Locating local bifurcations in optimal control problems of 4-dimensional ODE systems

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25.1.2000

Abstract

This paper presents a complete characterization of the local dynamics for optimal control problems of 4-dimensional systems of ordinary differential equations, by using geometrical methods. We prove that the particular structure of the Jacobian implies that the 8th order characteristic polynomial is equivalent to a composition of two lower order polynomials, which are solvable by radicals. The classification problem for local dynamics is addressed by finding partitions, over an intermediate 4-dimensional space, which are homomorphic to the sub-spaces tangent to the complex, center and stable sub-manifolds. Then we get local necessary conditions for the existence of 1- to 4-fold, Hopf, 1- and 2-fold-Hopf and Hopf-Hopf bifurcations.

Keywords: Optimal control problems; local dynamics; fold and Hopf bifurcations **JEL Classification:** C61; C62.

Submitted to the World Congress of the Econometric Society
August 2000, Seattle

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1 Introduction

This paper presents a complete characterization of the local dynamics for optimal control problems in systems governed by four non-linear ordinary differential equations (ODE). We focus on the infinite horizon, continuous time, autonomous and discounted problem.

The local dynamics of the Hamiltonian system for scalar and planar optimal control problems is well understood. For the scalar problem, (Hartl, 1987) showed that the stable sub-manifold has maximum dimension equal to one and that its tangent stable sub-space will only display monotonous trajectories. For the planar problem, (Dockner, 1985) and (Feichtinger, Novak, and Wirl, 1994) showed that the stable sub-manifold has maximum dimension equal to two and that the tangent sub-spaces can display monotonous or oscillatory paths. Fold and Hopf bifurcations may occur (see (Feichtinger et al., 1994) and (Brito, 1997), and for all these definitions see (Guckenheimer and Holmes, 1990) or (Kuznetsov, 1995)). For the 3-dimensional problem we showed (see (Brito, 1998)) that: (i) the stable sub-manifold has maximum dimension equal to three, (ii) several combinations of oscillatory and monotonous behaviors may occur, and, (iii) the set of possible local bifurcations includes: fold, Hopf, double fold and fold-Hopf bifurcations.

In this paper we add one further dimension. Though the characteristic equation for the Jacobian of the modified Hamiltonian dynamic system is a 8th order polynomial, we prove that it is equivalent to a composition of two polynomials of 2nd and 4th orders. Therefore, the eigenvalues can be explicitly determined.

The eigenvalues map the primitive parameter space into a six-dimensional complex field. Local bifurcation analysis consists in finding partitions over the primitive parameter space which are the sub-ranges of an inverse mapping from the local center, stable and oscillatory sub-manifolds. As the decomposition of the characteristic polynomial implies the existence of an intermediate 4-dimensional real field, then the classification problem can be both simplified and solved in general, by partitioning this intermediate space. However, from the galois field theorem, as general polynomials of order larger than four are not solvable by radicals (see (Brison, 1997)[p.110] or (Hungerford, 1974)[p.308] a theorem by Abel), then we deal in this paper with the largest system for which we may determine explicitly the eigenvalues for the associated Jacobian matrix, and therefore present a complete characterization of the variational system.

The paper proceeds as follows. Section 2 states the problem and our method for solving it. Section 3 characterizes the algebraic structure of the Jacobian and determines explicitly the eigenvalues as functions of the intermediate coefficients. In Section 4, we derive a taxonomy for local dynamics by performing three overlapping partitions on the domain of the intermediate mapping. Section 5 presents the local stability theorem and locates the possible local bifurcations.

2 The problem.

We deal with the optimal control problem, defining the value functional,

$$V(x_0, \phi, \delta) := \max_u \left\{ \int_0^{+\infty} F(u, x, \phi) e^{-\delta t} dt : \dot{x} = f(u, x, \phi), x(0) = x_0 \right\},$$

where $x \in \mathbb{R}^4$ denotes the state vector, $u \in \mathbb{R}^n$ denotes the control vector and $\phi \in \Phi^* \subset \mathbb{R}^m$ denotes the vector of given primitive parameters. Let the instantaneous rate of discount, δ, and the defining mappings, $(u, x, \phi) \mapsto F(u, x, \phi) \in \mathbb{R}$ and $(u, x, \phi) \mapsto f(u, x, \phi) \in \mathbb{R}^4$, verify the following assumption:

Assumption 1. *(1): The function F and the components of f are continuous and at least one of them is $C^r, r \geq 2$ in (u, x). (2): $\delta > 0$.*

Now, let $\varphi := (\phi, \delta) \in \Phi$ where $\Phi = \Phi^* \times \mathbb{R}^{++} \subset \mathbb{R}^m \times \mathbb{R}^{++}$. From the maximum principle of Pontriyagin, there are piecewise continuous co-state variables, $p \in \mathbb{R}^4$, such that the current-value Hamiltonian, $H(u, x, p, \varphi) := F(u, x, \varphi) + \langle p, f(x, u, \varphi) \rangle$ is maximized by $\hat{u} = \text{argmax}_u H(.)$, for each $t \in [0, +\infty)$. Additionally, $\{\hat{x}(t), p(t)\}_{t=0}^{+\infty}$ solves the canonical dynamic system, $(\dot{p}, \dot{x}) = (\delta.p - \hat{H}_x(x, p, \varphi), \hat{H}_p(x, p, \varphi))$, for $x(0) = x_0$ and $\lim_{t \to +\infty} \langle p, x \rangle e^{-\delta t} = 0$, where $\hat{H}(x, p, \varphi) = H(\hat{u}, x, p, \varphi)$. Let the modified Hamiltonian dynamic system (MHDS) be written as $\dot{y} = G(y, \varphi)$, where $y := (p, x) \in \mathbb{R}^8$. Assume that it has a non-empty local attractor set, $\bar{y} := \{y : G(y, \varphi) = 0, \ \varphi = \varphi_0 \in \Phi \ \text{given}\}$, which contains all the isolated equilibrium points and/or limit-cycles.

With $G(.)$ non-linear, the Jacobian of the variational system, $\dot{y} = D_y G(\bar{y}, \varphi_0)(y - \bar{y}) + \mathcal{O}(\mid y - \bar{y} \mid)$, is

$$D_y G(\bar{y}, \varphi_0) := \begin{bmatrix} \delta.I_4 - \hat{H}_{xp} & -\hat{H}_{xx} \\ \hat{H}_{pp} & \hat{H}_{px} \end{bmatrix}.$$

The vector sub-spaces, which are tangent to the local sub-manifolds, are the generalized eigenspaces corresponding to the union of all its eigenvalues (see (Guckenheimer and Holmes, 1990), chap. 3). Near an hyperbolic equilibrium point, the structure (dimension and cyclical properties) of the local sub-manifolds is, from Grobman-Hartman's theorem, topologically equivalent to the structure of the tangent vector sub-spaces. However, if an element of the attractor set $\bar{y}$ is non-hyperbolic then we will have a local bifurcation, whose local bifurcation theorem [1] involves not only necessary conditions related to the linear part of the variational system (number of eigenvalues with zero real parts) but also to the non-linear part. Therefore, from the tangent vector space we can both locate and give a first generic classification to a central manifold.

The eigenvalues define a mapping from the primitive parameter space into the complex field, $\lambda : \Phi \subset \mathbb{R}^m \to \mathbb{C}^8$ which transform $\varphi \mapsto \lambda(\varphi)$. Let n_+, n_0 and n_- be, respectively, the number of eigenvalues with positive, zero and negative real parts and let n^r be the number of real eigenvalues and n^c be the number of pairs of complex conjugate eigenvalues. From the fundamental theorem of Algebra, $n_+ + n_0 + n_- = 8$, and $n^r + 2n^c = 8$, and, then, $n^c \leq 4$

[1]See (Kuznetsov, 1995) for a recent systematic presentation.

and, as we prove ahead, $n_- \leq 4$ and $n_0 \leq 4$. Following a geometrical approach, we will present general classification results for local dynamics, characterizing the stable, center and complex vector sub-spaces, by the following overlapping partitions over the primitive parameter space: $\mathcal{S}_i := \{\varphi \in \Phi : n_- = i\}$, $\mathcal{C}_j := \{\varphi \in \Phi : n_0 = j\}$ and $\mathcal{I}_l := \{\varphi \in \Phi : n^c = l\}$ for $i, j, l = 0, \ldots, 4$ [2].

However, as the parameter set Φ cannot be defined independently of any particular application, we will prove that there is an intermediate field $\mathcal{K} \subset \mathbb{R}^4$ which is the domain of a mapping, l, such that $\lambda(\varphi)$ is equivalent to a composition of the mappings $k : \Phi \to \mathcal{K}$, transforming $\varphi \mapsto k(\varphi)$, and $l : \mathcal{K} \to \mathbb{C}^8$, transforming $k \mapsto l(k)$. Therefore, we can derive general classification results by defining the following overlapping partitions over $\mathcal{K}$: $\mathcal{I}_i^k := \{k \in \mathcal{K} : n^c = i\}$, $\mathcal{S}_j^k := \{k \in \mathcal{K} : n_- = j\}$ and $\mathcal{C}_l^k := \{k \in \mathcal{K} : n_0 = l\}$ for $i, j, l = 0, 1, 2, 3, 4$.

3 Eigenvalues of the variational system.

The characteristic equation of the jacobian matrix $D_yG(.)$ is the fourth-order polynomial over a generic eigenvalue, λ,

$$c(\lambda) = \det(D_yG - \lambda I_8) = \sum_{j=0}^{8} (-1)^j M_{8-j} \lambda^j = 0, \qquad (1)$$

where M_j is the sum of the principal minors of order $j$and $M_0 = 1$ (see Gantmacher (1960), p. 70). Hence,M_1 is the trace and M_8 is the determinant of D_yG. From the Galois field theory we know that a general eight-order polynomial is not solvable by radicals. However, from the symmetry properties of the MHDS' Jacobian matrix, the principal minors of odd order are linear combinations of the principal minors of even order, as the next result shows.

Lemma 1. *(Principal minors) If Assumption 1 holds then the principal minors of $D_yG(.)$ verify:*

$$\begin{aligned} M_1 &= 4\delta, \\ M_3 &= -14\delta^3 + 3\delta M_2, \\ M_5 &= 28\delta^5 - 5\delta^3 M_2 + 2\delta M_4, \\ M_7 &= -17\delta^7 + 3\delta^5 M_2 - \delta^3 M_4 + \delta M_6 \end{aligned}$$

[2] A given set $\mathcal{C}_j$ belongs to the common boundary between two adjacent $\mathcal{S}_{i+j}$ and $\mathcal{S}_i$ sets. In the boundary between two adjacent $\mathcal{I}_{i+j}$ and $\mathcal{I}_i$ sets there is multiplicity of order j. Note that there are also two other overlapping partitions, among the unstable manifolds, $\mathcal{U}_i := \{\varphi \in \Phi : n_+ = i\}$, $i = 0, \ldots, 4$ and the real sub-manifolds $\mathcal{R}_i := \{\varphi \in \Phi : n^r = i\}$, $i = 0, \ldots, 4$. But these can be determined residually. That is, locally, $\varphi \in \mathcal{C}_i \cap \mathcal{S}_j \cap \mathcal{U}_l$ where $i + j + l = 6$ and $\varphi \in \mathcal{R}_i \cap \mathcal{I}_j$ where $i + 2j = 8$.

