A Joint Model of Labor Supply and Consumption Decisions Under Uncertainty[†]

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Abstract

This paper presents a dynamic model of the joint labor/leisure and consumption/saving decision over the life cycle. Such a dynamic model provides a framework for considering the important policy experiments related to the reforms in Social Security. We address the role of labor supply in a life cyle utility maximization model formally, building upon recent work by Low (1998), and extending the classical optimal lifetime consumption problem under uncertainty first formalized in Phelps (1962) and later in Hakansson (1970). We begin by solving the finite horizon consumption/saving problem analytically and numerically and compare the two solutions. We also simulate this benchmark model. Once the labor choice is considered, the stochastic dynamic programming utility maximization problem of the individual is solved numerically, since analytical solutions are infeasible when the individual is maximizing utility over consumption and leisure, given non-linear marginal utility. We show how such a model captures changes in labor supply over the life cycle and that simulated consumption and wealth accumulation paths are consistent with empirical evidence. We also present a model of endogenously determined annuities in a consumption/saving framework under capital uncertainty and in the presence of bequest motives.

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1 Introduction

This paper presents a dynamic model of the joint labor/leisure and consumption/saving decision over the life cycle. We introduce several models of the life cycle decision making of the individual in increasing level of complexity and closeness to reality in order to provide a framework of policy analysis for considering the important policy experiments related to the reforms of Social Security.

We begin the analysis with a simple model, that ignores, for almost all purposes, the individual's labor supply decision. In this model, consumption and saving over the life of the individuals are analyzed in detail. Modigliani and Brumberg (1954, 1980), Friedman (1957), Beckman (1959), Phelps (1962), and Ando and Modigliani (1963) represent seminal contributions to the analysis of this classic problem in economics.

Phelps presents an infinite horizon model of the consumption/saving decision under investment uncertainty (providing the framework for the first model that we solve), and he derives closed-form solutions for several models with varying assumptions regarding the utility function. Hakansson (1970) provides a refinement and extension of Phelps' work, allowing for a choice among risky investment opportunities and the possibility to borrow and lend.[1]

We first present a finite horizon version of the simplest model and report closed-form solutions for the consumption decision rule. We then solve this model numerically with two objectives in mind: first, to validate the techniques that will be used exclusively in the more realistic model that introduces labor supply, and second, to determine whether an accurate characterization of the finite horizon problem is as difficult to obtain as it is for the infinite horizon case. Rust (1999a) discusses the complications involved in attempting to replicate Phelps' (1962) solutions using numerical dynamic programming. The unboundedness of the utility functions used complicates the numerical approach, and even when using the most sophisticated techniques and considering logarithmic utility, the problem remains quite challenging.

The numerical approach for the finite horizon case is, however, well behaved and not because of the introduction of a bequest function. Even in the absence of the bequest motive the numerical solution replicates the closed form solution, using either the logarithmic utility function or the CRRA utility function. We show both analytically and numerically that the finite horizon solution of the consumption/saving problem with bequests converges to the infinite horizon model (without

[1] Levhari and Srinivasan (1969) re-examine Phelp's model and include a dynamic portfolio choice. Merton (1969) generalizes Phelps's model to the continuous time case and also allows for a portfolio selection decision. Samuelson (1969) analyzes the lifetime portfolio selection problem in discrete time. Fama (1970), assuming "perfect markets," shows that a two-period model provides most of the insights of the multi-period model of consumption decisions.

bequests). We also show simulated solution paths for consumption and wealth accumulation.

Modified versions of this benchmark model of the consumption/saving decision has been used extensively in the literature with different objectives. Hubbard and Judd (1987) provide a partial and general equilibrium discussion of the importance of social insurance in a model with uncertainty and borrowing constraints. Thurow (1969) invokes credit market restrictions to reconcile the prediction of the life cycle model with the empirical evidence. Zeldes (1989a) and Deaton (1991) study the role of liquidity constraints using extensions of this model, in a finite and infinite horizon framework, respectively. Beckman (1959) presents a dynamic programming model that introduces income uncertainty (but with no labor decision), Sandmo (1970) explore the role of income and capital uncertainty in a two period consumption/saving model, and Miller (1974) presents the infinite horizon version of such a model concentrating on income uncertainty, and finds that agents would always consume less when income is stochastic. Nagatani (1972) also introduces income uncertainty to justify the close relationship between consumption and income in the data, and Zeldes (1989b) solves a similar model using numerical techniques given the unavailability of closed-form solutions when using a constant relative risk aversion utility function.

Skinner (1988) explores the importance of precautionary savings in a model with risky income, approximating the optimal consumption path via Taylor expansions. Carroll (1997, 1999), presents a theory of *buffer-stock saving* where individuals maintain contingency funds to hedge against income uncertainty. Some empirical evidence presented by Carroll (1994), and Carroll and Samwick (1997) seems to support certain implications of this theory. Hubbard et al. (1994, 1995) analyze a multi-period model of the consumption decision with multiple sources of uncertainty and solve it with numerical techniques. They emphasize the importance of precautionary savings and the role of social insurance. Attanasio and Weber (1995), Attanasio and Browning, and Attanasio et al. (1997) highlight the importance of considering the effects of changes in demographics and labor supply behavior in a life cycle model if we are to match the empirical evidence, however, they still model labor supply as exogenous. More recently Gourinchas and Parker (1999) estimate the consumption/saving model using simulation techniques, and Cagetti (1999) concentrates on wealth accumulation. Dynan et al. (1999) explore saving behavior across income groups, and Cifuentes (1999) uses the consumption/saving model to discuss the effects of Pension reform.[2]

None of these models considers explicitly the labor supply decision of the individual, and thus, our work can be considered an attempt to complement and extend those models by considering

[2] Browning and Lusardi (1996) present an comprehensive survey of the consumption/saving literature, and focus on saving behavior. See also Deaton (1992) for an illuminating presentation of consumption models.

labor decisions as indeed endogenous to the life cycle consumption/saving problem. Although this is not a completely novel consideration, our models attempt to incorporate realism by considering several sources of uncertainty.[3]

More recently, an increasing number of papers have incorporated the labor decision in general equilibrium models of the economy in their analysis of the effects of Social Security Reform. Huggett and Ventura (1997), Bütler (1998), and İmrohoroğlu et al. (1994, 1999a, 1999b) are just a few examples. However, since they do not focus on individual decision-making and the general equilibrium approach requires a number of strong assumptions to make the problem solvable, there are many aspects of the life cycle model still to be addressed.

At the heart of our work is the allowance of agents to make their labor/leisure decision along with their consumption/saving decision in a utility maximizing framework in finite horizon. Individuals can work full-time, part-time, or not at all at any point in time, and they can consume continuously subject to a budget constraint. They can also accumulate wealth over their life at an uncertain rate of return that we model as draws from a log-normal distribution. Following our piecemeal approach to solving these models, we first introduce wages as deterministic; that is, agents know their exact profile of wages from day one. This effectively maintains the value function as only dependent on wealth, making the model a fairly simple extension of the consumption/saving model. We consider an isoelastic and Cobb-Douglas utility function on consumption and leisure, and given the unavailability of closed-form solutions when the marginal utility is non-linear, we solve the problem numerically by backward induction using dynamic programming techniques. We will assume throughout most of the analysis that the constant relative risk aversion parameter is larger than one, effectively implying that consumption and leisure are substitutes.[4] We also have to parameterize the within-period valuation of consumption versus leisure, a parameter that has an important effect on the labor supply decisions, as will become clear from our discussion in the following sections. We show that this model already captures paths of consumption, labor, and wealth accumulation, consistent with the literature and empirical regularities.

We next introduce labor income uncertainty, allowing for the wages to be stochastic. We start by characterizing the wage realizations as independent and identically distributed draws from a log-normal distribution, with a mean at each point in time that matches both the deterministic profile considered previously and a standard deviation consistent with research on the variability of

[3] Heckman (1974), MaCurdy (1981), Heckman and MaCurdy (1980), and MaCurdy (1983) tackle this issue in a theoretical and empirical context.

[4] See the discussion in Low (1998).

income. This new source of uncertainty increases the computational burden of solving the model by a single order of magnitude, since now the value function also depends on the uncertain draws of wages. The numerical techniques used can still handle the problem, but computing time increases as the "curse of dimensionality" makes its appearance. We then allow for serial correlation in the wages following the empirical evidence on the topic. We solve models with different serial correlation factors and compare the results to those of the previous models.

These models are in some sense extensions of the consumption/saving models that allowed for labor income to exist, but with an acknowledgement that a labor decision is to be made by the agents. Once the models are solved, we simulate the solutions with certain starting values of the state variables and average out the simulations to compute a path for consumption, labor, and wealth accumulation over the life cycle.

Finally, we tackle the problem of introducing some sort of insurance instrument for these individuals that can be interpreted as a pseudo Social Security system. As of now, we have only introduced this additional complication in the consumption/savings problem and are working on generalizing it to the the extended model that accounts for the labor/leisure decision.

