Complex Unit Roots and Business Cycles: Are They Real?

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Abstract

In this paper the asymptotic properties of ARMA processes with complex-conjugate unit roots in the AR lag polynomial are studied. These processes behave quite di¤erently from regular unit root processes (with a single root equal to 1). In particular, the asymptotic properties of a standardized version of the periodogram for such processes are analyzed, and a nonparametric test of the complex unit root hypothesis against the stationarity hypothesis is derived. This test is applied to the annual change of the monthly number of unemployed in the US, in order to see whether this time series has complex unit roots in the business cycle frequencies.

1 Introduction

The current literature on non-seasonal unit root processes focuses almost entirely on the case of real unit roots (all equal to 1). A notable exception is the recent work by Gregoir (1999a,b,c). In the …rst two papers, Gregoir studies covariance stationary vector moving average (VMA) processes where the determinant of the lag polynomial matrix involved has multiple real and/or complex unit roots. In the third paper, Gregoir derives a parametric test for a pair of complex conjugate unit roots in an AR(2) process with white noise errors. In this paper, however, we will take a di¤erent route.

As is well known, AR processes with roots on the complex unit circle are non-stationary, and are actually more interesting than AR processes with a real valued unit root, because these processes display a persistent cyclical behavior. Thus, if there exist persistent business cycles, it seems that the data generating process involved is more compatible with an AR(MA) process with complex conjugate unit roots than with a real unit root and/or roots outside the complex unit circle.

In this paper we analyze the asymptotic properties of a standardized version of the periodogram for ARMA processes with complex unit roots in the AR lag polynomial, and derive a nonparametric test of the complex unit root hypothesis against the stationarity hypothesis. This test will be applied to US unemployment time series data[1] in order to see whether this series has complex unit roots in the business cycle frequencies.

2 AR(2) Processes with Complex Unit Roots

2.1 Introduction

Consider the AR(2) process

$$y_t = 2\cos(\phi)y_{t-1} - y_{t-2} + \mu + u_t; \tag{1}$$

where u_t is i.i.d. $(0;\sigma^2)$ with $E|u_t|^{2+\delta} < \infty$ for some $\delta > 0$; μ is a constant, and $\phi \in (0;\pi)$. Throughout this paper we assume that y_t is observable for $t = 1; \ldots; n$: The AR lag polynomial $\Psi(L) = 1 - 2\cos(\phi)L + L^2$ can be written as $\Psi(L) = (1 - \exp(i\phi)L)(1 - \exp(-i\phi)L)$, hence $\Psi(L)$ has two roots on the complex unit circle, $\exp(i\phi) = \cos(\phi) + i\sin(\phi)$; and its complex conjugate $\exp(-i\phi) = \cos(\phi) - i\sin(\phi)$; provided that $\sin(\phi) \neq 0$: The latter condition will be assumed throughout the paper, because otherwise either $\cos(\phi) = 1$; which implies that y_t is $I(2)$, or $\cos(\phi) = -1$; which implies that $y_t + y_{t-1}$ is $I(1)$: .

Note that (1) generates a persistent cycle of $2\pi/\phi$ periods. If $\phi \in (\pi; 2\pi)$; the cycle length is less than two periods. Such short cycles are unlikely to occur in macroeconomic time series, and if they occur, they are difficult, if not impossible, to distinguish from random variation. This is the reason for only considering the case $\phi \in (0;\pi)$:

[1] The empirical application involved has been conducted with the author's free software package EasyReg (Version 1.28), which is downloadable from web page:
http://econ.la.psu.edu/~hbierens/EASYREG.HTM
The monthly unemployment time series involved is included in the EasyReg database.

As is well known, the general solution of the difference equation (1) is a linear combination of a particular solution, which can be obtained by backwards substitution, and the solution of the homogenous equation $z_t = 2\cos(\phi)z_{t-1} - z_{t-2}$. As has been shown by Gregoir (1999a,b,c), using the operator

$$S_t(\phi)u_t = \sum_{j=1}^{t} \sin(\phi(t+1-j))\,u_j \tag{2}$$

for $t \geq 1$; the general solution is of the form:

LEMMA 1: Under data generating process (1), $y_t = S_t(\phi)u_t + d_t$; where and d_t is a deterministic process of the form

$$d_t = a\cos(\phi t) + b\sin(\phi t); \tag{3}$$

with a and b real valued time invariant random variables depending on initial conditions.

It is a standard calculus exercise to show that

$$\begin{aligned} S_t(\phi)u_t &= \sin(\phi(t+1))\sum_{j=1}^{t}\cos(\phi j)u_j - \cos(\phi(t+1))\sum_{j=1}^{t}\sin(\phi j)u_j \\ &= (\cos(\phi t); \sin(\phi t)) \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \\ &\times \begin{pmatrix} -\sum_{j=1}^{t} u_j\sin(\phi j) \\ \sum_{j=1}^{t} u_j\cos(\phi j) \end{pmatrix}. \end{aligned}$$

Moreover, it follows from the easy equalities

$$\sum_{j=1}^{t}\sin(2\phi j) = \frac{\cos(\phi)}{2\sin(\phi)}(1-\cos(2\phi(t+1))) - \frac{1}{2}\sin(2\phi(t+1)) \tag{4}$$

$$\sum_{j=1}^{t}\cos(2\phi j) = \frac{\cos(\phi)}{2\sin(\phi)}\sin(2\phi(t+1)) - \frac{1}{2}(1+\cos(2\phi(t+1))) \tag{5}$$

that for $\phi \in (0;\pi)$;

$$\frac{1}{n}\sum_{j=1}^{n}(\cos(\phi j))^2 = \frac{1}{2} + \frac{1}{2n}\sum_{j=1}^{n}\cos(2\phi j) = \frac{1}{2} + O(1/n)$$

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}(\sin(\phi j))^2 &= \frac{1}{2} - \frac{1}{2n}\sum_{j=1}^{n}\cos(2\phi j) = \frac{1}{2} + O(1/n); \\
\frac{1}{n}\sum_{j=1}^{n}\sin(\phi j)\cos(\phi j) &= \frac{1}{2n}\sum_{j=1}^{n}\sin(2\phi j) = O(1/n).
\end{aligned}
$$

Therefore, denoting[2]

$$
W^*_{1,n}(x) = -\frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{[xn]} u_j \sin(\phi j);\; W^*_{2,n}(x) = \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{[xn]} u_j \cos(\phi j); \qquad (6)
$$

for $x \in [0,1]$; it follows from Herrndorf's (1984) functional central limit theorem for α-mixing processes that jointly[3]

$$
W^*_{1,n} \Rightarrow W_1 \text{ and } W^*_{2,n} \Rightarrow W_2;
$$

where W_1 and W_2 are independent standard Wiener processes. See Billingsley (1968). The same applies to

$$
\begin{pmatrix} W_{1,n}(x) \\ W_{2,n}(x) \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} W^*_{1,n}(x) \\ W^*_{2,n}(x) \end{pmatrix}; \qquad (7)
$$

because the matrix involved is orthogonal. Consequently, we have

LEMMA 2: Under data-generating process (1),

$$
\begin{aligned}
y_t/\sqrt{n} &= \frac{\sigma}{\sin(\phi)\sqrt{2}}\left(\cos(\phi t)W_{1,n}(t/n) + \sin(\phi t)W_{2,n}(t/n)\right) \qquad (8) \\
&\quad + O_p(1/\sqrt{n});
\end{aligned}
$$

where

$$
\begin{pmatrix} W_{1,n} \\ W_{2,n} \end{pmatrix} \Rightarrow \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}
$$

on $[0,1]$, with W_1 and W_2 independent standard Wiener processes. Moreover, the $O_p(1/\sqrt{n})$ remainder term is uniform in $t = 1, \ldots, n$.

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[2] Thoughout this paper we adopt the convention that for $t < 1$ the sum $\sum_{j=1}^{t}$ is zero.

[3] Following Billingsley (1968), throughtout this paper the double arrow $\Rightarrow$ indicates weak convergence of random functions, or convergence in distribution in the case of random variables. The single arrow $\rightarrow$ indicates convergence in probability, unless otherwise stated.

