t(r)$ cannot take on values less than $-1/r$, while with normal errors there is always the possibility that the right-hand side of (1) may be less than $-1/r$. Davidson and MacKinnon (1985) argue that it is reasonable to ignore the problem especially if $E(y_t)$ is very large relative to σ, and the possibility that u_t may be so large and negative as to make the right-hand side of (1) unacceptably small can be safely ignored. We check the sensitivity of departures from the normality assumption in our Monte Carlo experiments.

Substituting (3) into (1) rewritten in vector form yields

$$(I - \lambda W)\ y^{(r)} = (I - \lambda W)\ X^{(r)}\beta + (I - \lambda W)\ Z\gamma + v \tag{4}$$

where $y^{(r)}$ is $n \times 1$, $X^{(r)}$ is $n \times K$, Z is $n \times S$ and β and γ are $K \times 1$ and $S \times 1$, respectively.

The loglikelihood function is given by

$$\begin{aligned} \log L \quad = \quad & -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) + \log|I - \lambda W| + (r-1)\sum_{i=1}^{n}\log(y_i) \\ & -\frac{1}{2\sigma^2}[(I - \lambda W)\ y^{(r)} - (I - \lambda W)\ X^{(r)}\beta - (I - \lambda W)\ Z\gamma]' \\ & [(I - \lambda W)\ y^{(r)} - (I - \lambda W)\ X^{(r)}\beta - (I - \lambda W)\ Z\gamma]. \end{aligned} \tag{5}$$

Note that $\log|I - \lambda W| = \sum_{i=1}^{n}\log(1 - \lambda\omega_i)$, where ω_i's are the eigenvalues of W, see Ord (1975) and Anselin (1988).

The first-order derivatives are given by

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}v'v \tag{6}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{1}{\sigma^2}[(I - \lambda W)X^{(r)}]'v \tag{7}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{1}{\sigma^2}[(I - \lambda W)Z]'v \tag{8}$$

$$\frac{\partial \log L}{\partial \lambda} = -\sum_{i=1}^{n}\frac{\omega_i}{1 - \lambda\omega_i} + \frac{1}{\sigma^2}v'W(y^{(r)} - \ X^{(r)}\beta - Z\gamma) \tag{9}$$

$$\frac{\partial \log L}{\partial r} = \sum_{i=1}^{n}\log(y_i) - \frac{1}{\sigma^2}v'(I - \lambda W)[C(y,r) - C(X,r)\beta] \tag{10}$$

where $C(y,r) = \frac{\partial y^{(r)}}{\partial r} = \frac{1}{r^2}(ry^r\log y - y^r + 1)$. The second-order derivatives of the loglikelihood function are given in Appendix A.

Let $\theta = (\sigma, \beta', \gamma', \lambda, r)'$, then the gradient is given by $G = \frac{\partial \log L}{\partial \theta}$, and the information matrix is given by $\mathcal{I} = E(-\frac{\partial^2 \log L}{\partial\theta\partial\theta'})$. The LM test statistic is given by

$$LM = \tilde{G}'\ \tilde{\mathcal{I}}^{-1}\ \tilde{G} \tag{11}$$

where $\tilde{G}$ and $\tilde{\mathcal{I}}$ denote the restricted gradient and information matrix evaluated under the null hypothesis, respectively. Following Efron and Hinkley (1978), we estimate the information matrix by the negative Hessian $-H(\tilde{\theta})$. This is recommended on the ground that is closer to the data than the corresponding expected value. Davidson and MacKinnon (1993) demonstrated that it is better to use $\tilde{\mathcal{I}}$ rather than $\tilde{H}$ using Monte Carlo experiments. However, the latter is difficult to compute in this case.

2.1 Joint Tests

Under the null hypothesis $H_0^a : \lambda = 0$ and $r = 0$, the model in (1) becomes a loglinear model with no spatial error dependence

$$\log y_i = \sum_{k=1}^{K} \beta_k \log x_{ik} + \sum_{s=1}^{S} \gamma_s \; z_{is} + u_i, \qquad i = 1, ..., n \tag{12}$$

with $y^{(r)} = \log y$, $C(y, 0) = \frac{1}{2}(\log y)^2$ and $\lim_{r \to 0} \frac{\partial C(y,r)}{\partial r} = \frac{1}{3}(\log y)^3$. The restricted OLS residuals from (12) are given by $\hat{v} = \log y - (\log X)\hat{\beta} - Z\hat{\gamma}$ with $\hat{\sigma}^2 = \hat{v}'\hat{v}/n$.

The gradient becomes

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} v'v \tag{13}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{1}{\sigma^2}[\log X]'v \tag{14}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{1}{\sigma^2} Z'v \tag{15}$$

$$\frac{\partial \log L}{\partial \lambda} = -\sum_{i=1}^{n} \omega_i + \frac{1}{\sigma^2} v'W(\log y - \; (\log X)\beta - Z\gamma) \tag{16}$$

$$\frac{\partial \log L}{\partial r} = \sum_{i=1}^{n} \log(y_i) - \frac{1}{\sigma^2} v'[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta] \tag{17}$$

and the second order derivatives of the loglikelihood function are given in Appendix B.1.

Under the null hypothesis $H_0^b : \lambda = 0$ and $r = 1$, the model in (1) becomes a linear model with no spatial error dependence

$$y_i - 1 = \sum_{k=1}^{K} \beta_k (x_{ik} - 1) + \sum_{s=1}^{S} \gamma_s \; z_{is} + u_i, \qquad i = 1, ..., n \tag{18}$$

with $y^{(r)} = y - 1$, $C(y, 1) = y \log y - y + 1$ and $\lim_{r\to 1} \frac{\partial C(y,r)}{\partial r} = y(\log y)^2 - 2y \log y + 2y - 2$. The restricted OLS residuals from (18) are given by $\tilde{v} = (y - \iota_n) - (X - J_{nK})\tilde{\beta} - Z\tilde{\gamma}$ with $\tilde{\sigma}^2 = \tilde{v}'\tilde{v}/n$, where ι_n and J_{nK} are $n \times 1$ and $n \times K$ matrix with all elements equal to 1, respectively.

The gradient becomes

$$
\begin{aligned}
\frac{\partial \log L}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} v'v && (19)\\
\frac{\partial \log L}{\partial \beta} &= \frac{1}{\sigma^2}(X - 1)'v && (20)\\
\frac{\partial \log L}{\partial \gamma} &= \frac{1}{\sigma^2} Z'v && (21)\\
\frac{\partial \log L}{\partial \lambda} &= -\sum_{i=1}^{n} \omega_i + \frac{1}{\sigma^2} v'W(y - \iota_n - \ (X - J_{nK})\beta - Z\gamma) && (22)\\
\frac{\partial \log L}{\partial r} &= \sum_{i=1}^{n} \log(y_i) - \frac{1}{\sigma^2} v'[y \log y - y + \iota_n - (X \log X - X + J_{nK})\beta] && (23)
\end{aligned}
$$

and the second-order derivatives of the loglikelihood function are given in Appendix B.2.

2.2 Conditional Tests

Joint tests are often criticized because they do not point out the “right” model we should adopt when the null hypothesis is rejected. In this section we will consider conditional LM tests. These tests account for the possible presence of spatial correlation when testing for functional form, or the possible misspecification of the functional form when testing for spatial correlation.

2.2.1 LM Tests for Spatial Dependence Conditional on a General Box-Cox Model

Under the null hypothesis H_0^g: $\lambda = 0$ |unknown r, the model in (1) becomes a general Box-Cox model with no spatial error dependence in the error term. The gradient becomes

$$
\begin{aligned}
\frac{\partial \log L}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} v'v = 0 && (24)\\
\frac{\partial \log L}{\partial \beta} &= \frac{1}{\sigma^2}[X^{(r)}]'v = 0 && (25)\\
\frac{\partial \log L}{\partial \gamma} &= \frac{1}{\sigma^2} Z'v = 0 && (26)\\
\frac{\partial \log L}{\partial \lambda} &= -\sum_{i=1}^{n} \omega_i + \frac{1}{\sigma^2} v'Wv && (27)
\end{aligned}
$$

$$\frac{\partial \log L}{\partial r} = \sum_{i=1}^{n} \log(y_i) - \frac{1}{\sigma^2} v'[C(y,r) - C(X,r)\beta] \tag{28}$$

The second order derivatives of the loglikelihood function are given in Appendix C.1.

2.2.2 LM Tests for Functional Form Conditional on Spatial Correlation

Next we consider tests for functional form, linear or loglinear against a general Box-Cox transformation, conditional on the presence of spatial dependence in the error term.

Loglinear with Spatial Correlation Under the null hypothesis H_0^h: $r = 0$ |unknown λ, the model in (1) becomes a loglinear model with spatial error dependence. Note that $u = \log y - (\log X)\beta - Z\gamma$ and $v = (I - \lambda W)u$.

The gradient is

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} v'v = 0 \tag{29}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{1}{\sigma^2}[(I - \lambda W)\log X]'v = 0 \tag{30}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{1}{\sigma^2}[(I - \lambda W)Z]'v = 0 \tag{31}$$

$$\frac{\partial \log L}{\partial \lambda} = -\sum_{i=1}^{n} \frac{\omega_i}{1 - \lambda\omega_i} + \frac{1}{\sigma^2} v'Wu \tag{32}$$

$$\frac{\partial \log L}{\partial r} = \sum_{i=1}^{n} \log(y_i) - \frac{1}{\sigma^2} v'(I - \lambda W)[C(y,0) - C(X,0)\beta] \tag{33}$$

The second order derivatives of the loglikelihood function are given in Appendix C.2.

Linear with Spatial Correlation Under the null hypothesis H_0^i: $r = 1$ |unknown λ, the model in (1) becomes a linear model with spatial error dependence. Note again that $u = (y-1)-(X-1)\beta-Z\gamma$ and $v = (I - \lambda W)u$.

The gradient is

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} v'v = 0 \tag{34}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{1}{\sigma^2}[(I - \lambda W)(X - 1)]'v = 0 \tag{35}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{1}{\sigma^2}[(I-\lambda W)Z]'v = 0 \tag{36}$$

$$\frac{\partial \log L}{\partial \lambda} = -\sum_{i=1}^{n}\frac{\omega_i}{1-\lambda\omega_i} + \frac{1}{\sigma^2}v'Wu \tag{37}$$

$$\frac{\partial \log L}{\partial r} = \sum_{i=1}^{n}\log(y_i) - \frac{1}{\sigma^2}v'(I-\lambda W)[C(y,1)-C(X,1)\beta] \tag{38}$$

The second order derivatives of the loglikelihood function are given in Appendix C.3.

2.3 Local Misspecification Robust Tests

Conditional LM tests do not ignore the possibility that r is not known when testing for $\lambda = 0$. Whereas simple LM tests for $\lambda = 0$ assume implicitly that $r = 0$ or 1 and this may lead to misleading inference. However, conditional LM tests are computationally more involved than the corresponding simple LM tests. The latter are usually based on least squares residuals.

Bera and Yoon (1993) showed that, under local misspecification, the simple LM test asymptotically converges to a *noncentral* chi-square distribution. They suggest a modified Rao-Score (RS) test which is robust to local misspecification. This modified RS test retains the computational simplicity of the simple LM test in that it is based on the same restricted MLE (usually OLS). However, it is more robust than the simple LM test because it guards against local misspecification. The idea is to adjust the one-directional score test by accounting for its non-centrality parameter. Bera and Yoon (1993) and Bera et al. (1998) showed using Monte Carlo experiments that these modified RS tests have good finite sample properties and are capable of detecting the right direction of departure from the null hypothesis. For our purposes, we consider four hypotheses:

$H_0^c : \lambda = 0$ assuming $r = 0$ (no spatial correlation assuming loglinearity);

$H_0^d : \lambda = 0$ assuming $r = 1$ (no spatial correlation assuming linearity);

$H_0^e : r = 0$ assuming $\lambda = 0$ (loglinearity assuming no spatial correlation);

$H_0^f : r = 1$ assuming $\lambda = 0$ (linearity assuming no spatial correlation).

