The Marginal Pricing Rule in Economies with Infinitely Many Commodities

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Abstract

In this paper, we consider an economy with infinitely many commodities and non-convex production sets. We propose a definition of the marginal pricing rule which allows us to encompass the case of smooth and convex production sets. We also show the link with the definition used in a finite dimensional setting where the marginal pricing rule is defined by means of the Clarke's normal cone. We prove the existence of a marginal pricing equilibrium under assumptions similar to the one given for an economy with a finite set of commodities.

Keywords: General equilibrium, increasing returns, infinitely many commodities, marginal pricing rule.

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1 Introduction

The marginal cost pricing rule was introduced in the thirties to obtain sufficient conditions for the Pareto optimal allocation. To quote Hotelling (1938), an optimum of welfare "corresponds to the sale of everything at marginal cost". In a seminal paper by Guesnerie (1975), a statement of the second welfare theorem is provided in a general equilibrium framework. It appears that the marginal cost pricing rule must be generalized by using the marginal pricing rule when the iso-production sets are not convex since, in this case, the cost may not be minimized at a Pareto optimal allocation. When the production set is smooth, the marginal pricing rule means that the relative prices must be equal either to the marginal rate of transformation or to the marginal rate of substitution. In other words, the producer fulfills a first-order

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necessary condition for profit maximization. Actually, by using the normal cone of Dubovickii and Miljutin to define the marginal pricing rule, Guesnerie encompasses nonsmooth production sets. Note that it is natural to define the marginal pricing rule through a normal cone since, when the boundary is smooth and the cost function is well defined and differentiable, the marginal cost pricing rule means that the price vector is orthogonal to the tangent space to the production set. This is also true with the marginal pricing rule when the cost function is not defined.

Several authors have generalized the result of Guesnerie in different frameworks by using different definitions of the normal cone which correspond to different marginal pricing rules (see, Bonnisseau-Cornet (1988b), Cornet (1986, 1990a), Jofré (1997), Khan (1998), Khan and Vohra (1987, 1988), Quinzii (1988), Yun (1984)). It is important to note that whatever is the definition of the marginal pricing rule, it coincides with the profit maximization rule when the production set is convex and it satisfies the equalities between relative prices and marginal rate when the production set is smooth.

For the problem of the existence of a marginal pricing equilibrium, the situation is not the same since the proof requires much more properties of the normal cone than the one used in the proof of the second welfare theorem. Actually, the different notions of normal cone given in the literature dealing with non-smooth analysis, are designed to obtain optimization results like the characterization of the Pareto optimal allocations. But it is well known that an equilibrium needs a simultaneous optimization and thus, stronger properties of the tools used in the proof. After Cornet (1990b), it appears that the right approach in a finite dimensional commodity space is to define the marginal pricing rule by means of Clarke's normal cone or the related concept of generalized gradient. A fundamental reason for this choice is that the marginal pricing rule satisfies then convexity and continuity properties under reasonable assumptions on the production sets. An existence result of marginal pricing equilibrium is presented in Bonnisseau-Cornet (1990b) and the proof reveals how the different properties of Clarke's normal cone are useful.

It appears that the Clarke's normal cone does not have sufficient continuity properties in infinite dimensional commodity space to obtain an equilibrium. Indeed, in Bonnisseau-Meddeb (1999), an existence result for general pricing rules with bounded losses is proved. It is assumed that the pricing rules satisfy a kind of continuity assumption which is crucial to get an equilibrium as a limit of equilibria in a sequence of auxiliary economies. The pricing rule defined by means of Clarke's normal cone does not satisfy this assumption. That is why we propose a definition of the marginal pricing rule by using a new normal cone. The main interest of our notion is that it allows us to obtain a version of the second theorem of welfare economics together with an existence result under reasonable assumptions.

The cost to be paid to obtain a general existence result is that the marginal pricing rule is less precise since the normal cone is larger than the Clarke's one. In a finite dimensional commodity space, the marginal pricing rule may also be to large as it is shown in Jouini (1988). Note that our definition of the marginal pricing rule satisfies some requirements to be consistent with the literature. When the production set is convex, one gets the standard maximization behavior for the producer and when the commodity space is finite dimensional, the notion coincides with the one defined by means of the Clarke's normal cone. Furthermore, when the production set is smooth as in Shannon (1997), there is only one price vector which satisfies the marginal pricing rule, that is the unique outward normal vector. Thus, our approach allows us to encompass economies with convex or smooth production sets and we do not need to have a specific treatment for each case.

From a technical point of view, note that the profit maximization rule needs a global information relative to the production set since it is included in an half space defined by the price vector and the production plan. The Clarke's normal cone takes into account not only the production plan but every production plan is a neighborhood. Our approach is similar but we consider weak*-open neighborhoods instead of open balls. Actually, since the weak*-open neighborhoods are never bounded, we consider the production plans which are not to far from the reference point. The use of weak topologies is natural in infinite dimensional spaces as it is explained in the survey of Mas-Colell and Zame in the Handbook of Mathematical Economics. When the production set is smooth, the information on a neighborhood is summarized by the unique outward normal vector.

As for our existence result, we consider the framework of Bewley (1972) and the proof is similar since we consider a generalized sequence of equilibria of auxiliary economies with increasing finite dimensional commodity spaces. Contrary to the existence of Walras equilibria, we cannot consider the Pareto optimal allocations since it is well known that a marginal pricing equilibrium may be not Pareto optimal (See, Beato and Mas-Colell (??)). We need to be careful with the so-called survival assumption which plays a key role in our proof since it has important consequences on the topology of the attainable allocations (Bonnisseau-Cornet(1990a,b), Cornet (1988), Kamiya(1988)). Indeed, even if the original economy is supposed to satisfy the survival assumption, this may not be true for the auxiliary economies contrary to the convex case. Thus, we begin the proof by a first limit argument which shows that the survival assumption holds true if the commodity space is large enough.

To compare our result with the one of Shannon(1997), note that her model allows with only one producer with a smooth production set and it is assumed that the unique normalized normal vector is continuous for the product of the strong and weak topologies which is a rather strong hypothesis. Indeed, it is not satisfied even if the set is convex. In the finite dimensional case, it is known that it is not possible to deduce the existence result for several producers from the one with one producer when the production sets are not supposed to be convex. The proofs are quite different since she uses a degree argument whereas we consider a generalized sequence of auxiliary economies with a finite dimensional commodity space.

The result given in Bonnisseau-Meddeb (1999) is more general since it encompasses general pricing rules. But, as in a finite dimensional setting, we remove the bounded losses assumption together with the continuity assumption on the pricing rules.

The note is organized as follows: in Section 2, we define the marginal pricing rule and its major properties. In Section 3, we state the existence result. Section 4 is devoted to the proof whereas the proofs of lemmas are given in Appendix.

2 The Marginal Pricing Rule

We consider an economy with infinitely many commodities represented by a σ-finite positive measure space $(M, \mathcal{M}, \mu)$. The bundles of goods are defined by essentially bounded, real-valued, measurable functions on $(M, \mathcal{M}, \mu)$. The commodity space is then $L = \mathcal{L}^\infty(M, \mathcal{M}, \mu)$ and we consider the standard norm defined on L denoted $\|.\|_\infty$[1]. L_+ is the positive cone of L. We give in Appendix the precise definitions.

A price system of the economy is a continuous linear mapping on the commodity space. Consequently, the price space is $\Pi = ba(M, \mathcal{M}, \mu)$, the space of bounded additive set functions on $(M, \mathcal{M})$ absolutely continuous with respect to μ, so that the value of a bundle x when the price system is given by π in Π is $\int_M x d\pi$. Note that elements of Π which are countably additive can be identified with elements of $\mathcal{L}^1 = \mathcal{L}^1(M, \mathcal{M}, \mu)$. In this case, the economic interpretation is easier since the value of a commodity bundle $x \in L$ at a price system $p \in \mathcal{L}^1$ is $\int_M p(m)x(m)d\mu(m)$.

In the following, the price vector is always fixed up to a multiplication with a positive real number and the forthcoming assumptions imply that it is non-negative that is in the the positive cone Π_+ of Π. Consequently, we only consider the prices in $S = \{p \in \Pi_+ \mid p(\chi) = 1\}$, where χ is the function equal to 1 for every m in M.

We consider the weak-star topology[2], σ^∞, and the Mackey topology[3], τ^∞, which are respectively the weakest and the strongest topology on L for which the topological dual is $\mathcal{L}^1$. The weak-star topology on Π is σ^{ba}.