Proof. If Assumption 1 holds, then it is trivial to prove that that $\hat{H}_{xp} = (\hat{H}_{px})^T$ and $\hat{H}_{xx}$ and $\hat{H}_{pp}$ are symmetric. Then, by direct computation we prove that the former relations between the principal minors hold. □

Let $k := (k_0, k_1, k_2)$ where

$$k_0 := \left(\frac{\delta}{2}\right)^{-8} M_8 \tag{2}$$

$$k_1 := \left(\frac{\delta}{2}\right)^{-6} \left(-17\delta^6 + 3M_2\delta^4 - M_4\delta^2 + M_6\right) \tag{3}$$

$$k_2 := \left(\frac{\delta}{2}\right)^{-4} \left(17\delta^4 - 3M_2\delta^2 + M_4\right) \tag{4}$$

$$k_3 := \left(\frac{\delta}{2}\right)^{-2} \left(-6\delta^2 + M_2\right). \tag{5}$$

As $M := (M_1, \ldots, M_8)$ define a mapping $M : \Phi \to \mathbb{R}^8$ such that $\varphi \mapsto M(\varphi)$ then, as $k = k \circ M$, we have just defined another mapping $k : \Phi \to \mathcal{K} \subset \mathbb{R}^4$ which transforms $\varphi \mapsto k(\varphi)$.

Next, we prove that there is a composition of a quadratic and a quartic polynomial which is equivalent to the characteristic eigth-order polynomial.

Lemma 2. *Consider the polynomials* $\omega(\lambda) := \left(\lambda - \frac{\delta}{2}\right)^2 \left(\frac{\delta}{2}\right)^{-2}$ *and*

$$g(\omega) = \omega^4 + k_{w3}\omega^3 + k_{w2}\omega^2 + k_{w1}\omega + k_{w0}. \tag{6}$$

Then $c(\lambda)\left(\frac{\delta}{2}\right)^{-8} = g \circ \omega$ *iff*

$$k_{w0} := -k_3 + k_2 - k_1 + k_0 + 1 \tag{7}$$

$$k_{w1} := 3k_3 - 2k_2 + k_1 - 4 \tag{8}$$

$$k_{w2} := -3k_3 + k_2 + 6 \tag{9}$$

$$k_{w3} := k_3 - 4. \tag{10}$$

Proof. Let $x := (\lambda - \frac{\delta}{2})(\frac{\delta}{2})^{-1}$, and consider the polynomial, $c(x) = \sum_{j=0}^{8}(-1)^j B_j x^j$. Then $c(\lambda) = \left(\frac{\delta}{2}\right)^8 c(x)$ iff the coefficients of the monomials of the new polynomial are: $B_1 = B_3 = B_5 = B_7 = 0$, $B_0 = 1385 - 61\left(\frac{\delta}{2}\right)^{-2} M_2 + 5\left(\frac{\delta}{2}\right)^{-4} M_4 - \left(\frac{\delta}{2}\right)^{-6} M_6 + \left(\frac{\delta}{2}\right)^{-8} M_8$, $B_2 = -1708 + 75\left(\frac{\delta}{2}\right)^{-2} M_2 - 6\left(\frac{\delta}{2}\right)^{-4} M_4 + \left(\frac{\delta}{2}\right)^{-6} M_6$, $B_4 = 350 - 15\left(\frac{\delta}{2}\right)^{-2} M_2 + \left(\frac{\delta}{2}\right)^{-4} M_4$ and $B_6 = -28 + \left(\frac{\delta}{2}\right)^{-2} M_2$. If $\omega := x^2$ then $g(\omega)\left(\frac{\delta}{2}\right)^6 = c(\lambda)$ iff $k_{w0} = B_0$, $k_{w1} = B_2$, $k_{w2} = B_4$ and $k_{w3} = B_6$, where k_{w0} to k_{w2} are given by equations (7), (8), (9) and (10). □

As the two lower order polynomials are solvable by radicals, we can determine the roots of the characteristic polynomial equation, $c(\lambda) = 0$.

Theorem 1. *The eigenvalues of the Jacobian $D_y G$ are*

$$\lambda_i^s = \frac{\delta}{2}(1 - \sqrt{\omega_i}) \qquad i = 1, \ldots, 4 \tag{11}$$

$$\lambda_i^u = \frac{\delta}{2}(1 + \sqrt{\omega_i}) \qquad i = 1, \ldots, 4, \tag{12}$$

where

$$\omega_{1,2} := -\frac{k_{w3}}{4} - \frac{1}{2}[\sqrt{z_1} \pm (\sqrt{z_2} + \sqrt{z_3})] \tag{13}$$

$$\omega_{3,4} := -\frac{k_{w3}}{4} + \frac{1}{2}[\sqrt{z_1} \pm (\sqrt{z_2} - \sqrt{z_3})] \tag{14}$$

where $i = \sqrt{-1}$, $z_1 := s_1 + s_2 - \frac{l_2}{3}$ *and* $z_{2,3} = -\frac{1}{2}(s_1 + s_2) - \frac{l_2}{3} \pm \frac{\sqrt{3}}{2}(s_1 - s_2)i$ *where* $s_{1,2} := \left(-\frac{l_0}{2} + \frac{l_1}{2}\frac{l_2}{3} - \left(\frac{l_2}{3}\right)^3 \pm \sqrt{\delta_s}\right)^{\frac{1}{3}}$,
and [3]

$$l_0 := -\left[\left(\frac{k_3}{2}\right)^3 - k_2\left(\frac{k_3}{2}\right) + k_1\right]^2 \tag{16}$$

$$l_1 := 3\left(\frac{k_3}{2}\right)^4 - k_3^2 k_2 + k_1 k_3 + k_2^2 - 4k_0 \tag{17}$$

$$l_2 := -3\left(\frac{k_3}{2}\right)^2 + 2k_2, \tag{18}$$

$$\delta_s(l) := \left(\frac{l_1}{3}\right)^3 - \frac{1}{3}\left(\frac{l_1}{3}\right)^2\left(\frac{l_2}{2}\right)^2 - l_0\left(\frac{l_2}{3}\right)^3 - l_0\frac{l_1}{2}\frac{l_2}{3} + \left(\frac{l_0}{2}\right)^2. \tag{19}$$

[3]The discriminant can also be written in terms of k as

$$\begin{aligned}\delta_s(k) := {}& 4\left(\frac{k_3}{2}\right)^4 k_0^2 + 8\left(\frac{k_3}{2}\right)^3\left(\frac{k_1}{3}\right)^3 - 4\left(\frac{k_3}{2}\right)^3\frac{k_2 k_1 k_0}{3} + 4\left(\frac{k_3}{2}\right)^2\left(\frac{k_2}{3}\right)^3 k_0 - \\ & -3\left(\frac{k_3}{2}\right)^2\left(\frac{k_2}{3}\right)^2\left(\frac{k_1}{3}\right)^2 + 2\left(\frac{k_3}{2}\right)^2\left(\frac{k_1}{3}\right)^2 k_0 - 16\left(\frac{k_3}{2}\right)^2\left(\frac{k_2}{3}\right)k_0^2 - \\ & -9\left(\frac{k_3}{2}\right)k_2\left(\frac{k_1}{3}\right)^3 + 40\left(\frac{k_3}{2}\right)\left(\frac{k_2}{3}\right)^2\left(\frac{k_1}{3}\right)k_0 + \frac{32}{3}\left(\frac{k_3}{2}\right)\left(\frac{k_1}{3}\right)k_0^2 - \\ & -12\left(\frac{k_3}{2}\right)^4 k_0 + \left(\frac{k_3}{2}\right)^3 k_1^2 + \frac{32}{3}\left(\frac{k_3}{2}\right)^2 k_0^2 - 4\left(\frac{k_3}{2}\right)k_1^2 k_0 + 4\left(\frac{k_1}{2}\right)^4 - \left(\frac{4k_0}{3}\right)^3.\end{aligned} \tag{15}$$

Proof. From Lemma 3, λ solves the characteristic polynomial equation iff $\omega := \left(\lambda\left(\frac{\delta}{2}\right)^{-1} - 1\right)^2$ solves the polynomial equation $g(\omega) = 0$. Then $\lambda_i^{s,u} = \left(\frac{\delta}{2}\right)\left(1 \mp \sqrt{\omega_i}\right)$ where ω_i, $i = 1, \ldots, 4$ are the solution of $g(\omega) = 0$. From Abramowitz and Stegun (1972) the roots of the polynomial are given by equations (13) and (14). z_1, z_2 and z_3 are the roots of the polynomial equation $z^3 + l_2 z^2 + l_1 z + l_0 = 0$ with coefficients $l_0 := -\left[\left(\frac{k_{w3}}{2}\right)^3 - k_{w2}\left(\frac{k_{w3}}{2}\right) + k_{w1}\right]^2$, $l_1 := 3\left(\frac{k_{w3}}{2}\right)^4 - k_{3w}^2 k_{w2} + k_{w1}k_{w3} + k_{2w}^2 - 4k_{w0}$ and $l_2 := -3\left(\frac{k_{w3}}{2}\right)^2 + 2k_{w2}$, and the discriminant is given by equation (19). Upon substitution of the expressions for k_{w0} to k_{w3}, given in equations (7) to (10), we can see that the coefficients l_0 to l_2 are formally identical function of k_0 to k_3 (see equations (16) to (18)). □

The next result follows naturally.

Corollary 1. *(Dimensions of the sub-manifolds) If $\delta > 0$ then $n_- \leq 4$, $n_0 \leq 4$ and $n^c \leq 4$.*

From equation (**??**), the eigenvalues $\lambda := (\lambda_1^s, \ldots \lambda_4^u)$ are equivalent to the composite $\omega \circ k_w \circ k$, where $k_w := (k_{w0}, k_{w1}, k_{w2}, k_{w3})$. Let $l := \omega \circ k_w$ define a mapping $l : \mathcal{K} \to \mathbb{C}^8$ which transforms $k \mapsto l(k)$. Then $\lambda = l \circ k$. The inverse mapping $k = l^{-1}(\lambda)$ allows us to define partitions over the field $\mathcal{K}$, which are homomorphic to the vector sub-spaces tangent to the complex, center and stable sub-manifolds. In the present paper we address the location of the center manifolds as subsets of $\mathcal{K}$.

4 Types of eigenvalues

From equations (11) and (12) it can be seen that both the number of complex eigenvalues and the magnitude their real parts depend on ω. The polynomial equation $g(\omega) = 0$, may have real or complex roots.