Let $\eta' = (\sigma, \beta', \gamma')$ so that $\theta' = (\sigma, \beta', \gamma', \lambda, r) = (\eta', \lambda, r)$ and partition the gradient and the information matrix such that

$$d(\theta) = \frac{\partial L(\theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial L(\theta)}{\partial \eta} \\ \frac{\partial L(\theta)}{\partial \lambda} \\ \frac{\partial L(\theta)}{\partial r} \end{bmatrix} \tag{39}$$

and

$$J(\theta) = -\left[\frac{\partial^2 L(\theta)}{\partial\theta\partial\theta'}\right] = \begin{bmatrix} J_\eta & J_{\eta\lambda} & J_{\eta r} \\ J_{\lambda\eta} & J_\lambda & J_{\lambda r} \\ J_{r\eta} & J_{r\lambda} & J_r \end{bmatrix} \tag{40}$$

To be more specific, consider the null hypothesis H_0^c. The general model is represented by the loglikelihood function $L(\eta', \lambda, r)$. For the null hypothesis H_0^c, the investigator sets $r = 0$ and tests $\lambda = 0$ using the loglikelihood function $L_1(\eta', \lambda) = L(\eta', \lambda, 0)$. The standard Rao-Score statistic based on $L_1(\eta', \lambda)$ is denoted by RS_λ. Let $\tilde{\eta}$ be the maximum likelihood estimator of η when $\lambda = 0$ and $r = 0$. If $L_1(\eta', \lambda)$ were the true model, it is well known that when $\lambda = 0$ the test statistic $\text{RS}_\lambda \rightarrow \chi_1^2(0)$ and the test will have the correct size and will be locally optimal. Now suppose that the true loglikelihood function is $L_2(\eta', r) = L(\eta', 0, r)$ so that the alternative $L_1(\eta', \lambda)$ is misspecified. Using a sequence of local values $r = \xi/\sqrt{n}$, the asymptotic distribution of RS_λ under $L_2(\eta', r)$ is $\chi_1^2(c_1)$ where c_1 is the non-centrality parameter, see Bera and Yoon (1993), Davidson and MacKinnon (1987) and Saikkonen (1989) for details. Due to the presence of this non-centrality parameter, RS_λ will over-reject the null hypothesis even when $\lambda = 0$. Therefore, the test will have an incorrect size. In light of this non-centrality parameter, Bera and Yoon (1993) suggested a modification to RS_λ so that the resulting test statistic is robust to the presence of r. The new test essentially adjusts the asymptotic mean and variance of the standard RS_λ.

Following equation (6) in Bera et al. (1998), we derive this modified RS test as follows

$$\begin{aligned} RS_\lambda^* \;=\; & \frac{1}{n}[d_\lambda(\tilde{\theta}) - J_{\lambda r\cdot\eta}(\tilde{\theta})J_{r\cdot\eta}^{-1}(\tilde{\theta})d_r(\tilde{\theta})]' \\ & [J_{\lambda\cdot\eta}(\tilde{\theta}) - J_{\lambda r\cdot\eta}(\tilde{\theta})J_{r\cdot\eta}^{-1}(\tilde{\theta})J_{r\lambda\cdot\eta}(\tilde{\theta})]^{-1} \\ & [d_\lambda(\tilde{\theta}) - J_{\lambda r\cdot\eta}(\tilde{\theta})J_{r\cdot\eta}^{-1}(\tilde{\theta})d_r(\tilde{\theta})] \end{aligned} \tag{41}$$

where $d_\lambda(\tilde{\theta})$ is the gradient for λ evaluated at the restricted MLE, $J_{\lambda\cdot\eta} \equiv J_{\lambda\cdot\eta}(\tilde{\theta}) = J_\lambda - J_{\lambda\eta}J_\eta^{-1}J_{\eta\lambda}$

and $J_{r\cdot\eta}$ is similarly defined. Also, $J_{\lambda r\cdot\eta} = J_{\lambda r} - J_{\lambda\eta}J_{\eta}^{-1}J_{\eta r}$ and $J_{r\lambda\cdot\eta}$ is similarly defined. All the above quantities are estimated under the null hypothesis $H_0^c : \lambda = 0$ assuming $r = 0$. The null hypothesis $H_0^d : \lambda = 0$ assuming $r = 1$ can be handled similarly with $r = 1$ rather than 0.

For the null hypotheses H_0^e and H_0^f, the modified RS test statistic is given by

$$\begin{aligned} RS_r^* \quad = \quad & \frac{1}{n}[d_r(\tilde{\theta}) - J_{r\lambda\cdot\eta}(\tilde{\theta})J_{\lambda\cdot\eta}^{-1}(\tilde{\theta})d_\lambda(\tilde{\theta})]' \\ & [J_{r\cdot\eta}(\tilde{\theta}) - J_{r\lambda\cdot\eta}(\tilde{\theta})J_{\lambda\cdot\eta}^{-1}(\tilde{\theta})J_{\lambda r\cdot\eta}(\tilde{\theta})]^{-1} \\ & [d_r(\tilde{\theta}) - J_{r\lambda\cdot\eta}(\tilde{\theta})J_{\lambda\cdot\eta}^{-1}(\tilde{\theta})d_\lambda(\tilde{\theta})]. \end{aligned} \tag{42}$$

where $d_r(\tilde{\theta})$, $J_{r\lambda\cdot\eta}(\tilde{\theta})$, and $J_{\lambda\cdot\eta}(\tilde{\theta})$ are computed as described below (66) under the respective null hypothesis.

In the Monte Carlo experiment in Section 4, we compute the simple LM test and the corresponding Bera-Yoon modified LM test for each hypothesis considered. The simple RS tests for no spatial correlation under linearity or loglinearity, i.e., H_0^c and H_0^d, are given in Anselin (1988) and Anselin and Bera (1998), while the simple RS tests for functional form assuming no spatial correlation, i.e., H_0^e and H_0^f, are given in Davidson and MacKinnon (1985).

Note that it is not possible to robustify tests in the presence of global misspecification (i.e., λ and r taking values far from their values under the null), see Anselin and Bera (1998). Also, it is important to note that the modified RS tests satisfy the following decomposition

$$RS_{\lambda r} = RS_\lambda^* + RS_r = RS_\lambda + RS_r^* \tag{43}$$

i.e., the joint test can be decomposed into the sum of the modified RS test of one type of alternative and the simple RS test for the other, see Bera and Yoon (1993).

3 Empirical Example

Anselin (1988) considered a simple relationship between crime and housing values and income in 1980 for 49 neighborhoods in Columbus, Ohio. The data are listed in Table 12.1, p. 189 of Anselin (1988). Crime is measured as per capita residential burglaries and vehicle thefts, and housing values and income are measured in thousands of dollars. The OLS regression gives

$$\begin{aligned} \text{Crime} &= \underset{(4.735)}{68.619} - \underset{(0.334)}{1.597}\,\text{Housing} - \underset{(0.103)}{0.274}\,\text{Income} \end{aligned}$$

where the standard errors are given in parentheses.

We apply the tests proposed in the last section to the crime data. The dependent and independent variables *Crime, Housing* and *Income* are subject to the Box-Cox transformation while the constant term is not. The results are reported in Table 1. The joint LM test statistic for $H_0^a : \lambda = 0$ and $r = 0$, is 54.06. This is distributed as χ_2^2 under H_0^a and is significant. The LM test statistic for $H_0^b : \lambda = 0$ and $r = 1$, is 13.53. This is distributed as χ_2^2 under H_0^b and has a p-value of 0.001. Both the linear and loglinear models without spatial autocorrelation are rejected in favor of a more general Box-Cox model with spatial autocorrelation. Assuming a loglinear model, one does not reject the absence of spatial correlation. However, assuming a linear model, one rejects the absence of spatial correlation. These outcomes are not changed by allowing for local misspecification using the corresponding Bera and Yoon (1993) adjusted LM statistics. In addition, if one assumes no spatial correlation, one rejects loglinearity but not linearity of the model. Again, both outcomes are not changed by allowing for local misspecification using the Bera and Yoon (1993) adjustment. Conditional on a general Box-Cox model, the hypothesis of no spatial correlation is rejected with a p-value of 0.006. Conditional on spatial correlation, the loglinear model is rejected with a p-value of 0.000 and the linear model is not rejected with a p-value of 0.602.

For this empirical example, one does not know the true model. However, the evidence is against a loglinear model and in favor of a linear model with spatial autocorrelation. In the next section, Monte Carlo experiments are performed, where we know the true model and we can report the empirical size and power performance of these tests.

4 Monte Carlo Results

The experimental design used in the Monte Carlo simulations follows those extensively used in other spatial studies (e.g. Anselin and Rey, 1991; Florax and Folmer, 1992; Anselin et al., 1996). The model considered is given by

$$y^{(r)} = X^{(r)}\beta + Z\gamma + u \tag{44}$$

where $u = \lambda W u + v$. We use the spatial weight matrix from the crime data in Anselin (1988). The number of observations is $n = 49$. The explanatory variables X, an $n \times 2$ matrix, are generated from a uniform (0,10) distribution and the coefficients β's are set to 1. The Z variable consists of constant term and γ is set equal to 4. The error term v is generated from a standard normal distribution. In addition to a normal error, a student t error term is generated as well, with mean and variance equal to that of the normal variates. The tests are evaluated at their asymptotic critical value for $\alpha = 0.05$ and the power is reported. The three conditional tests in this paper involve numerical maximum likelihood estimation. These are computationally more expensive compared to the unconditional or Bera-Yoon type LM tests. For each combination of parameter values, 1000 replications were carried out.

Figure 1 plots the frequency of rejections in 1000 replications using the joint LM statistic for no spatial correlation and loglinearity, i.e., $H_0^a : \lambda = 0$ and $r = 0$. This test tends to over-reject with size equal to 18.5% rather than 5%. The power of the test increases as λ or r depart from zero. In fact, if the true model is linear, the frequency of rejections of loglinearity with no spatial autocorrelation is 100%. Figure 2 gives the frequency of rejections in 1000 replications using the joint LM statistic for no spatial correlation and linearity, i.e., $H_0^b : \lambda = 0$ and $r = 1$. This test tends to over-reject with size equal to 11.3% rather than 5%. The power of the test increases as λ departs from zero or r departs from 1. In fact, if the true model is loglinear, the frequency of rejections is 100%. Figure 3 gives the frequency of rejections in 1000 replications using the simple Rao-Score statistic for no spatial correlation assuming a loglinear model, i.e., $H_0^c : \lambda = 0$ assuming that $r = 0$. This is the standard LM test statistic for no spatial correlation given by Anselin (1988) and Anselin and Bera (1998). The size of the test is equal to 7.4% rather than 5%. For $r = 0$, the power of this test increases as λ departs from zero. However, this test is sensitive to departures of r from 0. In fact, if the true model is linear, we reject that $\lambda = 0$ only 3.3% of the time when true λ is equal to 0.6. This rejection frequency is only 0.2% when true λ is equal to -0.6. Figure 4 gives the frequency of rejections in 1000 replications of the Bera and Yoon (1993) adjusted Rao-Score statistic for H_0^c. This Bera-Yoon adjustment helps increase the power of the test as clear from comparing Figure 3 to Figure 4. However, the size of this test is 11.7% and is sensitive to departures from local misspecification. In fact, for $r = 1$, this test rejects the null when true in 83% of the cases. Figure 5 gives the frequency of rejections in 1000 replications using the simple Rao-Score statistic for no spatial correlation assuming a linear model, i.e., $H_0^d : \lambda = 0$ assuming that $r = 1$. The size of the test is equal to 8.5% rather than 5%. For $r = 1$, the power of this test increases as λ departs from zero. However, this test is sensitive to departures of r from 1. In fact, if the true model is loglinear, we reject that $\lambda = 0$ only 27.7% of the time when true λ is equal to 0.6 and 15.2% of the time when true λ is equal to - 0.6. Figure 6 gives the frequency of rejections in 1000 replications of the Bera and Yoon (1993) adjusted Rao-Score statistic for H_0^d. This adjustment helps increase the power of the test as clear from comparing Figure 5 to Figure 6. However, the size of this test is 9.6% and is sensitive to departures from local misspecification. In fact, for $r = 0$, this test rejects the null when true in 38% of the cases. Figure 7 gives the frequency of rejections in 1000 replications using the simple Rao-Score statistic for loglinearity assuming no spatial correlation, i.e., $H_0^e : r = 0$ assuming that $\lambda = 0$. The size of the test is equal to 16.6% rather than 5% and tends to over-reject the null when in fact it is true. For $\lambda = 0$, the power of this test increases as r departs from zero. This test is not very sensitive to departures of λ from 0. Figure 8 gives the frequency of rejections in 1000 replications of the Bera and Yoon (1993) adjusted Rao-Score statistic for H_0^e. This adjustment helps increase the power of the test as clear from comparing Figure 7 to Figure 8. However, the size of this test is 17.5% and is sensitive to departures from local misspecification. In fact, for $\lambda = 0.6$, this test rejects the null when true in 32% of the cases. This rejection frequency is 22% when true λ is equal to -0.6. Figure 9 gives the frequency of rejections in 1000 replications using the simple Rao-Score statistic for linearity assuming no spatial correlation, i.e., $H_0^f : r = 1$ assuming that $\lambda = 0$. The size of the test is equal to 8.2% rather than 5% and tends to over-reject the null when in fact it is true. For $\lambda = 0$, the power of this test increases as r departs from one. This test is not very sensitive to departures of λ from 0. Figure 10 gives the frequency of rejections in 1000 replications of the Bera and Yoon (1993) adjusted Rao-Score statistic for H_0^f. This adjustment helps increase the power of the test as clear from comparing Figure 9 to Figure 10. However, the size of this test is 8.9% and is sensitive to departures from local misspecification. In fact, for $\lambda = 0.6$, this test rejects the null when true in 26% of the cases. This rejection frequency is 13% when true λ is equal to -0.6. Figure 11 gives the conditional LM frequency of rejections for no spatial correlation assuming a general Box-Cox model, i.e., $H_0^g : \lambda = 0$ assuming an unknown r. This test tends to over-reject the null when true. This over-rejection depends on the value of r. The power of this test increases as λ departs from zero. Figure 12 gives the conditional LM frequency of rejections for loglinearity assuming the presence of spatial correlation, i.e., $H_0^h : r = 0$ assuming an unknown λ. The size of the test varies between 7.9% and 9.3% depending on the value of λ. The power of this test increases as r departs from zero. Figure 13 gives the conditional LM frequency of rejections for linearity assuming the presence of spatial correlation, i.e., $H_0^i : r = 1$ assuming an unknown λ. The size of the test varies between 8.0% and 10.6% depending on the value of λ. The power of this test increases as r departs from one.