The interest of the marginal pricing rule comes from the second welfare theorem. In a model with infinite dimensional commodity spaces, we can find an exposition of this result with non-convex production sets in Bonnisseau-Cornet (1988b). This result has been generalized in several directions in Cornet(1986), (1990), Benoist(1990), Jofré(1997), Khan(1998) and Khan and Vohra(1987),(1988). Roughly speaking, it states that a non-zero price vector can be associated with a Pareto optimal allocation such that each producer satisfies the marginal pricing rule

[1] If x is an element of $\mathcal{L}^\infty(M, \mathcal{M}, \mu)$ and ε a positive real number, $B(x, \varepsilon)$ is the open ball of center x and radius ε and $\bar{B}(x, \varepsilon)$ is the closed ball of center x and radius ε. If C is a subset of $\mathcal{L}^\infty(M, \mathcal{M}, \mu)$, $\text{int}_\infty C$ denotes its interior and $\partial_\infty C$ its boundary for the norm topology.

[2] A generalized sequence $(x_\psi)_{(\psi \in \Psi)}$ converges to x for the σ^∞ topology if for all $p \in \mathcal{L}^1$, $(p(x_\psi))_{(\psi \in \Psi)}$ converges to $p(x)$.

[3] τ^∞ is the topology of uniform convergence on the weak compact subsets of $\mathcal{L}^1$.

with respect to this price. A central question is the concept of normal cone which is used in the definition of the marginal pricing rule. In Bonnisseau-Cornet(1988b), the authors use the Clarke's definition whereas the other results use different definitions which lead to smaller sets, hence to better results.

Our purpose is to propose a new definition for the marginal pricing rule which means a new concept of normal cone. We want to be able to prove a version of the second welfare theorem together with an existence theorem for marginal pricing equilibria. The need for a new concept comes from the problem of non-continuity of the mapping $(p, y) \in \Pi \times L \to p(y)$ in infinite dimensional spaces for the weak topologies. This question is relevant even with convex production sets (See Mas-Colell and Zame (1991) for more details). Furthermore, we need some closedness assumption on the marginal pricing rule in order to prove the existence of an equilibrium. If we consider the Clarke's normal cone, it has a closed graph for the strong topologies under Assumption (P). But, this is not enough since we consider weak topologies and then, the closedness property fails.

To be consistent with the literature, our notion must satisfy some minimal requirement. It must encompass the case of producers with a convex production set who maximize their profits taken the prices as given and the case of smooth production sets as it is defined in Shannon (1997). In this case, the normal cone is the half line generated by the unique outward normalized normal vector. When the commodity space L is finite dimensional (that is when M is a finite set), the marginal pricing rule must coincide with the usual one given by the Clarke's normal cone.

We now come to a precise definition. We consider a producer whose technological knowledge is represented by a production set Y which is a subset of L. We posit the following assumption on Y.

Assumption (P) Y is σ^∞-closed, $Y \cap L_+ = \{0\}$ and Y satisfies the free-disposal assumption that is, $Y - L_+ = Y$.

Contrary to the case of a finite dimensional space, it is simpler to introduce first the tangent cone and then, the normal cone is the polar cone of the tangent cone. We call our notion of tangent cone, small tangent cone, since we shall prove that it is included in the Clarke's tangent cone. Conversely, we call our notion of normal cone, large normal cone, since it contains the Clarke's normal cone.

Definition 2.1 *Let $y \in Y$. The small tangent cone of Y at y denoted $\mathcal{T}_Y(y)$ is the closure of the set of vector $v \in L$ which satisfies the following condition : for all $\rho > 0$, there exists $\eta > 0$, for all $r > 0$, there exists a weak*-open neighborhood U of y and $\varepsilon > 0$, such that for all $y' \in B(y, \rho) \cap U \cap Y$, for all $t \in (0, \varepsilon)$,*

$$[\{y'\} + tB(v + \eta(y - y'), r)] \cap Y \neq \emptyset$$

The large normal cone of Y *at* y *denoted* $\mathcal{N}_Y(y)$ *is the polar cone to* $\mathcal{T}_Y(y)$*, that is,* $p \in \mathcal{N}_Y(y)$ *if* $p(v) \leq 0$ *for all* $v \in \mathcal{T}_Y(y)$*.*

Taken a price vector $p \in S$ and a production plan $y \in \partial_\infty Y$, the producer follows the marginal pricing rule if $p \in \mathcal{N}_Y(y)$. In the following, for every $y \in \partial_\infty Y$, we denote by $MP(y)$ the set $\mathcal{N}_Y(y) \cap S$. Note that the free-disposal assumption implies that the boundary of Y is the set of weakly efficient production plans, that is, the set of production plans y such that $(\{y\} + \mathrm{int}_\infty L_+) \cap Y = \emptyset$. Consequently, the marginal pricing rule requires that the producer satisfies a minimal efficiency condition.

To situate our definition with respect to previous works, we give in the following proposition some properties of $\mathcal{T}_Y$ and $\mathcal{N}_Y$. In the following, T_Y and N_Y denote respectively the Clarke's tangent cone and the Clarke's normal cone.

Proposition 2.1 *Under Assumption (P), for every* $y \in Y$*,*

(i) $\mathcal{T}_Y(y)$ *and* $\mathcal{N}_Y(y)$ *are nonempty, convex and closed cones.*

(ii) $-L_+ \subset \mathcal{T}_Y(y)$ *and* $\mathcal{N}_Y(y) \subset \Pi_+$*.*

(iii) $\mathcal{T}_Y(y) \subset T_Y(y)$ *and* $N_Y(y) \subset \mathcal{N}_Y(y)$*. Furthermore if* Y *is convex or if* L *is finite dimensional, then* $\mathcal{T}_Y(y) = T_Y(y)$ *and* $N_Y(y) = \mathcal{N}_Y(y)$*.*

(iv) Let $((y^\gamma, p^\gamma)_{\gamma \in (\Gamma, \preceq)}$ *be a bounded generalized sequence of* $\partial_\infty Y \times S$ *which converges to* (y,p) *for the topologies* σ^∞ *and* σ^{ba}*, such that* $(p^\gamma(y^\gamma))$ *converges and* $p^\gamma \in N_Y(y^\gamma) \cap S$ *for all* γ*. Then* $p(y) \leq \lim p^\gamma(y^\gamma)$*. Furthermore, if* $p(y) = \lim p^\gamma(y^\gamma)$*, then* $p \in \mathcal{N}_Y(y)$*.*

Assertion (i) is a technical one. Assertion (ii) shows that a producer sets non negative prices when he follows the marginal pricing rule. This is a consequence of the free-disposal assumption on the production set. Assertion (iii) justifies our terminology. Furthermore, since the large normal cone is included in the Clarke's normal cone, it implies that the second welfare theorem holds true in the sense that we can associate to a Pareto optimal allocation, a price vector such that each producer satisfies the marginal pricing rule for this price vector under the standard assumption of the local nonsatiation of consumers. Furthermore, if a producer with a convex production set follows the marginal pricing rule, then he maximises its profit with respect to the price vector. Finally, our definition extends the usual one for finite dimensional commodity space. Assertion (iv) is a key property in the proof of the existence of marginal pricing equilibria since it allows us to deduce that a limit of marginal pricing equilibria for a sequence of auxiliary economies, is a marginal pricing equilibrium. Note that (iv) does not imply a closedness property of the graph of $\mathcal{N}_Y$ but, roughly speaking, it means that the closure of the graph of the Clarke's normal cone is included in the graph of $\mathcal{N}_Y$.

We now show that our definition of the marginal pricing rule coincides with the one of Shannon (1997) who considers smooth production sets.

Lemma 2.1 *Let Y be defined as follows : $Y = \{y \in L \mid g(y) \leq 0\}$ where g, from L to R is differentiable, $\nabla g(y) \in \Pi_+ \setminus \{0\}$ when $g(y) = 0$. Furthermore, for all $\bar{y} \in \partial_\infty Y$, for all $\varepsilon > 0$, there exists a weak*-open neighborhood U of $\bar{y}$ such that $\nabla g(y) \in B(\nabla g(\bar{y}), \varepsilon)$ for all $y \in U \cap \partial_\infty Y$. Then, for all $\bar{y} \in \partial_\infty Y$, $\mathcal{N}_Y(\bar{y}) = N_Y(\bar{y}) = \{t\nabla g(\bar{y}) \mid t \geq 0\}$.*

The proof of this lemma is given in Appendix. With a smooth production set, the interpretation of the marginal pricing rule is natural. Indeed, the relative prices are equal to the marginal rate of substitution between two inputs or the marginal rate of transformation.

3 Existence of marginal pricing equilibrium

The purpose of this section is to show that our concept of the marginal pricing rule allows us to prove the existence of an equilibrium under assumptions similar to the ones of a finite dimensional model.