Let

$$\nu_i := \left(\frac{\delta}{2}\right)^{-2} \lambda_i^s \lambda_i^u \tag{20}$$

As the eigenvalues involve square roots of the ω, then the following types of (pairs) of eigenvalues may occur:

- *type I* eigenvalues if ω_i is real and non-negative. Therefore, the associated eigenvalues, λ_i^{Is} and λ_i^{Iu}, are real, ν_i^I is real and $\text{sign}\{\nu_i^I\} = \text{sign}\{\lambda_i^{Is}\}$;
- *type II* eigenvalues if ω_i is real and negative. The associated pair of eigenvalues are complex with a positive real part, as $\lambda_i^{IIs}, \lambda_i^{IIu} = \frac{\delta}{2}\left(1 \mp \sqrt{\beta_i} i\right)$, where $\beta_i > 0$ and $i := \sqrt{-1}$. Then $\nu_i^{II} = 1 + \beta_i$ is real and larger than 1;

- *type III* eigenvalues if there is a pair of complex conjugate ω_i ($\overline{\omega}_i$). Therefore the associated pair of eigenvalues is complex, as $\lambda_i^{IIIs}, \lambda_i^{IIIu} = \frac{\delta}{2}\left[1 \mp \left(\sqrt{\alpha_i} + \sqrt{\beta_i} i\right)\right]$ and there is a conjugate complex pair of eigenvalues, $\overline{\lambda}_i^{IIIs}, \overline{\lambda}_i^{IIIu} = \frac{\delta}{2}\left[1 \mp \left(\sqrt{\alpha_i} - \sqrt{\beta_i} i\right)\right]$, where $\alpha_i > 0$, $\beta_i > 0$ and $i := \sqrt{-1}$. Then there is an associated complex conjugate pair ν_i^{III} and $\overline{\nu}_i^{III}$. However, their sum and product are real, as $\nu_i^{III} + \overline{\nu}_i^{III} = 2(1 - \alpha_i + \beta_i)$ and $\nu_i^{III}\overline{\nu}_i^{III} = (1-\alpha_i+\beta_i)^2 + 4\alpha_i\beta_i$. While the former sum has an ambiguous sign, the latter product is always positive.

This classification allows us to observe that a necessary condition for the existence of eigenvalues with zero real parts is that the eigenvalues should be of type I or III. In particular:

- zero eigenvalues will occur only if the pair of eigenvalues is of type I, if the associated ω_i is equal to 1. This implies that $\nu_i^I = 0$;
- a pair of complex conjugate eigenvalues with zero real parts will occur only if the pair of eigenvalues is of type III, if the associated α_i, that is the real part of the associated ω_i is equal to 1. This implies that $\nu_i^{III} + \overline{\nu}_i^{III} = 2\beta_i$ and that $\nu_i^{III}\overline{\nu}_i^{III} = \beta_i^2 + 4\beta_i$.

4.1 Analytical derivation of the $\mathcal{I}^w$-sets

Next we present necessary and sufficient conditions for the existence of zero to four pairs of complex eigenvalues.

Lemma 3. *(The $\mathcal{I}_i^w$ sets) Let $\mathcal{I}_i^w := \{k \in \mathcal{K} : n^{c_w} = i\}$ where $i = 0, \dots 2$ is the number of complex roots of $g(\omega) = 0$. Let k_{w0}, k_{w1}, k_{w2} and k_{w3} be given by equations (7) to (10). Then*

$$
\begin{aligned}
\mathcal{I}_0^w &= \{k: \ \delta_s \le 0, l_1 \ge 0, l_2 \le 0\} && (21)\\
\mathcal{I}_1^w &= \{k: \ \delta_s > 0\} \cup \{k: \ \delta_s = 0, l_1 < 0 \ \ or \ \ l_2 > 0\} && (22)\\
\mathcal{I}_2^w &= \{k: \ \delta_s < 0, l_1 < 0 \ \ or \ \ l_2 > 0\}. && (23)
\end{aligned}
$$

Proof. The roots of the polynomial equation $g(\omega) = 0$,

$$
\begin{aligned}
\omega_{1,2} &= -\frac{k_{w3}}{4} - \frac{1}{2}\left(\sqrt{z_1} \pm \left(\sqrt{z_2} + \sqrt{z_3}\right)\right) && (24)\\
\omega_{3,4} &= -\frac{k_{w3}}{4} + \frac{1}{2}\left(\sqrt{z_1} \pm \left(\sqrt{z_2} - \sqrt{z_3}\right)\right), && (25)
\end{aligned}
$$

where z_1, z_2 and z_3 are the roots of the reduced cubic polynomial equation $z^3 + l_2 z^2 + l_1 z + l_0 = 0$,

$$
\begin{aligned}
z_1 &= (s_1 + s_2) - \frac{l_2}{3} && (26)\\
z_{2,3} &= -\frac{1}{2}(s_1 + s_2) - \frac{l_2}{3} \pm \frac{\sqrt{3}}{2}(s_1 - s_2)i && (27)
\end{aligned}
$$

where $i := \sqrt{-1}$, $s_{1,2} = \sqrt[3]{r \pm \sqrt{\delta_s}}$ and $q := \frac{l_1}{3} - \left(\frac{l_2}{3}\right)^2$, $r := -\frac{l_0}{2} + \frac{l_1}{2}\frac{l_2}{3} - \left(\frac{l_2}{3}\right)^3$ and the discriminant is $\delta_s := q^3 + r^2$. According to the Fundamental theorem of Algebra the roots z_i verify:

$$\begin{aligned} z_1 + z_2 + z_3 &= -l_2 \\ z_1 z_2 + z_1 z_3 + z_2 z_3 &= l_1 \\ z_1 z_2 z_3 &= -l_0. \end{aligned}$$

First, $\delta_s < 0$ and $l_1 < 0$ or $l_2 > 0$ iff $n^{c_w} = 2$. Sufficiency: As $\delta_s < 0$ then the three z_i are real and also as $l_0 \leq 0$ then, v. g., $z_1 \geq 0$ and $\text{sign}(z_2) = \text{sign}(z_3)$. Assume that the last two are non-negative. Then $l_2 \leq 0$ and $l_1 \geq 0$ which contradicts the assumption. Therefore z_2 and z_3 are negative and their square roots are complex. As they are not complex conjugate then $\sqrt{z_2} + \sqrt{z_3}$ and $\sqrt{z_2} - \sqrt{z_3}$ are also complex and, therefore, there are two pairs of complex conjugate eigenvalues. Necessity: Assume that there are two pairs of complex conjugate eigenvalues $\lambda_{1,2} = \alpha_1 \pm \beta_1^2$ and $\lambda_{3,4} = \alpha_2 \pm \beta_2^2$. Then given equations (**??**) and (**??**), there is no complex conjugate pair of z_i and there should be at least one negative z_i. From equations (26) and (27), for having three real z then $\delta_s < 0$ and as $l_0 < 0$ there should be two negative z_i, v.g. $z_2, z_3 < 0$. As z_1 is non-negative, then $l_2 + z_2 + z_3 \leq 0$ and $l_1 + l_2(z_2 + z_3) < 0$. Then $l_1 < 0$ or $l_2 > 0$.

Second, $\delta_s > 0$ or $\delta_s = 0$ and $l_1 < 0$ or $l_2 > 0$ iff $n^c = 1$. Sufficiency: If $\delta_s > 0$ there is one real z (v.g. z_1) it is non-negative because $l_0 \leq 0$ and a pair of complex conjugate (v.g. $z_{2,3} = \alpha \pm \beta i$). Then $\sqrt{z_1}$ and $\sqrt{z_2} + \sqrt{z_3}$ is real and $\sqrt{z_2} - \sqrt{z_3}$ is complex. Therefore $n^c = 1$, that is, λ_1 and λ_2 are real and λ_3 and λ_4 are complex conjugate. If $\delta_s = 0$ then there are at least two real z (v.g. $z_2 = z_3 = z$). If $l_1 < 0$ or $l_2 > 0$ they are non-zero and negative, as we saw in the last point. However as their square roots are also equal then $\sqrt{z_2} - \sqrt{z_3} = 0$ and $\sqrt{z_2} + \sqrt{z_3}$ is complex and $n^c = 1$. Necessity: Assume that there are two real eigenvalues and a pair of complex conjugate. Then, for instance, between the two terms, $\sqrt{z_2} + \sqrt{z_3}$ and $\sqrt{z_2} - \sqrt{z_3}$, there is one real and one complex. There are only two possibilities: they are complex conjugate, $z_3 = \bar{z}_2$, or real, equal and negative, z_i, $z_2 = z_3 = z < 0$. In the first case $\delta_s < 0$ and in the second $\delta_s = 0$ and $l_2 \leq -2z$ and $l_1 + 2zl_2 < 0$, which implies that $l_1 < 0$ or $l_2 > 0$.

Third, $\delta_s \leq 0$, $l_1 \geq 0$ and $l_2 \leq 0$ iff $n^c = 0$. Sufficiency: If $\delta_s \leq 0$ then the three z_i are real. If additionally, $l_1 \geq 0$ and $l_2 \leq 0$ they are non-negative. Then, as their square roots are also non-negative, there are no complex eigenvalues, i.e., $n^c = 0$. Necessity: If the four eigenvalues are real then the three z_i are non-negative and then $\delta_s \leq 0$ and $l_1 \geq 0$ and $l_2 \leq 0$. □

4.2 The geometry of the $\mathcal{I}^w$-sets

Figure 1 and Figures 2.A to 2.N present implicit plots for $\delta_s = 0$. These figures must be read in the following way: first, a point in the space (k_0, k_1) is fixed in figure 1, second, the points labeled by a letter refer to a figure 2 labeled with the same letter. The letters in figure 1 refer to the qualitatively different cases. This representation allows for a direct comparison with analogous figures for the optimal control problems for 2-dimensional (see (Brito, 1997)) and 3-dimensional ODE systems (see (Brito, 1998)).

There are three main cases, separated by the axis for $k_0 = 0$ and by the curve $\chi := (16k_0)^3 - (2k_1)^4 = 0$, which is the locus in which $l_0 = l_1 = l_2 = 0$:

- if $k_0 \leq 0$ we will only have cases $\mathcal{I}_1^w$ and $\mathcal{I}_0^w$, that is, there will be no two pairs of complex ω. Additionally, if $k_0 = k_1 = 0$ (see figure 2.A), $\delta_s = 0$ will have two branches, one corresponding to the well known quadratic case and the k_3-axis; if $k_0 = 0$ and $k_1 \neq 0$ (see figures 2.B and 2.C) the discriminant will be asymmetric around the k_2 axis and will display two branches such that one will have a singularity giving birth to a $\mathcal{I}_0^w$-subset "inside" a $\mathcal{I}_1^w$-area; and if $k_0 < 0$ (see figures 2.D, 2.F and 2.G) then $\delta_s = 0$ will display two separate $\mathcal{I}_0^w$-branches starting at two singularity points spreading away from the k_2-axis when k_1 becomes different from zero. Again, the sign of k_1 will only control for the asymmetry of the singularity points around the k_2-axis;

- if $k_0 > 0$ and $\chi \geq 0$ then all cases, $\mathcal{I}_2^w$, $\mathcal{I}_1^w$ and $\mathcal{I}_0^w$, are possible (see figures 2.E, 2.H, 2.I, 2.J, 2.L and 2.N). Now we always have a new kind of singularity point (called double point) in which two tangents to $\delta_s = 0$ intercept, and which separates the three $\mathcal{I}^w$ subsets, and, in particular $\mathcal{I}_2^w$ from $\mathcal{I}_0^w$. Again, the sign of k_1 is only important to locate that singularity point as regards the k_2-axis (it will be located in the k_2-axis when $k_1 = 0$). In the borderline case (figures 2.J and 2.L), when $\chi = 0$ we will have a particular discontinuity point, $k_0 = \frac{1}{16}\left(\frac{k_3}{2}\right)^4$, $k_1 = \frac{1}{2}\left(\frac{k_3}{2}\right)^3$ and $k_2 = \frac{3}{2}\left(\frac{k_3}{2}\right)^2$, that will give birth to the behavior described next;

- if $k_0 > 0$ and $\chi < 0$ we may also have all cases, $\mathcal{I}_2^w$, $\mathcal{I}_1^w$ and $\mathcal{I}_0^w$ (see figures 2.K and 2.M). However, there will be a transition subset (whose location as regards the k_2-axis is again controlled by the sign of k_1) between the two former cases. This will be materialized by a closed $\mathcal{I}_0^w$ subset inside a $\mathcal{I}_2^w$ area.