We have also checked the sensitivity of our results to the normality assumption. A student t distribution with 3 degree of freedom was also considered with mean and variance equal to that of the normal variates. Except for differences in magnitudes of empirical size and power, the graphs for the t-distribution look the same as those for the normal distribution. The results are available upon request from the authors.

5 Conclusion

This paper derived joint, conditional and modified Rao-Score tests for functional form and spatial error correlation. They are illustrated using an empirical example. In addition, the power performance of these tests were compared using Monte Carlo experiments. Some of our findings are as follows: (i) Choosing the wrong functional form could lead to misleading inference regarding the presence or absence of spatial correlation. (ii) Ignoring spatial correlation when present could also lead to the wrong choice of functional form. Our experiments show that the power was more sensitive to functional form misspecification than misspecification of spatial error dependence. (iii) Bera and Yoon (1993) modified Rao-Score tests guard against local misspecification but their power deteriorate for large departures from the null hypothesis. (iv) Joint as well as conditional LM tests perform well in Monte Carlo experiments and are recommended.

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Table 1. Results for Crime Data

	Statistic	p-value
Joint LM Tests		
$H_0^a : \lambda = 0$ and $r = 0$	54.058	0.000
$H_0^b : \lambda = 0$ and $r = 1$	13.528	0.001
Rao Score Tests and Their Modified Forms		
$H_0^c : RS_{\lambda=0}$ assuming $r = 0$	2.063	0.151
$H_0^c : RS^*_{\lambda=0}$ assuming $r = 0$	0.304	0.581
$H_0^d : RS_{\lambda=0}$ assuming $r = 1$	11.442	0.001
$H_0^d : RS^*_{\lambda=0}$ assuming $r = 1$	13.504	0.000
$H_0^e : RS_{r=0}$ assuming $\lambda = 0$	53.754	0.000
$H_0^e : RS^*_{r=0}$ assuming $\lambda = 0$	51.995	0.000
$H_0^f : RS_{r=1}$ assuming $\lambda = 0$	0.024	0.878
$H_0^f : RS^*_{r=1}$ assuming $\lambda = 0$	2.086	0.149
Conditional LM Tests		
$H_0^g : \lambda = 0 \vert$unknown r	7.600	0.006
$H_0^h : r = 0 \vert$unknown λ	75.534	0.000
$H_0^i : r = 1 \vert$unknown λ	0.272	0.602

Appendix A: Hessain Matrix for the General Box-Cox Model with Spatial Correlation

From (4), one can write

$$v = (I - \lambda W)(y^{(r)} - X^{(r)}\beta - Z\gamma) = (I - \lambda W)u$$

and the second-order derivatives of the loglikelihood function given in (5) yield the following results:

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}v'v \tag{45}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\beta'} = -\frac{1}{\sigma^4}v'(I - \lambda W)X^{(r)} \tag{46}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\gamma'} = -\frac{1}{\sigma^4}v'(I - \lambda W)Z \tag{47}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\lambda} = -\frac{1}{\sigma^4}v'Wu \tag{48}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial r} = \frac{1}{\sigma^4}v'(I - \lambda W)(C(y,r) - C(X,r)\beta) \tag{49}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\sigma^2} = -\frac{1}{\sigma^4}[(I - \lambda W)X^{(r)}]'v \tag{50}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}[(I - \lambda W)X^{(r)}]'[(I - \lambda W)X^{(r)}] \tag{51}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\gamma'} = -\frac{1}{\sigma^2}[(I - \lambda W)X^{(r)}]'[(I - \lambda W)Z] \tag{52}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\lambda} = -\frac{1}{\sigma^2}(WX^{(r)})'v - \frac{1}{\sigma^2}[(I - \lambda W)X^{(r)}]'(Wu) \tag{53}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\beta\partial r} &= \frac{1}{\sigma^2}[(I - \lambda W)X^{(r)}]'[(I - \lambda W)(C(y,r) - C(X,r)\beta)] \\ &\quad + \frac{1}{\sigma^2}[(I - \lambda W)C(X,r)]'[(I - \lambda W)u]\end{aligned} \tag{54}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\sigma^2} = -\frac{1}{\sigma^4}[(I - \lambda W)Z]'v \tag{55}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\beta'} = -\frac{1}{\sigma^2}[(I - \lambda W)Z]'[(I - \lambda W)X^{(r)}] \tag{56}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\gamma'} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)Z] \tag{57}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\lambda} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'(Wu) - \frac{1}{\sigma^2}(WZ)'v \tag{58}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial r} = \frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)(C(y,r)-C(X,r)\beta)] \tag{59}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma^2} = -\frac{1}{\sigma^4}v'Wu \tag{60}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\beta'} = -\frac{1}{\sigma^2}v'WX^{(r)} - \frac{1}{\sigma^2}(Wu)'[(I-\lambda W)X^{(r)}] \tag{61}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\gamma'} = -\frac{1}{\sigma^2}v'WZ - \frac{1}{\sigma^2}[W(y^{(r)} - X^{(r)}\beta - Z\gamma)]'[(I-\lambda W)Z] \tag{62}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\lambda} = -\sum_{i=1}^{n}\frac{\omega_i^2}{(1-\lambda\omega_i)^2} - \frac{1}{\sigma^2}(Wu)'(Wu) \tag{63}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial r} = \frac{1}{\sigma^2}[(I-\lambda W)(C(y,r)-C(X,r)\beta)]'Wu + \frac{1}{\sigma^2}v'W(C(y,r)-C(X,r)\beta) \tag{64}$$

$$\frac{\partial^2 \log L}{\partial r\partial\sigma^2} = \frac{1}{\sigma^4}v'(I-\lambda W)(C(y,r)-C(X,r)\beta) \tag{65}$$

$$\frac{\partial^2 \log L}{\partial r\partial\beta'} = \frac{1}{\sigma^2}v'(I-\lambda W)C(X,r) + \frac{1}{\sigma^2}(C(y,r)-C(X,r)\beta)'(I-\lambda W)'[(I-\lambda W)X^{(r)}] \tag{66}$$

$$\frac{\partial^2 \log L}{\partial r\partial\gamma'} = \frac{1}{\sigma^2}[(I-\lambda W)(C(y,r)-C(X,r)\beta)]'[(I-\lambda W)Z] \tag{67}$$

$$\frac{\partial^2 \log L}{\partial r\partial\lambda} = \frac{1}{\sigma^2}(Wu)'(I-\lambda W)(C(y,r)-C(X,r)\beta) + \frac{1}{\sigma^2}v'W(C(y,r)-C(X,r)\beta) \tag{68}$$

$$\frac{\partial^2 \log L}{\partial r\partial r} = -\frac{1}{\sigma^2}[(I-\lambda W)(C(y,r)-C(X,r)\beta)]'(I-\lambda W)(C(y,r)-C(X,r)\beta) - \frac{1}{\sigma^2}v'(I-\lambda W)(C'(y,r)-C'(X,r)\beta) \tag{69}$$

where $C'(y,r) = \partial C(y,r)/\partial r = [r^2y^r(\log y)^2 - 2ry^r\log y + 2y^r - 2]/r^3$ and $C'(X,r) = \partial C(X,r)/\partial r$ is similarly defined.

Appendix B: Joint Tests

B.1 Hessian Matrix for the Loglinear Model with No Spatial Error Dependence

Under the hull hypothesis $H_0^a : \lambda = 0$ and $r = 0$, the second order derivatives of the loglikelihood function are given by

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} v' v \tag{70}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \beta'} = -\frac{1}{\sigma^4} v' (\log X) \tag{71}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \gamma'} = -\frac{1}{\sigma^4} v' Z \tag{72}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \lambda} = -\frac{1}{\sigma^4} v' W(\log y - (\log X)\beta - Z\gamma) \tag{73}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial r} = \frac{1}{\sigma^4} v' [\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2 \beta] \tag{74}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \sigma^2} = -\frac{1}{\sigma^4} (\log X)' v \tag{75}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \beta'} = -\frac{1}{\sigma^2} (\log X)' (\log X) \tag{76}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \gamma'} = -\frac{1}{\sigma^2} (\log X)' Z \tag{77}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \lambda} = -\frac{1}{\sigma^2} (W \log X)' v - \frac{1}{\sigma^2} (\log X)' [W(\log y - \ (\log X)\beta - Z\gamma)] \tag{78}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial r} = \frac{1}{\sigma^2} (\log X)' [\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2 \beta] + \frac{1}{\sigma^2} [\frac{1}{2}(\log X)^2]' [\log y - (\log X)\beta - Z\gamma] \tag{79}$$

$$\frac{\partial^2 \log L}{\partial \gamma \partial \sigma^2} = -\frac{1}{\sigma^4} Z' v \tag{80}$$

$$\frac{\partial^2 \log L}{\partial \gamma \partial \beta'} = -\frac{1}{\sigma^2} Z' (\log X) \tag{81}$$

$$\frac{\partial^2 \log L}{\partial \gamma \partial \gamma'} = -\frac{1}{\sigma^2} Z' Z \tag{82}$$

$$\frac{\partial^2 \log L}{\partial \gamma \partial \lambda} = -\frac{1}{\sigma^2} Z' [W(\log y - (\log X)\beta - Z\gamma)] - \frac{1}{\sigma^2} (WZ)' v \tag{83}$$

$$\frac{\partial^2 \log L}{\partial \gamma \partial r} = \frac{1}{\sigma^2} Z' [\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2 \beta] \tag{84}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma^2} = -\frac{1}{\sigma^4}v'W(\log y - (\log X)\beta - Z\gamma) \quad (85)$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\beta'} = -\frac{1}{\sigma^2}v'W\log X - \frac{1}{\sigma^2}[W(\log y - (\log X)\beta - Z\gamma)]'(\log X) \quad (86)$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\gamma'} = -\frac{1}{\sigma^2}v'WZ - \frac{1}{\sigma^2}[W(\log y - (\log X)\beta - Z\gamma)]'Z \quad (87)$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\lambda} = -\sum_{i=1}^{n}\omega_i^2 - \frac{1}{\sigma^2}[W(\log y - (\log X)\beta - Z\gamma)]'[W(\log y - (\log X)\beta - Z\gamma)] \quad (88)$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial r} = \frac{1}{\sigma^2}[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta]'Wv + \frac{1}{\sigma^2}v'W[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta] \quad (89)$$

$$\frac{\partial^2 \log L}{\partial r\partial\sigma^2} = \frac{1}{\sigma^4}v'[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta] \quad (90)$$

$$\frac{\partial^2 \log L}{\partial r\partial\beta'} = \frac{1}{\sigma^2}v'(\frac{1}{2}(\log X)^2) + \frac{1}{\sigma^2}[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta]'(\log X) \quad (91)$$

$$\frac{\partial^2 \log L}{\partial r\partial\gamma'} = \frac{1}{\sigma^2}[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta]'Z \quad (92)$$

$$\frac{\partial^2 \log L}{\partial r\partial\lambda} = \frac{1}{\sigma^2}[W(\log y - (\log X)\beta - Z\gamma)]'[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta] + \frac{1}{\sigma^2}v'W[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta] \quad (93)$$

$$\frac{\partial^2 \log L}{\partial r\partial r} = -\frac{1}{\sigma^2}[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta]'[\frac{1}{2}(\log y)^2 - \frac{1}{2}(\log X)^2\beta] - \frac{1}{\sigma^2}v'[\frac{1}{3}(\log y)^3 - \frac{1}{3}(\log X)^3\beta] \quad (94)$$

B.2 Hessian Matrix for the Linear Model with No Spatial Error Dependence

Under the null hypothesis $H_0^b : \lambda = 0$ and $r = 1$, the second order derivatives of the loglikelihood function are given by