3.1 Description of the economy

The economy has a finite number m of consumers denoted by the subscript i running from 1 to m. We denote by $X_i \subset L$ the set of possible consumption plans of the ith consumer, and the tastes of this consumer are described by a complete, reflexive, transitive, binary preference relation $\preceq_i$ on his consumption set X_i. The relation of strict preference $x \prec_i x'$ is then defined by $[x \preceq_i x'$ and not $x' \preceq_i x]$. Let ω_i in L be the initial endowment of agent i and $\omega = \sum_{i=1}^m \omega_i$ the total initial endowment of the economy. We make the following assumption on the consumption sector.

Assumption (C) For all i: $(\imath)$ X_i is a σ^∞-closed and convex subset of L_+, containing 0; $(\imath\imath)$ the preference relation $\preceq_i$ is convex, τ^∞-continuous and non-satiated[4].

This assumption is standard and usually considered in general equilibrium models with infinite dimensional commodity spaces. Just note that the σ^∞ and τ^∞ topologies have the same closed and convex sets so that it's equivalent to consider that the consumption sets are τ^∞-closed or σ^∞-closed.

[4]For all $x_i \in X_i$: $\{x \in X_i \mid x \preceq_i x_i\}$ and $\{x \in X_i \mid x_i \preceq_i x\}$ are τ^∞-closed and there exists $x_i' \in X_i$ such that $x_i \prec_i x_i'$. For all $(x_i, x_i') \in X_i \times X_i$, for all $t \in]0,1[$, if $x_i \prec_i x_i'$ then $x_i \prec_i tx_i + (1-t)x_i'$.

The production sector of this economy consists of a finite number n of producers represented by the subscript j running from 1 to n. The technological knowledge of the jth firm is represented by a production set Y_j. We assume that each production set satisfies Assumption (P). Note that we do not have any convexity assumption neither on the individual production sets nor on the global one.

The revenues of the agents are defined by wealth functions, (r_i), which depend on the value of the initial endowments and the profits or losses of the producers, that is, r_i is a function from R^{n+1} to R. For every $(p, (y_j)) \in S \times \prod_{j=1}^n Y_j$, for $i = 1, ..., m$, the wealth of the ith consumer is $r_i(p(\omega_i), (p(y_j)))$

A particular case of wealth distribution is the one corresponding to the private ownership economy, where each function r_i is given by $r_i(\nu_i, (\nu_j)) = \nu_i + \sum_{j=1}^n \theta_{ji}\nu_j$ where the (θ_{ji}) correspond to the shares of consumers in the profits of each firm. $(\sum_{i=1}^m \theta_{ji} = 1$ for $j = 1, ..., n$ and $\theta_{ji} \geq 0$ for all i, j.) We posit the following assumption on the revenue functions.

Assumption (R) The functions (r_i) from R^{n+1} to R are continuous. For every $((\nu_i), (\nu_j)) \in R^{m+n}$, $\sum_{i=1}^m r_i(\nu_i, (\nu_j)) = \sum_{i=1}^m \nu_i + \sum_{j=1}^n \nu_j$ and if $\sum_{i=1}^m \nu_i + \sum_{j=1}^n \nu_j > 0$ then for every $i = 1, ..., m$, $r_i(\nu_i, (\nu_j)) > 0$.

The distribution wealth structure is slightly more restrictive than the one considered in Bonnisseau-Cornet(1988a). This is due to the difficulty coming from the non continuity of the duality product with respect to the product of the weak-star topologies. This fact is well known even when one works with an exchange economy. The last part of Assumption (R) can be interpreted as the fact that the distribution of the income among the consumers is fair in the sense that each consumer has a positive income if the total wealth of the economy is positive. Nevertheless, behind this assumption, there is an institutional mechanism to support the losses of the nonconvex firms.

As usual in an economy with production, we need a boundedness assumption which means that with a finite quantity of inputs, the production sector is not able to produce an unbounded quantity of outputs. Contrary to the case of convex productions sets, we must consider the attainable allocations associated with larger initial endowments. We refer to Bonnisseau-Cornet(1988a) for an example of an economy with a bounded attainable set but without marginal pricing equilibrium since the following assumption does not hold.

Assumption (B) For every $\omega' \geq \omega$, the set

$$A(\omega') = \left\{ (y_j) \in \Pi_{j=1}^n Y_j \mid 0 \leq \sum_{j=1}^n y_j + \omega' \right\}$$

is norm bounded.

Note that the previous assumptions are almost the same as those in Bonnisseau-Meddeb(1999). The difference with this paper is the behavior of the producers since we consider the marginal pricing rule whereas more general pricing rules are considered in the quoted paper.

3.2 The existence result

We first give the definition of a marginal pricing equilibrium which actually extends the one in Bonnisseau-Cornet (1990a,b) to an infinite dimensional space.

Definition 3.1 $((x_i^*), (y_j^*), \pi^*)$ an element of $\prod_{i=1}^m X_i \times \prod_{j=1}^n \partial_\infty Y_j \times S$ is a marginal pricing equilibrium of the economy if:

(a) x_i^* is $\preceq_i$-maximal in $\{x \in X_i \mid \pi^*(x) \leq r_i(\pi^*(\omega_i), (\pi^*(y_j^*)))\}$ for $i = 1, ..., m$.

(b) $\pi^* \in \cap_{j=1}^n MP_j(y_j^*)$.

(c) $\sum_{i=1}^m x_i^* = \sum_{j=1}^n y_j^* + \omega$.

Condition (a) defines the behavior of consumers : each of them maximizes his preferences under his budget constraint. Condition (b) requires the production sector of the economy to be at equilibrium that is, each producer is at equilibrium for the same equilibrium price vector π^*. Condition (c) means that all markets clear or the demand is equal to the supply. This definition encompasses the concept of competitive equilibrium when the production sets are convex. A marginal pricing equilibrium is a special case of an equilibrium with general pricing rule as it is defined in Bonnisseau-Meddeb(1999).

Let's define the set of production equilibria:

$$PE = \{(p, (y_j)) \in S \times \prod_{j=1}^n \partial_\infty Y_j \mid \ p \in \cap_{j=1}^n MP_j(y_j)\}$$

We are now able to state the existence result.

Theorem 3.1 *The economy* $\mathcal{E} = ((X_i, \preceq_i, r_i), (Y_j), (\omega_i))$ *has a marginal pricing equilibrium if it satisfies Assumptions (C), (P), (R), (B) and:*

Assumption (SA) *For all* $(p, (y_j), t) \in PE \times R_+$, $\sum_{j=1}^n y_j + \omega + t\chi \geq 0$ *implies* $\sum_{j=1}^n p(y_j) + p(\omega) + t > 0$.

We postpone the proof to the next section. We have already discussed Assumptions (C), (P), (R), (B). As for Assumption (SA), we refer to the papers of Cornet (1988) and Kamiya (1988) which show its crucial importance in the proof of the existence. We recall that it is satisfied when the production sets are convex and the total initial endowments are in the interior of the positive cone of L. Note also that if $\sum_{j=1}^{n} y_j + \omega + t\chi \geq 0$, then $\sum_{j=1}^{n} p(y_j) + p(\omega) + t \geq 0$ for any positive price p. Consequently, we just require that the common price vector p which is given by the marginal pricing rule of each producer, does not lead to the smallest possible total wealth.

To compare this result with the literature, we first remark that it generalizes the result of Bewley(1972) since it considers nonconvex production sets and the marginal pricing rule leads to the same equilibria as the profit maximization rule when the production sets are convex. In Bonnisseau-Meddeb (1999), the behavior of the firms is defined through a general pricing rule which allows to encompass a large number of situations. Nevertheless, the existence result uses a bounded losses assumption as in the finite dimensional case, which is not necessary with the marginal pricing rule. Furthermore, the link of the normal cone with the geometry of production sets, allows to remove Assumption (PR) on the continuity of the pricing rules.

Our model is very close to the one of Shannon(1997). Nevertheless, the main difference comes from the fact that we consider several producers and our assumption on the marginal pricing rule is weaker. In the finite dimensional setting, it is well known that the gap between one and several producers is important when the production sets are not supposed to be convex. The proofs are very different since Shannon uses the degree theory whereas we use a limit argument.

4 Proof of Theorem 3.1

4.1 Finite dimensional auxiliary economies

In this subsection, we apply Theorem 2.1 of Bonnisseau-Cornet(1990b) to a family of auxiliary economies. The difficulty comes from the fact that the survival assumption is not satisfied for every production equilibria but only on a bounded set. Consequently, we have to check carefully the proof of the above quoted result to obtain a marginal pricing equilibrium of auxiliary economies when the commodity space is large enough.