Thus, $y_t/\sqrt{n}$ takes the form of a linear function of $\sin(\lambda t)$ and $\cos(\lambda t)$, with random coefficients $W_{1,n}(t/n)$ and $W_{2,n}(t/n)$, respectively, plus a vanishing remainder term. Consequently, the series y_t will display a rather smooth cyclical pattern, with a cycle of $2\pi/\lambda$ periods. A typical example is the artificial time series displayed in Figure 1. This time series is generated by $y_t = 1.9960534 y_{t-1} - y_{t-2} + u_t$, with u_t i.i.d. $N(0,1)$, for $t = 1, \ldots, 500$. This series has a cycle of 100 periods.

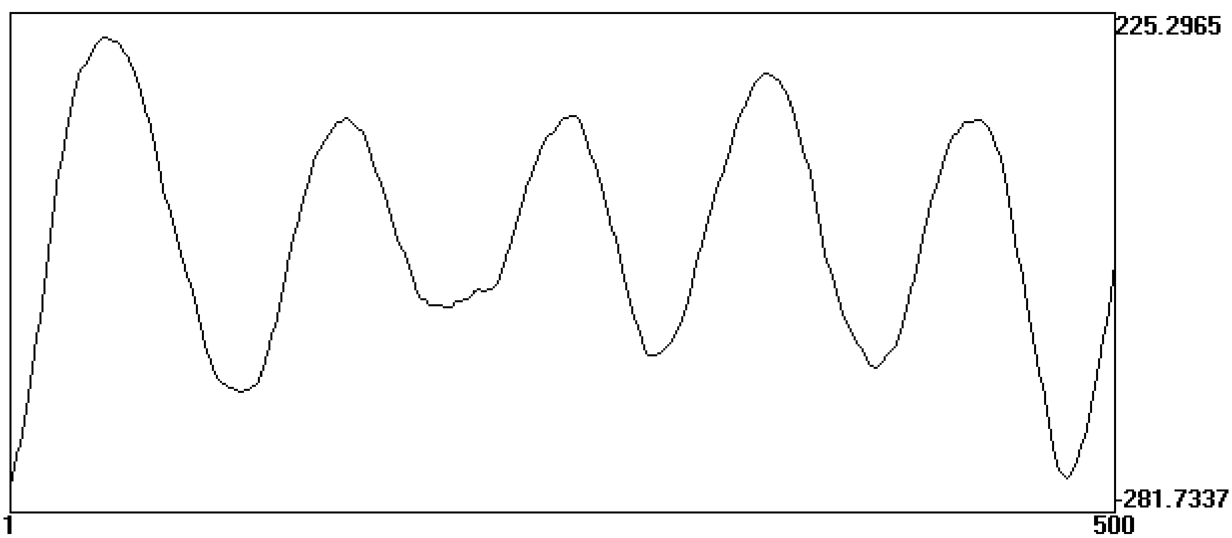

Figure 1: AR(2) process with complex unit roots and a cycle of 100 periods

2.2 Relaxing the i.i.d. error assumption

The assumption that the errors u_t in (1) are i.i.d. is not essential. We may replace it by:

ASSUMPTION 1: Let (1) hold, with u_t a zero-mean stationary ARMA process: $u_t = \psi(L)\varepsilon_t$, where ε_t is i.i.d. $(0,1)$; $E\left[|\varepsilon_t|^{2+\delta}\right] < \infty$ for some $\delta > 0$; $\psi(L) = \sum_{j=0}^{\infty} \psi_j L^j = \varphi_1(L)/\varphi_2(L)$ is a rational lag polynomial with all the roots of $\theta_2(L)$ outside the complex unit circle, and $\theta_1(e^{i\phi}) \neq 0$.[4]

Then we can write

$$\begin{aligned} u_t &= \eta(e^{i\phi})\varepsilon_t + \left(e^{i\phi} - L\right)\frac{\eta(L) - \eta(e^{i\phi})}{e^{i\phi} - L}\varepsilon_t \\ &= \eta(e^{i\phi})\varepsilon_t + e^{i\phi}w_t - w_{t-1}, \end{aligned} \tag{9}$$

where

$$w_t = \frac{\eta(L) - \eta(e^{i\phi})}{e^{i\phi} - L}\varepsilon_t = \rho(L)\varepsilon_t,$$

say. Since $\rho(L)$ is a rational lag polynomial: $\rho(L) = \rho_1(L)/\theta_2(L)$, where $\rho_1(L)$ is a finite-order lag polynomial, it follows that w_t is a (complex-valued) stationary process.

Next, observe from (9) that

$$\begin{aligned} \sum_{j=1}^{t}\exp(i\phi j)u_j &= \eta(e^{i\phi})\sum_{j=1}^{t}\exp(i\phi j)\varepsilon_j + \exp(i\phi(t+1))w_t \\ &\quad - \exp(i\phi)w_0, \end{aligned}$$

and consequently

$$\begin{aligned} \sum_{j=1}^{t}\cos(\phi j)u_j &= \mathrm{Re}\left(\eta(e^{i\phi})\right)\sum_{j=1}^{t}\cos(\phi j)\varepsilon_j \\ &\quad - \mathrm{Im}\left(\eta(e^{i\phi})\right)\sum_{j=1}^{t}\sin(\phi j)\varepsilon_j + O_p(1), \\ \sum_{j=1}^{t}\sin(\phi j)u_j &= \mathrm{Re}\left(\eta(e^{i\phi})\right)\sum_{j=1}^{t}\sin(\phi j)\varepsilon_j \\ &\quad + \mathrm{Im}\left(\eta(e^{i\phi})\right)\sum_{j=1}^{t}\cos(\phi j)\varepsilon_j + O_p(1), \end{aligned}$$

where the $O_p(1)$ term is due to the stationarity of w_t. Thus

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[4] Since $\theta_1(L)$ is real valued, all complex-valued roots come in conjugate pairs. Hence $\theta_1(e^{i\phi}) \neq 0$ implies $\theta_1(e^{-i\phi}) \neq 0$, and vice versa.

LEMMA 3: Let Assumption 1 hold. Redefine σ as

$$\sigma = \left|\eta(e^{i\varphi})\right|; \tag{10}$$

and redefine $W_{1;n}$ and $W_{1;n}$ as

$$\begin{pmatrix} W_{1;n}(x) \\ W_{2;n}(x) \end{pmatrix} = \frac{1}{\sigma}\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \times \begin{pmatrix} -\mathrm{Re}\left[\eta(e^{i\varphi})\right] & \mathrm{Im}\left[\eta(e^{i\varphi})\right] \\ \mathrm{Im}\left[\eta(e^{i\varphi})\right] & \mathrm{Re}\left[\eta(e^{i\varphi})\right] \end{pmatrix}\begin{pmatrix} W^{**}_{1;n}(x) \\ W^{**}_{2;n}(x) \end{pmatrix}; \tag{11}$$

where

$$W^{**}_{1;n}(x) = -\frac{\sqrt{2}}{\sqrt{n}}\sum_{j=1}^{[xn]}\varepsilon_j \sin(\varphi j); W^{**}_{2;n}(x) = \frac{\sqrt{2}}{\sqrt{n}}\sum_{j=1}^{[xn]}\varepsilon_j \cos(\varphi j) \tag{12}$$

Then the result of Lemma 2 goes through. Moreover, (10) is related to the spectral density

$$\delta(\xi) = \frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\cos(\xi s)\sum_{j=0}^{\infty}\eta_j\eta_{j+|s|} \tag{13}$$

of u_t, as follows: $\delta(\varphi) = \frac{1}{2\pi}|\eta(\exp(i\varphi))|^2 = \frac{\sigma^2}{2\pi}$.

The proof of the latter is a standard exercise, and therefore left to the reader.