4.3 The $\mathcal{I}^k$-sets

In some cases, related to the location of the local bifurcations it is important to know the number of pair of complex eigenvalues, defined over the $\mathcal{K}$-space. This is done in the next result.

Lemma 4. *Let $\mathcal{I}_i^k := \{k \in \mathcal{K} : n^c = i\}$ for $i = 0, \ldots 4$. Let k_{w0}, k_{w1}, k_{w2} and k_{w3} be given by equations (7) to (10). Then*

$$\mathcal{I}_0^k = \{k \in \mathcal{I}_0^w : k_{w0} \geq 0, k_{w1} \leq 0, k_{w2} \geq 0, k_{w3} \leq 0\} \tag{28}$$
$$\mathcal{I}_1^k := \{k \in \mathcal{I}_0^w : k_{w0} \leq 0, k_{w1} \geq 0, k_{w2} \leq 0, \ \ or \ \ k_{w2} \geq 0, k_{w3} < 0\} \tag{29}$$
$$\mathcal{I}_2^k = \{k \in \mathcal{I}_0^w : k_{w0} \geq 0, k_{w2} \leq 0 \ \ or \ \ k_{w2} \geq 0 \quad and \ \ k_{w1} > 0, k_{w3} \leq 0 \ \ or$$

$$\begin{aligned}
& \quad or\ k_{w1}\leq 0, k_{w3}>0\}\cup\{k\in\mathcal{I}^w_1: k_{w0}\geq 0, k_{w1}\leq 0, k_{w2}>0, k_{w3}\leq 0\} && (30)\\
\mathcal{I}^k_3 &= \{k\in\mathcal{I}^w_0: k_{w0}\leq 0, k_{w1}<0, k_{w2}<0,\ \ or\ k_{w2}>0, k_{w3}>0\}\cup\\
& \quad \cup\{k\in\mathcal{I}^w_1: k_{w0}<0\ \ or\ k_{w0}=0, k_{w1}>0\} && (31)\\
\mathcal{I}^k_4 &= \{k\in\mathcal{I}^w_0: k_{w0}>0, k_{w1}>0, k_{w2}>0, k_{w3}>0\}\cup\\
& \quad \cup\{k\in\mathcal{I}^w_1: k_{w0}>0, k_{w1}>0\ \ or\ k_{w1}\leq 0, k_{w2}<0\ \ or;\ k_{w2}>0, k_{w3}<0\}\cup\\
& \quad \cup\{k\in\mathcal{I}^w_2\}. && (32)
\end{aligned}$$

Proof. As we have seen, the number of pairs of complex eigenvalues depends upon the type of eigenvalues. These depend, first, on the types of ω_i, which are solutions of a quartic polynomial equation (6), which determine the sets $\mathcal{I}^w_0$ to $\mathcal{I}^w_2$ in equations (21) to (23). A necessary and sufficient condition for the existence of $j=0,...,2$ pairs of complex conjugate ω is that $k\in\mathcal{I}^w_j$. Then $k\in\mathcal{I}^w_2$ is a sufficient condition for the existence of four pairs of complex eigenvalues, ie for $n^c=4$. If $k\notin\mathcal{I}^w_2$ complex eigenvalues are related to type two eigenvalues which are associated to real but negative ω_i. Then we have to further partition the subsets $\mathcal{I}^w_0$ and $\mathcal{I}^w_1$ according to the number of real and negative ω. We will use the fact that

$$\sum_{i=1}^{4}\omega_i=-k_{w3}\quad \sum_{j>i=1}^{4}\omega_i\omega_j=k_{w2}\quad \sum_{l>j>i=1}^{4}\omega_i\omega_j\omega_l=-k_{w1}\quad \prod_{i=1}^{4}=k_{w0}. \qquad (33)$$

Let $k\in\mathcal{I}^w_1$, and assume, with no loss of generality that ω_1 and ω_2 are real and that $\omega_{3,4}=\alpha\pm\beta i$. Then n^c can be equal to 2, 3 or 4 if there are zero, one or two negative real ω. The system (33)becomes

$$\omega_1+\omega_2+2\alpha=-k_{3w} \qquad (34)$$
$$\omega_1\omega_1+2\alpha(\omega_1+\omega_2)+\alpha^2+\beta^2=k_{w2} \qquad (35)$$
$$2\alpha\omega_1\omega_2+(\omega_1+\omega_2)(\alpha^2+\beta^2)=-k_{w1} \qquad (36)$$
$$\omega_1\omega_2(\alpha^2+\beta^2)=k_{w0}. \qquad (37)$$

Note that the term $\alpha^2+\beta^2$ is always positive, even if $\alpha=0$. First, $k_{w0}<0$ or $k_{w0}=0$ and $k_{w1}>0$ iff $n^c=3$, i.e. if there is one negative ω, say $\omega_2<0$. As sign$(k_{w0})=$ sign$(\omega_1\omega_2)$ then $k_{w0}<0$ iff $\omega_1>0$ and $\omega_2<0$. And $\omega_1=0$ and $\omega_2<0$ iff $k_{w0}=0$ and $k_{w1}>0$. Second, $k_{w0}\geq 0$ is a necessary condition for both $n^c=2$ or $n^c=4$. In the first case, ω_1 and ω_2 are non-negative and in the second case they are negative. Basically, we have to determine the boundary between the two subsets. Obviously the point $k_{w0}=k_{w1}=0$ belongs to this boundary (and it implies $n^c=2$). And locally if k_{w0} is slightly positive and if $k_{w1}>0$ then $n^c=4$ and if $k_{w1}<0$ then $n^c=2$. Given the structure of the non-linear system (34) to (37), before determining the signs of ω_1 and ω_2 we have to know the sign of α. The locus $\alpha=0$ is equivalent to $\left(\frac{k_{w1}}{k_{w3}}\right)_{1,2}=\frac{k_{w2}}{2}\pm\sqrt{\left(\frac{k_{w2}}{2}\right)^2-k_{w0}}$ for $\left(\frac{k_{w2}}{2}\right)^2\geq k_{w0}\geq 0$, $k_{w2}>0$ and sign$(k_{w1})=$ sign(k_{w3}). Several things can be said about the $\alpha=0$-locus: (i) it belongs to subsets of $\mathcal{K}$ sets in which $k_{w1} > 0$, $k_{w2} > 0$ and $k_{w3} > 0$ or $k_{w1} > 0$, $k_{w2} > 0$ and $k_{w3} > 0$; (ii) for every value of k_{w2} and k_{w0}, verifying the former restrictions, it defines one subset, $\left(\left(\frac{k_{w1}}{k_{w3}}\right)_1, \left(\frac{k_{w1}}{k_{w3}}\right)_2\right)$, in the interior of the two former sets, (iii) the two limiting locus converge to a common point in the $k_{w0} = 0$-locus in which $k_{w2} > 0$. Then the sign of α inside and outside those two regions are symmetric. To determine the signs of α in the interior of those subsets note that we have the following sufficient conditions: if $k_{w0} > 0$, $k_{w1} > 0$, $k_{w2} > 0$ and $k_{w3} > 0$ then $n^c = 4$ and $\alpha < 0$ and if $k_{w0} > 0$, $k_{w1} < 0$, $k_{w2} > 0$ and $k_{w3} < 0$ then $n^c = 2$ and $\alpha > 0$. However as while the first condition define a closed (or empty) subset in the space $\mathcal{K}$, the second condition defines an open region. Then α will be negative if $k \in \left(\left(\frac{k_{w1}}{k_{w3}}\right)_1, \left(\frac{k_{w1}}{k_{w3}}\right)_2\right)$ and positive in the complement. Then, obviously, $k_{w0} \geq 0$, $k_{w1} \leq 0$, $k_{w2} > 0$ and $k_{w3} \leq 0$ iff $n^c = 2$. And k belongs to the complement of the set in which $k_{w0} > 0$ iff $n^c = 4$.

Now, let $k \in \mathcal{I}_0^w$ and the four ω_i are real and the number of them which are negative is equal to n^c. Let $k_{w0} \geq 0$. Then n^c is an even number equal to 0, 2, or 4. The following necessary conditions hold: if $n^c = 0$ then $k_{w0} \geq 0$, $k_{w1} \leq 0$, $k_{w2} \geq 0$ and $k_{w3} \leq 0$ and if $n^c = 4$ then $k_{w0} > 0$, $k_{w1} > 0$, $k_{w2} > 0$ and $k_{w3} > 0$. We will prove that they are indeed necessary and sufficient conditions by proving that they exclude the $n^c = 2$ case. With no loss of generality assume that ω_1 and ω_2 are negative and that ω_3 and ω_4 are non-negative. Will this case hold if $k_{w1} \leq 0$, $k_{w2} \geq 0$ and $k_{w3} \leq 0$? First, $k_{w3} \leq 0$ iff $\mid \omega_1+\omega_2 \mid \leq \mid \omega_3+\omega_4 \mid$. Second, the third equation in (33) will only be true when $k_{w1} \leq 0$ iff $\omega_3\omega_4 \geq \omega_1\omega_2$, which is consistent with the former inequality. At last taking into consideration the magnitudes of ω in the second equation of(33) we get and interval for k_{w2}, $\omega_1^2+\omega_2^2 \leq -k_{w2} \leq \omega_3^2+\omega_4^2$ which contradicts the assumption that k_{w2} is non-negative. Now, will it hold if $k_{w1} > 0$, $k_{w2} > 0$ and $k_{w3} > 0$? First, $k_{w3} > 0$ iff $\mid \omega_1 + \omega_2 \mid > \mid \omega_3 + \omega_4 \mid$. Then $\omega_1\omega_2 \geq \omega_3\omega_4$. Second, those relations imply that $\omega_1^2 + \omega_2^2 \geq -k_{w2} \geq \omega_3^2 + \omega_4^2$ which again contradicts the assumption that k_{w2} is positive. Summing up: the former conditions for $n^c = 4$ and $n^c = 0$ are indeed necessary and sufficient and $n^c = 2$ is defined in their $\mathcal{K}$-complement in which $k_{w0} \geq 0$.

At last, let $k_{w0} \leq 0$. Then n^c is an odd number equal to 1 or 3. With no loss of generality assume that $\omega_1 \geq 0$, $\omega_2 < 0$ and that $\text{sign}(\omega_3) = \text{sign}(\omega_4)$. If it is negative then $n^c = 3$ and if it is non-negative then $n^c = 1$. There is not a natural starting point for looking at the system (33). However, solving the first two equations of (33) for $\omega_1 + \omega_2$ and $\omega_1\omega_2$ and substituting the values into the third and fourth equations of (33), we get $(\omega_3+\omega_4)\left(\frac{k_{w0}-\omega_3^2\omega_4^2}{\omega_3\omega_4}\right) = -k_{w1}+k_{w3}(\omega_3+\omega_4)$ and $k_{w2}+k_{w3}(\omega_3+\omega_4)+(\omega_3^2+\omega_3\omega_4+\omega_4^2) \leq 0$. As $k_{w0} \leq 0$ and $\text{sign}(\omega_3) = \text{sign}(\omega_4)$ then those equations will only hold if

$$\text{sign}(\omega_3 + \omega_4) = -\text{sign}(-k_{w1} + \omega_3\omega_4 k_{w3}) \tag{38}$$

$$k_{w2} + k_{w3}(\omega_3 + \omega_4) \leq 0. \tag{39}$$

Next we prove that $n^c = 1$ iff $k_{w1} \geq 0$ and $k_{w2} \leq 0$ or $k_{w2} \geq 0$ and $k_{w3} \leq 0$. Let $k_{w2} \geq 0$ and $n^c = 1$. Then conditions (38) and (39) will only be met if $k_{w3} \leq 0$ irrespective of the sign of $k_{w1} \leq 0$. If $k_{w2} \geq 0$ and $n^c = 1$ then condition (38) can hold for any sign of k_{w3}. Additionally, condition (39) will only be met if $k_{w1} \geq 0$ [4] The case $n^c = 3$ holds in the complement. □

5 The location and characterization of local bifurcations

After characterizing the subsets of the space $\mathcal{K}$ according to the numeric features of the eigenvalues we can locate the two main sources of bifurcations, fold and Hopf bifurcations. The first are located by zero real eigenvalues and the latter to complex eigenvalues with zero real parts. In a system of this dimension we may also have higher-dimensional fold and Hopf bifurcations and combinations between them, as well.