The strategy is to introduce in the c/s model with bequests the possibility of partial or total annuitization by individuals. We allow for endogenous annuitization of a fraction of wealth by agents. Agents can annuitize at any point of their lives part or all their wealth; that is, they can purchase a sure income stream for the remainder of their lives at a price that takes into account the rate of return on the income purchased.[5] The cost of the annuity cannot exceed current wealth in the period that they annuitize, and the decision to annuitize is unique and non-reversible. This last assumption effectively means that they can only annuitize once in their life.[6] We do not, however, force them to do so at a given age or stage of their lives.[7]

To solve this model we have to take into account that agents are choosing their optimal time and size of the annuity along with their consumption/saving decision, forcing us to carry the annuity value as another state variable of the problem. This is an important exercise because we introduce this kind of social insurance in the simplest possible stochastic model of lifetime decision making and show that agents do react to the availability of this insurance. We also analyze the effects of higher interest differentials between the uncertain rates of returns on wealth and the return of the annuities, as well as the effect of having a higher bequest motive. This model provides important

[5] The literature refers to this type of annuity as *single premium immediate life annuity.*
[6] This is a fairly realistic assumption as emphasized in TIAA-CREF (1999).
[7] This model complements and extends the framework introduced in Friedman and Warshawsky (1990).

insights into the role of social insurance in our changing policy environment, revealing issues that will become especially important once we generalize this model by making the labor/leisure decision endogenous. In more general terms, the model also provides insights into the classic and important question of whether social schemes affect the behavior of individuals.[8] Finally, this model of endogenous annuities provides some insights into the "annuity puzzle," that is, the reason why the annuity market is very narrow. We suggest that the low rates of annuitization can be the result of optimal decision making by individuals that already have a sizable portion of their wealth in Social Security.

In the next section we solve analytically and numerically the finite horizon version of the consumption/saving benchmark model and simulate its implied consumption and wealth accumulation paths. Section 3 introduces the endogenous labor/leisure model, presents a numerical solution, and provides a discussion of its results. In section 4, we extend the life cycle model of consumption/saving decisions to allow for an endogenous annuity and show how the decision rule is affected by the possibility of life time insurance against investment uncertainty. Section 5 summarizes the main results and discusses extensions currently being considered and implemented.

2 The Consumption/Saving Model

In this section we solve a finite horizon version of the consumption/saving problem analyzed in Phelps (1962).[9] Agents choose consumption in the following utility maximizing framework:

$$\max_{0 \leq c_s \leq w} E_t \left[\sum_{t=0}^{T} \beta^{s-t} u(c_s) \right] , \tag{1}$$

where β is the discount factor, which in principle includes the mortality probabilities. Utility depends only on consumption, and savings accumulate at an uncertain rate of return.[10] We can express and solve this problem using Dynamic Programming and Bellman's principle of optimality. We solve it by backward induction starting in the last period of life. In that period the individual is solving

$$V_T(w) = \max_{0 \leq c \leq w} log(c) + K\ log(w-c) , \tag{2}$$

[8] Rust (1999b) provides a survey of models that try to incorporate uncertainty and insurance mechanisms in models of social insurance.

[9] He solved the infinite horizon problem analytically assuming no labor income and using different forms of the utility function.

[10] This is the standard characterization of the utility function. In a slightly different set up Alessie and Lusardi (1997b) introduce habit formation, that is, they consider a utility function that also depends on past consumption. See also Deaton (1992).

assuming a logarithmic utility function where K is the bequest factor, characterized as a number between zero and one.[11] By deriving the first order condition with respect to consumption we find that

$$c_T = \frac{w}{1+K} \ , \tag{3}$$

and from this we can write the analytical expression for the last period value function:

$$V_T(w) = log(\frac{w}{1+K}) + K\ log(\frac{wK}{1+K}) \ . \tag{4}$$

We can then iterate by backward induction and write the next to last period value function as follows:

$$V_{T-1}(w) = \max_{0 \le c \le w} log(c) + \beta\ E\ V_T(w-c) \ , \tag{5}$$

where the second term in the right hand side can be written as

$$E\ V_T(w-c) = \int_r V_T(\tilde{r}(w-c))fr\ dr \ . \tag{6}$$

Then we can write

$$V_{T-1}(w) = \max_{0 \le c \le w} log(c) + \beta\ E\ log(\tilde{r}\ (\frac{w-c}{1+k})) + \beta\ K\ E\ log(\tilde{r}\ (\frac{(w-c)K}{1+K})) \ . \tag{7}$$

Here the log utility simplifies the problem. Again taking first order conditions with respect to consumption, after some algebraic manipulation we can obtain an expression for the consumption rule in the next to last period of life:

$$c_{T-1} = \frac{w}{1+\beta+\beta K} \ . \tag{8}$$

We then have an expression for V_{T-1} in the following form:

$$V_{T-1}(w) = log(\frac{w}{1+\beta+\beta K}) + \beta\ log(\frac{w\beta}{1+\beta+\beta K}) + \beta\ K\ log(\frac{w\beta K}{1+\beta+\beta K}) + \Upsilon \ , \tag{9}$$

where Υ gathers all the terms that do not depend on w. From here we can write V_{T-2} and again derive first order conditions, resulting in:

$$c_{T-2} = \frac{w}{1+\beta+\beta^2+\beta^2 K} \ . \tag{10}$$

[11] Agents in this model only care about the absolute size of their bequests, this has been called the "egoistic" model of bequests. The importance of bequest motives is still an open issue in the literature. Here we take the position of acknowledging that bequests do exist and explore the implications of changing the importance of the bequest motive in the utility function. Hurd (1987, 1989), Bernheim (1991), Modigliani (1988), Wilhem (1996) and Laitner and Juster (1996) are some of the main references on the debate on the importance of bequests and altruism in the life cycle model. Kotlikoff and Summers (1981) stress the importance of intergenerational transfers in aggregate capital accumulation.

Through backward induction, we can continue iterating to find c_{T-k}

$$c_{T-k} = \frac{w}{1 + \beta + \beta^2 + \beta^3 + \ldots + \beta^k + \beta^k K} \,. \tag{11}$$

for any $k < T$. From these decision rules, we can observe that as T grows large, the finite horizon solution with bequests converges to the infinite horizon solution, since the influence of the bequest parameter becomes less important as the time horizon increases. In the infinite horizon case with logarithmic utility and no non-labor income, the simplified decision rule is $c = (1 - \beta)w$, as shown in Phelps (1962). The derivation of the decision rules in the case of the CRRA utility function is similar but a bit more involved and we present it in the Appendix.

Our ability to derive an analytical solution for this model allows us to evaluate the effectiveness of our numerical methods, which are all that we have available in more complicated models. The exercise of solving the model numerically is also interesting on its own given that the infinite horizon version of this model has been shown to be quite difficult to replicate using numerical methods, even with the logarithmic utility function, as discussed in Rust (1999a).

The numerical procedure is by nature very similar to the analytical approach, involving backward recursion starting in the last period of life. We discretize wealth and compute the optimal value of consumption for all those wealth levels using bisection. Bisection is an iterative algorithm with all the components of a nonlinear equation solver. It makes a guess, computes the iterative value, checks if the value is an acceptable solution, and if not, iterates again. The stopping rule depends on the desired precision given that the solution is bracketed by the nature of the algorithm and that the round-off errors will probably not allow us to increase the precision beyond a certain limit. In each iteration of this procedure, except for the last one where all uncertainty has been eliminated, we have to compute the expectation in equation (6), which is potentially the most computationally demanding step. For this we use Gaussian Quadrature. We also compute the derivative of this expectation using numerical differentiation, which also requires quadrature as part of its routine. In this case the analytical derivatives are simple to compute, but in more complicated models this might not be the case. We therefore want to check how accurate the numerical strategy actually is.

Gaussian Quadrature approximates the integral through sums using orthogonal polynomials with respect to the density function of the variable we are integrating over, which in this case are the draws of the interest rates following a log-normal distribution. The points and weights are selected in such a way that finite-order polynomials can be exactly integrated using quadrature formulae. The weights used have the natural interpretation of probabilities associated with intervals around the quadrature points.[12] At this point we are dealing with a one dimensional problem, for which quadrature methods have been shown to be very accurate compared with other techniques of computing expectations (integrals) like Montecarlo integration or weighted sums.[13]

This all amounts to manipulating (6) through a change of variables such that we can write it as an integral in the $(0,1)$ interval and then approximate it by a series of sums that depend on the quadrature weights and quadrature abscissae that we compute recursively, following readily available routines (e.g. Press et al. 1992).[14]

An additional numerical technique that we use to solve the model completely is function approximation by interpolation. Since savings in a given period are accumulated at a stochastic interest rate, next period's wealth will not necessarily fall in one of the grid points for which we have the value of the function already calculated. Ideally we would solve the next period's problem for any wealth level, but this is computationally infeasible. Therefore, we use linear interpolation to find the corresponding value of the function given the values in the nearest grid points.[15]

The bisection algorithm that uses the quadrature and interpolation procedures eventually converges to a maximum of the lifetime consumption problem for a given value of wealth in a given period (or reaches the pre-decided tolerance level). This procedure is repeated until the problem in the first period of life is also solved.

Once we have solved the model, we have a decision rule for every level of wealth in our initial grid. In this case we have chosen a grid space of 500 points, to gain accuracy more of these points are concentrated at low wealth levels where the function is changing rapidly. Figures 1 and 2 show the decision rule of the consumption/saving problem for wealth ranging from 0 to 200,000. In this case we have solved for expositional purposes just an 11-period model.

Figure 1 plots several decision rules given a logarithmic utility function. It first plots the numerical solutions for different time periods, denoted C_1, C_2, and so on. It also plots the true solution of the problem for C_T and for C_{10} denoted CT_{10}, which are to be compared with C_{11} and C_{10} in the graph. Finally, we also plot the solution of the infinite horizon problem borrowing from Phelps (1962), denoted by CIN in the figure. We have chosen a discount factor of 0.98 and a

[12] For a detailed characterization of quadrature methods we refer the reader to Tauchen and Hussey (1991), Judd (1998), and Burnside (1999).

[13] For an analysis of how different techniques perform in applied problems see Rust (1997).