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}v'v \quad (95)$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\beta'} = -\frac{1}{\sigma^4}v'(X - J_{nK}) \quad (96)$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\gamma'} = -\frac{1}{\sigma^4}v'Z \quad (97)$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\lambda} = -\frac{1}{\sigma^4}v'W(y - \iota_n - (X - J_{nK})\beta - Z\gamma) \quad (98)$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial r} = \frac{1}{\sigma^4}v'(C(y,1) - C(X,1)\beta) \quad (99)$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\sigma^2} = -\frac{1}{\sigma^4}(X - J_{nK})'v \tag{100}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}(X - J_{nK})'(X - J_{nK}) \tag{101}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\gamma'} = -\frac{1}{\sigma^2}(X - J_{nK})'Z \tag{102}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\lambda} = -\frac{1}{\sigma^2}[W(X - J_{nK})]'v - \frac{1}{\sigma^2}(X - J_{nK})'[W(y - \iota_n - (X - J_{nK})\beta - Z\gamma)] \tag{103}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\beta\partial r} &= \frac{1}{\sigma^2}(X - J_{nK})'[(C(y,1) - C(X,1)\beta)] \\ &\quad + \frac{1}{\sigma^2}[C(X,1)]'[y - \iota_n - (X - J_{nK})\beta - Z\gamma]\end{aligned} \tag{104}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\sigma^2} = -\frac{1}{\sigma^4}Z'v \tag{105}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\beta'} = -\frac{1}{\sigma^2}Z'(X - J_{nK}) \tag{106}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\gamma'} = -\frac{1}{\sigma^2}Z'Z \tag{107}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\lambda} = -\frac{1}{\sigma^2}Z'W[y - \iota_n - (X - J_{nK})\beta - Z\gamma] - \frac{1}{\sigma^2}(WZ)'v \tag{108}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial r} = \frac{1}{\sigma^2}Z'(C(y,1) - C(X,1)\beta] \tag{109}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma^2} = -\frac{1}{\sigma^4}v'W(y - \iota_n - (X - J_{nK})\beta - Z\gamma) \tag{110}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\beta'} = -\frac{1}{\sigma^2}v'W(X - J_{nK}) - \frac{1}{\sigma^2}[W(y - \iota_n - (X - J_{nK})\beta - Z\gamma)]'(X - J_{nK}) \tag{111}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\gamma'} = -\frac{1}{\sigma^2}v'WZ - \frac{1}{\sigma^2}[W(y - \iota_n - (X - J_{nK})\beta - Z\gamma)]'Z \tag{112}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\lambda\partial\lambda} &= -\sum_{i=1}^{n}\omega_i^2 \\ &\quad -\frac{1}{\sigma^2}[W(y - \iota_n - (X - J_{nK})\beta - Z\gamma)]'[W(y - \iota_n - (X - J_{nK})\beta - Z\gamma)]\end{aligned} \tag{113}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\lambda\partial r} &= \frac{1}{\sigma^2}[C(y,1) - C(X,1)\beta]'W(y - \iota_n - (X - J_{nK})\beta - Z\gamma) \\ &\quad + \frac{1}{\sigma^2}v'W(C(y,1) - C(X,1)\beta)\end{aligned} \tag{114}$$

$$\frac{\partial^2 \log L}{\partial r \partial \sigma^2} = \frac{1}{\sigma^4} v'(C(y,1) - C(X,1)\beta) \tag{115}$$

$$\frac{\partial^2 \log L}{\partial r \partial \beta'} = \frac{1}{\sigma^2} v' C(X,1) + \frac{1}{\sigma^2}(C(y,1) - C(X,1)\beta)'(X - J_{nK}) \tag{116}$$

$$\frac{\partial^2 \log L}{\partial r \partial \gamma'} = \frac{1}{\sigma^2}[C(y,1) - C(X,1)\beta]' Z \tag{117}$$

$$\begin{aligned} \frac{\partial^2 \log L}{\partial r \partial \lambda} &= \frac{1}{\sigma^2}[W(y - \iota_n - (X - J_{nK})\beta - Z\gamma)]'(C(y,1) - C(X,1)\beta) \\ &\quad + \frac{1}{\sigma^2} v' W (C(y,1) - C(X,1)\beta) \end{aligned} \tag{118}$$

$$\frac{\partial^2 \log L}{\partial r \partial r} = -\frac{1}{\sigma^2}[C(y,1) - C(X,1)\beta]'[C(y,1) - C(X,1)\beta] - \frac{1}{\sigma^2} v'(C'(y,1) - C'(X,1)\beta) \tag{119}$$

Appendix C: Conditional Tests

C.1 Hessian Matrix for the null hypothesis H_0^g: $\lambda = 0$ |unknown r

Under the null hypothesis H_0^g: $\lambda = 0$ |unknown r, the second order derivatives of the loglikelihood function are given by

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} v'v \tag{120}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \beta'} = -\frac{1}{\sigma^4} v' X^{(r)} \tag{121}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \gamma'} = -\frac{1}{\sigma^4} v' Z \tag{122}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial \lambda} = -\frac{1}{\sigma^4} v' W u \tag{123}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2 \partial r} = \frac{1}{\sigma^4} v'(C(y,r) - C(X,r)\beta) \tag{124}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \sigma^2} = -\frac{1}{\sigma^4}[X^{(r)}]' v \tag{125}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \beta'} = -\frac{1}{\sigma^2}[X^{(r)}]'[X^{(r)}] \tag{126}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \gamma'} = -\frac{1}{\sigma^2}[X^{(r)}]' Z \tag{127}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \lambda} = -\frac{1}{\sigma^2}[W X^{(r)}]' v - \frac{1}{\sigma^2}[X^{(r)}]'(Wu) \tag{128}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial r} = \frac{1}{\sigma^2}[X^{(r)}]'[(C(y,r) - C(X,r)\beta)] + \frac{1}{\sigma^2}[C(X,r)]' u \tag{129}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\sigma^2} = -\frac{1}{\sigma^4}Z'v \tag{130}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\beta'} = -\frac{1}{\sigma^2}Z'[X^{(r)}] \tag{131}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\gamma'} = -\frac{1}{\sigma^2}Z'Z \tag{132}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\lambda} = -\frac{1}{\sigma^2}Z'(Wu) - \frac{1}{\sigma^2}(WZ)'v \tag{133}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial r} = \frac{1}{\sigma^2}Z'[(C(y,r) - C(X,r)\beta)] \tag{134}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma^2} = -\frac{1}{\sigma^4}v'Wu \tag{135}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\beta'} = -\frac{1}{\sigma^2}v'WX^{(r)} - \frac{1}{\sigma^2}(Wu)'[X^{(r)}] \tag{136}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\gamma'} = -\frac{1}{\sigma^2}v'WZ - \frac{1}{\sigma^2}[W(y^{(r)} - \ X^{(r)}\beta - Z\gamma)]'Z \tag{137}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\lambda} = -\sum_{i=1}^{n}\omega_i^2 - \frac{1}{\sigma^2}(Wu)'(Wu) \tag{138}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial r} = \frac{1}{\sigma^2}[C(y,r) - C(X,r)\beta]'Wu + \frac{1}{\sigma^2}v'W(C(y,r) - C(X,r)\beta) \tag{139}$$

$$\frac{\partial^2 \log L}{\partial r\partial\sigma^2} = \frac{1}{\sigma^4}v'(C(y,r) - C(X,r)\beta) \tag{140}$$

$$\frac{\partial^2 \log L}{\partial r\partial\beta'} = \frac{1}{\sigma^2}v'C(X,r) + \frac{1}{\sigma^2}(C(y,r) - C(X,r)\beta)'[X^{(r)}] \tag{141}$$

$$\frac{\partial^2 \log L}{\partial r\partial\gamma'} = \frac{1}{\sigma^2}[(C(y,r) - C(X,r)\beta)]'Z \tag{142}$$

$$\frac{\partial^2 \log L}{\partial r\partial\lambda} = \frac{1}{\sigma^2}(Wu)'(C(y,r) - C(X,r)\beta) + \frac{1}{\sigma^2}v'W(C(y,r) - C(X,r)\beta) \tag{143}$$

$$\frac{\partial^2 \log L}{\partial r\partial r} = -\frac{1}{\sigma^2}[(C(y,r) - C(X,r)\beta)]'(C(y,r) - C(X,r)\beta) - \frac{1}{\sigma^2}v'(C'(y,r) - C'(X,r)\beta) \tag{144}$$

C.2 Hessian Matrix for the null hypothesis H_0^h: $r = 0$ |unknown λ

Under the null hypothesis H_0: $r = 0$ |unknown λ, the second order derivatives of the loglikelihood function are given by

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}v'v \tag{145}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\beta'} = -\frac{1}{\sigma^4}v'(I-\lambda W)\log X \tag{146}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\gamma'} = -\frac{1}{\sigma^4}v'(I-\lambda W)Z \tag{147}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\lambda} = -\frac{1}{\sigma^4}v'Wu \tag{148}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial r} = \frac{1}{\sigma^4}v'(I-\lambda W)(C(y,0)-C(X,0)\beta) \tag{149}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\sigma^2} = -\frac{1}{\sigma^4}[(I-\lambda W)\log X]'v \tag{150}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}[(I-\lambda W)\log X]'[(I-\lambda W)\log X] \tag{151}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\gamma'} = -\frac{1}{\sigma^2}[(I-\lambda W)\log X]'[(I-\lambda W)Z] \tag{152}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\lambda} = -\frac{1}{\sigma^2}(W\log X)'v - \frac{1}{\sigma^2}[(I-\lambda W)\log X]'(Wu) \tag{153}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\beta\partial r} &= \frac{1}{\sigma^2}[(I-\lambda W)\log X]'[(I-\lambda W)(C(y,0)-C(X,0)\beta)] \\ &\quad +\frac{1}{\sigma^2}[(I-\lambda W)C(X,0)]'[(I-\lambda W)u]\end{aligned} \tag{154}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\sigma^2} = -\frac{1}{\sigma^4}[(I-\lambda W)Z]'v \tag{155}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\beta'} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)\log X] \tag{156}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\gamma'} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)Z] \tag{157}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\lambda} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'(Wu) - \frac{1}{\sigma^2}(WZ)'v \tag{158}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial r} = \frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)(C(y,0)-C(X,0)\beta)] \tag{159}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma^2} = -\frac{1}{\sigma^4}v'Wu \tag{160}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\beta'} = -\frac{1}{\sigma^2}v'W\log X - \frac{1}{\sigma^2}(Wu)'[(I-\lambda W)\log X] \tag{161}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\gamma'} = -\frac{1}{\sigma^2}v'WZ - \frac{1}{\sigma^2}[W(\log y - \beta\log X \ - Z\gamma)]'[(I-\lambda W)Z] \tag{162}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\lambda} = -\sum_{i=1}^{n}\frac{\omega_i^2}{(1-\lambda\omega_i)^2} - \frac{1}{\sigma^2}(Wu)'(Wu) \tag{163}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial r} = \frac{1}{\sigma^2}[(I-\lambda W)(C(y,0)-C(X,0)\beta)]'Wu + \frac{1}{\sigma^2}v'W(C(y,0)-C(X,0)\beta) \tag{164}$$

$$\frac{\partial^2 \log L}{\partial r\partial\sigma^2} = \frac{1}{\sigma^4}v'(I-\lambda W)(C(y,0)-C(X,0)\beta) \tag{165}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial r\partial\beta'} &= \frac{1}{\sigma^2}v'(I-\lambda W)C(X,0) \\ &+\frac{1}{\sigma^2}(C(y,0)-C(X,0)\beta)'(I-\lambda W)'[(I-\lambda W)\log X]\end{aligned} \tag{166}$$

$$\frac{\partial^2 \log L}{\partial r\partial\gamma'} = \frac{1}{\sigma^2}[(I-\lambda W)(C(y,0)-C(X,0)\beta)]'[(I-\lambda W)Z] \tag{167}$$

$$\frac{\partial^2 \log L}{\partial r\partial\lambda} = \frac{1}{\sigma^2}(Wu)'(I-\lambda W)(C(y,0)-C(X,0)\beta) + \frac{1}{\sigma^2}v'W(C(y,0)-C(X,0)\beta) \tag{168}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial r\partial r} &= -\frac{1}{\sigma^2}[(I-\lambda W)(C(y,0)-C(X,0)\beta)]'(I-\lambda W)(C(y,0)-C(X,0)\beta) \\ &-\frac{1}{\sigma^2}v'(I-\lambda W)(C'(y,0)-C'(X,0)\beta)\end{aligned} \tag{169}$$

C.3 Hessian Matrix for the null hypothesis H_0^i: $r = 1$ |unknown λ

Under the null hypothesis H_0: $r = 1$ |unknown λ, the second order derivatives of the loglikelihood function are given by