2.3 Filtering

The argument in the previous subsection also implies that, for example, differencing of y_t does not eliminate the cycle, because the difference operator $1 - L$ changes $\eta(L)$ to $\eta_*(L) = (1 - L)\eta(L)$; which still satisfies Assumption 1. The same applies to any other linear filter $\tau(L)$ with $\tau(e^{i\varphi}) \neq 0$: In order to show how (8) changes if y_t is replaced by $\tau(L)y_t$, denote $\eta_*(L) = \tau(L)\eta(L)$. Then

$$\begin{aligned} \eta_*(e^{i\varphi}) &= \mathrm{Re}\left[\eta_*(e^{i\varphi})\right] + i\,\mathrm{Im}\left[\eta_*(e^{i\varphi})\right] \\ &= \mathrm{Re}\left[\tau(e^{i\varphi})\right]\mathrm{Re}\left[\eta(e^{i\varphi})\right] - \mathrm{Im}\left[\tau(e^{i\varphi})\right]\mathrm{Im}\left[\eta(e^{i\varphi})\right] \\ &\quad + i\left(\mathrm{Re}\left[\tau(e^{i\varphi})\right]\mathrm{Im}\left[\eta(e^{i\varphi})\right] + \mathrm{Im}\left[\tau(e^{i\varphi})\right]\mathrm{Re}\left[\eta(e^{i\varphi})\right]\right); \end{aligned}$$

hence, denoting

$$\sigma_* = \left|\eta_*(e^{i\phi})\right| = \left|\eta(e^{i\phi})\tau(e^{i\phi})\right| = \sigma\left|\tau(e^{i\phi})\right|;$$

we have

$$\begin{aligned} &\frac{1}{\sigma_*}\begin{pmatrix} -\mathrm{Re}\left[\eta_*(e^{i\phi})\right] & \mathrm{Im}\left[\eta_*(e^{i\phi})\right] \\ \mathrm{Im}\left[\eta_*(e^{i\phi})\right] & \mathrm{Re}\left[\eta_*(e^{i\phi})\right] \end{pmatrix} \\ = &\frac{1}{|\tau(e^{i\phi})|}\begin{pmatrix} \mathrm{Re}\left[\tau(e^{i\phi})\right] & \mathrm{Im}\left[\tau(e^{i\phi})\right] \\ -\mathrm{Im}\left[\tau(e^{i\phi})\right] & \mathrm{Re}\left[\tau(e^{i\phi})\right] \end{pmatrix} \\ &\times\frac{1}{\sigma}\begin{pmatrix} -\mathrm{Re}\left[\eta(e^{i\phi})\right] & \mathrm{Im}\left[\eta(e^{i\phi})\right] \\ \mathrm{Im}\left[\eta(e^{i\phi})\right] & \mathrm{Re}\left[\eta(e^{i\phi})\right] \end{pmatrix}; \end{aligned}$$

Therefore, it follows from Lemma 3, with $\eta(L)$ replaced by $\tau(L)\eta(L)$:

LEMMA 4: Let $\tau(L)$ be a linear filter satisfying $\tau(e^{i\phi}) \neq 0$. Under Assumption 1,

$$\begin{aligned} \tau(L)y_t/\sqrt{n} &= \frac{\sigma\left|\tau(e^{i\phi})\right|}{\sin(\phi)\sqrt{2}}\left(\cos(\phi t)\tilde{W}_{1,n}(t/n) + \sin(\phi t)\tilde{W}_{2,n}(t/n)\right) \\ &\quad + O_p(1/\sqrt{n}); \end{aligned}$$

where $\sigma = \left|\eta(e^{i\phi})\right|$, and

$$\begin{pmatrix} \tilde{W}_{1,n} \\ \tilde{W}_{2,n} \end{pmatrix} \Rightarrow \begin{pmatrix} W_1 \\ W_2 \end{pmatrix};$$

on $[0,1]$; with W_1 and W_2 the same as before.

Strictly speaking, the result in Lemma 4 also applies to the double difference filter $\tau(L) = (1 - L)^2 = 1 - 2L + L^2 = \Delta^2$. However, in practice this filter would wipe out a complex unit root in $\Delta^2 y_t$ if the complex unit root involved corresponds to a business cycle frequency. For example, the AR(2) lag polynomial of the process y_t displayed in Figure 1 is $1 - 1.9960534L + L^2$, which is numerically too close to $1 - 2L + L^2$ to be distinguishable, hence the AR and MA lag polynomials of the resulting ARMA(2,2) process $\Delta^2 y_t$ will approximately cancel out, causing $\Delta^2 y_t$ to look like a white noise process.

3 Frequency Analysis

The periodogram $\rho_n(\xi)$, say, of a time series y_t is defined by

$$\rho_n(\xi) = \frac{2}{n}\left[\left(\sum_{t=1}^{n} y_t\cos(\xi t)\right)^2 + \left(\sum_{t=1}^{n} y_t\sin(\xi t)\right)^2\right],$$

for $\xi \in (0,\pi)$ and odd n. See Fuller (1976, Chapter 7).

If y_t is a stationary linear process, say:

$$y_t = \mu + \psi(L)\varepsilon_t, \text{ where } \psi(L) \text{ and } \varepsilon_t \text{ are the same as in Assumption 1,} \quad (14)$$

then for fixed $\varphi \in (0,\pi)$,

$$\rho_n(\varphi) \Rightarrow 2\pi f(\varphi)\chi^2_2, \quad (15)$$

where $f(\varphi)$ is the spectral density of y_t. See (13) and Fuller (1976, Theorem 7.1.2, p. 280). As is not hard to verify, this result is due to the fact that under the stationarity hypothesis (14), $\rho_n(\xi) \Rightarrow \left|\psi\left(e^{i\varphi}\right)\right|^2 (W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2)$ pointwise in $\xi \in (0,\pi)$, where $W_{1,\xi}$ and $W_{2,\xi}$ are independent standard Wiener processes depending on ξ, which are also independent across the ξ's, and $\left|\psi\left(e^{i\xi}\right)\right|^2 = 2\pi f(\xi)$. Moreover, for $\xi = \varphi$, $W_{1,\xi}$ and $W_{2,\xi}$ are the same as W_1 and W_2 in the complex unit root case. Furthermore, since

$$\left|\psi\left(e^{i\varphi}\right)\right|^2 = \left(\sum_{j=0}^{\infty}\psi_j\cos(\xi j)\right)^2 + \left(\sum_{j=0}^{\infty}\psi_j\sin(\xi j)\right)^2$$
$$\leq \sum_{j=0}^{\infty}\psi_j^2\cos^2(\xi j) + \sum_{j=0}^{\infty}\psi_j^2\sin^2(\xi j) = \sum_{j=0}^{\infty}\psi_j^2 = \mathrm{var}(y_t),$$

it follows that under the stationarity hypothesis,

$$\hat{\rho}(\xi) = \frac{\rho_n(\xi)}{\hat{\sigma}_y^2} \Rightarrow \frac{\left|\psi\left(e^{i\varphi}\right)\right|^2}{\mathrm{var}(y_t)}\left(W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2\right) \quad (16)$$
$$\leq W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2,$$

where

$$\hat{\sigma}_y^2 = \frac{1}{n-1}\sum_{t=1}^{n}(y_t - \bar{y})^2$$

is the sample variance, with $\bar{y}$ the sample mean.

The main idea in this paper is to use the standardized periodogram $\hat{\phi}(\xi)$ as the basis for a nonparametric test of the complex unit root hypothesis against the stationarity hypothesis, because in the complex unit root case the properties of $\hat{\phi}(\xi)$ are quite different from the stationary case. This is illustrated in Figures 2 and 3. Figure 2 displays the periodogram of the complex unit root process plotted in Figure 1. Figure 3 displays the periodogram of the stationary Gaussian AR(2) process $y_t = 1.411423y_{t-1} - .5y_{t-2} + u_t$, $t = 1, ..., 500$, where the u_t's are i.i.d. $N(0, 1)$. The lag polynomial of this AR(2) process has complex roots outside the unit circle, corresponding to a (vanishing) cycle of 100 periods.