[14] We can write $\int_r V(r)fr\ dr$ after a change of variables as $\int_0^1 V(F^{-1})du$, which can then be approximated by $\sum_{i=1}^{N} w_i V(F^{-1}(u_i))$, where w_i are the quadrature weights and u_i are the quadrature abscissae.

[15] More sophisticated interpolation procedures can be used such as splines or Chebyshev interpolation. They are not considered here, but we plan to conduct a sensitivity analysis of the procedures used at each step of the numerical computations.

bequest parameter of 0.1.

The first reassuring feature to notice is that the true solution and the computed solution coincide with each other for the last period of analysis, and also for the next to last period, C_{10}. We also see that the consumption rule increases with wealth and time periods passed and that in very few periods we are fairly close to the solution of the infinite problem. Figure 2 plots the decision rule when we consider a CRRA, with risk aversion parameter equal to 1.5, $\beta = 0.95$, and bequest parameter equal to 0.1. We show the true solution for C_T (CTC in the figure) and C_{T-1} (CTC_{10} in the figure), and again coincide with the computed solution. We conclude that the performance of the numerical technique is excellent, something that comes as a striking contrast to the problematic infinite horizon model.

In Figures 3 and 4 we simulate this model using the numerical solution for the CRRA utility function. We report the results of 5,000 simulations of an 11-period model with 500 grid points for wealth and plot consumption and wealth paths with initial wealth level of 10,000. We also consider several values for the parameters of interest. The first specification considers a γ of 1.5 (the parameter of relative risk aversion), and the second increases this parameter to 2.5 (hg lines in the plots). Then we increase the bequest parameter to 0.6 (leaving $\gamma = 1.5$; bq lines in the plots), and finally, we decrease the relative risk aversion parameter to 0.7 (lg lines in the figures).

We observe that people consume less at the beginning of their lives given the uncertainty about the interest rates, which are represented by draws from a truncated log-normal distribution, and in the last periods of life consumption is higher. Consumption is, however, decreasing if the risk aversion parameter is less than 1. If we focus on the pattern of wealth accumulation we see that individuals deaccumulate their wealth gradually. We also see that changing the relative risk aversion parameter has the effect of making consumption less smooth (with higher wealth accumulation) when increased and leading to more smoothing (with lower wealth accumulation) when decreased from the benchmark value of 1.5. We can also observe the expected effect of the bequest parameter: those with a higher concern for their offsprings, represented by a higher valuation of their consumption in the utility function, consume uniformly less over the life cycle than those with a lower bequest parameter. This latter population also accumulates more and for a longer period. These results on the effect of the bequest motives are consistent with, and in fact extend, the theoretical model of Hurd (1987) to the case where agents with different levels of bequest are considered.

This model is meant to serve as a benchmark for the models discussed next and especially for the introduction of the insurance device in Section 4.

3 Introducing the Labor/Leisure Decision

We next tackle the issue of extending the model of Section 2 to allow for the endogenous labor supply decision. The agents are now going to maximize a utility function containing consumption and leisure and are going to choose consumption and labor (leisure) optimally in every period of their lives. Effectively, individuals maximize

$$\max_{c_s, l_s} E_t \left[\sum_{t=0}^{T} \beta^{s-t} u(c_s, l_s) \right] , \tag{12}$$

again in finite horizon. The within-period utility function is assumed to be Isoelastic and Cobb-Douglas between consumption and leisure in time t:

$$u(c_t, l_t) = \frac{(c_t^{\eta} l_t^{1-\eta})^{1-\gamma}}{1-\gamma} , \tag{13}$$

where γ is the coefficient of RRA and η is the valuation of consumption versus leisure. Consumption and leisure are substitutes or complements depending on the value of γ as discussed in Heckman (1974) and Low (1998), with the cut approximately equal to 1.[16] In most of our analysis we will assume values of γ larger than 1, implicitly assuming substitutability between consumption and leisure. We will assume that the agent only has three choices with respect to the labor decision, part-time work, full-time work, or out of the labor force.[17] It is also important to emphasize that for computational convenience we have chosen a lower bound on leisure equal to 20% of the available time during a given period.

3.1 Deterministic Wages

First we will assume that wages follow a deterministic path which peaks around age 50 and then smoothly decreases. Given that we allow for consumption and leisure to influence each other using a CRRA utility function and that we are concerned about corner solutions for the labor decision, the model can only be solved numerically. To do so we employ the techniques presented in the last section.

[16] Heckman presents a model of perfect foresight and shows that by introducing the labor supply decision it is possible to reconcile the empirical evidence on consumption paths with the life cycle framework, without resorting to credit market restrictions or uncertainty. Low's work is fairly close in nature to our analysis, he abstracts from capital uncertainty but allows borrowing.

[17] We solve in this case a 15-period model to reduce the computational burden of the solution process, but we plan to work with a 65 period model in the near future.

We can then use Dynamic Programming to characterize this problem and again solve by backward induction. The individual in the last period now solves

$$V_T(w) = \max_{(0 \le c \le w + \omega(1-l), l)} U(c,l) + K\, U(w + \omega(1-l) - c)] \,, \tag{14}$$

where ω represent wages and labor is chosen between the three possible states. Once we obtain the optimal decision rules using the bisection algorithm, we then solve recursively. We can write the value function in the next to last period as

$$V_{T-1}(w) = \max_{(0 \le c \le w + \omega(1-l), l)} U(c,l) + \beta\, E\, V_T(w + \omega(1-l) - c) \,. \tag{15}$$

The value function still remains unidimensional since there is no uncertainty about the wages. We solve this model again by bisection, computing the expectations by quadrature and interpolating the values of the next periods value functions.

Once we have solved the model, we simulate it given starting wealth values. The capital uncertainty is characterized by draws from a truncated log-normal distribution. Figures 5-10 present plots of the paths of consumption, labor supply, and wealth accumulation resulting from this 15-period model that we map into an age profile for expositional purposes. In Figures 5-7 we consider simulations with initial wealth equal to 10,000 units and with varying levels of the RRA parameter, bequest motive, and the valuation of consumption versus leisure in the utility function.

These results have several interesting features. First, as can be seen from Figure 5, consumption tracks income for a fair amount of time up to age 40 where the consumption path begins to flatten, finally decreasing by the end of the life cycle. We can also see that those who value leisure more ($\eta = 0.5$ *versus* $\eta = 0.7$) receive lower wages because they work only part-time, although they are able to maintain a consumption level higher than their wage level starting at about age 45. The pattern of labor supply is equally interesting. Agents with a high valuation of consumption seem to work full time most of their lives, except at the beginning when their wages are low and they have initial wealth to smooth consumption. Later in life, our model is able to pick up the decrease in labor supply due to lower wages. It is important to emphasize that those with higher bequest motives (bq in the figures, bequest parameter equal to 0.6 versus 0.1 for the other curves) save more and work more on average than those with lower bequest parameters. In Figure 7 we show the wealth accumulation over the life cycle implied by the model. The pattern here is fairly close to the estimated, simulated, and reported results of several papers (e.g., Hubbard et al. 1994, Attanasio and Weber 1995, Attanasio et al. 1997, Alessie and Lusardi 1997a, Alessie et al. 1997, and Cagetti 1999) reflecting empirical data quite closely. We see little accumulation early in life, and then after age 40, agents begin to accumulate higher levels of wealth which only decreases near the end of life. We can also see from the graph that those with higher bequest motives accumulate more wealth and those with higher valuation for leisure start to accumulate earlier in life. This model is broadly consistent with some features of the data that show very low savings rates among young individuals, with an increase only later in life. Figures 8-10 essentially show that when starting wealth is higher, the model predicts very similar behavior, except at the beginning of life when wealthy individuals delay their entrance into the labor force.

3.2 Stochastic and Serially Correlated Wages

We now make the model more realistic by introducing income uncertainty, while maintaining the endogeneity of the labor/leisure decision. We start by introducing stochastic i.i.d. wages from a log-normal distribution with a changing mean that follows the deterministic profile used above.

This feature complicates the model because the value functions now depend on the uncertain wage realizations. We now write the problem that the agents solve in the last period of life as

$$V_T(w,\omega) = \max_{(0\leq c\leq w+\omega(1-l),l)} U(c,l) + K\, U(w+\omega(1-l)-c)]\;, \tag{16}$$

where again labor is chosen between the three possible states. Once we obtain the decision rules numerically we can write the value function in the next to last period

$$V_{T-1}(w,\omega) = \max_{(0\leq c\leq w+\omega(1-l),l)} U(c,l) + \beta\, E\, V_T(w+\omega(1-l)-c,\omega)\;. \tag{17}$$

And recursively the earlier periods. The expectation $E\ V_t(\omega(1-l)+w-c,\omega)$ in the value functions in the different periods can be written as follows:

$$\int_r \int_\omega V(\tilde{r}(w+\tilde{\omega}(1-l)-c),\tilde{\omega})\, f\omega d\omega\, frdr\;. \tag{18}$$

Additionally, the interpolation of the values of the next period value function has to be carried out in two dimensions, a slightly more cumbersome and slower procedure. The double integrals are again solved by Gaussian Quadrature, but we use iterated integration since we are assuming independence of wages and interest rates.[18]

Figures 11-13 show the consumption, labor, and wealth accumulation paths for this model. The main difference from the case of deterministic wages is that individuals start to save and accumulate

[18] Given that the value function depends on wealth and wages we need to discretize both variables in order to approximate the integrals. We choose to use 50 points for wealth and 50 points for wages. We found that using less points significantly affected the accuracy of the calculations leading to possible erroneous conclusions.

earlier in life, and ultimately accumulate more wealth.[19] The labor supply behavior is very similar to that observed under deterministic wages.