5.1 Analytical derivation

The former results on the classification of the complex sub-manifolds allows us to observe that a necessary condition for the existence of eigenvalues with zero real parts is that the eigenvalues should be of type I or III. In particular:

- zero eigenvalues will occur only if the pair of eigenvalues is of type I, if the associated ω_i is equal to 1. This implies that $\nu_i^I = 0$;
- a pair of complex conjugate eigenvalues with zero real parts will occur only if the pair of eigenvalues is of type III, if the associated α_i, that is the real part of the associated ω_i is equal to 1. This implies that $\nu_i^{III} + \overline{\nu}_i^{III} = 2\beta_i$ and that $\nu_i^{III}\overline{\nu}_i^{III} = \beta_i^2 + 4\beta_i$.

Now, consider all the roots of the polynomial $g(\omega) = 0$. As, if $k \in \mathcal{I}_2^w$ there are 2 pairs of complex conjugate ω_i, if $k \in \mathcal{I}_1^w$ there is one pair of complex conjugate and two real ω_i and if $k \in \mathcal{I}_0^w$ there are four real ω_i, then:

- $k \in \mathcal{I}_2^w$ is a necessary condition for the existence of Hopf or Hopf-Hopf bifurcations;
- $k \in \mathcal{I}_1^w$ is a necessary condition for the existence of 1- or 2-fold, Hopf or 1- or 2-fold-Hopf bifurcations;
- $k \in \mathcal{I}_0^w$ is a necessary condition for the existence of 1- to 4-fold bifurcations.

In this section we will characterize the following subsets

$$\mathcal{C}_i^k := \{k \in \mathcal{K} : n_0 = i\} \qquad i = 0, \ldots, 4. \tag{40}$$

In all the next proves we will use the following result

[4]Note that if $k_{w0} = k_{w1} = k_{w2} = 0$ then $n^c = 1$ iff $k_{w3} > 0$. The case $k_{w0} = k_{w1} = k_{w2} = k_{w3}0$ belongs to the set $\mathcal{I}_0^k$.

$$
\begin{aligned}
k_3 &= \sum_{i=1}^{4} \nu_i && (41)\\
k_2 &= \sum_{j>i=1}^{4} \nu_i \nu_j && (42)\\
k_1 &= \sum_{k>j>i=1}^{4} \nu_i \nu_j \nu_k && (43)\\
k_0 &= \prod_{i=1}^{4} \nu_i. && (44)
\end{aligned}
$$

Lemma 5. *A necessary condition for the existence of a Hopf-Hopf bifurcation is that* $k \in \mathcal{C}^k_{h4}$, *where*

$$
\mathcal{C}^k_{h4} = \{k \in \mathcal{I}^w_2 : h_4(k) = \mathbf{0}\} \tag{45}
$$

and

$$
\begin{aligned}
h_4(k) = \Bigg\{ & k_1 - \left[k_2 - \left(\frac{k_3}{2}\right)^2 - 2k_3\right]\left(8 + \frac{k_3}{2}\right),\\
& k_0 - \frac{1}{4}\left[k_2 - \left(\frac{k_3}{2}\right)^2 - 2k_3\right]\left[k_2 - \left(\frac{k_3}{2}\right)^2 + 2k_3 + 32\right] :\\
& \left(\frac{k_3}{2}\right)^2 + 2k_3 < k_2 \leq \frac{3}{2}\left(\frac{k_3}{2}\right)^2 + 2k_3,\ k_3 > 0,\ k_2 > 0,\ k_1 > 0,\ k_0 > 0 \Bigg\}
\end{aligned} \tag{46}
$$

Proof. The linear part of the Hopf-Hopf bifurcation (see (Kuznetsov, 1995)) has two pairs of complex eigenvalues with zero real part. Therefore we should have four pairs of eigenvalues of type III,associated with two pairs of complex ω with a real part equal to 1. Therefore $k \in \mathcal{I}^w_2$, and without loss of generality we may write system (41) to (44) as

$$
\begin{aligned}
k_3 &= 2(1-\alpha_1+\beta_1+1-\alpha_2+\beta_2) && (47)\\
k_2 &= (1-\alpha_1+\beta_1)^2 + 4\alpha_1\beta_1 + 4(1-\alpha_1+\beta_1)(1-\alpha_2+\beta_2) + (1-\alpha_2+\beta_2)^2 + 4\alpha_2\beta_2 && (48)\\
k_1 &= 2(1-\alpha_2+\beta_2)[(1-\alpha_1+\beta_1)^2 + 4\alpha_1\beta_1] + 2(1-\alpha_1+\beta_1)[(1-\alpha_2+\beta_2)^2 + 4\alpha_2\beta_2] && (49)\\
k_0 &= [(1-\alpha_1+\beta_1)^2 + 4\alpha_1\beta_1][(1-\alpha_2+\beta_2)^2 + 4\alpha_2\beta_2], && (50)
\end{aligned}
$$

where β_1 and β_2 should be real and positive. Four the eistence of two pairs of eigenvalues with zero real part then $\alpha_1 = \alpha_2 = 1$ and the system (47) to (50) becomes

$$
k_3 = 2(\beta_1 + \beta_2) \tag{51}
$$

$$
\begin{aligned}
k_2 &= \beta_1^2 + 4\beta_1 + 4\beta_1\beta_2 + \beta_2^2 + 4\beta_2 && (52)\\
k_1 &= 2\beta_2(\beta_1^2 + 4\beta_1) + 2\beta_1(\beta_2^2 + 4\beta_2) && (53)\\
k_0 &= (\beta_1^2 + 4\beta_1)(\beta_2^2 + 4\beta_2), && (54)
\end{aligned}
$$

where β_1 and β_2 should be real and positive. Then $k >> 0$. Solving equations (51) and (52) we get $\beta_1 = \frac{k_3}{2} - \beta_2$ and $\beta_2^2 - \frac{k_3}{2}\beta_2 + \frac{1}{2}\left(k_2 - \left(\frac{k_3}{2}\right)^2 - 2k_3\right) = 0$. The conditions for the existence of positive and real β_i are that: $k_3 > 0$, $k_2 - \left(\frac{k_3}{2}\right)^2 - 2k_3 > 0$ and that $\frac{3}{2}\left(\frac{k_3}{2}\right)^2 + 2k_3 - k_2 \geq 0$. These conditions define restrictions on the values of k_2 and k_3. As equations (53) and (54) should also hold, if we substitute the values for β_1 and β_2 then we find the two equations which define $h_4 = 0$. □

The set $h_4 = 0$ defines a 2-dimensional manifold over the 4-dimensional space $\mathcal{K}$. Therefore, we may fix two co-ordinates and get particular values for the other two.

Lemma 6. *A necessary condition for the existence of a Hopf bifurcation is that $k \in \mathcal{C}_{h2}^k$, where*

$$
\mathcal{C}_{h2}^k = \{k\mathcal{I}_2^w : h_2(k) = 0\} \qquad (55)
$$

and

$$
\begin{aligned}
h_2(k) = \Bigg\{ & a_{10}a_{21}^2 - a_{11}a_{21}a_{20} + a_{12}a_{20}^2 : \ \frac{a_{20}}{a_{21}} < 0,\ 1 - \frac{a_{20}}{a_{21}} - \frac{k_3}{2} < 0,\ k_0 > 0 \\
& 0 < -\frac{a_{20}}{a_{21}} - \frac{k_2}{4} - \frac{1}{2}\left(\frac{a_{20}}{a_{21}}\right)^2 + \left(\frac{k_3}{2}\right)^2 - \frac{a_{20}k_3}{4a_{21}} \leq \frac{1}{4}\left(1 - \frac{a_{20}}{a_{21}} - \frac{k_3}{2}\right)^2 \\
& \frac{1}{2}\left(1 - \frac{a_{20}}{a_{21}} - \frac{k_3}{2}\right) \mp \left[1 - k_3 + 2\frac{a_{20}}{a_{21}}(1 + k_3) + 3\left(\frac{a_{20}}{a_{21}}\right)^2 + k_2\right]^{\frac{1}{2}} \neq 1 \Bigg\} \qquad (56)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{10} &= -4k_0\left(8(32 + k_2 + 22k_3) - (16 + 3k_3)(80 + k_3)\right) && (57)\\
a_{11} &= (3k_1 + 16k_2)\left(8(32 + k_2 + 22k_3) - (16 + 3k_3)(80 + k_3)\right) - \\
&- (80 + k_3)\left(16k_0 - k_1(80 + k_3)\right) && (58)\\
a_{12} &= -2(32 + k_2 + 22k_3)\left(8(32 + k_2 + 22k_3) - (16 + 3k_3)(80 + k_3)\right) \\
&- (80 + k_3)\left(2(80 + k_3)(k_2 + 2k_3) - 4(3k_1 + 16k_2)\right) && (59)\\
a_{20} &= a_{12}\left(2(80 + k_3)(k_2 + 2k_3) - 4(3k_1 + 16k_2)\right) - \\
&- a_{11}\left(8(32 + k_2 + 22k_3) - (16 + 3k_3)(80 + k_3)\right) && (60)\\
a_{21} &= a_{12}\left(16k_0 - k_1(80 + k_3)\right) - a_{10}\left(8(32 + k_2 + 22k_3) - (16 + 3k_3)(80 + k_3)\right) && (61)
\end{aligned}
$$

Proof. Again, we take the system (47) to (50) and set, without loss of generality $\alpha_1 = 1$ and $\alpha_2 \neq 1$ and $\beta_1 > 0$ and $\beta_2 > 0$. Then, we have three unknowns and four equations. From equation (47) we get $\alpha_2 = 1 + \beta_1 + \beta_2 - \frac{k_3}{2}$, from equation (48), upon substitution of α_2, we get $16\beta_2^2 + (16 - 8k_3 + 16\beta_1)\beta_2 + 16\beta_1 - 4k_2 - 8\beta_1^2 + k_3^2 + 4\beta_1 k_3 = 0$ and, again upon substitution, from equations (49) and (50), we get

$$3\beta_1^4 + (8 - 2k_3)\beta_1^3 + (k_2 - 8(2 + k_3))\beta_1^2 + 4k_2\beta_1 - k_0 = 0 \quad (62)$$

$$4\beta_1^3 - (3k_3 + 16)\beta_1^2 + 2(k_2 + 2k_3)\beta_1 - k_1 = 0. \quad (63)$$

From this system we get both a solution for $\beta_1 = -\frac{a_{20}}{a_{21}}$ and a restriction over k, which is the 3-dimensional manifold which defines $h_2(k)$. The restrictions on the values α_2, β_1 and β_2 determine the restrictions on the space in which that manifold is defined, as in equation (82). □

The set $h_2(k) = 0$ defines a 3-dimensional manifold over $\mathcal{K}$.