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}v'v \tag{170}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\beta'} = -\frac{1}{\sigma^4}v'(I-\lambda W)(X-1) \tag{171}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\gamma'} = -\frac{1}{\sigma^4}v'(I-\lambda W)Z \tag{172}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial\lambda} = -\frac{1}{\sigma^4}v'Wu \tag{173}$$

$$\frac{\partial^2 \log L}{\partial\sigma^2\partial r} = \frac{1}{\sigma^4}v'(I-\lambda W)(C(y,1)-C(X,1)\beta) \tag{174}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\sigma^2} = -\frac{1}{\sigma^4}[(I-\lambda W)(X-1)]'v \tag{175}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}[(I-\lambda W)(X-1)]'[(I-\lambda W)(X-1)] \tag{176}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\gamma'} = -\frac{1}{\sigma^2}[(I-\lambda W)(X-1)]'[(I-\lambda W)Z] \tag{177}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\lambda} = -\frac{1}{\sigma^2}[W(X-1)]'v - \frac{1}{\sigma^2}[(I-\lambda W)(X-1)]'(Wu) \tag{178}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\beta\partial r} &= \frac{1}{\sigma^2}[(I-\lambda W)(X-1)]'[(I-\lambda W)(C(y,1)-C(X,1)\beta)] \\ &\quad +\frac{1}{\sigma^2}[(I-\lambda W)C(X,1)]'[(I-\lambda W)u]\end{aligned} \tag{179}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\sigma^2} = -\frac{1}{\sigma^4}[(I-\lambda W)Z]'v \tag{180}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\beta'} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)(X-1)] \tag{181}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\gamma'} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)Z] \tag{182}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial\lambda} = -\frac{1}{\sigma^2}[(I-\lambda W)Z]'(Wu) - \frac{1}{\sigma^2}(WZ)'v \tag{183}$$

$$\frac{\partial^2 \log L}{\partial\gamma\partial r} = \frac{1}{\sigma^2}[(I-\lambda W)Z]'[(I-\lambda W)(C(y,1)-C(X,1)\beta)] \tag{184}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma^2} = -\frac{1}{\sigma^4}v'Wu \tag{185}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\beta'} = -\frac{1}{\sigma^2}v'W(X-1) - \frac{1}{\sigma^2}(Wu)'[(I-\lambda W)(X-1)] \tag{186}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\gamma'} = -\frac{1}{\sigma^2}v'WZ - \frac{1}{\sigma^2}[Wu]'[(I-\lambda W)Z] \tag{187}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial\lambda} = -\sum_{i=1}^{n}\frac{\omega_i^2}{(1-\lambda\omega_i)^2} - \frac{1}{\sigma^2}(Wu)'(Wu) \tag{188}$$

$$\frac{\partial^2 \log L}{\partial\lambda\partial r} = \frac{1}{\sigma^2}[(I-\lambda W)(C(y,1)-C(X,1)\beta)]'Wu + \frac{1}{\sigma^2}v'W(C(y,1)-C(X,1)\beta) \tag{189}$$

$$\frac{\partial^2 \log L}{\partial r\partial\sigma^2} = \frac{1}{\sigma^4}v'(I-\lambda W)[C(y,1)-C(X,1)\beta] \tag{190}$$

$$\frac{\partial^2 \log L}{\partial r\partial\beta'} = \frac{1}{\sigma^2}v'(I-\lambda W)C(X,1)$$

$$+\frac{1}{\sigma^2}(C(y,1)-C(X,1)\beta)'(I-\lambda W)'[(I-\lambda W)(X-1)] \tag{191}$$

$$\frac{\partial^2 \log L}{\partial r \partial \gamma'} = \frac{1}{\sigma^2}[(I-\lambda W)(C(y,1)-C(X,1)\beta)]'[(I-\lambda W)Z] \tag{192}$$

$$\frac{\partial^2 \log L}{\partial r \partial \lambda} = \frac{1}{\sigma^2}(Wu)'(I-\lambda W)(C(y,1)-C(X,1)\beta)+\frac{1}{\sigma^2}v'W(C(y,1)-C(X,1)\beta) \tag{193}$$

$$\frac{\partial^2 \log L}{\partial r \partial r} = -\frac{1}{\sigma^2}[(I-\lambda W)(C(y,1)-C(X,1)\beta)]'(I-\lambda W)(C(y,1)-C(X,1)\beta) -\frac{1}{\sigma^2}v'(I-\lambda W)(C'(y,1)-C'(X,1)\beta) \tag{194}$$

Figure 1. Joint Test H_o^a: r=0 and λ=0

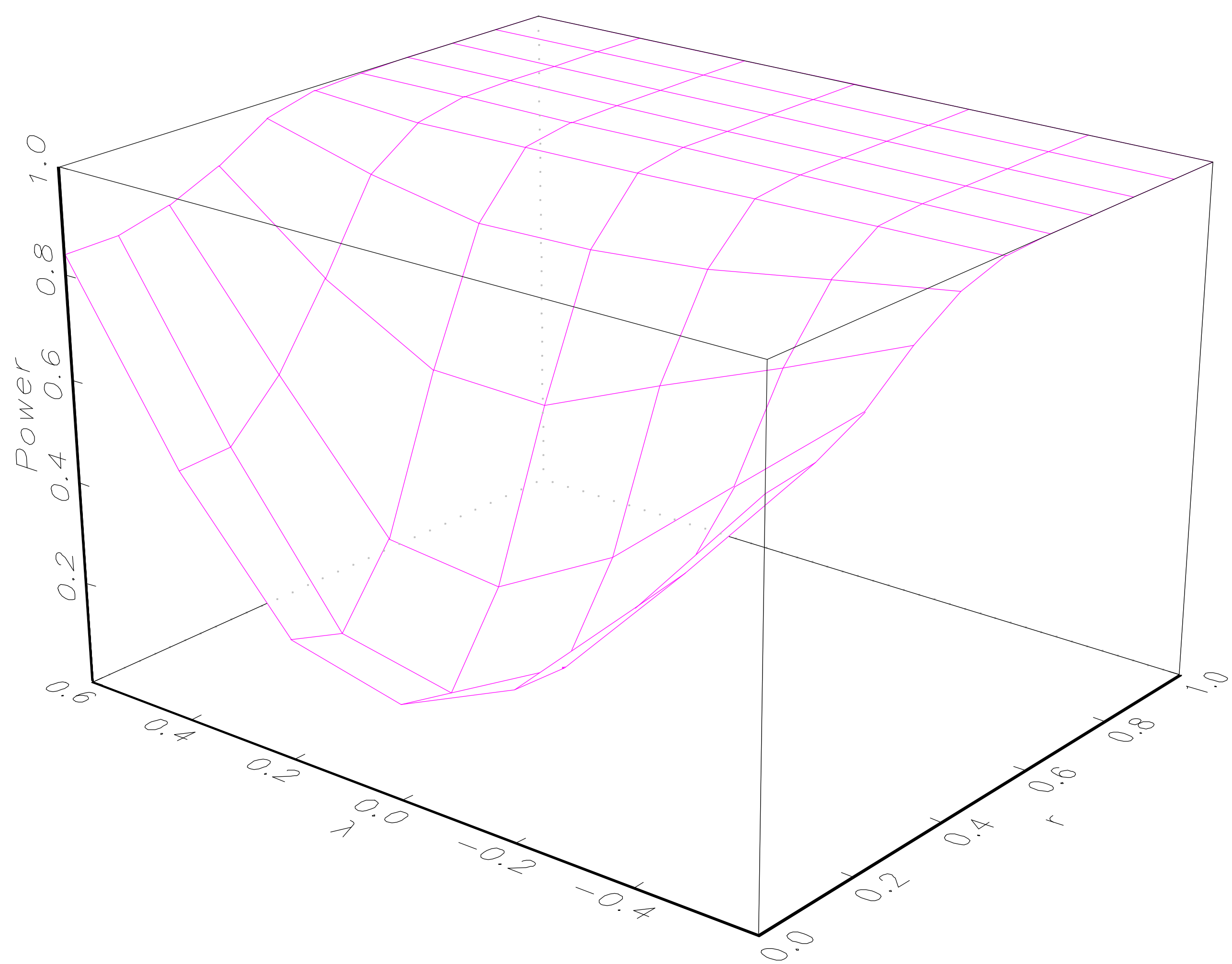
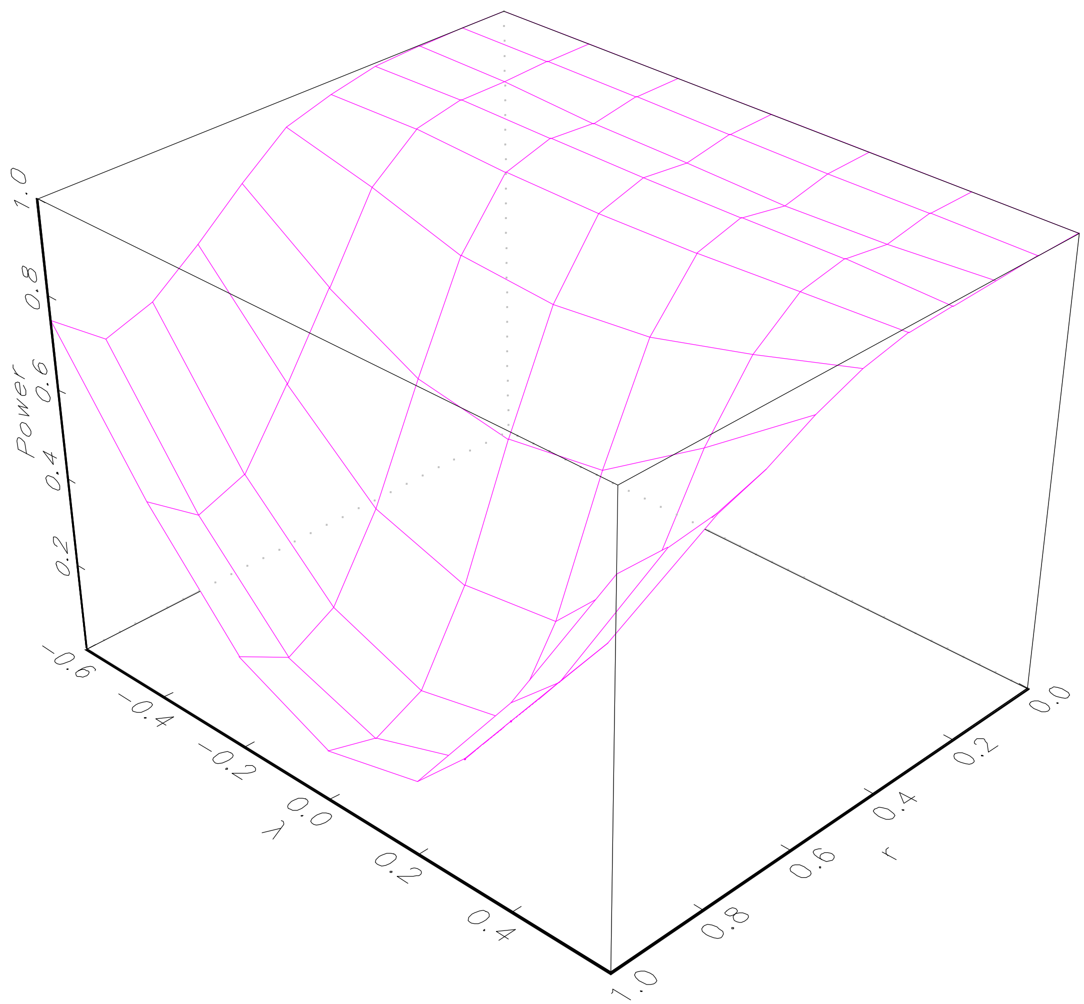