Finally, we introduce serially correlated wages, such that

$$ln\, \omega_t \;=\; \alpha + \rho\, ln\, \omega_{t-1} \;+\; \epsilon\,, \tag{19}$$

where the ϵ are i.i.d. draws from a log-normal distribution and ρ equaling 0 essentially returns us to the case of i.i.d. wages. The solution method does not change significantly from the last model, and only the careful manipulation of the serially correlated component has to be considered.

Figures 14-16 show the paths of the relevant variables. Our results are consistent with the idea that risk averse individuals would want to accumulate more in the presence of serially correlated uncertain wages given that they want to be prepared in case their labor income suffers a very bad draw, whose influence will last a large number of periods. Here we plot the paths for different values of the serial correlation parameter. With low serial correlation, the model is similar to that of stochastic wages, but it shows higher savings, and much higher wealth accumulation. With high correlation, we plot the case of individuals starting with wealth of 10,000 units and initial wages of 30,000 units. An interesting feature of the model is related to the labor supply of the agents. Those who face low serial correlation in their wages work full time for most of their lives, and then phase out late in the life cycle. However, those who face a high serial correlation parameter, prefer to work part-time, and only increase their labor supply in mid-life when the increase in wages makes it more attractive to work. We can also see from Figure 15 that those with a lower valuation of consumption only work full time for a relatively short period in mid-life and start exiting the labor force earlier than the other agents.

From the solution and simulation of these models we can conclude that a life cycle model that endogenizes labor supply behaves quite consistently with the empirical data on wealth accumulation and consumption profiles and that wealth accumulation seems to increase as wages become more uncertain. Additionally, such a model captures the phasing out of the labor force by older individuals who become less productive in the market and who have a low serial correlation of wages once they reach a certain age. This model seems well-suited for analyzing important policy issues regarding the effects on savings and labor supply of reforms in social insurance programs.

[19] This can be considered evidence of the importance of income uncertainty. Lusardi (1998) presents empirical results of the role of the variance of income in a consumption/saving model.

4 Endogenously chosen Annuities

In this section we introduce a modification in the model presented in Section 2, the consumption/saving model, by allowing individuals to purchase an annuity with a fraction or all of their wealth at any point in their lives. We effectively endogenize the annuitization decision by providing the agents with the possibility of exchanging a certain number of dollars today for a stream of income over the rest of their lives. The annuity has a given rate of return we assume to be fixed. The cost of the annuity, calculated as the net present value of the promised stream of income, cannot exceed the total wealth of the agent at the time of the purchase of the annuity. This is effectively a single premium immediate life annuity. The decision to annuitize is unique and non-reversible. These last two assumptions mean that individuals can only annuitize once in their lives. We do not, however, place any restriction on the timing of this annuity.[20]

The model presented here is similar to that of Friedman and Warshawsky (1990), although they focus on older individuals and on the issue of annuity pricing in order to explain the almost non-existence of a market for these instruments. Another difference is that they force individuals to invest a proportion of their wealth in an actuarially fair social annuity, without considering investment uncertainty. Brugiavini (1993) focuses on the role of longevity uncertainty in the purchase of annuities in a two/three period model. She also considers a model that allows for income uncertainty, and the different behavior of employees and entrepreneurs. Mitchell et al. (1999) use the term structure of interest rate, instead of a fixed interest rate, to calculate the expected present discounted value of the annuities in a model of uncertain lifetime; they find that retirees should value annuities even if they are not actuarially fair. Brown (1999a) extends the model of Yaari (1965), who focus on the role of annuities when individuals face an uncertain lifetime, using data on older Americans constructing a measure of the consumer's valuation of additional annuitization. However, his model abstracts from capital uncertainty and does not endogenize the annuity decision in the general sense that we do. Brown (1999b) uses data on older individuals to test and reject the "Annuity Offset Model," that is, the hypothesis that old individuals purchase term insurance to offset the excessive annuitization imposed by the government social programs. Kotlikoff and Spivak (1981) also use a Yaari type model to emphasize the important role of the family as an incomplete annuities market, in their model the annuity decision is made at exogenous points in time. Eichenbaum and Peled (1987) use a two period model to underline the over accumulation of

[20] We do not consider at this point the role of taxes in the decision to annuitize, see Gentry and Milano (1998) for a discussion of the effects of taking taxes into account.

private capital in a model of competitive annuities with adverse selection.[21]

The agents are again trying to maximize consumption over their lifetime but now have the choice of converting part of their wealth to an annuity. This annuity has a rate of return *lower* than the mean of the market rate and provides a stream of income until the time of death, which can be considered uncertain given that the mortality probabilities are embedded in the discount rate. The annuity cost $A(a)$, where a is the annuity received every period and β is the rate at which we discount it, is equal to

$$A = a[\frac{1-\beta^{k+1}}{1-\beta}] \ , \tag{20}$$

assuming that agents receive the first payment in the same period that they annuitize. We again solve this model by backward induction using numerical Dynamic Programming techniques. The decision in the last period of life is very similar to that of the simple consumption/saving model, but now the value function depends not only on wealth but also on the value of the annuity, which enters the budget constraint:

$$V_T(w,a) = \max_{0 \le c \le w - A(a) + a} U(c) + K\, U(w-c) \ , \tag{21}$$

where $A(a) \le w$. In this last period we do not allow for the annuity decision to occur, since annuitizing would return exactly what they put into the annuity, assuming no transaction costs. But even if agents do not actually decide to annuitize it is possible that they have annuitized earlier in their lives, and therefore, we have to solve for the value function under as many combinations of wealth and annuity values as possible.[22] In the simulation part of the model, if an agent reaches the last period of life without having annuitized he will not annuitize in the last period. Recall that the agent still faces capital uncertainty.

We can then write the next to last period value function as follows:

$$V_{T-1}(w,a) = \max_{0 \le c \le w - A(a) + a} U(c) + \beta\, E\, V_T(w,a) \ . \tag{22}$$

Here things are a bit more complex. If the agent has already annuitized, she will receive a stream a and subsequently find the optimal consumption rule. If the agent has not already annuitized, she is then able to decide what portion of her wealth will be put into the annuity.

In order to solve this model we conduct a maximization in stages. First, for a given value of the annuity we compute the optimal consumption rule via bisection, and again use quadrature

[21] Walliser (1997, 1998) discusses the role of annuities in a social insurance framework.

[22] As in the previous section we discretize the two variables that enter the value function in order to approximate the integrals, and again choose 50 grid points for each variable.

and interpolation to calculate the expectations (the integrals in the model). This is embedded in another bisection algorithm for calculating the optimal fraction of wealth to annuitize and the implied annuity to be received in the periods ahead, possibly 0. This can be written as

$$\max_{a} \max_{0 \leq c \leq w - A(a)} [U(c) + \beta\, E\, V(a, \tilde{r}(w - A(a) - c))] \; , \tag{23}$$

and again $A(a) \leq w$. The first order condition for an optimum in the inner maximization is

$$U'(c_a) - \beta\, E\, [r\, V'(a, \tilde{r}(w - A(a) - c_a))] = 0 \; . \tag{24}$$

We solve this by the same methods explained above. The outer maximization solution method is very similar, but now the first order condition results from deriving

$$U(c(a,w)) + \beta\, E\, [V\ (a, \tilde{r}(w - A(a) - c(a,w)))] \; , \tag{25}$$

with respect to a. This results in the following first order condition:

$$U'(c(a,w))\frac{\delta c}{\delta a} + \beta\, E\, [\, \frac{\delta V}{\delta a} - \tilde{r}\frac{\delta V}{\delta \tilde{w}}\, [A'(a) + \frac{\delta c}{\delta a}]\,] \; , \tag{26}$$

which by the envelope condition reduces to this intuitively plausible f.o.c.:

$$E\, [\frac{\delta V_{t+1}(a, \tilde{w})}{\delta a}] = E\, [\tilde{r}\, \frac{\delta V_{t+1}(a, w')}{\delta \tilde{w}}\, A'(a)] \; , \tag{27}$$

where $\tilde{w}$ is wealth next period. The left hand side of this expression can be understood as the marginal value of an additional unit of annuity and the right hand side as its marginal cost. The agent will try to set these equal when calculating the optimal annuity in every period.

Bisection effectively searches over the values of the annuity (which imply optimal levels of consumption that are calculated in the inner bisection algorithm), using quadrature to calculate the expectations and again interpolating the values of the next period value function. The interpolation has to be performed in two dimensions, which complicates and slows down the procedure slightly.

Once we have solved this model, we simulate it and construct consumption, wealth accumulation, and annuity paths over the life cycle. The results are quite striking. In Figure 17 we replicate the model of Section 2, for a starting wealth value of 10,000 units, in a 30-period model, which we then map into a lifetime age profile. Consumption is again increasing over the lifetime due to the investment uncertainty, and the mortality uncertainty embedded in the discount rate. Figure 18 shows the consumption path resulting from averaging 500 simulations for individuals with a starting wealth value of 10,000 units. We can see the smoothness of the path compared with the one of Figure 17, for the same starting value of wealth and same parameter values. In fact, consumption is practically flat around 600 units what is consistent considering that we actually solve a 30-period model. The contrast is sharper the higher the parameter of relative risk aversion.