In order to define auxiliary economies with a finite dimensional space of commodity, we introduce some notations. Let $\mathcal{F}$ be the directed set of finite dimensional subspaces of L containing ω_i for all i and χ. Let $F \in \mathcal{F}$. Then $F_+ = F \cap L_+$ is the positive cone of F. Note that the interior of F_+, is equal to $F \cap \mathrm{int} L_+$ and thus it is non-empty since χ is an element of this interior. Since F_+ is a pointed cone ($F_+ \cap (-F_+) = \{0\}$), we can choose an Euclidean structure on F such that the orthogonal space to χ, $\chi^{\perp_F}$ satisfies $\chi^{\perp_F} \cap F_+ = \{0\}$ and the norm of χ is equal to 1. In the following, $\langle .,. \rangle_F$ denotes the inner product of F and $\mathrm{proj}_{\chi^{\perp_F}}$ the orthogonal projection on $\chi^{\perp_F}$. With this Euclidean structure, we identify the dual space of F to itself. Let $S^F = \{p \in F_+^\circ \mid \langle p, \chi \rangle_F = 1\}$ where F_+° is the positive polar cone of

F_+. With a little abuse, if π is an element of S, we let $\pi_{|F}$ be the element of S^F such that the restriction of π to F is $\langle \pi_{|F}, . \rangle_F$.

For every $F \in \mathcal{F}$, let us consider the economy $\mathcal{E}^F = \{(X_i^F = X_i \cap F, \preceq_i^F, r_i), (Y_j^F = Y_j \cap F), (\omega_i)\}$, where $\preceq_i^F$ is the preorder induced on X_i^F by $\preceq_i$. For $j = 1, ..., n$ and $y_j \in \partial Y_j^F$, we define:

$$MP_j^F(y_j) = N_{Y_j^F}(y_j) \cap S^F,$$

and

$$PE^F = \{(p, (y_j)) \in S^F \times \prod_{j=1}^{n} \partial Y_j^F \mid p \in \cap_{j=1}^{n} MP_j^F(y_j)\}$$

Since the two notions coincide in finite dimensional spaces, we define above the marginal pricing rule as usual by means of Clarke's normal cone. Our first lemma gives an important link between the finite economies and the original one which is used in the proof.

Lemma 4.1 *Let Y_j be a subset of L satisfying Assumptions (P). For every $F \in \mathcal{F}$, for every $y_j \in \partial Y_j^F$, for every $p \in MP_j^F(y_j)$, there exists $\pi \in N_{Y_j}(y_j)$ such that $\pi_{|F} = p$.*

The proof of this lemma is given in Appendix. We now remark that the economy $\mathcal{E}^F$ satisfies Assumptions (C), (P), (R) and (B) except the non satiation of the preferences. The following lemma shows that weak versions of Assumption (SA) and of the non satiation of the preferences are also satisfied if the commodity space F is large enough. Let us now introduce the elements for the statement of the lemma. Let $\bar{\eta}$ an arbitrary positive real number. From Assumption (B), there exists $\bar{r} > 0$ such that $A(\omega + \bar{\eta}\chi) \subset B(0, \bar{r})^n$.

Lemma 4.2 *Under the Assumptions of Theorem 3.1, there exists $\bar{F} \in \mathcal{F}$ such that for all $F \in \mathcal{F}$, if $\bar{F} \subset F$, then the economy $\mathcal{E}^F$ satisfies :*

Assumption (SA') *For all $(p, (y_j), t) \in PE^F \times [0, 4n\bar{r} + \|\omega\|_\infty]$, $\sum_{j=1}^n y_j + \omega + t\chi \geq 0$ implies $\langle p, \sum_{j=1}^n y_j + \omega \rangle_F + t > 0$.*

Assumption (NS') *For every $((x_i), (y_j)) \in \prod_{i=1}^m X_i^F \times \prod_{j=1}^n Y_j^F$, if $\sum_{i=1}^m x_i \leq \sum_{j=1}^n y_j + \omega$, then there exists $(x_i') \in \prod_{i=1}^m X_i^F$ such that $x_i \prec_i^F x_i'$ for every i.*

The proof of this lemma is given in Appendix. It is important to consider only bounded values for t since it implies that the production plans remain in a bounded hence relatively weakly compact set. Note also that the convexity of the preferences implies that they are locally non satiated if they are non satiated.

Proposition 4.1 *Under the Assumptions of Theorem 3.1, if $\bar{F} \subset F$, then $\mathcal{E}^F$ has a marginal pricing equilibrium $((x_i^F), (y_j^F), p^F)$.*

Proof. Let F such that $\bar{F} \subset F$. To compare with the proof of Bonnisseau-Cornet(1990b), note that the Euclidean structure on F is not the same as the one on R^ℓ but it suffices to replace the vector e by χ. For the non satiation of the preferences, we remark that it is used only in Claim 4 for an attainable allocation so that Assumption (NS$'$) of Lemma 4.2 is enough to conclude.

We now come to the main difference that is the fact that the survival assumption is true only on a bounded set. Note that the proof of Bonnisseau-Cornet(1990b) starts with an arbitrary fixed parameter $\varepsilon > 0$. We replace it by $\bar{\eta}$ as given before Lemma 4.2. We remark that in the proof of Bonnisseau-Cornet(1990b), the survival assumption is used to prove the existence of a retraction and in the proofs of Claims 2 and 3. For the last points, it is applied only for production plans close to the attainable allocation in the sense that $\sum_{j=1}^n y_j + \omega + \bar{\eta}\chi \geq 0$ with the notations of this paper. From the definition of $\bar{r}$, $\bar{\eta} < 4n\bar{r} + \|\omega\|_\infty$. Consequently, Assumption (SA$'$) is enough to conclude.

To prove that Assumption (SA$'$) is sufficient for the existence of the retraction, we recall how it is built. Let $\Lambda_j^F(s_j) = s_j - \lambda_j^F(s_j)\chi$ be the lipeomorphism between the hyperplan $\chi^{\perp_F}$ and the boundary of the production set Y_j^F and $\Lambda_0^F(s_j) = s_j - \lambda_0^F(s_j)\chi$ be the lipeomorphism between the $\chi^{\perp_F}$ and the boundary of F_+. We remark that s_j and $\Lambda_j^F(s_j)$ are neither in $\mathrm{int}_\infty F_+$ nor in $-\mathrm{int}_\infty F_+$ since we have suitably chosen the Euclidean structure of F, the production sets satisfy the free disposal assumption and 0 is on their boundaries. Furthermore, $s_j - \Lambda_j^F(s_j)$ is colinear to χ. Consequently, one deduces that $\|\Lambda_j^F(s_j)\|_\infty \leq 2\|s_j\|_\infty$. Conversely, with the same arguments, if $y_j \in \partial Y_j^F$, then $\|\mathrm{proj}_{\chi^{\perp_F}}(y_j)\|_\infty \leq 2\|y_j\|_\infty$. For $s \in (\chi^{\perp_F})^n$, let

$$\theta^F(s) = \sum_{j=1}^n \lambda_j^F(s_j) + \lambda_0^F(-\sum_{j=1}^n s_j - \mathrm{proj}_{\chi^{\perp_F}}\omega) - \langle \omega, \chi \rangle_F$$

From the definition of λ_j^F and λ_0^F, $\theta^F(s) \leq t$ is equivalent to $\sum_{j=1}^n \Lambda_j^F(s_j) + \omega + t\chi \geq 0$. If we look carefully to the proof of Lemma 4.1 of Bonnisseau-Cornet(1990b), we just need to check that the Clarke's generalized gradient of θ^F does not contain 0 for s in the convex hull of $M_{\bar{\eta}} = \{s \in (\chi^{\perp_F})^n \mid \theta^F(s) \leq \bar{\eta}\}$.

For all $s \in (\chi^{\perp_F})^n$, if $\theta^F(s) \leq \bar{\eta}$, then $(\Lambda_j^F(s_j)) \in A(\omega + \bar{\eta}\chi)$. Consequently, from our choice of $\bar{r}$, $\|\Lambda_j^F(s_j)\|_\infty < \bar{r}$ and from the above remark, $\|\mathrm{proj}_{\chi^{\perp_F}}(\Lambda_j^F(s_j))\|_\infty = \|s_j\|_\infty < 2\bar{r}$. Consequently, $M_{\bar{\eta}} \subset (B(0, 2\bar{r}) \cap \chi^{\perp_F})^n$. Thus the convex hull of $M_{\bar{\eta}}$ is included in $(B(0, 2\bar{r}) \cap \chi^{\perp_F})^n$. Now, let $s \in (\bar{B}(0, 2\bar{r}) \cap \chi^{\perp_F})^n$. Then $\|\Lambda_j^F(s_j)\|_\infty \leq 4\bar{r}$ for all j which implies that $\Lambda_j^F(s_j) \geq -4\bar{r}\chi$. Thus, $\sum_{j=1}^n \Lambda_j^F(s_j) + \omega + (4n\bar{r} + \|\omega\|_\infty)\chi \geq 0$. Using the same argument than the one in Bonnisseau-Cornet (1990b), one deduces that Assumption (SA$'$) implies that the Clarke's generalized gradient of θ^F at s does not contain 0. ∎

4.2 From the finite to the infinite dimensional commodity space

We consider the generalized sequence $((x_i^F),(y_j^F),p^F)$ given by Proposition 4.1. From Lemma 4.1, there exist price vectors (π_j^F) in S such that for all j, $\pi_j^F \in N_{Y_j}(y_j^F)$ and $\pi_{j|F}^F = p^F$. Note that Proposition 4.1, implies that $((x_i^F),(y_j^F))$ is an attainable allocation. Hence, from Assumption (B) and from Alaoglu-Bourbaki Theorem, it remains in a weak-star compact subset of L^{m+n}. We can do the same remark for (π_j^F) since S is σ^{ba} compact. This also implies that the real numbers $(\pi_j^F(y_j^F))$ remain in a bounded set.