Figure 2. Joint Test H_o^b: r=1 and λ=0



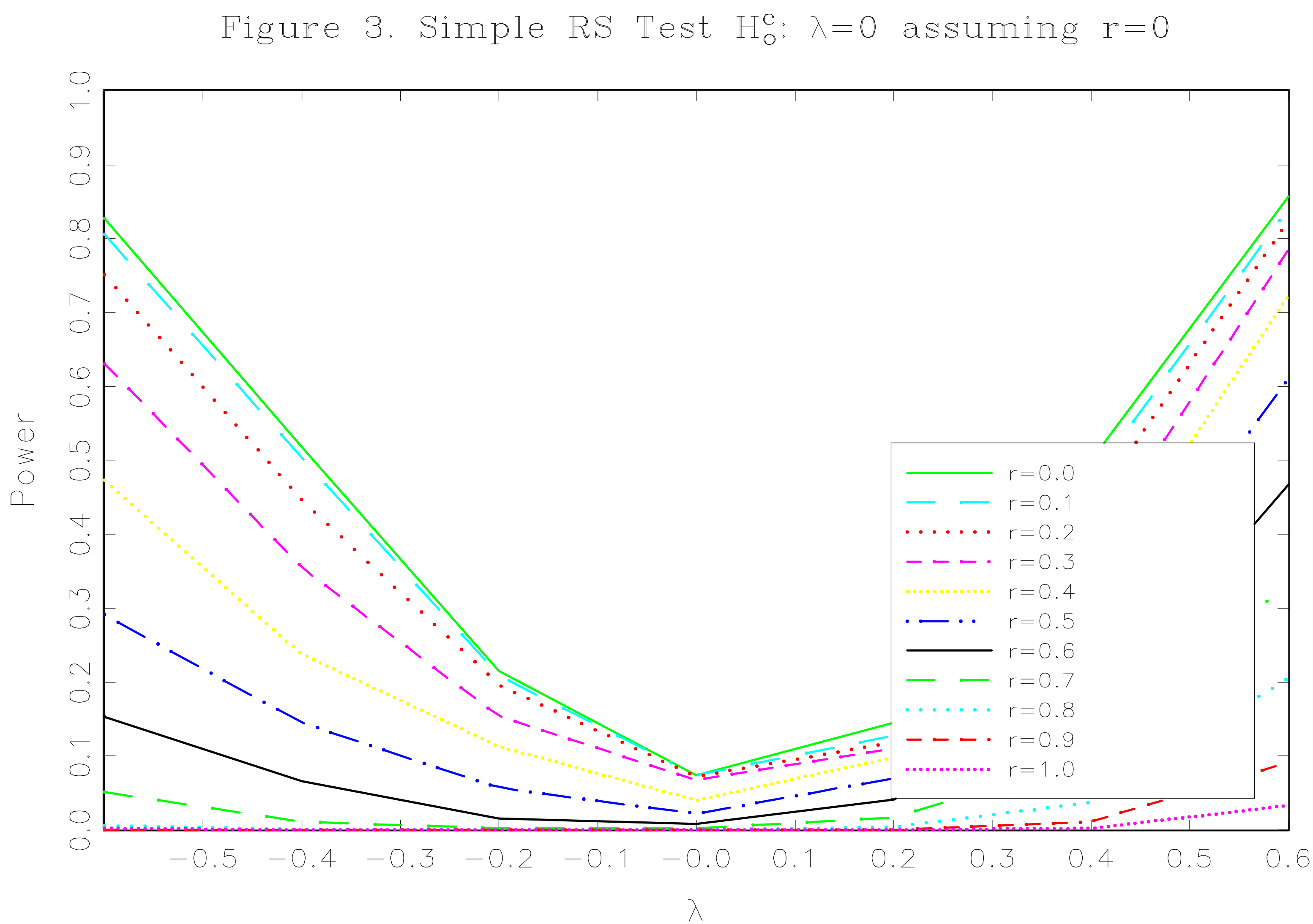


Figure 3. Simple RS Test H_o^c: λ=0 assuming r=0
Power
λ
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
−0.5
−0.4
−0.3
−0.2
−0.1
−0.0
0.1
0.2
0.3
0.4
0.5
0.6
r=0.0
r=0.1
r=0.2
r=0.3
r=0.4
r=0.5
r=0.6
r=0.7
r=0.8
r=0.9
r=1.0

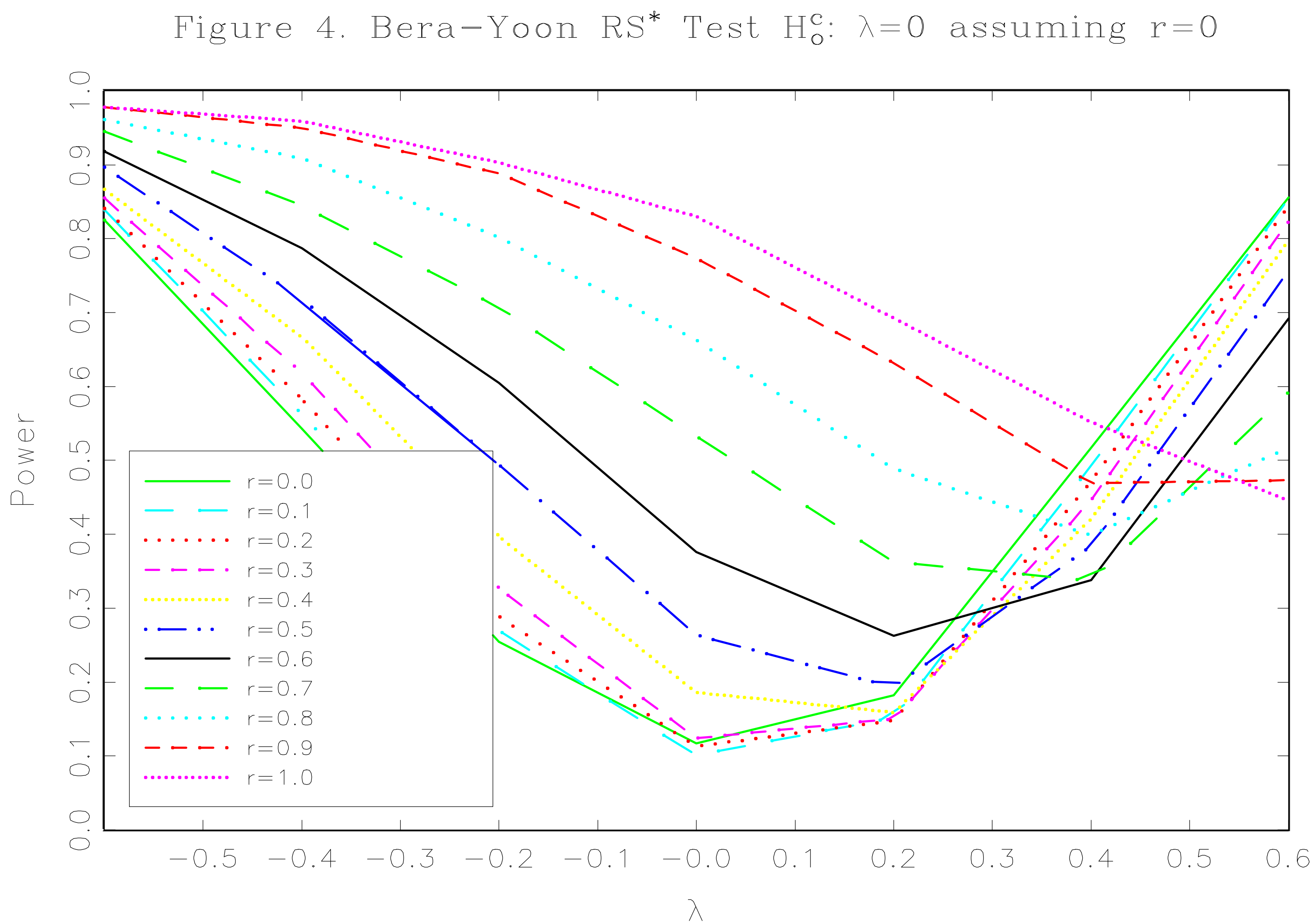


Figure 4. Bera−Yoon RS* Test H_o^c: λ=0 assuming r=0
Power
λ
r=0.0
r=0.1
r=0.2
r=0.3
r=0.4
r=0.5
r=0.6
r=0.7
r=0.8
r=0.9
r=1.0

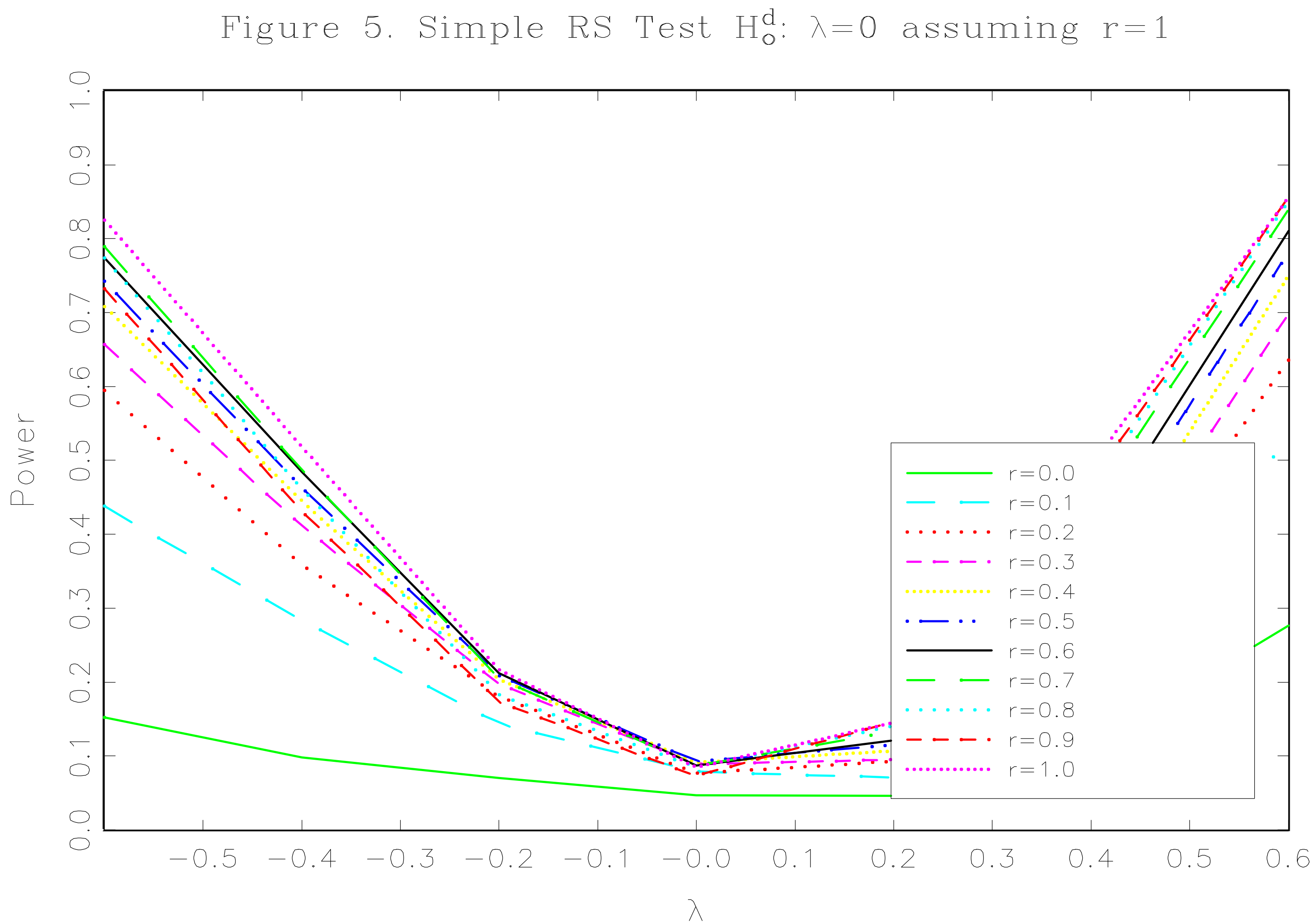


Figure 5. Simple RS Test H_o^d: λ=0 assuming r=1
Power
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
−0.5
−0.4
−0.3
−0.2
−0.1
−0.0
0.1
0.2
0.3
0.4
0.5
0.6
λ
r=0.0
r=0.1
r=0.2
r=0.3
r=0.4
r=0.5
r=0.6
r=0.7
r=0.8
r=0.9
r=1.0