Also in Figure 18 we also show the average annuity value received, which changes a bit at the beginning but is very flat during most of the lifetime. We report four different specifications: the first has a RRA parameter of 1.5, the second (*hg* in the plot) has $\gamma = 2.5$, the third (*lr* in the plot) is simulated with a 1% average interest rate of the annuity (2 percentage points lower than the benchmark), and the fourth (*bq* in the plot) considers a higher bequest motive, 0.6 versus 0.1 in the benchmark case. We can observe that when the rate of return is lower the annuitization and lifetime consumption are also lower, as might be expected and consistent with Friedman and Warshawsky (1990). We find that increasing the bequest motive results in a slightly smaller level of average annuities received at the beginning of life but has little effect thereafter, but consumption is less smooth. The implications for wealth accumulation are that an individual accumulates higher balances late in life when has a higher valuation of bequests. A higher relative risk aversion does not have a very significant effect, but from the figures we can conclude that consumption is less smooth and wealth accumulation is higher in the last third of the lifetime, something consistent with the idea that higher risk aversion should lead to less smooth paths for lifetime consumption and higher paths of accumulation.

Figure 20 reports wealth accumulation and the evolution of the value annuitized at each stage of the life cycle. The difference between this wealth path and the one in Figure 19, which replicates the model of Section 2, is clear, implying that once individuals are able to annuitize they run down their wealth fairly smoothly over their life, again more risk averse individuals benefit the most. The annuitization happens very early in this endowment consumption/saving model with investment uncertainty, and for most agents, it seems to amount to a third of their wealth in the initial periods. Depending on the realizations of interest rates, which are again draws from a truncated log-normal distribution with mean higher than the rate of return of the annuity, agents sometimes annuitize later in life and in a lower proportion, a seemingly reasonable result. These results are also consistent with Mitchell et al. (1999), what suggests that our model extends their simplified stochastic life cycle model to a full dynamic characterization of the annuitization decision in the presence of bequest motives and capital uncertainty, allowing for annuitization to happen at any point in the life cycle and for any fraction of the individual's wealth.

In Figure 21 we present the simulations of the annuity model if we make the annuity actuarially fair, in the sense that its rate of return is equal to the average return of the stochastic market return. The proportion of wealth annuitized in this case is not larger than 50%, and does not always happen in the first period of life. We can also see that the patterns of wealth accumulation and consumption are fairly different from the model with less than actuarially fair annuities, given that people accumulate higher balances at the end of their life what allows them to consume more at the end of the life cycle.

These results have several interesting implications. First, in a simple model of consumption and saving decisions with income uncertainty the possibility of annuitizing wealth is used by individuals to smooth their consumption stream almost entirely. If we interpret this mechanism as a pseudo social insurance system, there is no doubt as to the importance of the effects that such a scheme has on the microeconomic behavior of agents. However, we do not explore the reasons for the lack of availability of such annuities in the current capital markets due to such phenomena as adverse selection (which are analyzed extensively in the literature). Our main point is that our model captures the reaction of individuals to an insurance scheme. We also claim that this simple model provides some insights into the "annuity puzzle," that is, the reason why the annuity market is very narrow. If it is optimal for individuals to only annuitize between 30% and 50% of their wealth, as our model suggests, and Social Security accounts for approximately that proportion of their wealth (Friedman and Warshawsky 1990) it is very likely that the demand for annuities that we observe is the result of an optimal decision making by individuals. For some individuals Social Security would provide a smaller than optimal amount of annuities, therefore they would buy some additional annuity notes in the market, for others S.S would lead to over-annuitization and they might react buying life insurance that offsets the imposed annuity purchases through the social insurance system.

An extension of this model to incorporate the labor decision in a fashion similar to that of the previous section will likely deliver important results. A model of endogenous annuities and endogenous labor/leisure decisions could help resolve some long standing questions such as the effect of Social Security on the micro behavior of agents, and provide further insights on the "annuity puzzle". The conjecture at this point is that once we introduce labor supply we would see the annuity decision delayed in the life cycle, given that individuals use their labor as an insurance instrument when they are young and healthy. The extensions of this model will also help to shed some light on the effects of changes to the current Social Systems since they can be interpreted as modifications to this benchmark model. All of these possibilities are currently being considered and will be the subject of our future research.

5 Conclusions

This paper has presented several models of life cycle consumption/savings and labor/leisure decision making under uncertainty. We first present a benchmark finite horizon consumption/saving problem and solve it analytically, and then use numerical dynamic programming techniques to validate the methodology used throughout the paper. Interestingly, we find that the decision rule of the finite horizon model with bequests converges to the infinite horizon solution. We also find that numerical methods can approximate the finite horizon version of Phelps (1962) model much better than the original infinite horizon model.

We then present a model that endogenizes labor supply, allowing first for deterministic wages, and then introducing income uncertainty. The conclusions are that the model is consistent with consumption and wealth accumulation profiles in the data and that precautionary savings can even increase as uncertainty increases, even considering that labor supply (another source of accumulating precautionary balances) is endogenous, a result consistent with Low (1998). The model also shows the reduction of labor force participation at the end of the life cycle.

Finally, we introduce the possibility of endogenously choosing annuities in a consumption/saving framework with capital uncertainty and bequest. Agents can choose to annuitize part or all their wealth at any point of their lives, but they can do this only once. This model can be understood as a privatized system with no mandatory contributions but with a once in a lifetime chance to annuitize. The solution is consistent with some early results in the literature and in a sense generalizes those models. We find that agents do choose to annuitize part of their wealth and that they do it early in life, allowing them to smooth consumption considerably compared with the behavior observed in the benchmark model. We also analyze the effects of higher interest differentials between the uncertain rates of returns on wealth and the return on the annuities, as well as the effect of having a higher bequest motive. We also claim that this model provides some important insights into the "annuity puzzle" since the lack of demand for annuities can be the result of optimal behavior given the proportion of individuals' wealth that Social Security accounts for.

There are several possible extensions of the model(s) presented here. First, we are already working towards extending the endogenous annuity model of Section 4 to consider the labor/leisure decision. Such a model is quite challenging computationally, although feasible, and has the potential to provide important insights into such important issues as the role of social insurance in the simplest possible realistic dynamic model of consumption and labor decisions. Such an extension would open the door for policy experimentation, constructing modifications of this model that fit more closely the Social Security systems now in place in the U.S and other countries, and then modifying them according with suggested reforms or new possible reforms. We are also planning to allow for added uncertainty through health shocks that can be correlated with wages, and also mortality uncertainty based on life tables, instead of embedding it in the discount factor. Another extension would explicitly consider borrowing and borrowing constraints as in the consumption/saving literature. The model could eventually also allow for private pensions. This model can also be used to estimate underlying parameter values following the simulation techniques in Gourinchas and Parker (1999) given data on the variables of interest.

Finally, another possible extension of this model would be an attempt to integrate the job search decision into the life cycle dynamic maximization framework introduced here. Both young and older workers search for new jobs while out of work and on the job in non-trivial proportions. This activity should be taken into account in a life cycle model given the importance of the outcomes for the future path of earnings, wealth accumulation, and lifetime utility. Such a unifying framework would extend the life cycle utility maximization model and reconcile these two bodies of literature, which although theoretically intertwined, have evolved in different directions.