Now we will present the bifurcations associated with $\mathcal{I}_1^w$.

Lemma 7. *A necessary condition for the existence of a 2-fold-Hopf bifurcation is that* $k \in \mathcal{C}_{h2f2}^k$, *where*

$$\mathcal{C}_{h2f2}^k = \{k \in \mathcal{I}_1^w : h_2 f_2(k) = 0\} \quad (64)$$

and

$$h_2 f_2(k) = \left\{ k_2 - \left(\frac{k_3}{2}\right)^2 - 2k_3 : \; k_3 > 0, \; k_2 > 0, \; k_1 = k_0 = 0 \right\} \quad (65)$$

Proof. Now, as the bifurcations will be related to zero real parts of eigenvalues of types I and III, system (41) to (44) will be in general

$$k_3 = \nu_1^I + \nu_2^I + 2(1 - \alpha + \beta) \quad (66)$$

$$k_2 = \nu_1^I \nu_2^I + 2(\nu_1^I + \nu_2^I)(1 - \alpha + \beta) + (1 - \alpha + \beta)^2 + 4\alpha\beta \quad (67)$$

$$k_1 = 2\nu_1^I \nu_2^I (1 - \alpha + \beta) + (\nu_1^I + \nu_2^I)[(1 - \alpha + \beta)^2 + 4\alpha\beta] \quad (68)$$

$$k_0 = \nu_1^I \nu_2^I [(1 - \alpha + \beta)^2 + 4\alpha\beta]. \quad (69)$$

As, in this case we should have $\nu_1^I = \nu_2^I = 0$ and $\alpha = 1$, then the system simplifies to $k_0 = k_1 = 0$, $\beta^2 + 4\beta = k_2$ and $k_3 = \frac{\beta}{2}$ for $\beta > 0$. Then the manifold and the restrictions on the space of k in which it is defined, represented by set $h_2 f_2$, are easily derived. □

Now, set $h_2 f_2 = 0$ defines a 1-dimensional manifold in the space (k_2, k_3) and the corresponds to a point in the space $(k_1, k_0) = (0, 0)$.

The next result locates the fold-Hopf bifurcation.

Lemma 8. *A necessary condition for the existence of a fold-Hopf bifurcation is that* $k \in \mathcal{C}^k_{f1h2}$*, where*

$$\begin{aligned}\mathcal{C}^k_{h2f1} \;=\; & \Big\{ k \in \mathcal{I}^w_1 : h_2 f_1(k) = 0, -\frac{k_3}{2} \neq \frac{a_{20}}{a_{21}} < 0, k_1 \neq 0 \textit{ if } k \in \mathcal{I}^k_2 \\ & \textit{or } \frac{1-k_3}{2} < \frac{a_{20}}{a_{21}} < 0, k_1 > 0, k_2 > 0, k_3 > 1 \textit{ if } k \in \mathcal{I}^k_3 \Big\} \end{aligned} \tag{70}$$

and

$$h_2 f_1(k) = \left\{ a_{10} a_{21}^2 - a_{11} a_{21} a_{20} + a_{12} a_{20}^2 : \; k_0 = 0 \right\} \tag{71}$$

where

$$\begin{aligned} a_{10} &= 3k_1 && (72)\\ a_{11} &= -(2k_2 + 12k_3) && (73)\\ a_{12} &= 32 + k_3 && (74)\\ a_{20} &= 9k_1 - k_2(32 + k_3) && (75)\\ a_{21} &= 2(2 + k_3)(32 + k_3) - 3(2k_2 + 13k_3). && (76) \end{aligned}$$

Proof. For the existence of one zero and one pair of complex eigenvalues with zero real parts we sould have one type I and a pair of type III eigenvalues. The other eigenvalue may be of type I or II. In the first case $k \in \mathcal{I}^k_2$ and the system (47) to (50) becomes

$$\begin{aligned} \nu^I &= k_3 - 2\beta && (77)\\ -3\beta^2 + 2(k_3 + 2)\beta - k_2 &= 0 && (78)\\ -2\beta^3 + (k_3 - 8)\beta^2 + 4k_3\beta - k_1 &= 0 && (79)\\ k_0 &= 0 && (80) \end{aligned}$$

with the restrictions $\nu \neq 0$ and $\beta > 0$. Then we get the 2-dimensional manifold $h_2 f_1$, a solution for β and the restrictions which define the set $\mathcal{C}^k_4$: However, in the second case $k \in \mathcal{I}^k_3$ and the system and letting $\nu^{II} = 1 + \gamma$ we get $\gamma = k_3 - 1 - 2\beta > 0$, equations (78) and (79) and $k_0 = 0$, with the restrictions $\gamma > 0$ and $\beta > 0$. □

We may also have Hopf bifurcations when $k \in \mathcal{I}^w_1$.

Lemma 9. *A necessary condition for the existence of a Hopf bifurcation is that* $k \in \mathcal{C}^k_{h2}$*, where*

$$\begin{aligned} \mathcal{C}^k_{h2} \;=\; & \{ k \in \mathcal{I}^w_1 : \; h_2(k) = 0 \textit{ and} \\ & \frac{k_3}{2} + \frac{a_{20}}{a_{21}} \pm \left[\left(\frac{k_3}{2} + \frac{a_{20}}{a_{21}} \right)^2 - k_2 + 3\left(\frac{a_{20}}{a_{21}}\right)^2 - 2\frac{a_{20}}{a_{21}}(2 + k_3) \right]^{\frac{1}{2}} \neq 0 \textit{ if } k \in \mathcal{I}^k_2 \textit{ or} \\ & 1 + \frac{k_3}{2} + \frac{a_{20}}{a_{21}} \pm \left[\left(1 + \frac{k_3}{2} + \frac{a_{20}}{a_{21}} \right)^2 - k_2 + 3\left(\frac{a_{20}}{a_{21}}\right)^2 - 2\frac{a_{20}}{a_{21}}(1 + k_3) + k_3 - 1 \right]^{\frac{1}{2}} > 0 \\ & \textit{if } k \in \mathcal{I}^k_3 \cup \mathcal{I}^k_4 \Big\} \end{aligned} \tag{81}$$

and

$$h_2(k) = \{a_{10}a_{21}^2 - a_{11}a_{21}a_{20} + a_{12}a_{20}^2 : \; k_0 \neq 0, k_1 \neq 0\} \tag{82}$$

where a_{10} to a_{21} are formally identical to equations (57) to (61).

Proof. The demonstration is done by using system (47) to (50) and by stating $\beta > 0$ always and ν_1^I and ν_2^I different from zero when there are no type II eigenvalues, $\nu^I \neq 0$ and $\nu^{II} = 1 + \gamma > 1$ when there is a pair of type I and a pair of type II eigenvalues and $\nu_1^{II} = 1 + \gamma_1 > 1$ and $\nu_2^{II} = 1 + \gamma_2 > 1$ when there are two pairs of type II eigenvalues. We also get the 3-dimensional manifold h_2. □

Lemma 10. *Necessary conditions for the existence of a 1 to 4 fold bifurcations is that $k \in \mathcal{C}_{f_i}^k$ for $i = 1, \ldots, 4$, where*

$$\mathcal{C}_{f1}^k = \{k \in \mathcal{I}_1^w \cup \mathcal{I}_0^w : k_0 = 0, k_1 \neq 0, k_2 \neq 0, k_3 \neq 0\} \tag{83}$$

$$\mathcal{C}_{f2}^k = \{k \in \mathcal{I}_1^w \cup \mathcal{I}_0^w : k_0 = k_1 = 0, k_2 \neq 0, k_3 \neq 0\} \tag{84}$$

$$\mathcal{C}_{f3}^k = \{k \in \mathcal{I}_1^w \cup \mathcal{I}_0^w : k_0 = k_1 = k_2 = 0, k_3 \neq 0\} \tag{85}$$

$$\mathcal{C}_{f4}^k = \{k \in \mathcal{I}_1^w \cup \mathcal{I}_0^w : k_0 = k_1 = k_2 = k_3 = 0\}. \tag{86}$$

Proof. Obvious. □

The main results of the paper are gathered in the next result on the characterization of the center sub-manifolds.

Theorem 2. *Let $\mathcal{C}_i^k := \{k \in \mathcal{K} : n_0 = i\}$ for $i = 0, \ldots, 4$ be a central manifold with dimension equal to n_0. Then*

$$\mathcal{C}_1^k = \mathcal{C}_{f1}^k \tag{87}$$

$$\mathcal{C}_2^k = \mathcal{C}_{f2}^k \cup \mathcal{C}_{h2}^k \tag{88}$$

$$\mathcal{C}_3^k = \mathcal{C}_{f3}^k \cup \mathcal{C}_{h2f1}^k \tag{89}$$

$$\mathcal{C}_4^k = \mathcal{C}_{f4}^k \cup \mathcal{C}_{h2f2}^k \cup \mathcal{C}_{h4}^k. \tag{90}$$

5.2 The geometry of the $\mathcal{C}^k$-sets

Again, we fix several representative values for k_0 and k_1 in figure 1. The letter which labels each point refers to the related figure 2.

Let us start with the fold bifurcations. These are the simplest, and are related to the number of consecutive k_i, starting with $i = 0$, which are equal to zero. Therefore: figure 2.A, which is associated with $k_0 = k_1 = 0$ contains the locations from 2-fold (in all points except the k_3-axis and the curve $h_2 f_2 = 0$) 3-fold (in the k_3-axis with the exception of the origin) and 4-fold (in the origin) bifurcations and figures 2.B and 2.C, and in general all the analogous figures related to the k_1-axis in figure 1, contain the locations of 1-fold bifurcations (in all points with the exception of the $h_2 f_1 = 0$-locus. Geometrically, the 4-fold corresponds to a point in the k-space, the 3-fold corresponds to a line, the 2-fold to a surface and the 1-fold to a cube.

The Hopf bifurcation theorem has as a necessary condition that there should be one (and only one) pair of complex eigenvalues with zero real parts and no zero eigenvalues. Geometrically, we may have Hopf bifurcations for any value of k_1 and for any non-zero k_0, except for the points corresponding to the Hopf-Hopf bifurcation that we will discuss later. Also, as $h_2(k) = 0$ defines a 2-dimensional manifold in the 4-dimensional $\mathcal{K}$ space, then it is represented geometrically by a folded plane. As the values for k_0 and k_1 represent particular "slices" in the four-dimensional parameter space then the projections in the (k_2, k_3)-space will be given by particular curves. Therefore we have a projection of the $h_2 = 0$-locus in all figures 2 with the exception of those related to $k_0 = 0$-values, figures 2.D to 2.N. When $k_0 < 0$ (see figures 2.D, 2.F and 2.G) it seems that the locus has only one branch which cuts vertically the k_3-axis in a way which is dependent upon the sign of k_1. When $k_0 > 0$ (see figures 2.E and 2.H to 2.N) the locus $h_2 = 0$ is multivalued with two branches, and these branches are almost everywhere located in the $k_2 > 0$-orthants. If k_1 changes sign, they tend to change their position as regards the upper part of the k_2-axis.