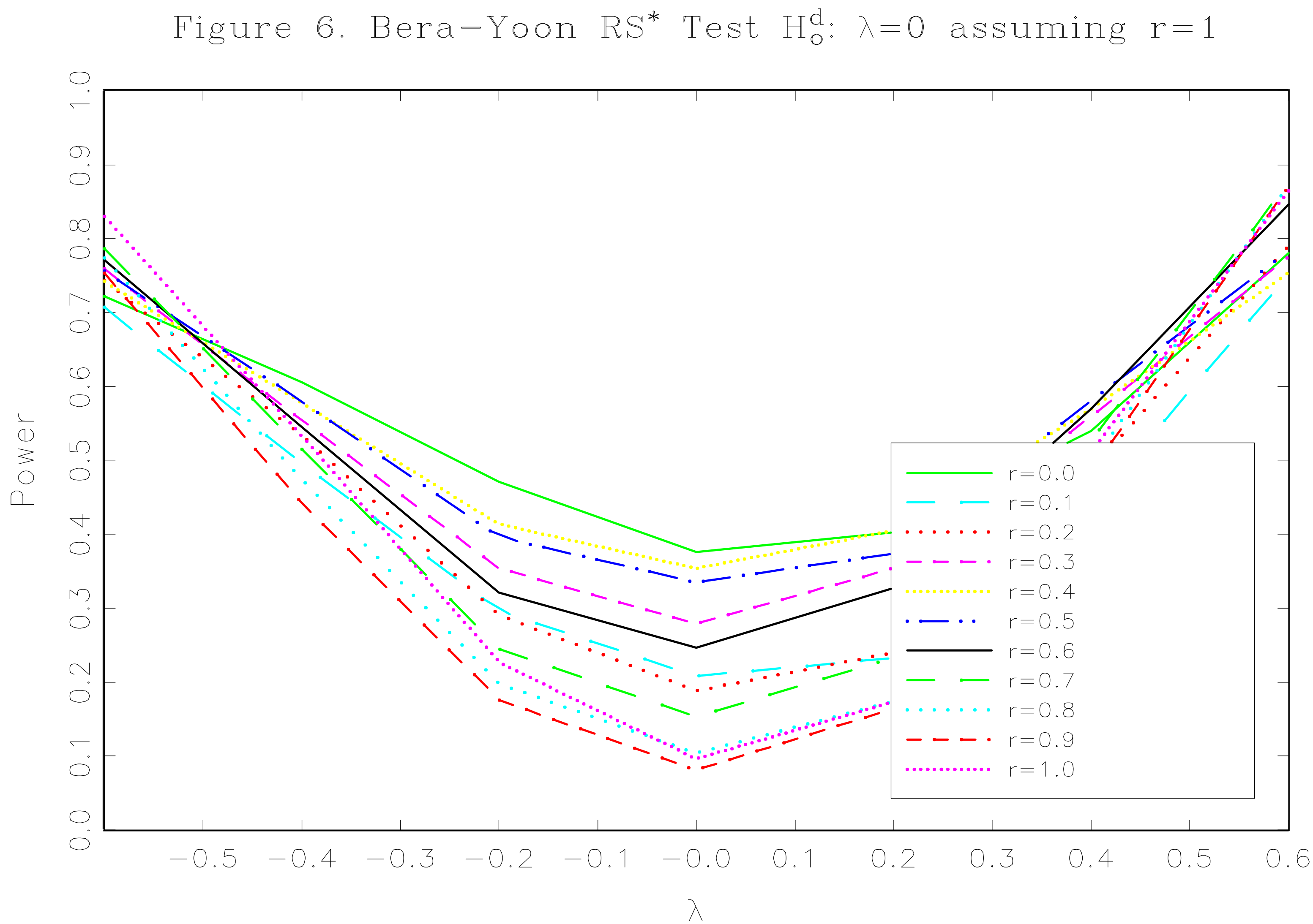
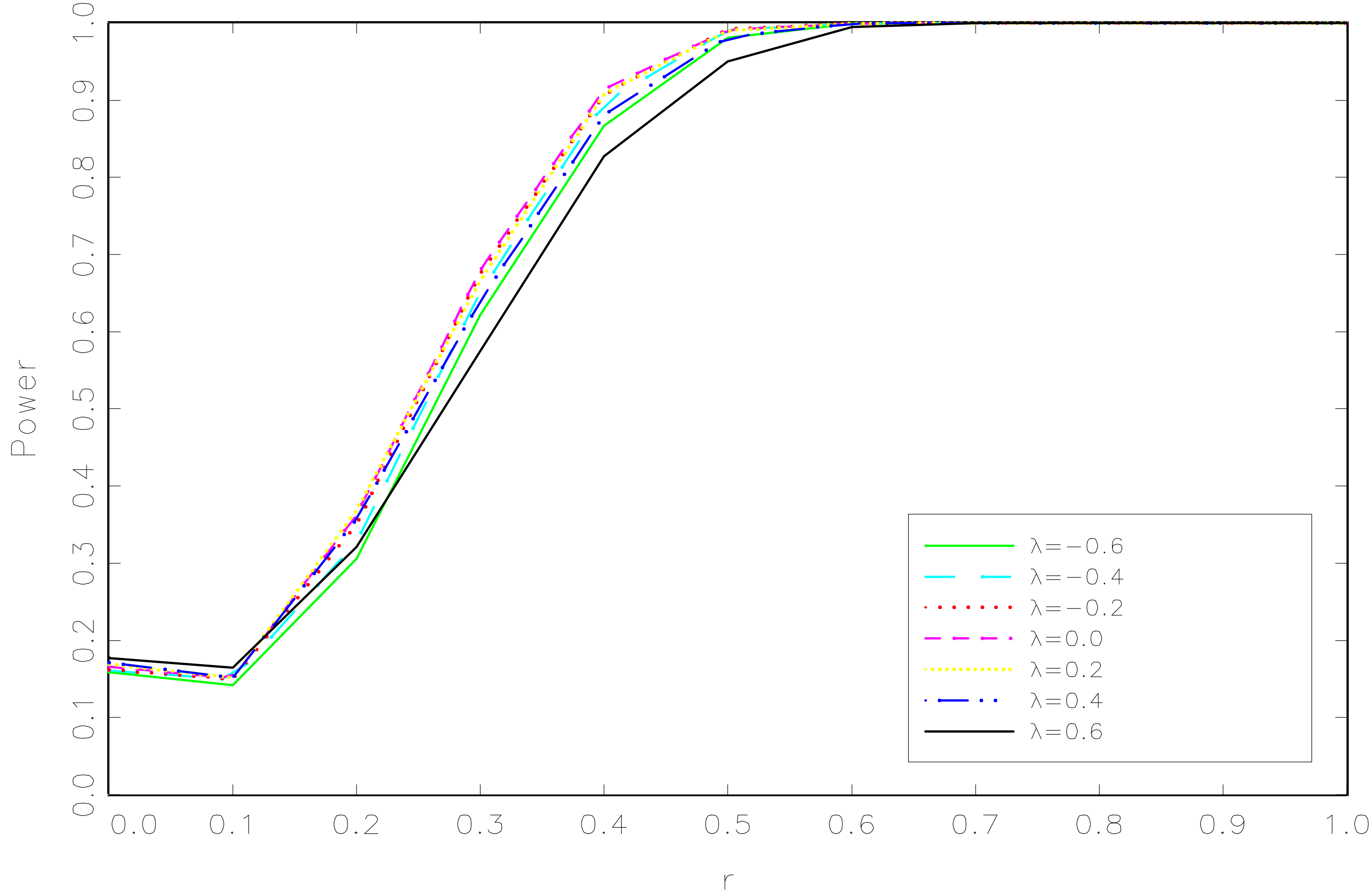


Figure 6. Bera-Yoon RS* Test H_o^d: λ=0 assuming r=1
Power
λ
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
−0.5
−0.4
−0.3
−0.2
−0.1
−0.0
0.1
0.2
0.3
0.4
0.5
0.6
r=0.0
r=0.1
r=0.2
r=0.3
r=0.4
r=0.5
r=0.6
r=0.7
r=0.8
r=0.9
r=1.0

Figure 7. Simple RS Test H_o^e: $r=0$ assuming $\lambda=0$



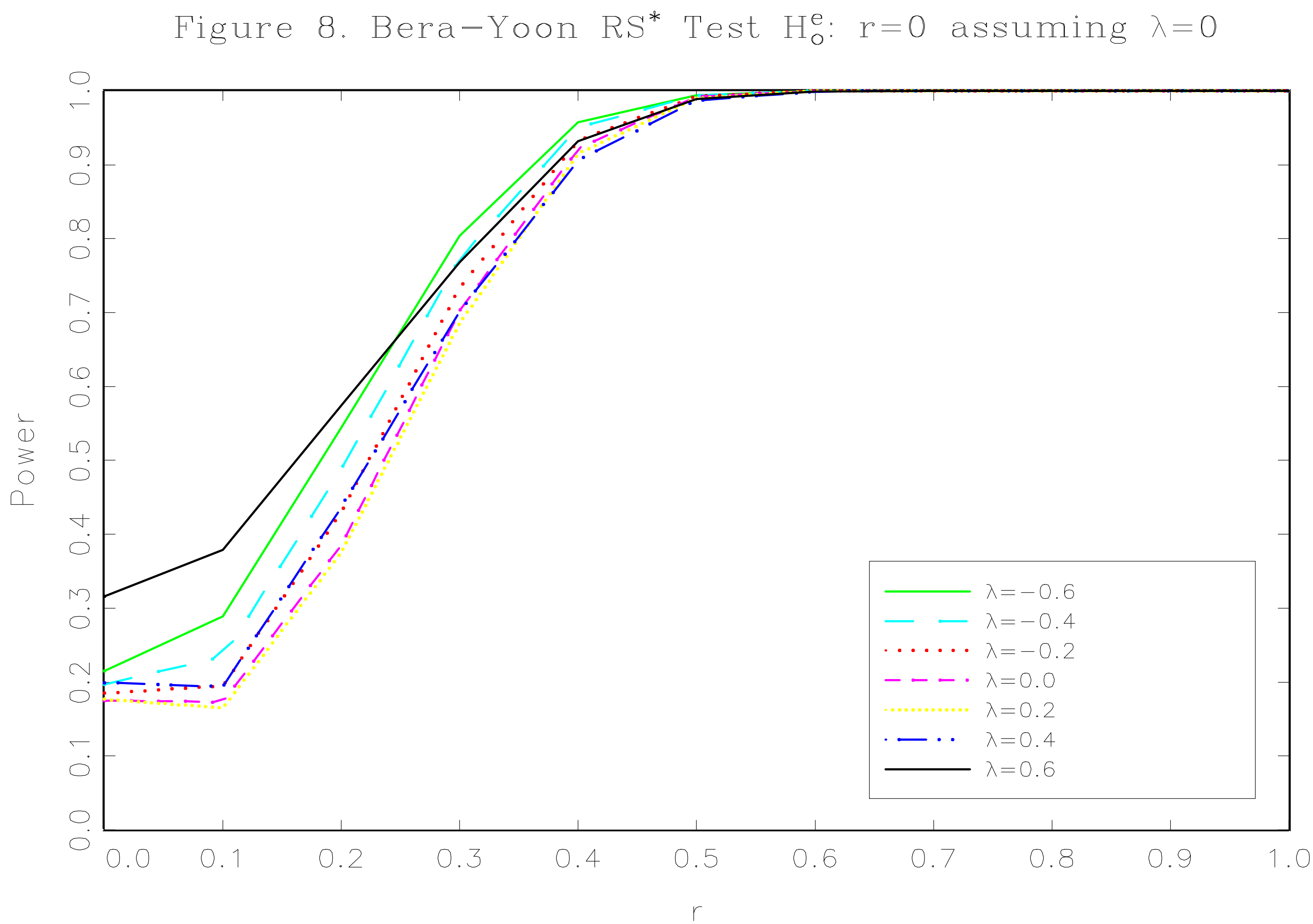


Figure 8. Bera-Yoon RS* Test H_o^e: r=0 assuming λ=0
Power
r
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
λ=−0.6
λ=−0.4
λ=−0.2
λ=0.0
λ=0.2
λ=0.4
λ=0.6