Consequently, there exists $((x_i^{F(\psi)}),(y_j^{F(\psi)}),(\pi_j^{F(\psi)}))_{(\psi\in(\Psi,\geq))}$ a generalized subsequence which converges to $((\bar{x}_i),(\bar{y}_j),(\bar{\pi}_j))$ for the product of weak-star topologies and such that for $j=1,...n$, the generalized sequences $(\pi_j^{F(\psi)}(y_j^{F(\psi)}))$ are converging to limits denoted $(\bar{\nu}_j)$.

The purpose of the following claims is to prove that $((\bar{x}_i),(\bar{y}_j),\bar{\pi}_1)$ is a marginal pricing equilibrium of the economy $\mathcal{E}$. Note that $\sum_{i=1}^m \bar{x}_i = \sum_{j=1}^n \bar{y}_j + \omega$.

Claim 1. For $j=2,\dots,n$, $\bar{\pi}_j = \bar{\pi}_1$.

Proof. Let $j \in \{2,\dots,n\}$. Let x in L. There exists a finite dimensional space $F \in \mathcal{F}$ containing x. There exists $\psi_0 \in \Psi$ such that $\psi > \psi_0$ implies $F \subset F(\psi)$. As $\pi_{j|F(\psi)}^{F(\psi)} = \pi_{1|F(\psi)}^{F(\psi)} = p^{F(\psi)}$, we deduce that for $\psi > \psi_0$, $\langle p^{F(\psi)}, x\rangle_{F(\psi)} = \pi_1^{F(\psi)}(x) = \pi_j^{F(\psi)}(x)$. The limit of $(\pi_1^{F(\psi)}(x))$ is $\bar{\pi}(x)$ and the limit of $(\pi_j^{F(\psi)}(x))$ is $\bar{\pi}_j(x)$, thus $\bar{\pi}_j(x) = \bar{\pi}_1(x)$. Since this equality holds for all $x \in L$, this leads to the result. ∎

Claim 2. For $i=1,\dots,m$, for all $x_i \in X_i$, if $\bar{x}_i \preceq_i x_i$, then $\bar{\pi}_1(x_i) \geq r_i(\bar{\pi}_1(\omega_i),(\bar{\nu}_j))$.

Proof. Let $x_i \in X_i$ such that $\bar{x}_i \preceq_i x_i$. From Assumption (C), there exists x_i' arbitrarily close to x_i such that $x_i \prec_i x_i'$. We consider the set $\bar{P}_i(x_i') = \{\tilde{x}_i \in X_i \mid x_i' \preceq_i \tilde{x}_i\}$. From the continuity and the convexity of the preferences, the set $\bar{P}_i(x_i')$ is convex (see, for example, Debreu (1959)). It is also τ^∞-closed, then it's σ^∞-closed.

As $\bar{x}_i \notin \bar{P}_i(x_i')$ and $(x_i^{F(\psi)})$ converges to $\bar{x}_i$ for the σ^∞ topology, there exists $\psi_0 \in \Psi$, for all $\psi \geq \psi_0$, $x_i^{F(\psi)} \prec_i x_i'$ and $x_i' \in F(\psi)$. Since $((x_i^{F(\psi)}),(y_j^{F(\psi)}),p^{F(\psi)})$ is an equilibrium, one has $\langle p^{F(\psi)}, x_i'\rangle_{F(\psi)} > r_i(\langle p^{F(\psi)},\omega_i\rangle_{F(\psi)},(\langle p^{F(\psi)}, y_j^{F(\psi)}\rangle_{F(\psi)}))$.

Recalling the facts that $\langle p^{F(\psi)}, x_i'\rangle_{F(\psi)} = \pi_1^{F(\psi)}(x_i')$, $\langle p^{F(\psi)},\omega_i\rangle_{F(\psi)} = \pi_1^{F(\psi)}(\omega_i)$, $\langle p^{F(\psi)}, y_j^{F(\psi)}\rangle_{F(\psi)}) = \pi_j^{F(\psi)}(y_j^{F(\psi)})$, the weak-star convergence of $(\pi_1^{F(\psi)})$ to $\bar{\pi}_1$ and the continuity of r_i (Assumption (R)), imply that

$$\bar{\pi}_1(x_i') \geq r_i(\bar{\pi}_1(\omega_i),(\bar{\nu}_j))$$

Since this inequality holds for x_i' arbitrarily close to x_i, one deduces that it holds for x_i which ends the proof of the claim. ∎

Claim 3. $\bar{\pi}_1 \in \cap_{j=1}^{n} MP_j(\bar{y}_j)$ and, for all $i = 1, \ldots, m$, $\bar{\pi}_1(\bar{x}_i) = r_i(\bar{\pi}_1(\omega_i), (\bar{\pi}_1(\bar{y}_j))) = r_i(\bar{\pi}_1(\omega_i), (\bar{\nu}_j))$.

Proof. From Proposition 2.1(iv), one has $\bar{\pi}_1(\bar{y}_j) \le \bar{\nu}_j$ for all j. From Claim 2, for all i, $\bar{\pi}_1(\bar{x}_i) \ge r_i(\bar{\pi}_1(\omega_i), (\bar{\nu}_j))$. Consequently, $0 = \bar{\pi}_1(\sum_{i=1}^{m} \bar{x}_i - \sum_{j=1}^{n} \bar{y}_j - \omega) \ge \sum_{i=1}^{m} r_i(\bar{\pi}_1(\omega_i), (\bar{\nu}_j)) - \sum_{j=1}^{n} \bar{\nu}_j - \bar{\pi}_1(\omega) = 0$. The last equality comes from Assumption (R). This implies that $\bar{\pi}_1(\bar{y}_j) = \bar{\nu}_j$ for all j and $\bar{\pi}_1(\bar{x}_i) = r_i(\bar{\pi}_1(\omega_i), (\bar{\nu}_j))$ for all i.

Finally, one deduces that $\bar{\pi}_1 \in \cap_{j=1}^{n} MP_j(\bar{y}_j)$ from Proposition 2.1(iv). ∎

Claim 4. For $i = 1, ..., m$, $\bar{x}_i$ is $\preceq_i$-maximal in the budget set $\{x_i \in X_i \mid \bar{\pi}_1(x_i) \le r_i(\bar{\pi}_1(\omega_i), (\bar{\pi}_1(\bar{y}_j)))\}$.

Proof. In view of Claim 3, we have to show that for every agent i, if $\bar{x}_i \prec_i x_i$ then $\bar{\pi}_1(x_i) > \bar{\pi}_1(\bar{x}_i)$. From Claim 2, one has $\bar{\pi}_1(x_i) \ge \bar{\pi}_1(\bar{x}_i)$. Suppose $\bar{\pi}_1(x_i) = \bar{\pi}_1(\bar{x}_i)$. From Assumptions (SA) and (R) and Claim 3, we know that $\bar{\pi}_1(\bar{x}_i) > 0$. For $t > 0$ close enough to 0, $(1-t)x_i \in X_i$ and since the preferences are continuous, $\bar{x}_i \prec_i (1-t)x_i$. From Claim 2 and 3, $\bar{\pi}_1((1-t)x_i) = (1-t)\bar{\pi}_1(\bar{x}_i) \ge \bar{\pi}_1(\bar{x}_i)$. But, this contradicts $\bar{\pi}_1(\bar{x}_i) > 0$. Consequently, $\bar{\pi}_1(x_i) > \bar{\pi}_1(\bar{x}_i)$ which ends the proof of the claim. ∎

Appendix A

Let us first defined precisely the space L and Π. Let $(M, \mathcal{M}, \mu)$ be a σ-finite positive measure space, that is, μ is a non-negative real valued, countably additive set function defined on the σ-algebra $\mathcal{M}$ of subsets of M. Let $L^\infty(M, \mathcal{M}, \mu)$ be the set of all μ-essentially bounded $\mathcal{M}$-measurable functions on M.