When these two branches intercept, and if that interception is located in the $\mathcal{I}_2^w$-space we would have a Hopf - Hopf bifurcation (see figure 2.N). We saw that the set $\langle_4 = 0$ defines a 2-dimensional manifold over the 4-dimensional space $\mathcal{K}$. From the presentation in Lemma (82) we may see that: first, the Hopf-Hopf bifurcation will be represented by a point in the space (k_0, k_1) and a point in the space (k_2, k_3); second, in the space (k_2, k_3) it will be located between lines $k_2 =\leq \frac{3}{2}\left(\frac{k_3}{2}\right)^2 + 2k_3$ and $k_2 = \left(\frac{k_3}{2}\right)^2 + 2k_3$; third, as it is a 2-dimensional manifold, the restrictions which were expressed over space (k_2, k_3), may be mapped into space (k_0, k_1). As our representation presupposes taking (k_0, k_1) parametrically, then $h_4 = 0$ is represented as a 0-dimensional manifold, that is, a point. After some tedious algebra we found that, for $k_0 < 192$, the following two conditions should be met $k_0 - 2k_1 < 0$ and $16k_1 - (k_3^*)^2(k_3^* + 16 \leq 0$ where $k_3^* = \frac{8(2k_0 - 3k_1) + k_1\sqrt{64 + k_0 - 2k_1}}{2k_1 - k_0}$ for $k_1 > 0$. The geometrical analogs are curves $H4_M$ and $H4_m$ in Figure 1. Intuitively, two pairs of imaginary eigenvalues will only exist if k_0 and k_1 are inside the two lines, like in point N corresponding to Figure 2.N.

By now, the geometrical location of the fold-Hopf bifurcations in the $\mathcal{K}$-space should be straightforward: the 1-fold-Hopf bifurcation is located along the k_1-axis with the exception of the origin as in figures 2.B and 2.C and the 2-fold-Hopf bifurcation is located in the origin of figure 1 and in the semi-parabola in figure 2.A.

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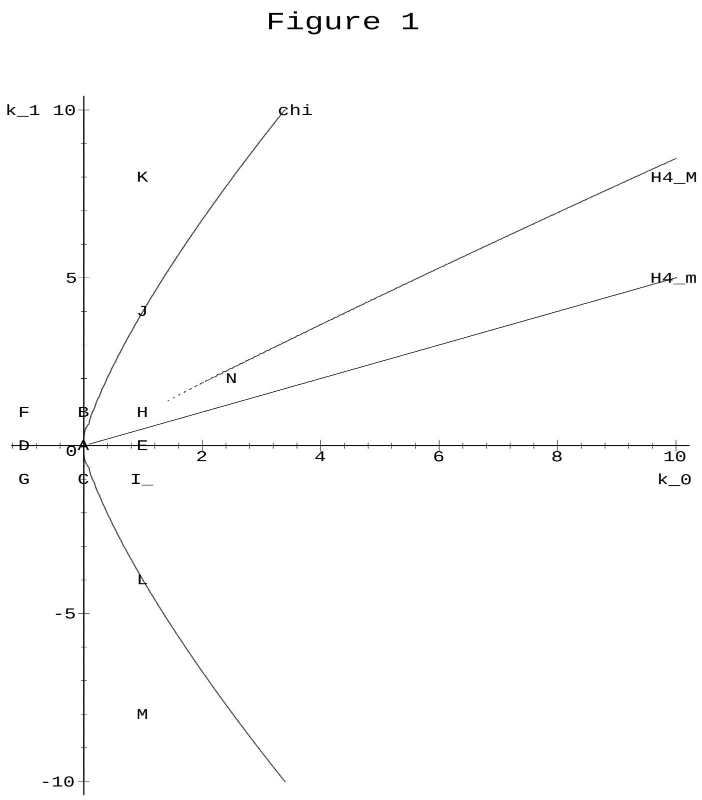
Figure 1: k_0 and k_1 values



The letter refers to a particular set of values for k_0 and k_1 which were used in a particular figure 2.