Figure 1: Consumption Decision Rule. Log Utility

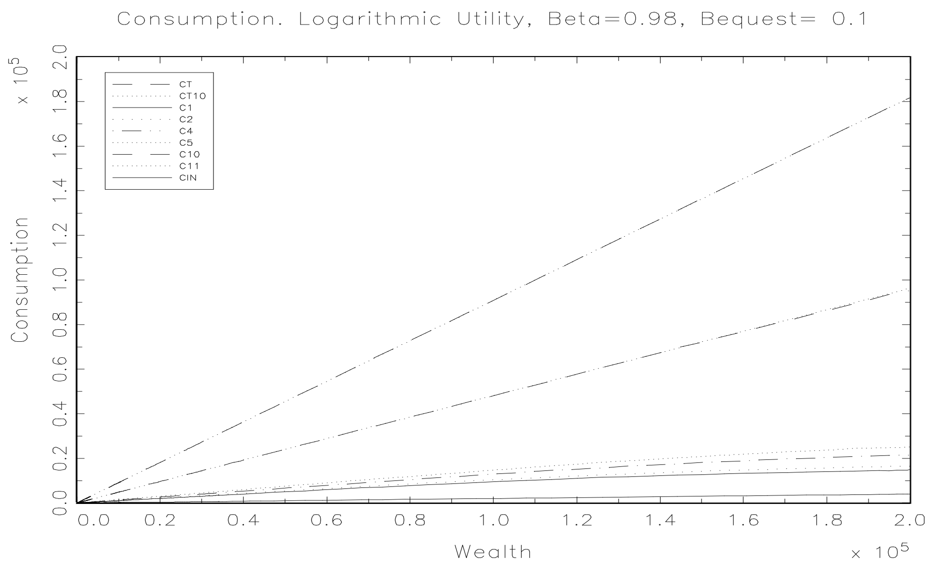


Figure 2: Consumption Decision Rule. CRRA Utility

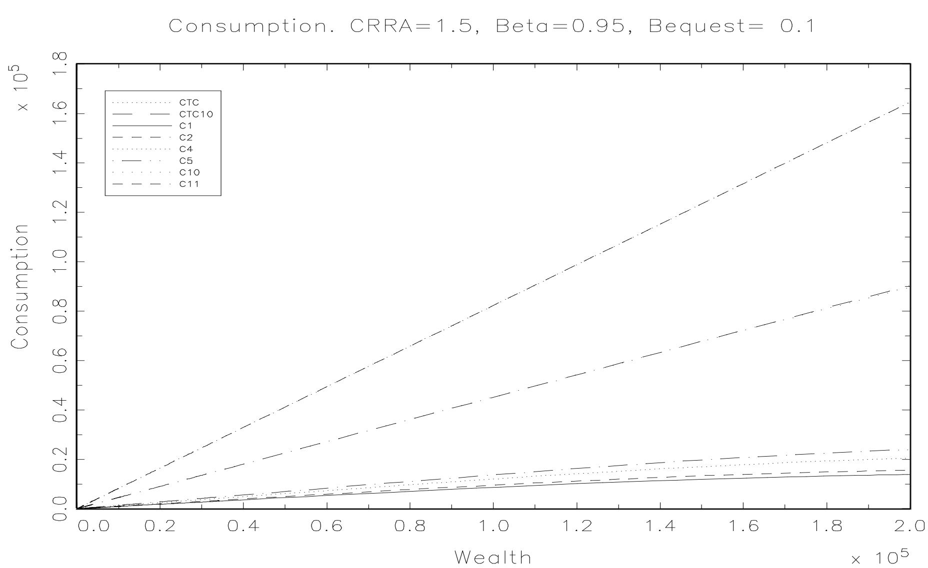


Figure 3: Simulated Consumption. CRRA utility

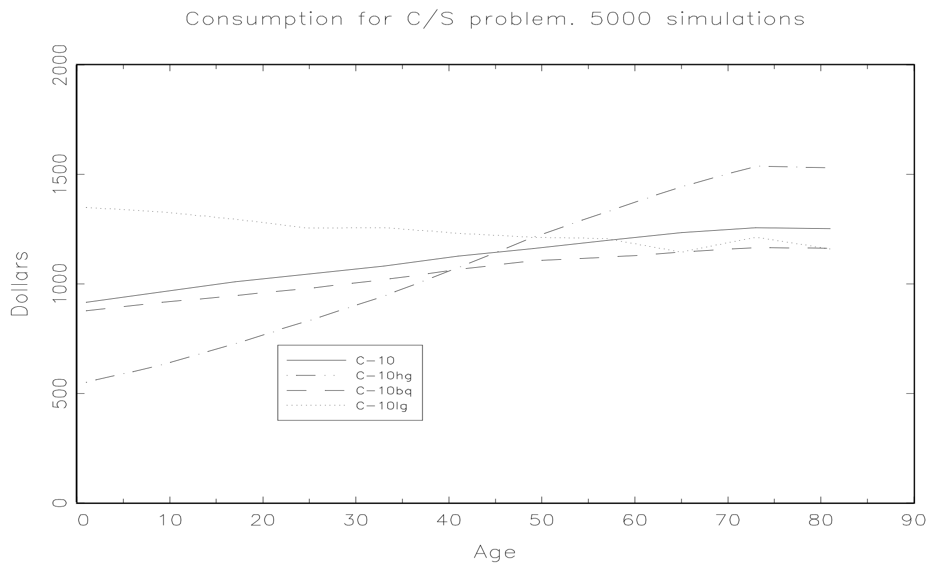


Figure 4: Simulated Wealth Accumulation. CRRA utility

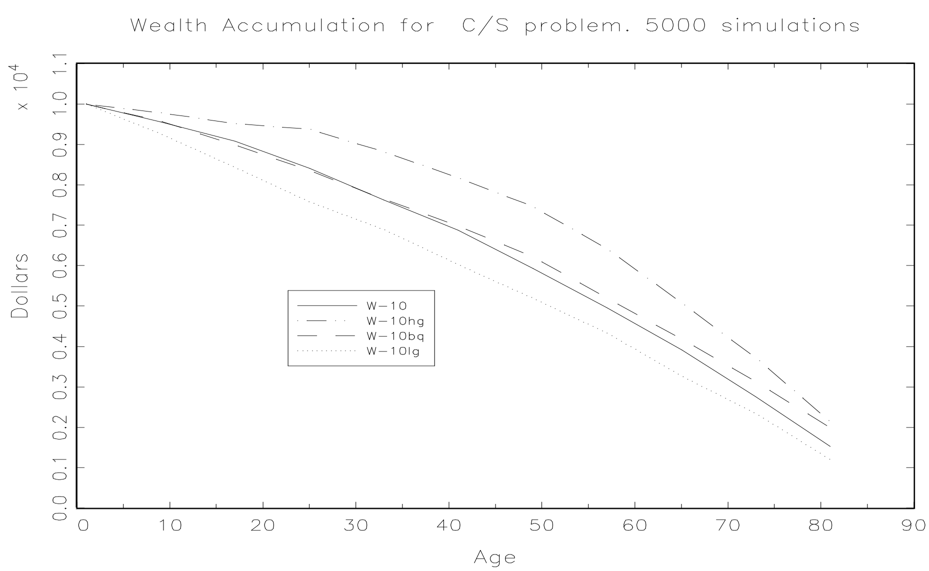


Figure 5: Simulated Consumption. Deterministic Wages

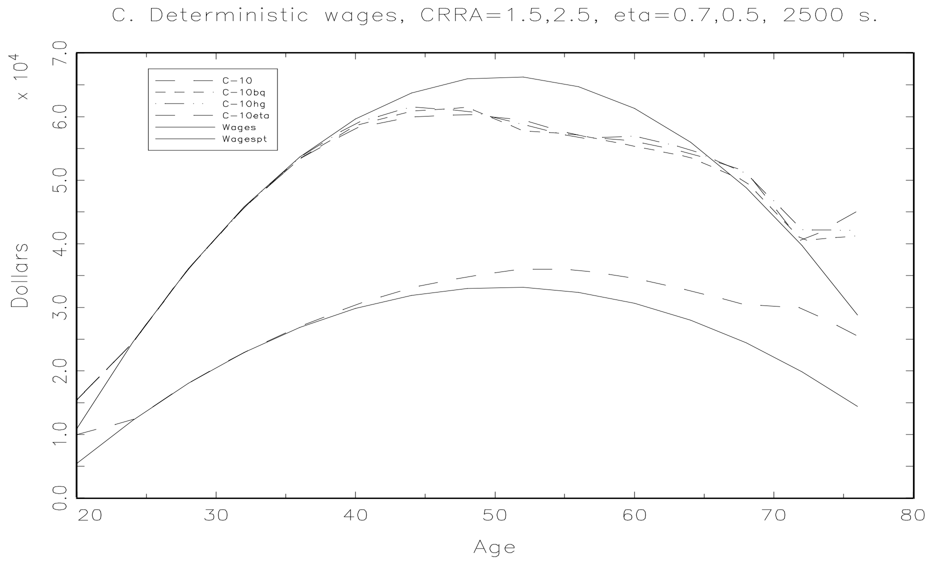


Figure 6: Simulated Labor Supply. Deterministic Wages

Figure 7: Simulated Wealth. Deterministic Wages

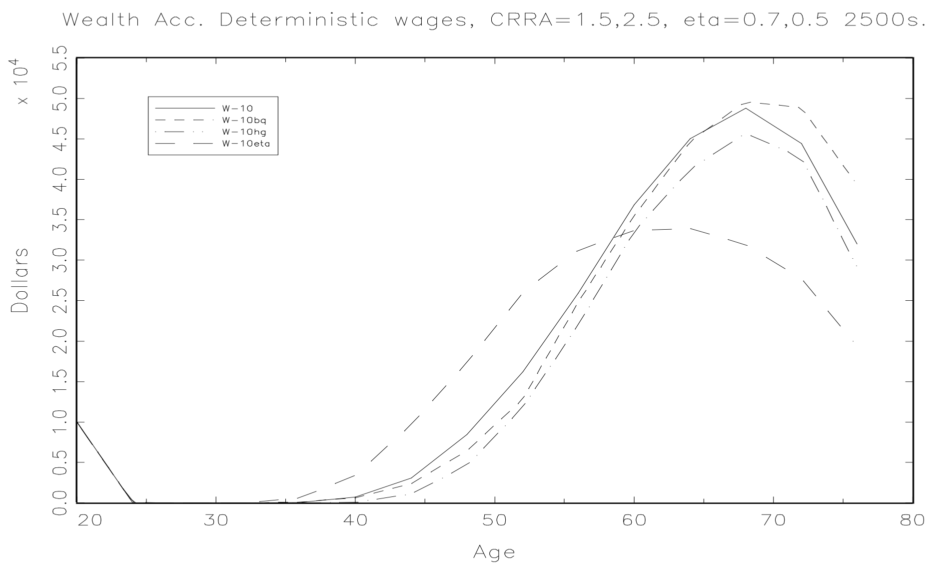


Figure 8: Simulated Consumption. Deterministic Wages

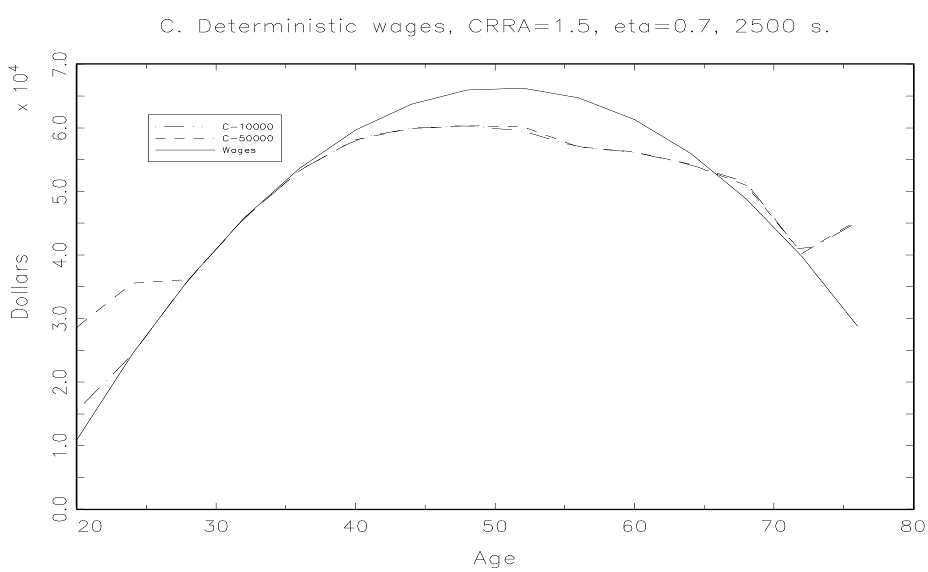


Figure 9: Simulated Labor Supply. Deterministic Wages

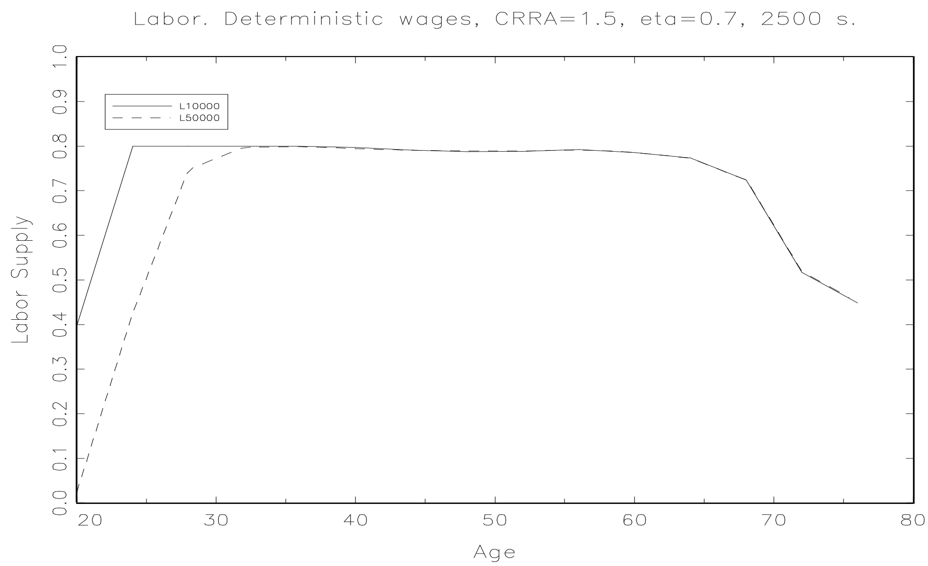


Figure 10: Simulated Wealth. Deterministic Wages

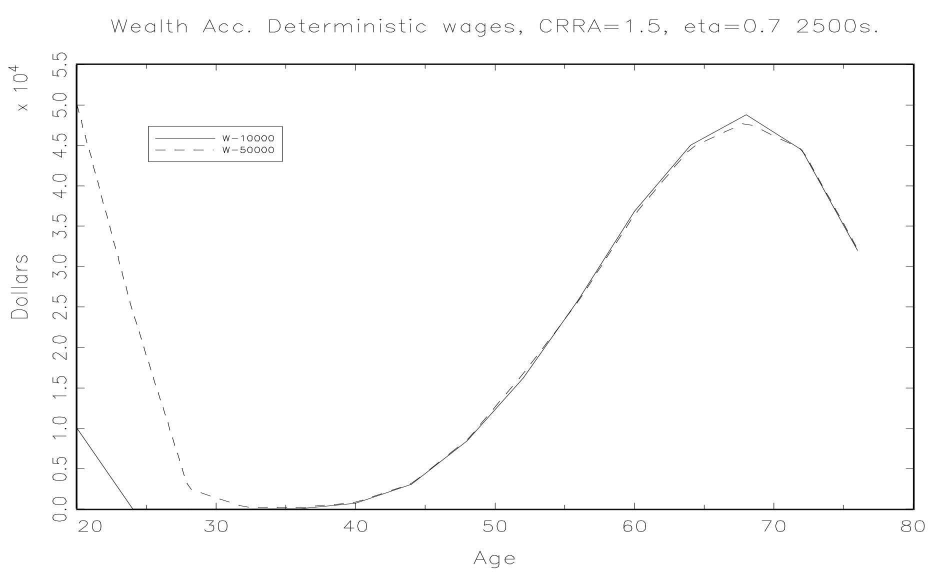


Figure 11: Simulated Consumption. Stochastic Wages

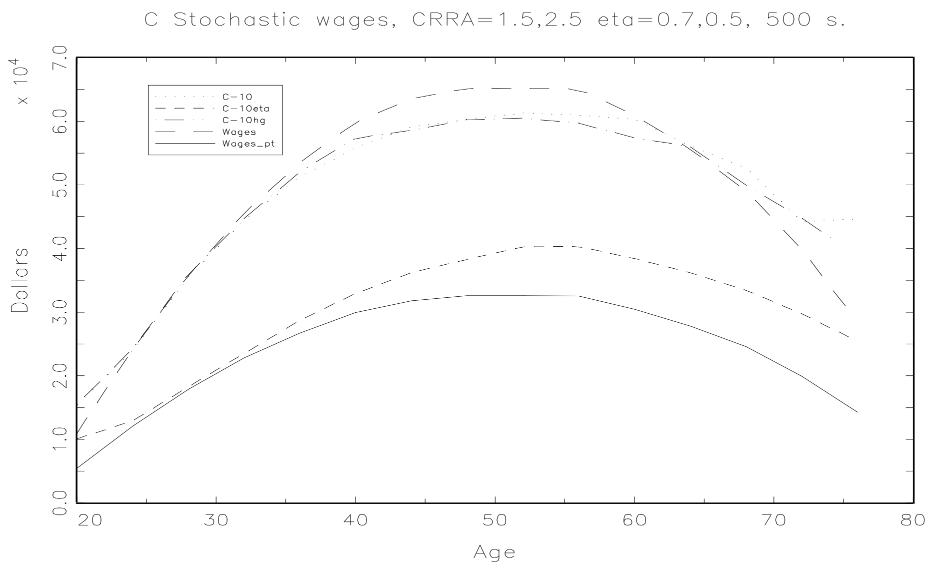


Figure 12: Simulated Labor Supply. Stochastic Wages

Figure 13: Simulated Wealth. Stochastic Wages

Figure 14: Simulated Consumption. Serially Correlated Wages

Figure 15: Simulated Labor Supply. Serially Correlated Wages

Figure 16: Simulated Wealth. Serially Correlated Wages

Figure 17: Simulated Consumption. CRRA utility

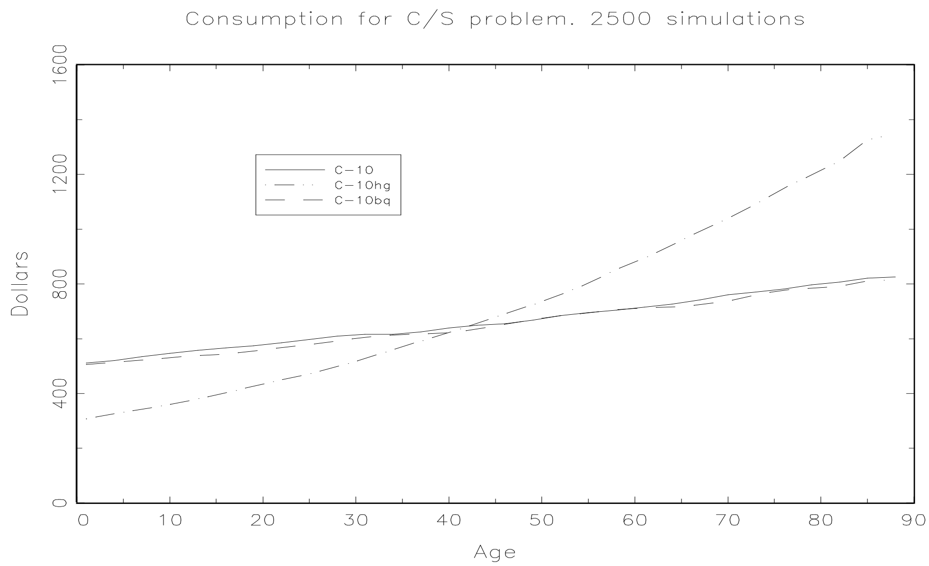


Figure 18: Simulated Consumption and Annuities. C/S Problem

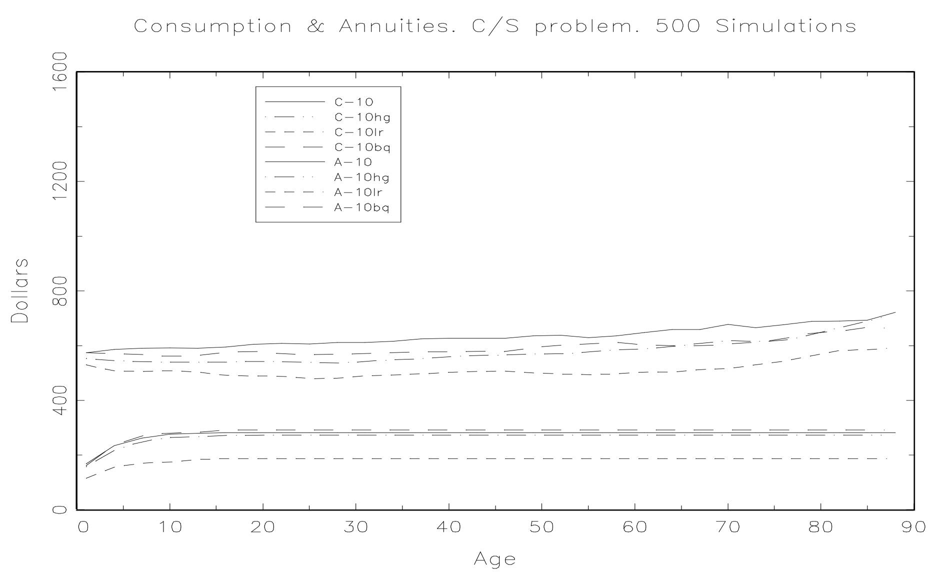


Figure 19: Simulated Wealth Accumulation. CRRA utility

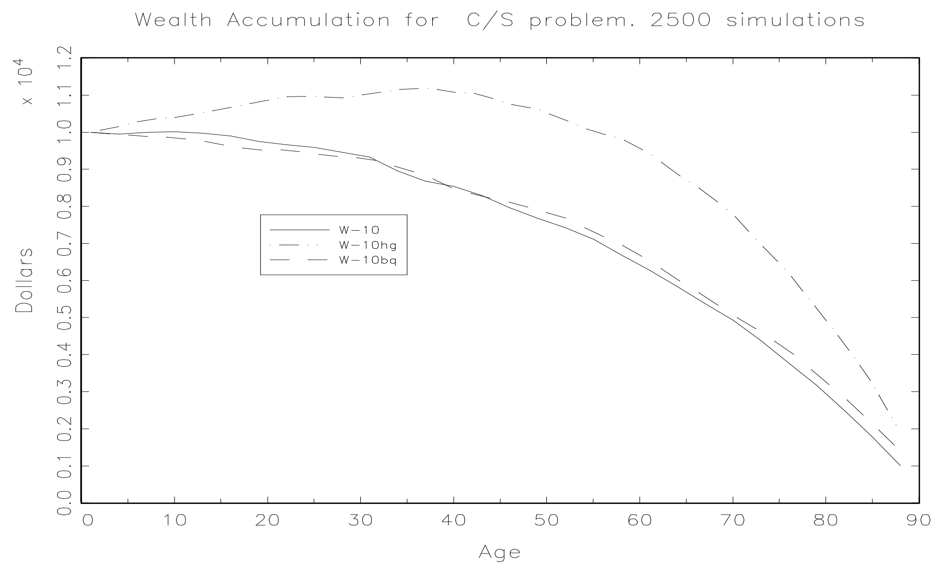


Figure 20: Simulated Wealth and Annuity Costs. C/S Problem

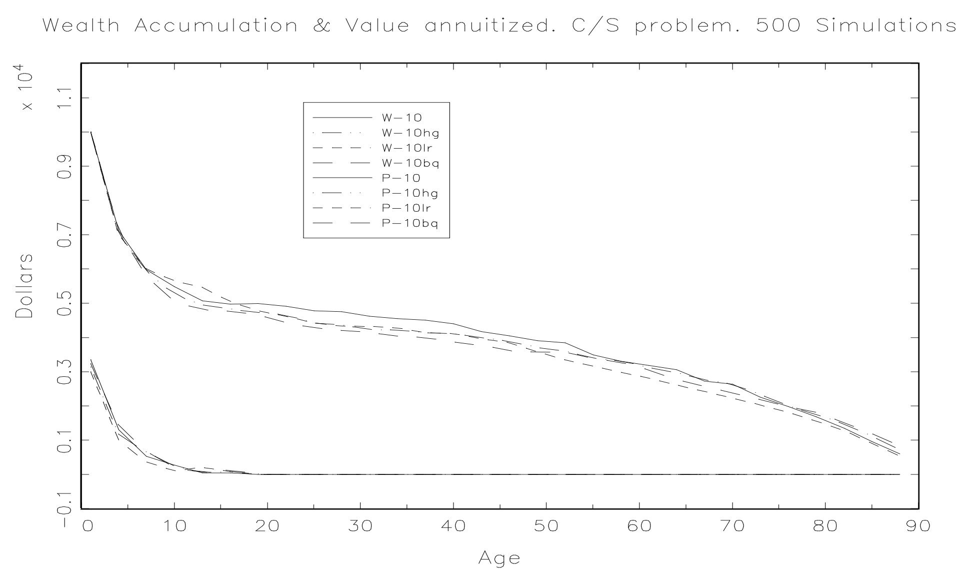
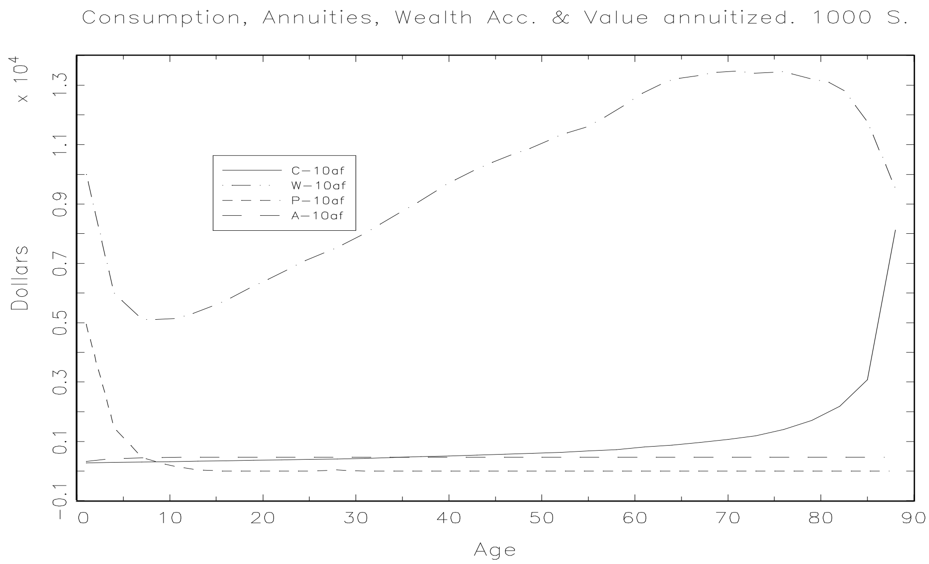


Figure 21: C/S problem with actuarially fair annuities. CRRA utility



Appendix

In this Appendix we derive the closed form solution of the finite horizon version of Phelps (1962) consumption/saving problem assuming a CRRA utility function. Our derivation is also close in nature to the one performed in Levhari and Srinivasan (1969). We can again solve this problem relying on Dynamic Programming and Bellman's principle of optimality, using backward induction. In the last period of life agents solve