Figure 9. Simple RS Test H_o^f: r=1 assuming λ=0

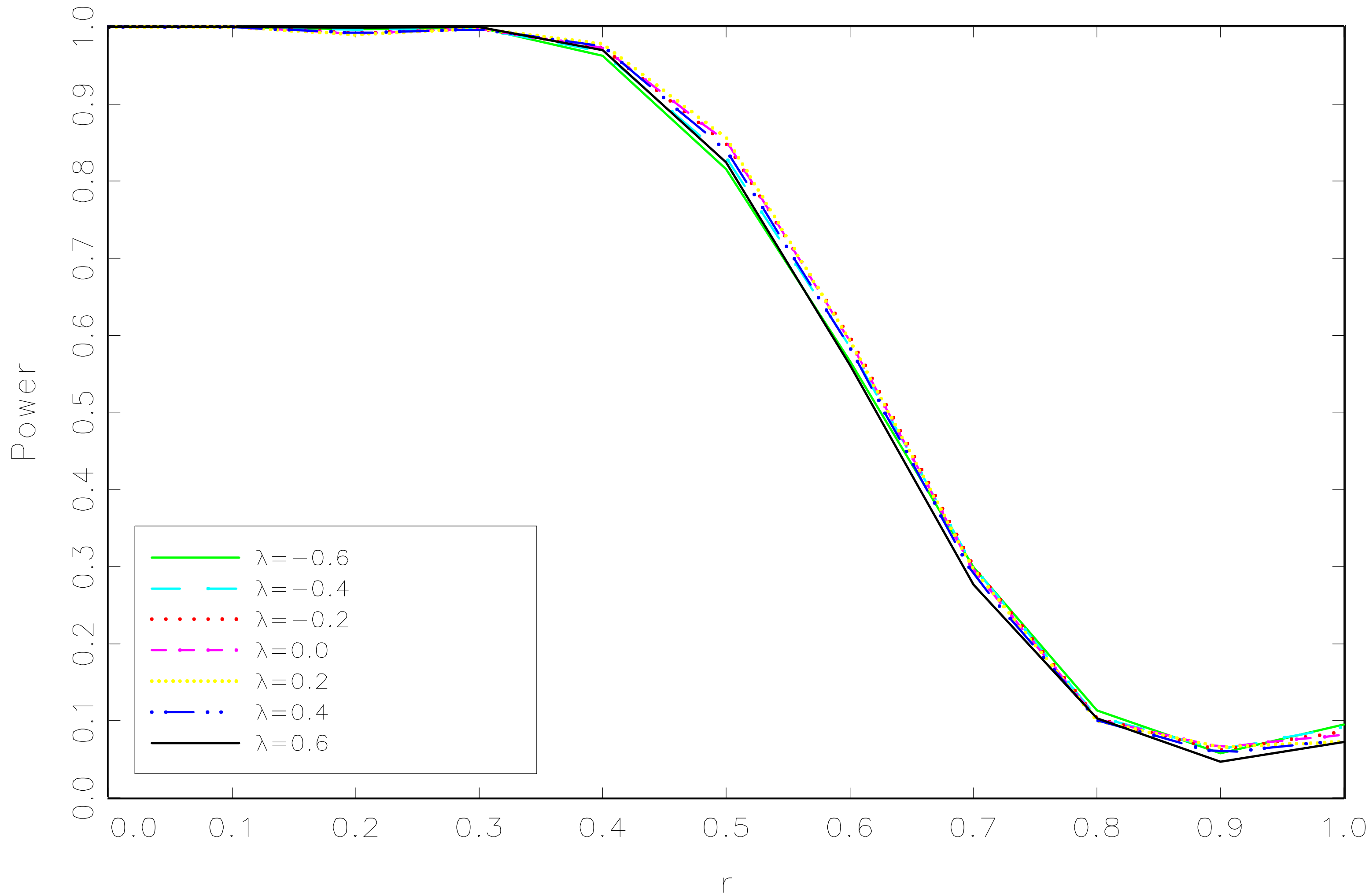


Figure 10. Bera−Yoon RS^* Test H_o^f: r=1 assuming λ=0

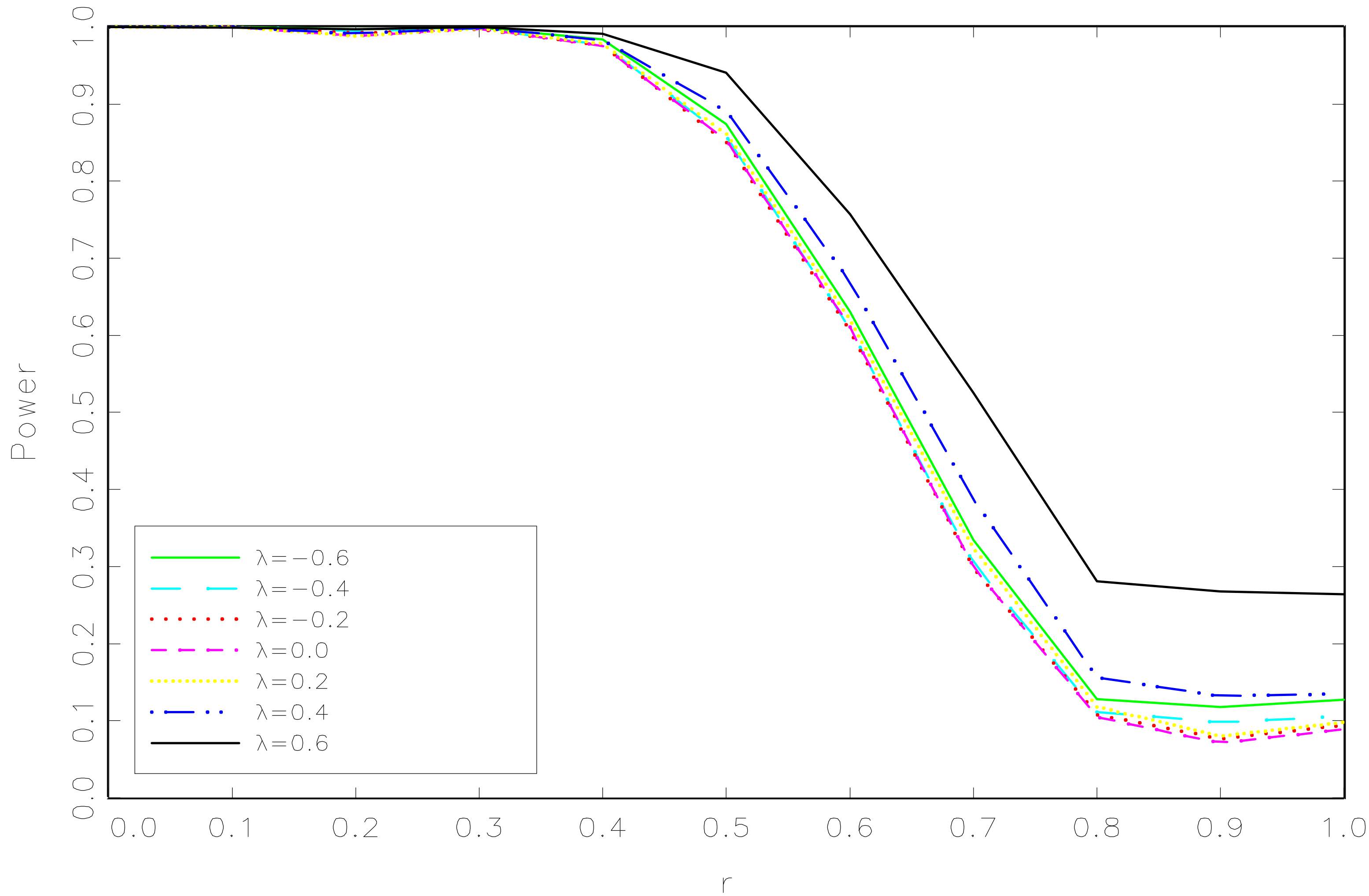


Figure 11. Conditional Test H_o^g: $\lambda=0$ | unknown r

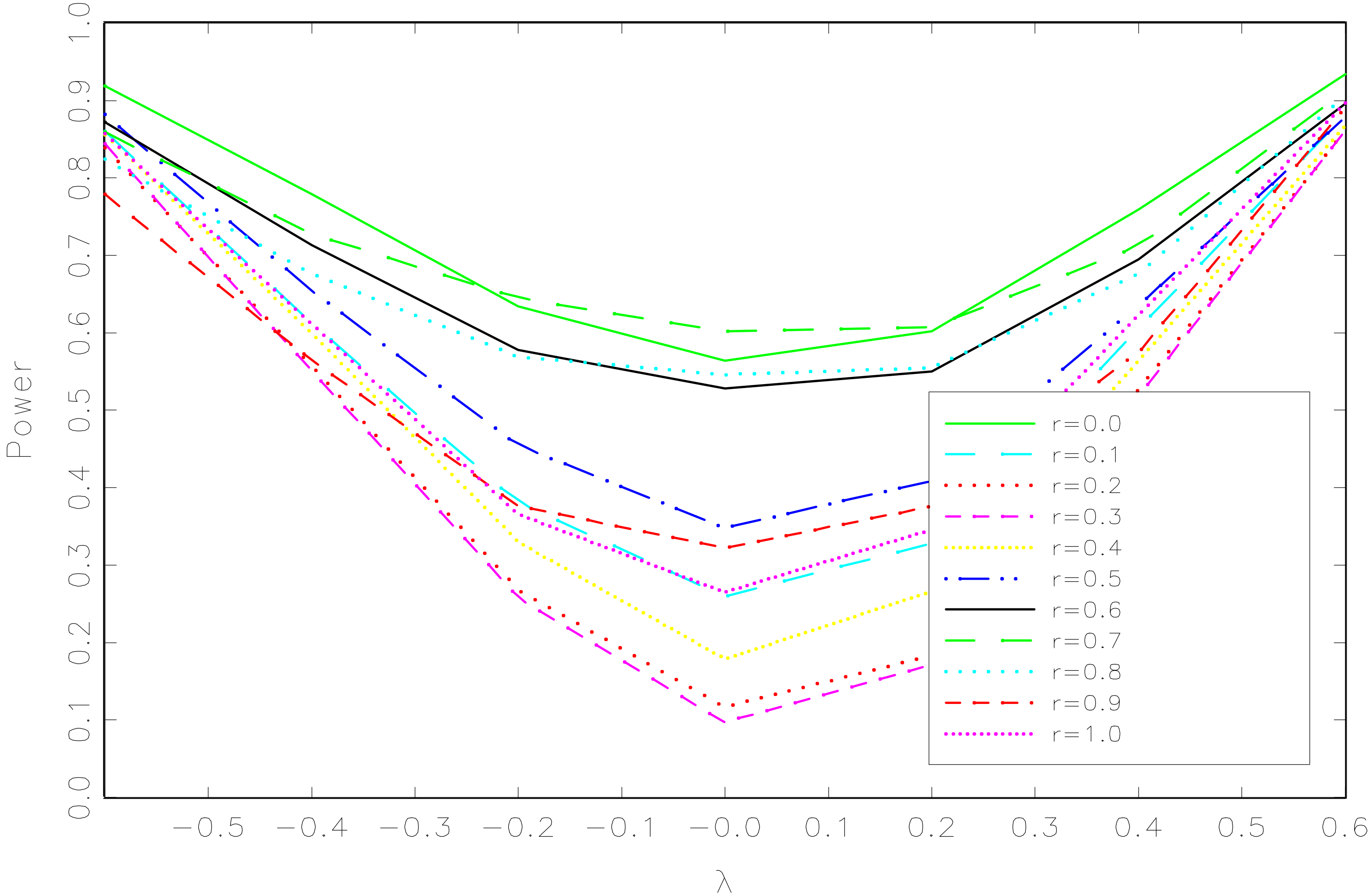


Figure 12. Conditional Test H_o^h: r=0 | unknown λ

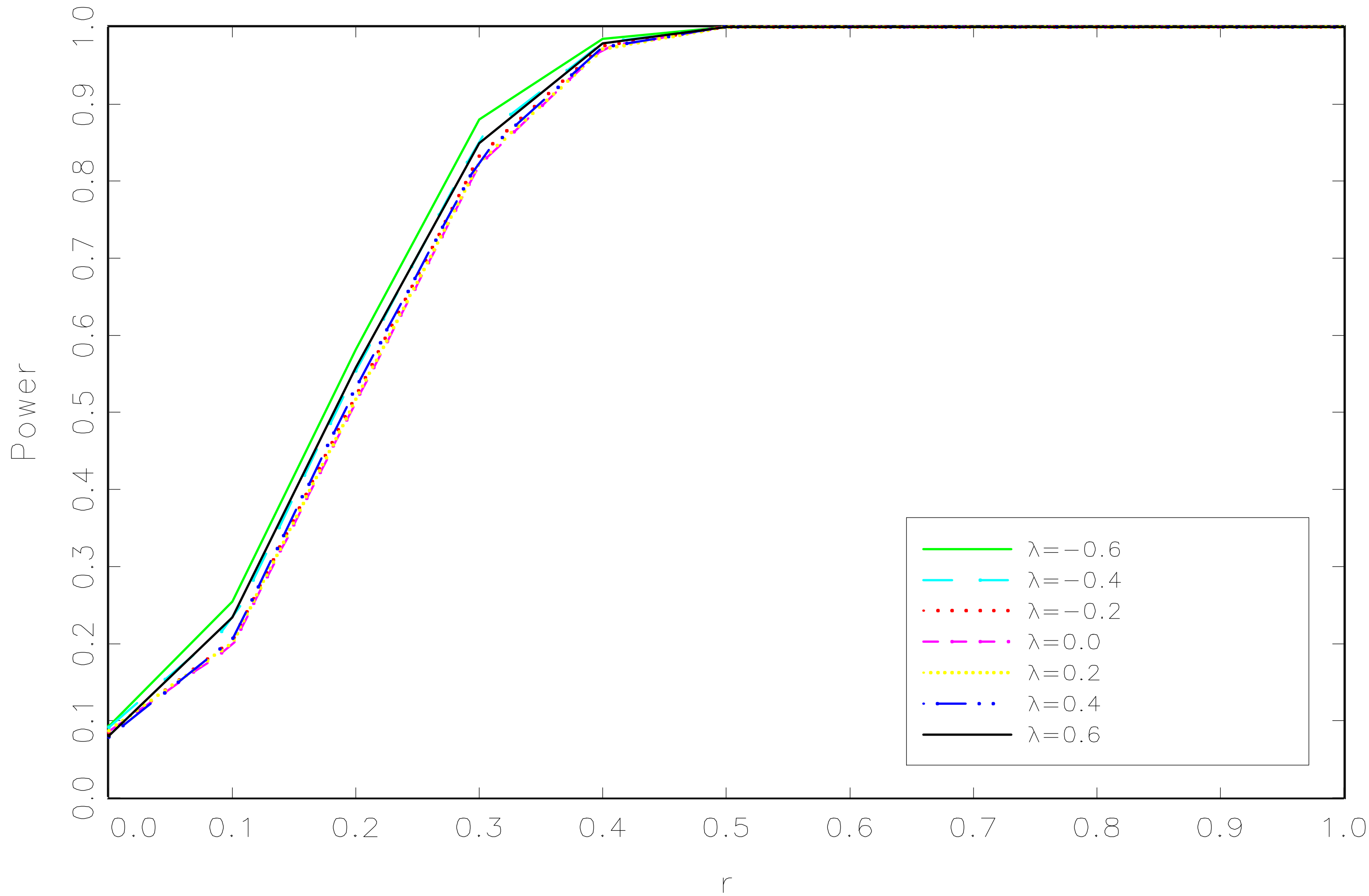


Figure 13. Conditional Test H_o^i: r=1 | unknown λ