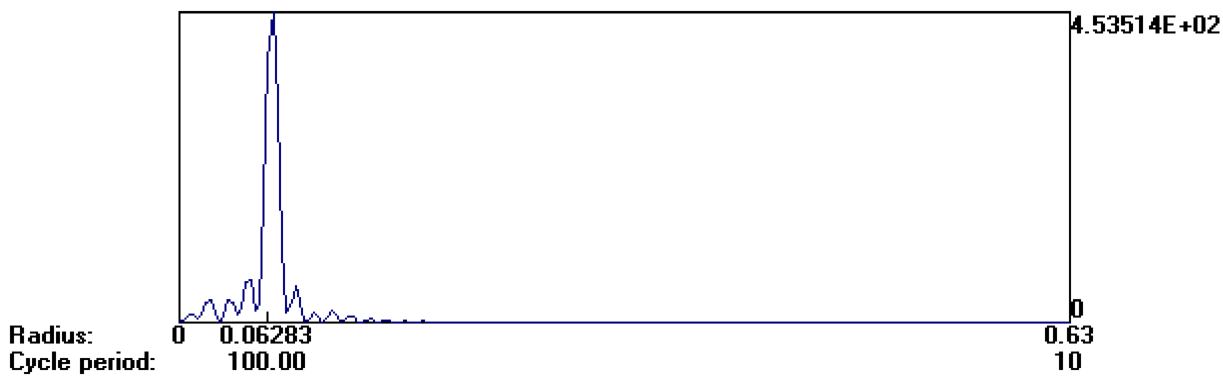

Figure 2: Periodogram of the complex unit root process plotted in Figure 1

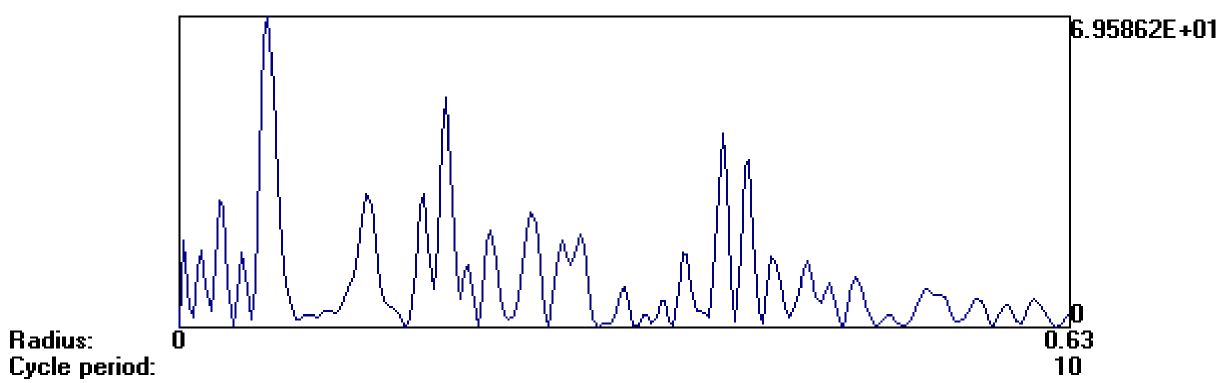

Figure 3: Periodogram of a stationary AR(2) process with complex roots and a cycle of 100 periods

We see that the two periodograms are very distinct, in shape as well as in scale. In particular, the periodogram of the stationary process has many more, and more widely spread, peaks than the periodogram of the complex unit root process, and the peaks are much lower than in the latter case.

The following lemma, which is proved in the appendix, explains the differences between these two cases.

LEMMA 5: Under Assumption 1,

$$\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t = O_p(1); \tag{17}$$

and

$$\frac{1}{n^2}\sum_{t=1}^{n} y_t^2 \Rightarrow \frac{\sigma^2}{4\sin^2(\phi)}\left(\int_0^1 W_1(x)^2dx + \int_0^1 W_2(x)^2dx\right): \tag{18}$$

Moreover,

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} y_t \begin{pmatrix} \cos(\phi t) \\ \sin(\phi t) \end{pmatrix} \Rightarrow \frac{\sigma}{2\sqrt{2}\sin(\phi)}\begin{pmatrix} \int_0^1 W_1(x)dx \\ \int_0^1 W_2(x)dx \end{pmatrix}: \tag{19}$$

Furthermore, for fixed $\xi \in (0;\phi) \cup (\phi;\pi)$;

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} y_t \begin{pmatrix} \cos(\xi t) \\ \sin(\xi t) \end{pmatrix} = O_p(1/\sqrt{n}): \tag{20}$$

Lemma 5 implies that in the complex unit root case $\hat{I}(\xi)/n$ has a sharp spike at $\xi = \phi$; with height asymptotically distributed as

$$\frac{\left(\int_0^1 W_1(x)dx\right)^2 + \left(\int_0^1 W_2(x)dx\right)^2}{\int_0^1 W_1(x)^2dx + \int_0^1 W_2(x)^2dx};$$

and asymptotically zero elsewhere, whereas (15) implies that under the stationarity hypothesis, $\hat{I}(\xi)$ is bounded away from zero, and asymptotically bounded from above by independent χ^2_2 random variables, pointwise in $\xi \in (0,\pi)$.

Summarizing, we have shown:

THEOREM 1: Consider the standardized periodogram

$$\hat{I}(\xi) = \frac{2}{n\hat{\sigma}_y^2}\left(\left(\sum_{t=1} y_t\cos(\xi t)\right)^2 + \left(\sum_{t=1} y_t\sin(\xi t)\right)^2\right); \ \xi \in (0,\pi);$$

where $\hat{\sigma}_y^2$ is the sample variance. Under Assumption 1,

$$\frac{\hat{I}(\xi)}{n} \Rightarrow \frac{\left(\int_0^1 W_1(x)dx\right)^2 + \left(\int_0^1 W_2(x)dx\right)^2}{\int_0^1 W_1(x)^2dx + \int_0^1 W_2(x)^2dx} \quad \text{if } \xi = \phi;$$

$$\frac{\hat{I}(\xi)}{n} = O_p\left(1/\sqrt{n}\right) \quad \text{if } \xi \neq \phi;$$

pointwise in $\xi \in (0,\pi)$; where W_1 and W_2 are independent standard Wiener processes.

Under the stationarity hypothesis (14),

$$\hat{I}(\xi) \Rightarrow \frac{\left|\lambda\left(e^{i\phi}\right)\right|^2}{var(y_t)}\left(W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2\right) \le W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2;$$

pointwise in $\xi \in (0,\pi)$; where $\{(W_{1,\xi}, W_{2,\xi}), \xi \in (0,\pi)\}$ is a collection of independent bivariate standard Wiener processes. Moreover, for $\xi = \phi$, $W_{1,\xi} = W_1$ and $W_{2,\xi} = W_2$.

4 Multiple Cycles

4.1 The state space case

The periodograms of macroeconomic time series often display multiple peaks in the business cycle frequencies. If k of these peaks are due to complex unit roots, then one way of modelling the process involved is as an AR(2k) process with all the roots of the AR lag polynomial on the complex unit circle. However, as is already clear from Figure 1, the plots of such processes are very smooth; much smoother than for most economic time series. Therefore, in first instance we propose to model these time series as a state space model of aggregates of ARMA processes with different single pairs of complex-conjugate unit roots, plus a stationary ARMA process representing the noise. The AR(2k) case will be considered in the next subsection.

ASSUMPTION 2: The data-generating process is: $y_t = \sum_{j=0}^{k} y_{j,t}$, where $y_{0,t} = \mu_0 + \eta_0(L)\varepsilon_{0,t}$ satisfies the conditions in (14), and for $j = 1, \ldots, k$, $y_{j,t} = 2\cos(\phi_j)y_{j,t-1} - y_{j,t-2} + \mu_j + \eta_j(L)\varepsilon_{j,t}$, with $0 < \phi_1 < \ldots < \phi_k < \pi$. The lag polynomials $\eta_j(L)$ are rational: $\eta_j(L) = \eta_{1,j}(L)/\eta_{2,j}(L)$, with $\eta_{2,j}(L)$ having all its roots outside the unit circle, and the $(\varepsilon_{1,t}, \ldots, \varepsilon_{k,t})'$'s are i.i.d. $(0, I)$, with $E|\varepsilon_{j,t}|^{2+\delta} < \infty$ for some $\delta > 0$.

Admittedly, the assumption that the $\varepsilon_{j,t}$'s are uncorrelated across the j's is quite restrictive, but is needed in order to derive nuisance-free asymptotic null distributions of the tests we are going to propose.

The process $y_{0,t}$ will only play a role under the alternative hypothesis of stationarity, which corresponds to the case $k = 0$.