$$V_T(w) \quad = \quad \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + K \, \frac{(w-c)^{1-\gamma}}{1-\gamma} \, ,$$

where γ is the coefficient of relative risk aversion and K is the bequest factor, characterized as a number between zero and one.[23] By deriving the first order condition with respect to consumption it is straightforward to show that

$$c_T \quad = \quad \frac{w}{1+K^{\frac{1}{\gamma}}} \, ,$$

we can then write the analytical expression for the last period value function:

$$V_T(w) \quad = \quad \frac{\left(\frac{w}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K \, \frac{\left(\frac{wK^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} \, .$$

Then the problem that agents solve in the next to last period of life is:

$$V_{T-1}(w) \quad = \quad \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta \, E \, V_T(w-c) \, ,$$

Using the previous results we can write

$$V_{T-1}(w) \quad = \quad \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta \, E \left[\frac{\left(\frac{\tilde{r}(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K \left[\frac{\left(\frac{\tilde{r}(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} \right] \right] \, .$$

Here in order to derive the first order condition with respect to consumption we assume, as in Lavhari and Srinivasan (1969), that the value function is differentiable and that the differential and expected value operators can be interchanged. The f.o.c is then,

$$c^{-\gamma} - \beta \, E \, (\tilde{r}^{1-\gamma}) \left[\left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}} \right)^{-\gamma} \frac{1}{1+K^{\frac{1}{\gamma}}} + K \left[\left(\frac{(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}} \right)^{-\gamma} \frac{K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}} \right] \right] \quad = \quad 0 \, .$$

[23] We also follow in this case the "egoistic" model of bequests.

Then some algebraic manipulation allows us to write the f.o.c as