$$L^\infty(M, \mathcal{M}, \mu) = \{f : M \to R \mid f \text{ is measurable and } \|f\|_\infty < \infty\}$$

where $\|f\|_\infty = \sup\{\alpha \ge 0 \mid \mu\{m \in M \mid |f(m)| \ge \alpha\} > 0\}$. Consider now the equivalence relation $\sim$ defined by: f and f' are real-valued measurable functions on M, $f \sim f'$ if $\mu\{m \in M \mid f(m) \ne f'(m)\} = 0$. $\mathcal{L}^\infty(M, \mathcal{M}, \mu)$ is $L^\infty_\sim(M, \mathcal{M}, \mu)$ the set of equivalence classes. We let $\mathcal{L}^\infty_+(M, \mathcal{M}, \mu)$ be the positive cone of $\mathcal{L}^\infty(M, \mathcal{M}, \mu)$ defined as follows:

$$\mathcal{L}^\infty_+(M, \mathcal{M}, \mu) = \{f \in \mathcal{L}^\infty(M, \mathcal{M}, \mu) \mid f(m) \ge 0 \text{ a.e. }\}$$

We also define $L^1(M, \mathcal{M}, \mu) = \{f : M \to R \mid f \text{ is measurable and } \int_M |f(m)| d\mu(m) < \infty\}$. $\mathcal{L}^1(M, \mathcal{M}, \mu) = L^1_\sim(M, \mathcal{M}, \mu)$.

$ba(M, \mathcal{M}, \mu)$ is the space of bounded additive set functions on $(M, \mathcal{M})$ absolutely continuous with respect to μ, that is, π in $ba(M, \mathcal{M})$ is such that $\pi(E) = 0$ for all

$M' \in \mathcal{M}$ such that $\mu(M') = 0$. The norm of $ba(M, \mathcal{M}, \mu)$ is the total variation. If $\pi \in ba(M, \mathcal{M}, \mu)$, $\|\pi\| = \sup\{\sum_{i=1}^{n} \mid \pi(E_i) \mid \mid E_1, ..., E_n$ disjoint sets of $\mathcal{M}\}$. Every $\pi \in ba^+(M, \mathcal{M}, \mu)$ can be decomposed $\pi = \pi_c + \pi_p$ where π_c is an element of $\mathcal{L}^1_+(M, \mathcal{M}, \mu)$ and π_p is positive and purely finitely additive, that is, for all $p \in \mathcal{L}^1(M, \mathcal{M}, \mu)$, $0 \leq p \leq \pi_p$ implies $p = 0$.

Proof of Proposition 2.1 In order to simplify the notations, for all $\rho > 0$, we denote by $\mathcal{T}^{\rho}_Y(y)$ the set of vecteur v such that there exists $\eta > 0$, for all $r > 0$, there exists a weak*-open neighborhood U of y and $\varepsilon > 0$ such that for all $y' \in B(y, \rho) \cap U \cap Y$, for all $t \in (0, \varepsilon)$,

$$[\{y'\} + tB(v + \eta(y - y'), r)] \cap Y \neq \emptyset$$

Consequently, $\mathcal{T}_Y(y)$ is the closure of $\cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$.

We recall the definition of the Clarke's tangent cone to Y at y. An element $v \in L$ is in $T_Y(y)$, if for all $r > 0$, there exists $\varepsilon > 0$, such that for all $y' \in B(y, \varepsilon) \cap Y$, for all $t \in (0, \varepsilon)$,

$$[\{y'\} + tB(v, r)] \cap Y \neq \emptyset$$

The Clarke's normal cone, $N_Y(y)$, is the polar cone of the Clarke's tangent cone.

(i) Taken into account Definition 2.1, it suffices to prove that $\cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$ is a nonempty convex cone. Let $\alpha > 0$. We first prove that $-\alpha\chi$ belongs to $\mathcal{T}_Y(y)$. Let $\rho > 0$, $\eta < \frac{\alpha}{\rho}$, $U = L$ and $\varepsilon = 1$. Then for all $r > 0$, for all $y' \in Y \cap B(y, \rho)$, one has $\|\eta(y - y')\| \leq \alpha$, hence $-\alpha\chi + \eta(y - y') \leq 0$. Consequently, $y' + t(v + \eta(y - y')) \leq y'$, hence, from the free-disposal assumption, it belongs to Y and obviously to $\{y'\} + tB(v + \eta(y - y'), r)$ which shows that $-\alpha\chi$ belongs to $\mathcal{T}^{\rho}_Y(y)$.

We now prove that $\cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$ is a cone. Let $v \in \cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$ and $\tau > 0$. Let $\rho > 0$. There exists $\eta > 0$, for all $r > 0$, there exists a weak*-open neighborhood U of y and $\varepsilon > 0$ such that for all $y' \in B(y, \rho) \cap U \cap Y$, for all $t \in (0, \varepsilon)$,

$$[\{y'\} + tB(v + \eta(y - y'), r)] \cap Y \neq \emptyset$$

Let $\eta' = \tau\eta$, and for all $r > 0$ let U and ε associated for v to $\frac{r}{\tau}$. Let $\varepsilon' = \frac{\varepsilon}{\tau}$. Then, for all $y' \in B(y, \rho) \cap U \cap Y$, for all $t \in (0, \varepsilon')$, $tB(\tau v + \eta'(y - y'), r) = \tau t B(v + \eta(y - y'), \frac{r}{\tau})$. Since $\tau t < \varepsilon$, from our choice of U and ε, one has

$$[\{y'\} + tB(\tau v + \eta'(y - y'), r)] \cap Y \neq \emptyset$$

Since it is true for all ρ, this implies $\tau v \in \cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$.

We end the proof by showing that $\cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$ is stable by addition which implies that it is convex. Let v and w in $\cap_{\rho>0}\mathcal{T}^{\rho}_Y(y)$. Let $\rho > 0$ and $r > 0$. From the definition of $\mathcal{T}^{\rho+1}_Y(y)$, there exists $\eta > 0$, $\eta' > 0$, $\varepsilon > 0$, $\varepsilon' > 0$ and weak*-open neighborhoods U and U' of y such that for all $y' \in B(y, \rho + 1) \cap U \cap Y$, for all $t \in (0, \varepsilon)$,

$$[\{y'\} + tB(v + \eta(y - y', \frac{r}{3})] \cap Y \neq \emptyset$$

and for all $y' \in B(y, \rho+1) \cap U' \cap Y$, for all $t \in (0, \varepsilon')$,

$$[\{y'\} + tB(w + \eta'(y - y'), \frac{r}{3})] \cap Y \neq \emptyset$$

There exists $\alpha \in (0,1)$ and U'' a weak*-open neighborhood[5] of y such that $U'' + B(0, \alpha) \subset U \cap U'$.

Let $\varepsilon'' > 0$ smaller than ε, ε', $\frac{r}{\eta'(3\|v\|+3\rho\eta+r)}$ and $\frac{3\alpha}{3\|v\|+3\eta\rho+r}$. For all $y' \in B(y, \rho) \cap U'' \cap Y$, for all $t \in (0, \varepsilon'')$, there exists $\zeta \in L$ such that $\|\zeta\| \leq \frac{r}{3}$ and $z = y' + t(v + \eta(y - y') + \zeta) \in Y$. Note that $\|z - y'\| \leq \varepsilon''(\|v\| + \eta\rho + \frac{r}{3}) < \alpha < 1$. Consequently, $z \in B(y, \rho+1) \cap U' \cap Y$. Thus, there exists $\zeta' \in L$ such that $\|\zeta'\| \leq \frac{r}{3}$ and $z' = z + t(w + \eta'(y - z) + \zeta') \in Y$. We remark that $z' = y' + t(v + w + (\eta + \eta')(y - y') + \zeta + \zeta' + \eta'(y' - z))$ and $\eta'\|y' - z\| \leq \eta'\varepsilon''(\|v\| + \eta\rho + \frac{r}{3}) < \frac{r}{3}$. Thus $z' \in Y \cap [\{y'\} + tB(v + w + (\eta + \eta')(y - y'), r)]$ which implies that $v + w \in \mathcal{T}_Y^\rho(y)$. Since it is true for all $\rho > 0$, this ends the proof.

(ii) It suffices to show that the strictly negative element of L are in $\cap_{\rho>0}\mathcal{T}_Y^\rho(y)$ but one easily checks from the definition and the free-disposal assumption on Y that, for all $\rho > 0$, $\mathcal{T}_Y^\rho(y) - L_+ = \mathcal{T}_Y^\rho(y)$ and we proved above that $-\alpha\chi \in \mathcal{T}_Y^\rho(y)$ for all $\alpha > 0$.