It follows straightforwardly from Lemma 3 that under Assumption 2,

$$y_t/\sqrt{n} = \sum_{j=1}^{k} \frac{\sigma_j}{\sqrt{2}\sin(\phi_j)}\left(\cos(\phi_j t)W_{1,j,n}(t/n) + \sin(\phi_j t)W_{2,j,n}(t/n)\right) + O_p(1/\sqrt{n});$$

hence (17) still holds, and (18) becomes

$$\frac{1}{n^2}\sum_{t=1}^{n} y_t^2 \Rightarrow \frac{1}{4}\sum_{j=1}^{k}\frac{\sigma_j^2}{\sin^2(\phi_j)}\left(\int_0^1 W_{1,m}(x)^2dx + \int_0^1 W_{2,m}(x)^2dx\right);$$

where

$$\sigma_j = |\eta_j(\exp(i\phi_j))|;$$

and

$$(W_{1,j,n}, W_{2,j,n})' \Rightarrow W_j = (W_{1,j}, W_{2,j})'$$

jointly, with $W_1, \ldots, W_k$ independent bivariate standard Wiener processes. Note that without the assumption that the $\varepsilon_{j,t}$'s are uncorrelated across the j's, the W_j's would be dependent, but that is the only difference.

Except for parts (22) and (23), the following results follow straightforwardly from Lemma 5:

THEOREM 2: Let $\omega_j = (1/\sqrt{2})\sigma_j / \sin(\phi_j)$. Under Assumption 2,

$$y_t/\sqrt{n} = \sum_{m=1}^{k} \omega_m \left(\cos(\phi_m t) W_{1,m,n}(t/n) + \sin(\phi_m t) W_{2,m,n}(t/n)\right) + O_p\left(1/\sqrt{n}\right). \quad (21)$$

and consequently,

$$\frac{\hat{\rho}(\phi_j)}{n} \Rightarrow \psi_k(\phi_j) = \frac{\omega_j^2 \left(\left(\int_0^1 W_{1,j}(x)dx\right)^2 + \left(\int_0^1 W_{2,j}(x)dx\right)^2 \right)}{\sum_{m=1}^{k} \omega_m^2 \left(\int_0^1 W_{1,m}(x)^2 dx + \int_0^1 W_{2,m}(x)^2 dx\right)},$$

jointly for $j = 1, \ldots, k$. Hence, $\max_{j=1,\ldots,k} \hat{\rho}(\phi_j)/n \Rightarrow \max_{j=1,\ldots,k} \psi_k(\phi_j)$ and $\min_{j=1,\ldots,k} \hat{\rho}(\phi_j)/n \Rightarrow \min_{j=1,\ldots,k} \psi_k(\phi_j)$. Moreover,

$$\max_{j=1,\ldots,k} \psi_k(\phi_j) \geq B_k, \quad \min_{j=1,\ldots,k} \psi_k(\phi_j) \leq B_k, \quad (22)$$

where

$$B_k = \left(\sum_{m=1}^{k} \frac{\int_0^1 W_{1,m}(x)^2 dx + \int_0^1 W_{2,m}(x)^2 dx}{\left(\int_0^1 W_{1,m}(x)dx\right)^2 + \left(\int_0^1 W_{2,m}(x)dx\right)^2} \right)^{-1}. \quad (23)$$

Furthermore,

$$\frac{\hat{\rho}(\xi)}{n} = O_p(1/\sqrt{n}),$$

pointwise in $\xi \in (0, \pi) \setminus \{\phi_1, \ldots, \phi_k\}$.

Theorem 2 suggests to test the complex unit root hypothesis:

$$H_0: \text{Assumption 2 holds for given } k \text{ and } \phi_1 = \phi_{0,1}, \ldots, \phi_k = \phi_{0,k}, \quad (24)$$

by using the test statistic

$$\hat{B}_k = \max_{j=1,\ldots,k} \hat{\rho}(\phi_{0,j})/n \quad (25)$$

with $\alpha \times 100\%$ critical values $\underline{c}_k(\alpha)$, say, based on the lowerbound B_k of the asymptotic null distribution of $\hat{B}_k$:

$$P\left(B_k \le \underline{c}_k(\alpha)\right) = \alpha.$$

In Table 1 we present the critical values $\underline{c}_k(\alpha)$ for $k = 1, \dots, 10$, and $\alpha = 0.05$, 0.10, which have been computed by Monte Carlo simulation.[5]

Table 1: Values of $\underline{c}_k(\alpha)$

k	$\alpha = 0.05$	$\alpha = 0.10$
1	0.1403	0.2411
2	0.0667	0.1146
3	0.0441	0.0732
4	0.0313	0.0519
5	0.0249	0.0409
6	0.0210	0.0337
7	0.0177	0.0287
8	0.0154	0.0250
9	0.0137	0.0222
10	0.0120	0.0196

Given that k and $\phi_{0,1}, \dots, \phi_{0,k}$ are specified in advance, this test is consistent against the stationarity hypothesis, as well as the hypothesis that none of the given values of $\phi_{0,1}, \dots, \phi_{0,k}$ correspond to the ones in Assumption 2. In particular, for the specified frequencies $\phi_{0,1}, \dots, \phi_{0,k}$ under the null hypothesis, we have under stationarity that for $M > 0$, and independent χ_2^2 variates $\chi_2^2(1), \dots, \chi_2^2(k)$,

$$\begin{aligned} P\left(n\hat{B}_k \ge M\right) &\to P\left(\max_{j=1,\dots,k} \frac{|\eta(\exp(i\phi_{0,j}))|}{\mathrm{var}(y_t)}\chi_2^2(j) \ge M\right) \\ &\le P\left(\max_{j=1,\dots,k} \chi_2^2(j) \ge M\right) \\ &= 1 - (1 - \exp(-M/2))^k, \end{aligned} \tag{26}$$

[5] The critical values $\underline{c}_k(\alpha)$, $k = 1, \dots, 20$, have been computed by Monte Carlo simulation, on the basis of 10,100 replications of 20 independent Gaussian random walks z_t, $t = 1, \dots, n = 5{,}000$, $z_0 = 0$, and the well-known convergence results $(1/n)\sum_{t=1}^n z_t/\sqrt{n} \Rightarrow \int_0^1 W(x)dx$, $(1/n^2)\sum_{t=1}^n z_t^2 \Rightarrow \int_0^1 W(x)^2 dx$, where W is a standard Wiener process.

where the last step follows from the fact that the χ^2_2 distribution is the same as the exponential distribution with expected value 2. Setting

$$\alpha = 1 - \left(1 - \exp(-\bar{c}_k(\alpha)/2)\right)^k, \quad \bar{c}_k(\alpha) = -2\ln\left(1 - (1-\alpha)^{1/k}\right) \tag{27}$$

then yields an upperbound $\bar{c}_k(\alpha)$ of the asymptotic $\alpha \times 100\%$ critical value of a test of the stationarity hypothesis, with test statistic $n\hat{B}_k$.

4.2 The AR(2k) case

Consider the AR(2k) model with k pairs of complex conjugate unit roots:

ASSUMPTION 3: $\left[\prod_{j=1}^{k}(1 - 2\cos(\psi_{k+1-j})L + L^2)\right] y_t = \mu + \theta(L)\varepsilon_t$, where $\theta(L)$ and ε_t are the same as in Assumption 1, and $0 < \psi_1 < \ldots < \psi_k < \pi$.

Let $u_t = \theta(L)\varepsilon_t$. It follows similarly to Lemma 1 that

$$y_t = S_t(\psi_k)S_t(\psi_{k-1}).....S_t(\psi_1)u_t + d_t,$$

where $S_t(\psi)$ is defined by (2), for each pair ψ_1, ψ_2,

$$S_t(\psi_2)S_t(\psi_1) = \sum_{j=1}^{t} \sin(\psi_2(t+1-j))\, S_j(\psi_1);$$

and d_t is a deterministic process of the type (3).