$$c^{-\gamma} \;=\; \beta\, E\left(\tilde{r}^{1-\gamma}\right) \left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma} .$$

Some more tedious algebra leads to the following expression for the decision rule in the next to last period

$$c_{T-1} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}\left[1+K^{\frac{1}{\gamma}}\right]} ,$$

that can be rewritten as

$$c_{T-1} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}+\beta^{\frac{1}{\gamma}}\,\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}\,K^{\frac{1}{\gamma}}} ,$$

Assuming next that the interest rate, $\tilde{r}$, follows a log-normal distribution with mean μ and variance σ^2, then given that $E(\tilde{r}) = e^{\,\mu+\frac{\sigma^2}{2}}$ and denoting $E(\tilde{r})$ as $\overline{r}$ we can write

$$E\left(\tilde{r}^{1-\gamma}\right) \;=\; \overline{r}^{1-\gamma} e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}} ,$$

We then substitute back in the formula for c_{T-1} and obtain

$$c_{T-1} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\beta^{\frac{1}{\gamma}}\,K^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}} ,$$

given the similarity with expression (8) in the text it is easy to see how backward induction would lead us to the decision rules for the rest of the periods, for example we can write c_{T-k} as

$$c_{T-k} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\beta^{\frac{2}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\ldots+\beta^{\frac{k}{\gamma}}\,K^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}} ,$$

We can also see that if γ is equal to 1 we are back to the logarithmic utility case and the expression for c_{T-1} above is equivalent to (8), which is a special case of the expression above. It is also important to emphasize that this expression is the finite horizon counterpart to the one obtained in Levhari and Srinivasan (1969) once a bequest motive is introduced, and that their results regarding the effects of uncertainty (decreasing proportion of wealth consumed as the uncertainty grows if $\gamma > 1$) go through in this case.

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