(iii) Since $T_Y(y)$ is closed, it suffices to prove that $\cap_{\rho>0}\mathcal{T}_Y^\rho(y) \subset T_Y(y)$. Let $\rho > 0$ and $v \in \mathcal{T}_Y^\rho(y)$. Let $r > 0$. There exists $\eta > 0$, $\varepsilon > 0$ and U associated to ρ and $\frac{r}{2}$. Let $\varepsilon' > 0$, smaller than ε, $\frac{r}{2\eta}$ and such that $B(y, \varepsilon') \subset U$. Then, for all $y' \in B(y, \varepsilon') \cap Y$, one has $y' \in U$ and consequently, for all $t \in (0, \varepsilon')$, $[\{y'\} + tB(v + \eta(y - y'), \frac{r}{2})] \cap Y \neq \emptyset$. But, $\|\eta(y - y')\| \leq \varepsilon'\eta < \frac{r}{2}$. Thus, $[\{y'\} + tB(v, r)] \cap Y \neq \emptyset$, hence $v \in T_Y(y)$.

If Y is convex, we just need to prove that $Y - \{y\} \subset \cap_{\rho>0}\mathcal{T}_Y^\rho(y) \subset T_Y(y)$ since $T_Y(y)$ is a closed, convex cone. Let $z \in Y$. Let $\rho > 0$, $r > 0$, $\eta = \varepsilon = 1$ and $U = L$. For all $y' \in B(0, \rho) \cap Y$, for all $t \in (0,1)$, $y' + t(z - y + (y - y')) = (1-t)y' + tz \in Y$ since Y is convex. Clearly, $y' + t(z - y + (y - y')) \in [\{y'\} + tB(z - y + (y - y'), r)]$, hence $z - y \in \mathcal{T}_Y^\rho(y)$.

If L is finite dimensional, then the weak*-open neighborhood are the open neighborhood for the norm topology. Let $y \in Y$ and $v \in T_Y(y)$. Let $r > 0$, then there exists $\varepsilon > 0$ such that for all $y' \in B(y, \varepsilon) \cap Y$, for all $t \in (0, \varepsilon)$, $[\{y'\} + tB(v, \frac{r}{2})] \cap Y \neq \emptyset$. Thus, for $\rho > 0$, let $\eta = 1$ and $\varepsilon' > 0$ smaller than ρ, ε and $\frac{r}{2}$. Let $U = B(y, \varepsilon')$. Note that $B(y, \rho) \cap U \cap Y \subset B(y, \varepsilon)$. For all $y' \in B(y, \rho) \cap U \cap Y$, one has $[\{y'\} + tB(v, \frac{r}{2}) \subset [\{y'\} + tB(v + (y - y'), r)]$. Consequently, $\emptyset \neq [\{y'\} + tB(v, \frac{r}{2})] \cap Y \subset [\{y'\} + tB(v + (y - y'), r)] \cap Y$. This implies that $v \in \mathcal{T}_Y^\rho(y)$, thus $v \in \mathcal{T}_Y(y)$ since it is true for all $\rho > 0$.

(iv) We first state a lemma which is the key argument of the proof.

[5] If U is a weak* open neighborhood of y, there exists a finite family $(f_i)_{i \in I}$ of $\mathcal{L}^1$ and $\alpha > 0$ such that $\{x \in L \mid |f_i(x - y)| < \alpha, \ \forall i \in I\}$ is included in U. Let $\beta = \frac{\alpha}{2\max\{\|f_i\| | i \in I\}}$. Let $U' = \{x \in L \mid |f_i(x - y)| < \frac{\alpha}{2}, \ \forall i \in I\}$. Then, for all $i \in I$, for all $x \in U'$ and for all $x' \in B(0, \beta)$, $|f_i(x + x' - y)| \leq |f_i(x - y)| + |f_i(x')| \leq \frac{\alpha}{2} + \beta\|f_i\| \leq \alpha$. Consequently, $x + x' \in U$.

Lemma. *Let* $\rho > 0$, $v \in \mathcal{T}_Y^\rho(y)$, $r > 0$ *and* η, U *and* ε *as given by the definition of* $\mathcal{T}_Y^\rho(y)$. *Then, for all* $y' \in B(y,\rho) \cap U \cap Y$, $v + \eta(y - y') - 2r\chi \in T_Y(y)$.

Proof. Let $\varepsilon' > 0$ smaller than ε, $\frac{r}{\eta}$ and such that $B(y', \varepsilon') \subset B(y,\rho) \cap U$. For all $z \in B(y', \varepsilon') \cap Y$, for all $t \in (0, \varepsilon')$, there exists ζ such that $\|\zeta\| < r$ and $z + t(v + \eta(y-z) + \zeta) \in Y$. Note that $\|\zeta + \eta(y'-z)\| < 2r$, hence $\zeta + \eta(y'-z) \geq -2r\chi$. Since $z+t(v+\eta(y-z)+\zeta) = z+t(v+\eta(y-y')+\zeta+\eta(y'-z)) \geq z+t(v+\eta(y-y')-2r\chi)$, one deduces that, for all $t \in (0,\varepsilon')$, $z + t(v + \eta(y - y') - 2r\chi) \in Y$. Since it is true for all $z \in B(y', \varepsilon') \cap Y$, this implies that $v + \eta(y - y') - 2r\chi \in T_Y(y')$. ∎

Let $\rho > 0$ such that $y^\gamma \in B(0,\rho)$ for all $\gamma \in \Gamma$. For all $v \in \mathcal{T}_Y^\rho(y)$, let $\eta > 0$ as given by the definition of $\mathcal{T}_Y^\rho(y)$. Let $r > 0$. From the above lemma, there exists a weak*-open neighborhood U of y such that for all $y' \in B(y,\rho) \cap U \cap Y$ $v + \eta(y - y') - 2r\chi \in T_Y(y')$. Since (y^γ) converges to y, there exists $\bar{\gamma}$ such that for all $\gamma \succeq \bar{\gamma}$, $y^\gamma \in v + \eta(y - y') - 2r\chi \in T_Y(y')$. Consequently, since $p^\gamma \in N_Y(y^\gamma) \cap S$, $p^\gamma(v + \eta(y - y') - 2r\chi) \leq 0$. This implies that $p^\gamma(v) + p^\gamma(y) - p^\gamma(y^\gamma) \leq 2r$. Taking the limit, one obtains $p(v) + p(y) - \lim p^\gamma(y^\gamma) \leq 2r$. Since this inequality is true for all $r > 0$, one has $p(v) + p(y) - \lim p^\gamma(y^\gamma) \leq 0$.

For all $\alpha > 0$, $-\alpha\chi \in T_Y^\rho(y)$, thus the above inequality implies that $p(y) \leq \lim p^\gamma(y^\gamma) + \alpha$. Since it is true for all $\alpha > 0$, one obtains the desired inequality, $p(y) \leq \lim p^\gamma(y^\gamma)$.

If $p(y) = \lim p^\gamma(y^\gamma)$, then one obtains for all $v \in T_Y^\rho(y)$, $p(v) \leq 0$. Since $\mathcal{T}_Y(y) \subset T_Y^\rho(y)$, this implies that $p \in \mathcal{N}_Y(y)$. ∎

Proof of Lemma 2.1 Note that $\partial_\infty Y = \{y \in Y \mid g(y) = 0\}$. Since g is continuously differentiable, the tangent cone at $y \in \partial_\infty Y$ is then $\{u \in L \mid \nabla g(y)(v) \leq 0\}$ (See, Clarke(1983), Corollary 2 of Theorem 2.4.7.).