Next, let

$$C_t(\psi)u_t = \sum_{j=1}^{t} \cos(\psi_1(t+1-j))\, u_t;$$

and let for each pair ψ_1, ψ_2,

$$S_t(\psi_2)C_t(\psi_1) = \sum_{j=1}^{t} \sin(\psi_2(t+1-j))\, C_j(\psi_1).$$

Then we have:

LEMMA 6:

$$\begin{aligned}
S_t(\phi_2)S_t(\phi_1) &= (\gamma_1(\phi_2;\phi_1) - \delta_1(\phi_2;\phi_1))\,(C_t(\phi_2) - C_t(\phi_1)) \\
&\quad - \gamma_2(\phi_2;\phi_1)\,(S_t(\phi_2) - S_t(\phi_1)) \\
&\quad + \delta_2(\phi_2;\phi_1)\,(S_t(\phi_2) + S_t(\phi_1))\,;
\end{aligned}$$

$$\begin{aligned}
S_t(\phi_2)C_t(\phi_1) &= -(\gamma_2(\phi_2;\phi_1) + \delta_2(\phi_2;\phi_1))\,(C_t(\phi_2) - C_t(\phi_1)) \\
&\quad - \gamma_1(\phi_2;\phi_1)\,(S_t(\phi_2) - S_t(\phi_1)) \\
&\quad - \delta_1(\phi_2;\phi_1)\,(S_t(\phi_2) + S_t(\phi_1))\,;
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1(\phi_2;\phi_1) &= \frac{1}{2}\frac{\cos(\phi_2) - \cos(\phi_1)}{(\cos(\phi_2) - \cos(\phi_1))^2 + (\sin(\phi_2) - \sin(\phi_1))^2}; \\
\gamma_2(\phi_2;\phi_1) &= \frac{1}{2}\frac{\sin(\phi_2) - \sin(\phi_1)}{(\cos(\phi_2) - \cos(\phi_1))^2 + (\sin(\phi_2) - \sin(\phi_1))^2}; \\
\delta_1(\phi_2;\phi_1) &= \frac{1}{2}\frac{\cos(\phi_2) - \cos(\phi_1)}{(\cos(\phi_2) - \cos(\phi_1))^2 + (\sin(\phi_2) + \sin(\phi_1))^2} \\
\delta_2(\phi_2;\phi_1) &= \frac{1}{2}\frac{\sin(\phi_2) + \sin(\phi_1)}{(\cos(\phi_2) - \cos(\phi_1))^2 + (\sin(\phi_2) + \sin(\phi_1))^2}.
\end{aligned}$$

The proof of Lemma 6 is pretty tedious, but involves only elementary trigonometric operations, and is therefore omitted.

Lemma 6 implies that y_t can we written as

$$y_t = \sum_{j=1} \gamma_j S_t(\phi_j)u_t + \sum_{j=1} \delta_j C_t(\phi_j)u_t + d_t;$$

where the γ_j's and δ_j's are constants depending on the ϕ_j's. Moreover, similar to Lemma 2 it follows that there exist orthogonal 2×2 matrices $Q_1, \dots, Q_k$ and constants κ_j such that

$$\begin{aligned}
&\gamma_m S_t(\phi_m)u_t + \delta_m C_t(\phi_m)u_t \\
&= \kappa_m\,(\cos(\phi t), \sin(\phi t))\,Q_m \begin{pmatrix} \sum_{j=1}^{t} u_j \sin(\phi_m j) \\ \sum_{j=1}^{t} u_j \cos(\phi_m j) \end{pmatrix}.
\end{aligned}$$

Furthermore, it follows from Lemma 3 that there exist orthogonal 2×2 matrices $R_1, \ldots, R_k$ such that

$$\begin{pmatrix} \sum_{j=1}^{t} u_j \sin(\phi_m j) \\ \sum_{j=1}^{t} u_j \cos(\phi_m j) \end{pmatrix} = |\eta(\exp(i\phi_m))| R_m \begin{pmatrix} \sum_{j=1}^{t} \varepsilon_j \sin(\phi_m j) \\ \sum_{j=1}^{t} \varepsilon_j \cos(\phi_m j) \end{pmatrix}.$$

Therefore, defining

$$\begin{pmatrix} W_{1,m}(x) \\ W_{1,m}(x) \end{pmatrix} = Q_m R_m \begin{pmatrix} \left(\sqrt{2}/\sqrt{n}\right) \sum_{j=1}^{t} \varepsilon_j \sin(\phi_m j) \\ \left(\sqrt{2}/\sqrt{n}\right) \sum_{j=1}^{t} \varepsilon_j \cos(\phi_m j) \end{pmatrix},$$

it follows that there exist constants ω_j such that (21) carries over. Consequently,

THEOREM 3: Apart from the definition of the constants ω_j, Theorem 2 holds under Assumption 3 as well.

This result also holds if we combine Assumptions 2 and 3, i.e.

ASSUMPTION 4: Let $y_t = \sum_{j=0}^{K} y_{j,t}$, where $y_{0,t}$ is the same as in Assumption 2, and for $j = 1, \ldots, K$, $\left[\prod_{j=1}^{k} (1 - 2\cos(\phi_{k+1-j})L + L^2)\right] y_{j,t} = \mu_j + \eta_j(L)\varepsilon_{j,t}$, where $\eta_j(L)$ and $\varepsilon_{j,t}$ are the same as in Assumption 2, and $0 < \phi_1 < \ldots < \phi_k < \pi$.

Thus, in this case the processes $y_{j,t}$, $j = 1, \ldots, K$, have common complex-conjugate unit roots. The condition in Assumption 2 that the $\varepsilon_{j,t}$'s are uncorrelated across the j's is now no longer needed, because if the variance matrix of $(\varepsilon_{1,t}, \ldots, \varepsilon_{K,t})'$ is Σ, say, we may without loss of generality replace $(y_{1,t}, \ldots, y_{K,t})'$ by $Q'(y_{1,t}, \ldots, y_{K,t})'$, where Q is the $K \times K_*$ matrix of eigenvectors of Σ corresponding to the K_* positive eigenvalues. Thus, without loss of generality we may assume that $\Sigma = I$.

Under Assumption 4 there exist constants $\omega_{j,m}$ such that (21) becomes

$$y_t/\sqrt{n} = \sum_{j=1}^{K} \sum_{m=1}^{k} \omega_{j,m} \left(\cos(\phi_m t) W_{1,j,m,n}(t/n) + \sin(\phi_m t) W_{2,j,m,n}(t/n)\right) + O_p\left(1/\sqrt{n}\right).$$

where jointly for $i = 1,2;\ j = 1,\ldots,K;\ m = 1,\ldots,k$, the $W_{i,j,m,n}$'s converge weakly to independent standard Wiener processes $W_{i,j,m}$. Denoting

$$W_{i,m,n}(x) = \frac{\sum_{j=1}^{K} \omega_{j,m} W_{i,j,m,n}(x)}{\omega_m},$$

where

$$\omega_m = \sqrt{\sum_{j=1}^{K} \omega_{j,m}^2},$$

we now have that

THEOREM 4: Theorem 3 carries over under Assumption 4.

5 Are Business Cycles Due to Complex Unit Roots?

In conducting the test for complex unit roots, it is tempting to formulate the null hypothesis (24) by looking at the periodogram of the time series involved, and selecting the frequencies $\phi_{0,1},\ldots,\phi_{0,k}$ corresponding to the k highest peaks. However, this is akin to pretesting, and will affect the actual size and power of the test. The correct way of conducting the test is to formulate the null hypothesis prior to looking at the data. But all information about business cycles is based on empirical investigations [see for example Diebold and Rudebush 1999 and the references therein], so that even if we would choose $\phi_{0,1},\ldots,\phi_{0,k}$ corresponding to the NBER business cycle dates and durations listed in Diebold and Rudebush (1999, Table 2.1, p.39), prior to looking at the periodogram, we would indirectly commit a pretesting-type of sin as well. In testing for seasonal unit roots this problem does not occur, of course, but is virtually impossible to avoid when testing for complex unit roots in the business cycle frequencies. In our empirical application we will therefore ignore this problem, and look at the periodogram first, in order to determine potential complex unit root frequencies.

The time series we analyze is the monthly number of civilian unemployed for 15 weeks or more in the US, times 1000, from 1948.01 to 1999.07. This series is seasonally adjusted, but in order to be sure that there are no seasonal unit roots left, and to eliminate a possible unit root 1 as well, we have transformed the series to annual changes. The plot of the transformed series is displayed in Figure 4.

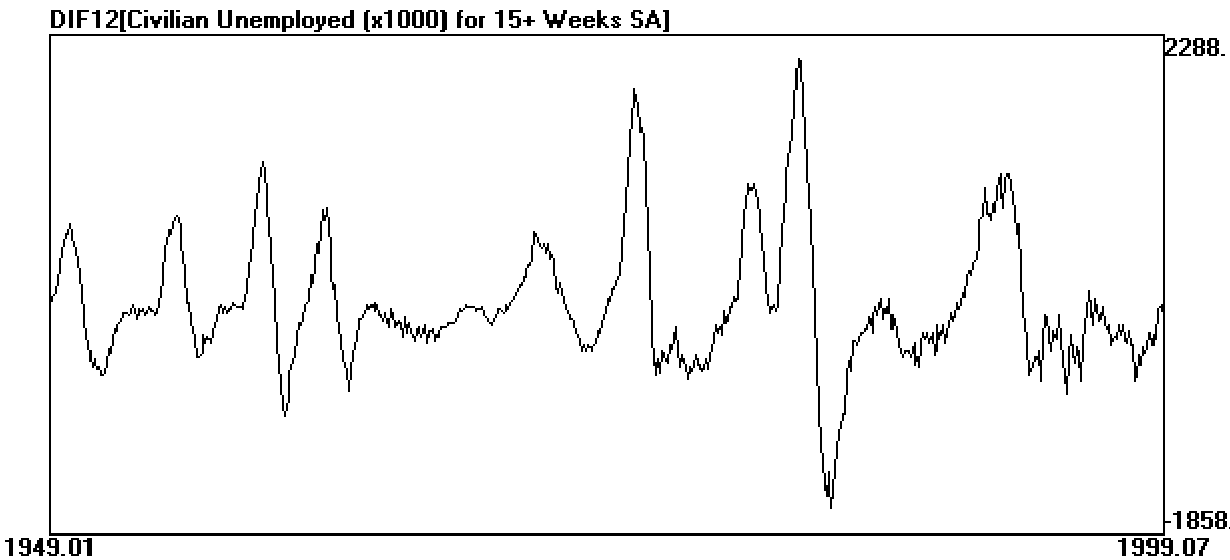

Figure 4: The data

The standardized periodogram $\hat{b}(\lambda)$ is displayed in Figure 5. The dashed lines are the 90% and 95% pointwise confidence bands under the stationarity hypothesis, based on (16).

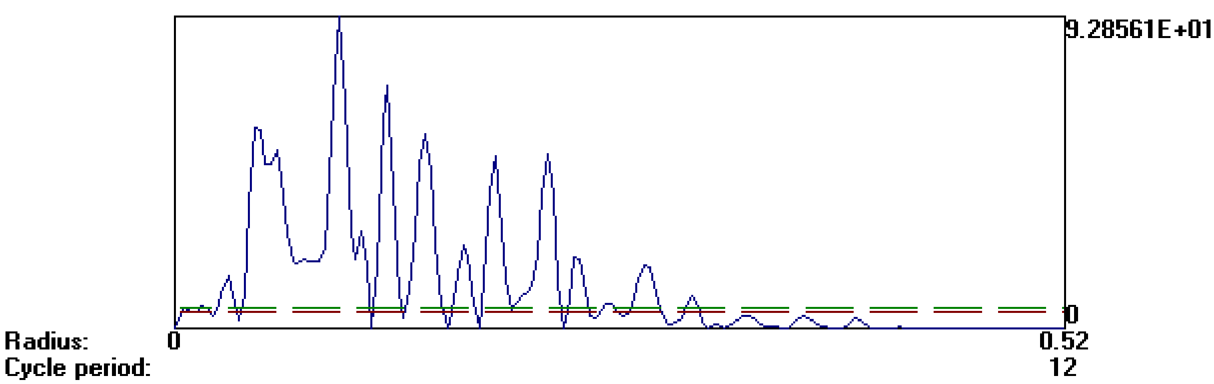

Figure 5: Standardized periodogram $\hat{W}(\xi)$ of the data

The first peak (with a little dip in the top) corresponds to a cycle duration between 104 and 133 months. The second, and highest, peak corresponds to a cycle of 65 months, and the four next highest peaks correspond to cycles of 50, 43, 33 and 28 months, respectively. These cycle durations are pretty close to the post-WW-II NBER business cycle (trough to trough) durations listed in Diebold and Rudebush (1999, Table 2.1, p.39). The longest post-war NBER cycle duration is 117 month, which corresponds to the little dip in the top of the first peak.

We now test the null hypothesis that this series has 6 pairs of complex conjugate unit roots, with frequencies corresponding to cycles of 117, 65, 50, 43, 33 and 28 months:

Tabel 2: Null hypothesis and test results

j	$\phi_{0,j}$	cycle	$\hat{W}(\phi_{0,j})/n$
1	0.05370	117	0.07238
2	0.09666	65	0.13934
3	0.12566	50	0.10855
4	0.14612	43	0.08387
5	0.19040	33	0.06667
6	0.22440	28	0.04361

Test statistic = $\max_{j=1,...,6} \hat{W}(\phi_{0,j})/n$ =	0.13934
10% critical region =	(0, 0.03366)
5% critical region =	(0, 0.02095)
$p-$value $\approx$	1

Clearly, the complex unit root hypothesis involved is not rejected.

If the time series is actually stationary, then given the specified frequencies $\phi_{0,1},...,\phi_{0,6}$, $\max_{j=1,...,6} \hat{W}(\phi_{0,j})$ is asymptotically bounded from above by a random variable which is distributed as the maximum of 6 independent χ^2_2 variates. See (26). Dividing the corresponding critical values $\bar{c}_6(\alpha)$ [see (27)] by n then yields the critical values of $\max_{j=1,...,6} \hat{W}(\phi_{0,j})/n$ under the stationarity hypothesis:

Table 3: Stationarity test results

Test statistic = $\max_{j=1,...,6} \hat{W}(\phi_{0,j})/n$ =	0.13934
5% critical region =	$(0.01570, \infty)$
10% critical region =	$(0.01335, \infty)$
$p-$value $\approx$	0

Clearly, the stationarity hypothesis is firmly rejected, given the "a priori" chosen frequencies $\phi_{0,j}, j = 1, \ldots, 6$, under the null hypothesis.

These results provide evidence that business cycles may indeed be due to complex unit roots. Whether this evidence is compelling depends on how one weighs the pretesting problem mentioned before.

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APPENDIX

Proof of (17): First, observe that there exist functions $a(\xi), b(\xi), c(\xi)$, and $d(\xi)$, not depending on t, such that for $t = 1, 2, \ldots,$

$$\cos(\lambda t) = a(\lambda)\int_t^{t+1} \cos(\lambda.x)dx + b(\lambda)\int_t^{t+1} \sin(\lambda.x)dx; \tag{28}$$
$$\sin(\lambda t) = c(\lambda)\int_t^{t+1} \cos(\lambda.x)dx + d(\lambda)\int_t^{t+1} \sin(\lambda.x)dx.$$

Therefore, it follows from (8) that

$$\frac{\sqrt{2}\sin(\phi)}{\sigma n\sqrt{n}} \sum_{t=1} y_t \tag{29}$$
$$= \frac{1}{n}\sum_{t=1} \cos(\phi t)W_{1,n}(t/n) + \frac{1}{n}\sum_{t=1} \sin(\phi t)W_{2:n}(t/n) + O_p(1/\sqrt{n})$$
$$= a(\phi)\int_0^1 \cos(n\phi.x)W_{1,n}(x)dx + b(\lambda)\int_0^1 \sin(n\phi.x)W_{1,n}(x)dx$$
$$c(\phi)\int_0^1 \cos(n\phi.x)W_{1,n}(x)dx + d(\phi)\int_0^1 \sin(n\phi.x)W_{1,n}(x)dx$$
$$+O_p(1/\sqrt{n}).$$

Moreover, it is not hard to verify that

$$E\left[W_{1,n}(x)W_{1,n}(y)\right] = \min(x, y) + O(1/n).$$

Therefore

$$E\left(\int_0^1 \cos(n\phi x)W_{1,n}(x)dx\right)^2$$
$$= \int_0^1\int_0^1 \cos(n\phi x)\cos(n\phi y)\min(x, y)dxdy + O(1/n)$$
$$= O(1/n),$$

where the last equality is an elementary calculus result. Thus,

$$\int_0^1 \cos(n\phi x)W_{1,n}(x)dx = O_p(1/\sqrt{n}).$$

Along the same lines it can be shown that the other terms in (29) are $O_p(1/\sqrt{n})$. Q.E.D.