To prove that $\mathcal{T}_Y(y) = T_Y(y)$, it suffices to show that for all v such that $\nabla g(y)(v) < 0$, for all $\rho > 0$, $v \in \mathcal{T}_Y^\rho(y)$. Let $\rho > 0$ and $\beta = \nabla g(\bar{y})(v)$. Let $\alpha < \frac{-\beta}{2\|v\|}$. From the continuity of ∇g, there exists a weak*-open neighborhood U of y such that for all $y' \in Y \cap U$, $\|\nabla g(y') - \nabla g(y)\| < \alpha$. There exist a weak*-open neighborhood U' of y and $\delta > 0$ such that $U' + B(0,\delta) \subset U$. Let $\eta < \frac{-\beta}{2\rho(\|\nabla g(y)\| + \alpha)}$ and $\varepsilon < \frac{\delta}{\|v\| + \eta\rho}$. Then, for all $y' \in U' \cap B(y,\rho) \cap Y$, for all $t \in (0,\varepsilon)$, one has

$$g(y' + t(v + \eta(y - y')) = g(y') + t\nabla g(y'')(v + \eta(y - y'))$$

where $y'' \in [y, y' + t(v + \eta(y - y')]$. We first remark that $\|y'' - y'\| \leq \varepsilon(\|v\| + \eta\rho) < \delta$. Consequently $y'' \in U$. Thus

$$\nabla g(y'')(v + \eta(y - y')) \leq \beta + (\nabla g(y'') - \nabla g(y))(v) + \eta \nabla g(y'')(y - y')$$

One remarks that $(\nabla g(y'') - \nabla g(y))(v) \leq \|\nabla g(y'') - \nabla g(y)\|\|v\| \leq \alpha\|v\| < \frac{-\beta}{2}$ and $\eta\nabla g(y'')(y - y') \leq \eta\|\nabla g(y'')\|\|(y - y')\| \leq \eta(\|\nabla g(y)\| + \alpha)\rho < \frac{-\beta}{2}$. Consequently, $\nabla g(y'')(v + \eta(y - y')) < 0$ and since $g(y') \leq 0$, one deduces that $g(y' + t(v + \eta(y - y')) \leq 0$ or equivalently, that $y' + t(v + \eta(y - y')) \in Y$. This implies that $v \in \mathcal{T}_Y^\rho(y)$. ∎

Proof of Lemma 4.1 Let $F \in \mathcal{F}$, $y_j \in \partial Y_j^F$ and $p \in MP_j^F(y_j)$. We first remark that $\text{int}_\infty T_{Y_j}(y_j) \cap F \subset \text{int}T_{Y_j^F}(y_j)$. Indeed, if $v_j \in \text{int}_\infty T_{Y_j}(y_j) \cap F$, from a result of Rockafellar (see Clarke(1983), Theorem 2.4.8) it is hypertangent to Y_j at y_j which means that there exists $\varepsilon > 0$ such that $y'_j + tv' \in Y_j$ for all $y'_j \in B(y_j, \varepsilon) \cap Y_j$, $v' \in B(v, \varepsilon)$ and $t \in]0, \varepsilon[$. This clearly implies that v_j is hypertangent to Y_j^F at y_j, hence it belongs to $\text{int}T_{Y_j^F}(y_j)$.

Since $p \in MP_j^F(y_j)$, one deduces that the kernel of $\langle p, .\rangle_F$, does not intersect $\text{int}T_{Y_j^F}(y_j)$. Thus, from the previous remark, it does not intersect $\text{int}_\infty T_{Y_j}(y_j)$. It now suffices to apply the standard separation theorem between $\text{int}_\infty T_{Y_j}(y_j)$ and the kernel of $\langle p, .\rangle_F$ to obtain the existence of π. ■

Proof of Lemma 4.2 We first prove the existence of $\bar{F} \in \mathcal{F}$ such that $\mathcal{E}^F$ satisfies Assumption (SA′) for each F containing $\bar{F}$. If it is not true, for all $F \in \mathcal{F}$, there exist $F' \in \mathcal{F}$ and $(p^F, (y_j^F), t^F) \in PE^{F'} \times [0, 4\bar{r} + \|\omega\|_\infty]$ such that $F \subset F'$, $\sum_{j=1}^n y_j^F + \omega + t^F\chi \geq 0$ and $\langle p^F, \sum_{j=1}^n y_j^F + \omega\rangle_F + t^F = 0$. From Lemma 4.1, for each j, there exists $\pi_j^F \in N_{Y_j}(y_j^F)$ such that $\pi_{j|F'}^F = p^F$. From Assumption (B), $((\pi_j^F), (y_j^F), t^F, (\pi_j^F(y_j^F)))$ remain in a compact set for the product of the weak-star topologies and the topology of R^{1+n}. Consequently, there exists a generalized subsequence $((\pi_j^{F(\psi)}), (y_j^{F(\psi)}), t^{F(\psi)}, (\pi_j^{F(\psi)}(y_j^{F(\psi)})))_{(\psi \in (\Psi, \geq))}$ which converges to $((\bar{\pi}_j), (\bar{y}_j), \bar{t}, (\bar{\nu}_j))$.

Since L_+ is weak star closed, $\sum_{j=1}^n \bar{y}_j + \omega + \bar{t}\chi \geq 0$. We now prove that $\bar{\pi}_1 = \bar{\pi}_j$ for $j = 2, \ldots, n$. Actually the proof is similar to the one of Claim 1 above. Let $j \in \{2, \ldots, n\}$. Let x in L. There exists a finite dimensional space $F \in \mathcal{F}$ containing x. There exists $\psi_0 \in \Psi$ such that $\psi > \psi_0$ implies $F \subset F(\psi)$. As $\pi_{j|F'(\psi)}^{F(\psi)} = \pi_{1|F'(\psi)}^{F(\psi)} = p^{F(\psi)}$, we deduce that for $\psi > \psi_0$, $\langle p^{F(\psi)}, x\rangle_{F(\psi)} = \pi_1^{F(\psi)}(x) = \pi_j^{F(\psi)}(x)$. The limit of $(\pi_1^{F(\psi)}(x))$ is $\bar{\pi}(x)$ and the limit of $(\pi_j^{F(t)}(x))$ is $\bar{\pi}_j(x)$, thus $\bar{\pi}_j(x) = \bar{\pi}_1(x)$. Since this equality holds for all $x \in L$, this leads to the result.

From Proposition 2.1(iv), one has $\bar{\pi}_1(\bar{y}_j) \leq \bar{\nu}_j$ for all j. Since $\langle p^F, \sum_{j=1}^n y_j^F + \omega\rangle_F + t^F = 0$, one deduces that $\sum_{j=1}^n \bar{\nu}_j + \bar{\pi}_1(\omega) + \bar{t} = 0$. Since $\bar{\pi}_1 \geq 0$, $0 \leq \sum_{j=1}^n \bar{\pi}_1(\bar{y}_j) + \bar{\pi}_1(\omega) + \bar{t} \leq \sum_{j=1}^n \bar{\nu}_j + \bar{\pi}_1(\omega) + \bar{t} = 0$. Hence $\bar{\pi}(y_j) = \bar{\nu}_j$ for all j. Again from Proposition 2.1(iv), $\bar{\pi}_1 \in \cap_{j=1}^n MP_j(\bar{y}_j)$. From Assumption (SA), one deduces that $\sum_{j=1}^n \bar{\pi}_1(\bar{y}_j) + \bar{\pi}_1(\omega) + \bar{t} > 0$ which leads to a contradiction.

To complete the proof of the Lemma, it suffices to prove that there exists $F \in \mathcal{F}$ such that the economy $\mathcal{E}^F$ satisfies Assumption (NS′). If it is not true, for all $F \in \mathcal{F}$, there exist $F' \in \mathcal{F}$ and $((x_i^F), (y_j^F)) \in \prod_{i=1}^m X_i^{F'} \times Y_j^{F'}$ such that $F \subset F'$, $\sum_{i=1}^m x_i^F \leq \sum_{j=1}^n y_j^F + \omega$ and for some i, it does not exist $\xi_i^F \in X_i^F$ such that $x_i^F \prec_i \xi_i^F$. From Assumption (B), $((x_i^F), (y_j^F))$ remain in a weakly compact set hence it has a generalized subsequence $((x_i^{F(\psi)}), (y_j^{F(\psi)}))_{(\psi \in (\Psi, \geq))}$ which converges to $(\bar{x}_i), (\bar{y}_j)) \in \prod_{i=1}^m X_i \times Y_j$. From the non satiation of the preferences (Assumption (C)), there exists $(\bar{\xi}_i) \in \prod_{i=1}^m X_i$, such that $\bar{x}_i \prec_i \bar{\xi}_i$ for every i. There exists a finite dimensional space $F \in \mathcal{F}$ containing every $\bar{\xi}_i$. There exists $\psi_0 \in \Psi$ such that $\psi \geq \psi_0$

implies $F \subset F(\psi)$. From Assumption (C), the sets $\bar{P}_i(\bar{\xi}_i) = \{x_i \in X_i \mid \bar{\xi}_i \preceq_i x_i\}$ are σ^∞ closed and $\bar{x}_i \notin \bar{P}_i(\bar{\xi}_i)$. Consequently, since $(x_i^{F(\psi)})$ weakly converges to $(\bar{x}_i)$, there exists $\psi_1 \geq \psi_0$ such that $\psi \geq \psi_1$ implies $x_i^{F(\psi)} \notin \bar{P}_i(\bar{\xi}_i)$ for every i. In other words, for $\psi \geq \psi_1$, $\bar{\xi}_i \in F \subset F(\psi)$ and $x_i^{F(\psi)} \prec_i \bar{\xi}_i$ for every i. This contradicts the fact that for some i it does not exist $\xi_i^F \in X_i^F$ such that $x_i^F \prec_i \xi_i^F$. This ends the proof of the lemma. ■

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