Proof of (18): It follows from (8) that

$$
\begin{aligned}
\frac{1}{n^2}\sum_{t=1} y_t^2 &= \frac{\sigma^2}{2\sin^2(\phi)}\frac{1}{n}\sum_{t=1}\Big(\cos^2(\phi t)W_{1,n}(t/n)^2 + \sin^2(\phi t)W_{2,n}(t/n)^2 \\
&\quad + 2\cos(\phi t)\sin(\phi t)W_{1,n}(t/n)W_{2,n}(t/n)\Big) + O_p(1/\sqrt{n}) \\
&= \frac{\sigma^2}{4\sin^2(\phi)}\Bigg(\frac{1}{n}\sum_{t=1} W_{1,n}(t/n)^2 + \frac{1}{n}\sum_{t=1} W_{2,n}(t/n)^2 \\
&\quad + \frac{1}{n}\sum_{t=1}\cos(2\phi t)W_{1,n}(t/n)^2 - \frac{1}{n}\sum_{t=1}\cos(2\phi t)W_{2,n}(t/n)^2 \\
&\quad + 2\frac{1}{n}\sum_{t=1}\sin(2\phi t)W_{1,n}(t/n)W_{2,t}(t/n)\Bigg).
\end{aligned}
$$

It is easy to show that

$$
\begin{aligned}
\frac{1}{n}\sum_{t=1} W_{1,n}(t/n)^2 &= \int_0^1 W_{1,n}(x)^2dx + O_p(1/n); \\
\frac{1}{n}\sum_{t=1} W_{2,n}(t/n)^2 &= \int_0^1 W_{2,n}(x)^2dx + O_p(1/n);
\end{aligned}
$$

hence by the continuous mapping theorem [see Billingsley (1968)],

$$
\frac{1}{n}\sum_{t=1} W_{1,n}(t/n)^2 + \frac{1}{n}\sum_{t=1} W_{2,n}(t/n)^2 \Rightarrow \int_0^1 W_1(x)^2dx + \int_0^1 W_2(x)^2dx.
$$

Moreover, it follows similarly to (29) that

$$
\begin{aligned}
&\frac{1}{n}\sum_{t=1}\cos(2\phi t)W_{1,n}(t/n)^2 \qquad (30)\\
&= a(2\phi)\int_0^1 \cos(2n\phi x)W_{1,n}(x)^2dx \\
&\quad + b(2\phi)\int_0^1 \sin(2n\phi x)W_{1,n}(x)^2dx + O_p(1/n);
\end{aligned}
$$

$$\frac{1}{n}\sum_{t=1}^{n}\sin(2\pi t)W_{1,n}(t/n)W_{2,t}(t/n) \tag{31}$$
$$= c(2\pi)\int_0^1 \cos(2n\pi x)W_{1,n}(x)W_{2,n}(x)dx$$
$$+d(2\pi)\int_0^1 \sin(2n\pi x)W_{1,n}(x)W_{2,n}(x)dx + O_p(1/n).$$

In analyzing the asymptotic properties of continuous functions of $W_{1,n}$ and/or $W_{2,n}$, it often suffices to analyze the properties of the same functions of the independent standard Wiener processes W_1, W_2, because of the Skorohod (1956), Dudley (1968), and Wichura (1970) representation theorem. See also Gaenssler, P. (1983, p. 83). Loosely speaking, this representation theorem states that there exist versions $\overline{W}_n = (\overline{W}_{1,n}, \overline{W}_{2,n})'$ and $\overline{W} = (\overline{W}_1, \overline{W}_2)'$ of $W_n = (W_{1,n}, W_{2,n})'$ and $W = (W_1, W_2)'$, respectively, such that $\overline{W}_n$ has the same distribution as W_n, $\overline{W}$ has the same distribution as W (namely a bivariate standard Wiener process), and $\overline{W}_n \to \overline{W}$ a.s.[6]

Due to the representation theorem, the limiting distribution of
$$\int_0^1 \cos(2n\pi x)W_{1,n}(x)^2dx$$
is the same as the limiting distribution of
$$\int_0^1 \cos(2n\pi x)W_1(x)^2dx.$$
The latter limited distribution is constant zero, because.
$$E\left(\int_0^1 \cos(2n\pi x)W_1(x)^2dx\right)^2$$
$$= \int_0^1\int_0^1 \cos(2n\pi x)\cos(2n\pi y)E\left[W_1(x)^2W_1(y)^2\right]dxdy$$

---

[6] More precisely,
$$P\left[\lim_{n\to\infty} \rho\left(\overline{W}_n, \overline{W}\right)\right] = 1,$$
where ρ is the Skorohod norm on the space $D^2[0,1]$ of right-continuous mappings from $[0,1]$ into R^2. See Billingsley (1968).

$$= \int_0^1\int_0^1 \cos(2n\phi x)\cos(2n\phi y)\left(2(\min(x;y))^2 + xy\right)dxdy$$
$$= O(1/n).$$

The second equality is a standard Wiener measure calculus result, and the last equality is an easy calculus exercise. Thus by Chebishev's inequality

$$\int_0^1 \cos(2n\phi x)W_{1;n}(x)^2dx \to 0. \qquad (32)$$

The same applies to the sinus case. Along the same lines it can be shown that

$$\int_0^1 \cos(2n\phi x)W_{1;n}(x)W_{2;n}(x)dx \to 0; \qquad (33)$$

and the same applies to the sinus case. Q.E.D.

Proof of (19): It follows from (8) that

$$\begin{aligned}
\frac{1}{n\sqrt{n}}\sum_{t=1} y_t\cos(\phi t) &= \frac{\sigma}{\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}\cos^2(\phi t)W_{1;n}(t/n) \\
&\quad + \frac{\sigma}{\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}\cos(\phi t)\sin(\phi t)W_{2;n}(t/n) \\
&= \frac{\sigma}{2\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1} W_{1;n}(t/n) \\
&\quad + \frac{\sigma}{2\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}\cos(2\phi t)W_{1;n}(t/n) \\
&\quad + \frac{\sigma}{2\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}\sin(2\phi t)W_{2;n}(t/n) \\
&= \frac{\sigma}{2\sqrt{2}\sin(\phi)}\int_0^1 W_{1;n}(x)dx + O_p(1/\sqrt{n}).
\end{aligned}$$

The last step follows similarly to the proof of (17). Similarly,

$$\frac{1}{n\sqrt{n}}\sum_{t=1} y_t\sin(\phi t) = \frac{\sigma}{2\sqrt{2}\sin(\phi)}\int_0^1 W_{2;n}(x)dx + O_p(1/\sqrt{n}). \qquad (34)$$

Part (19) of Lemma 5 follows now from the continuous mapping theorem. Q.E.D.

Proof of (20): Similarly to the proof of (19). Q.E.D.

Proof of Theorem 2: We only need to prove the parts (22) and (23), because the other results in Theorem 2 follow straightforwardly from Lemma 5 and its proof.

Denote

$$a_m = \left(\int_0^1 W_{1;m}(x)dx\right)^2 + \left(\int_0^1 W_{2;m}(x)dx\right)^2;$$
$$b_m = \int_0^1 W_{1;m}(x)^2dx + \int_0^1 W_{2;m}(x)^2dx$$

Then

$$\rho_k(\varphi_j) = \frac{\omega_j^2 a_j}{\sum_{m=1}^{k} \omega_m^2 b_m}; \qquad (35)$$

hence for $j = 1, 2, ..., k$,

$$\omega_j^2 = \frac{\omega_1^2 a_1 \rho_k(\varphi_j)}{a_j \rho_k(\varphi_1)}. \qquad (36)$$

Substituting (36) in (36) yields

$$\sum_{m=1} \rho_k(\varphi_m)\frac{b_m}{a_m} = 1;$$

hence

$$\min_{m=1,...,k} \rho_k(\varphi_m) \sum_{m=1} \frac{b_m}{a_m} \leq 1 \leq \max_{m=1,...,k} \rho_k(\varphi_m) \sum_{m=1} \frac{b_m}{a_m}.$$

Q.E.D.