Figure 2: k_2 and k_3 for $k_0 = k_1 = 0$

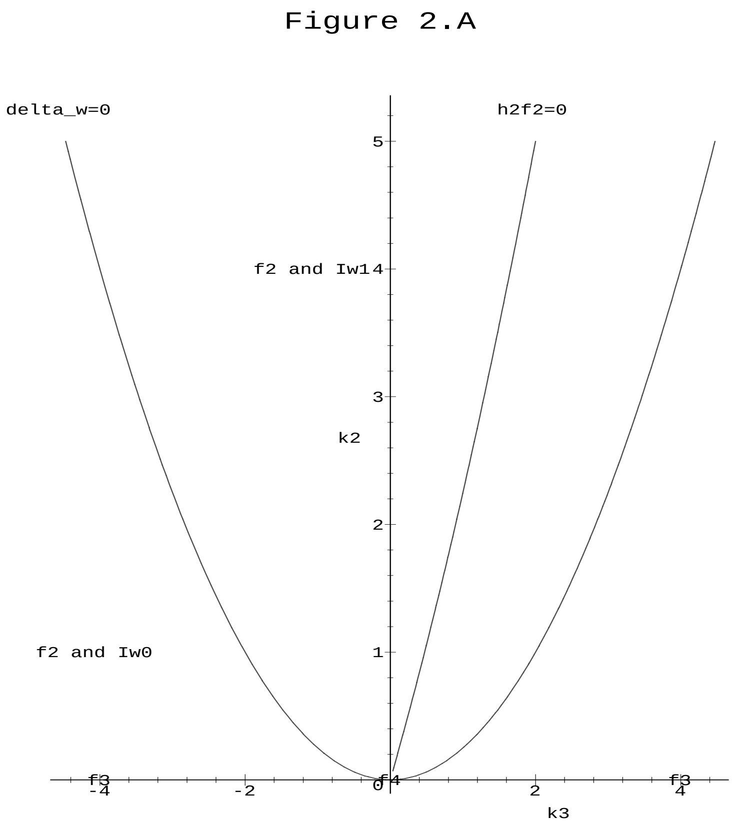


Figure 3: k_2 and k_3 for $k_0 = 0$ and $k_1 = 1$

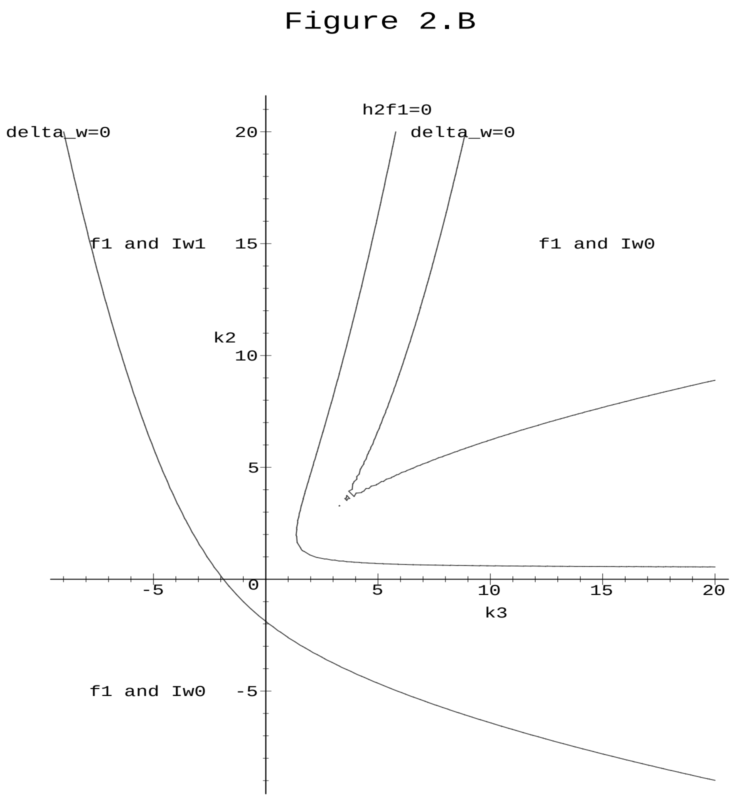


Figure 4: k_2 and k_3 for $k_0 = 0$ and $k_1 = -1$

Figure 2.C

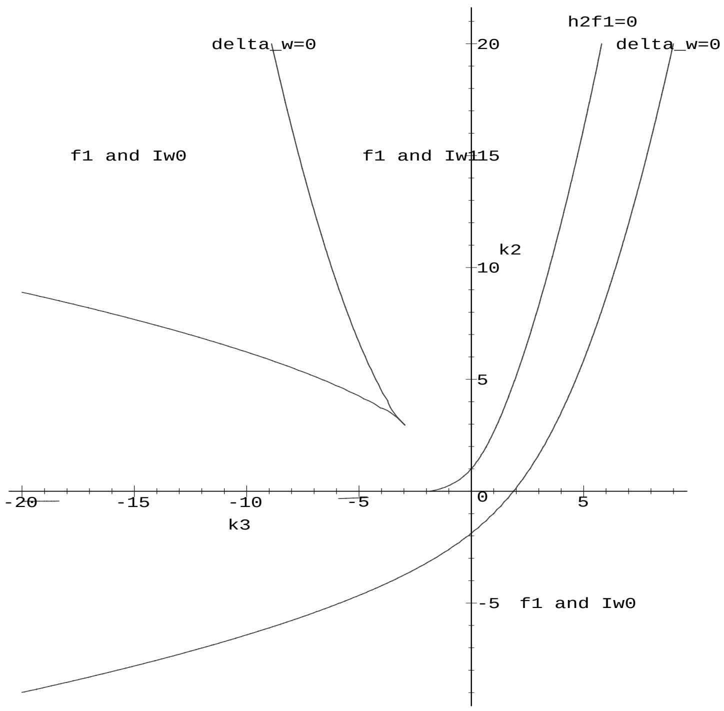


Figure 5: k_2 and k_3 for $k_0 = -1$ and $k_1 = 0$

Figure 2.D

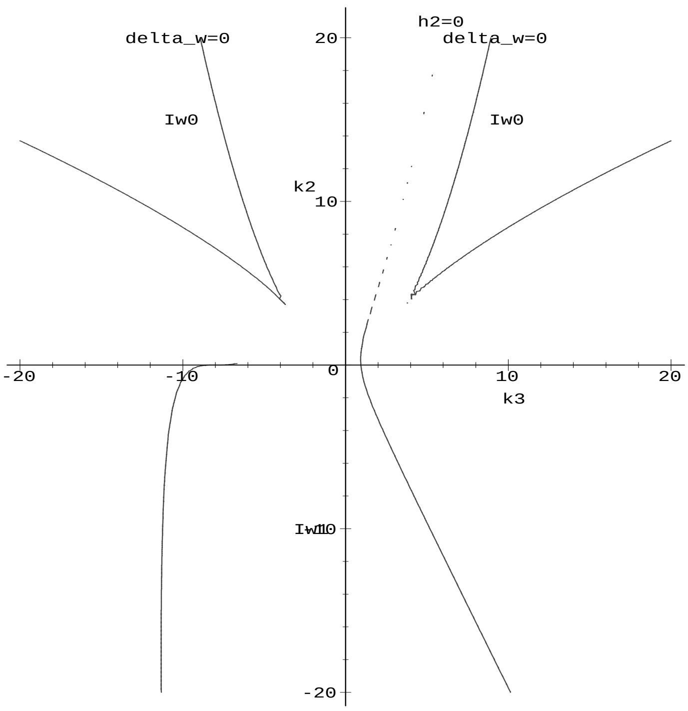


Figure 6: k_2 and k_3 for $k_0 = 1$ and $k_1 = 0$

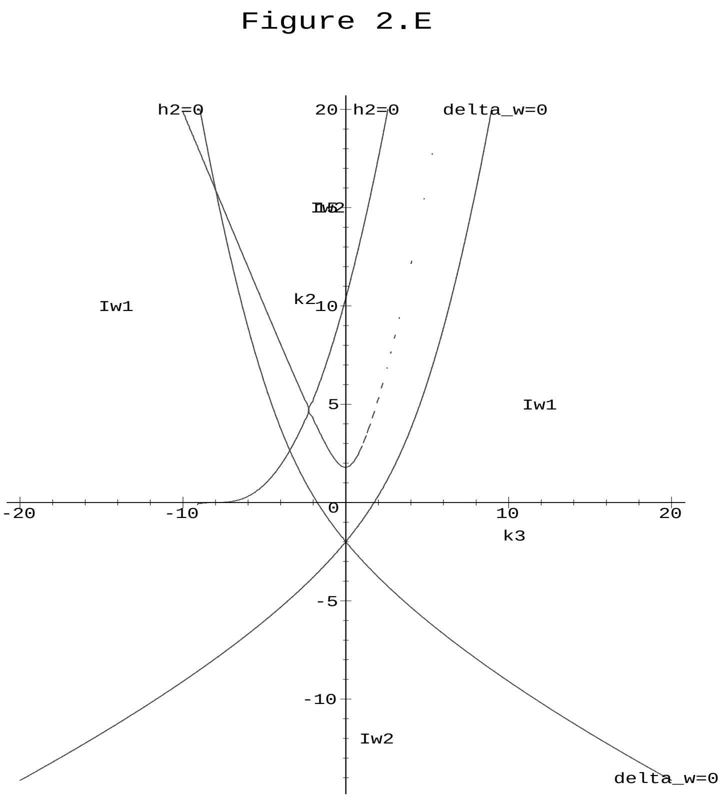


Figure 7: k_2 and k_3 for $k_0 = -1$ and $k_1 = 1$

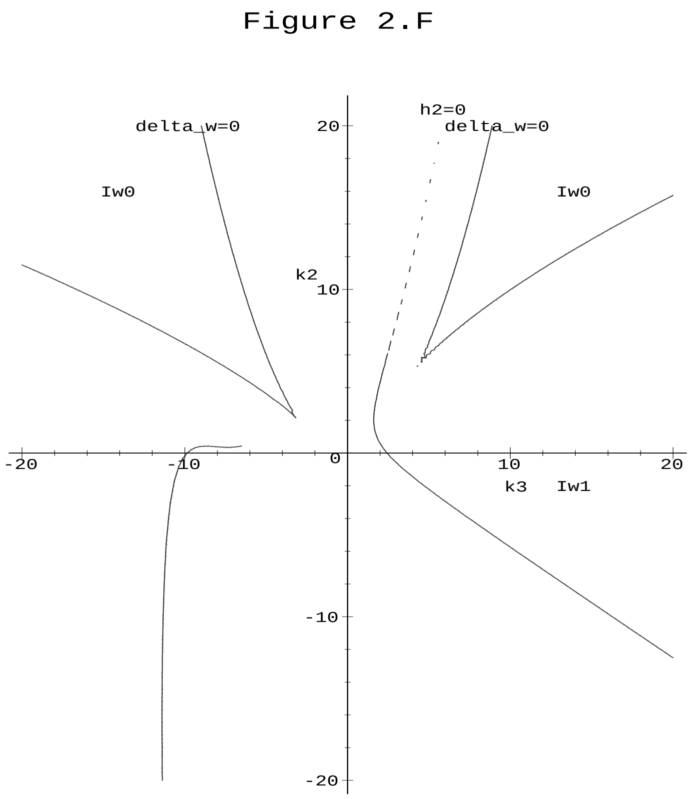


Figure 8: k_2 and k_3 for $k_0 = -1$ and $k_1 = -1$

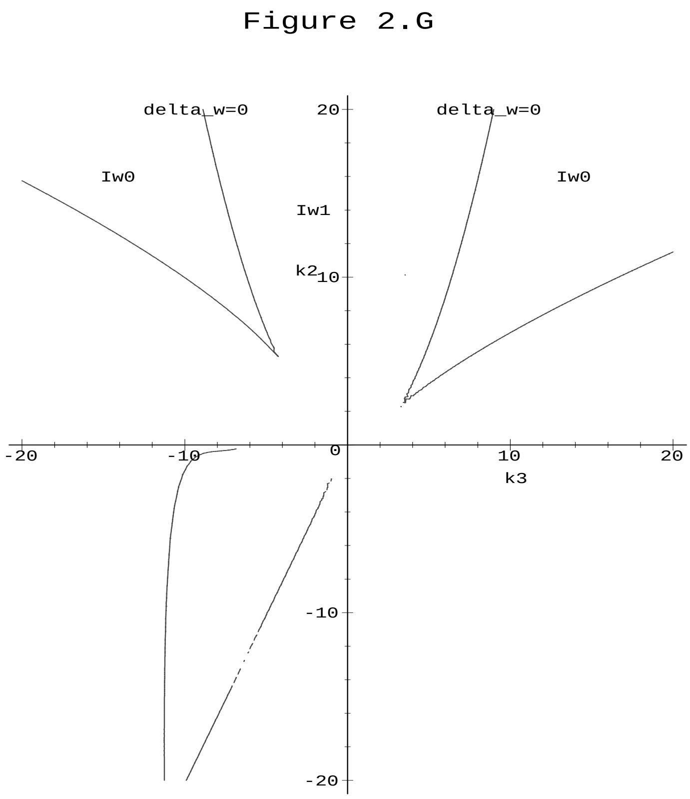


Figure 9: k_2 and k_3 for $k_0 = k_1 = 1$

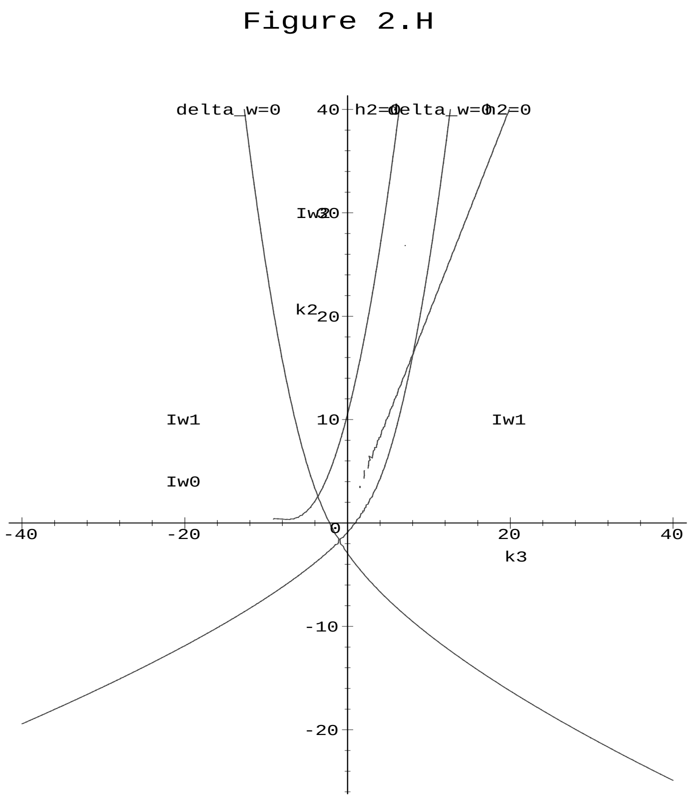


Figure 10: k_2 and k_3 for $k_0 = 1$ and $k_1 = 4$

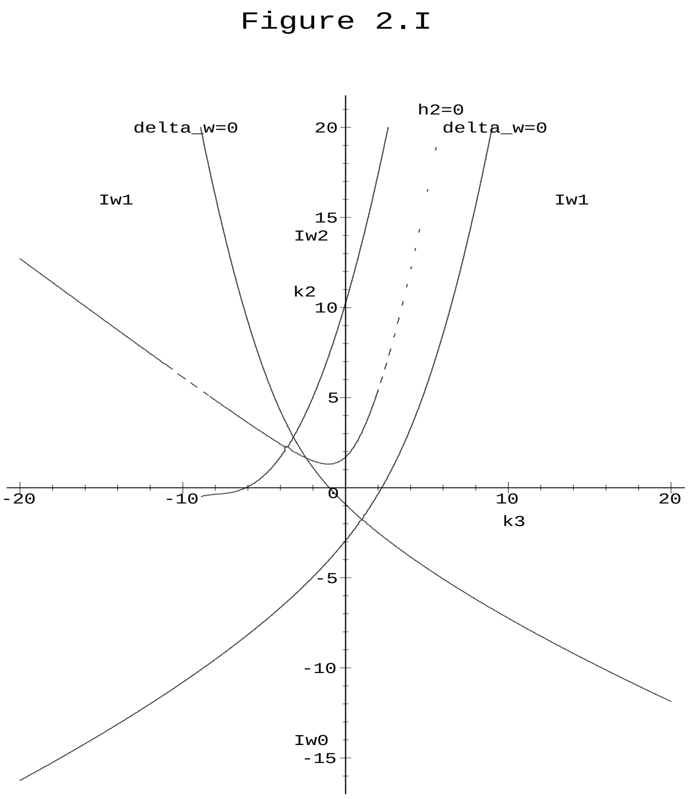


Figure 11: k_2 and k_3 for $k_0 = 1$ and $k_1 = 4$

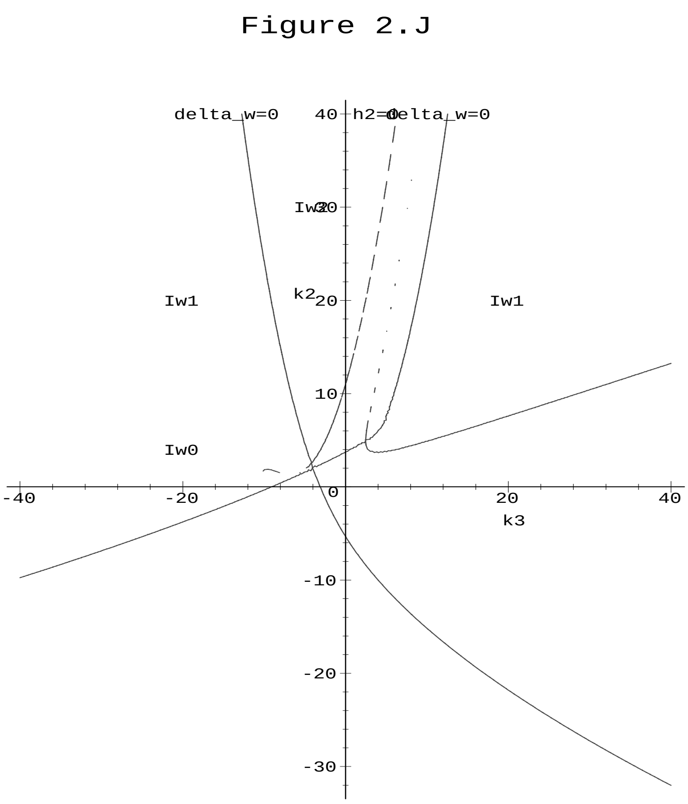


Figure 12: k_2 and k_3 for $k_0 = 1$ and $k_1 = 8$

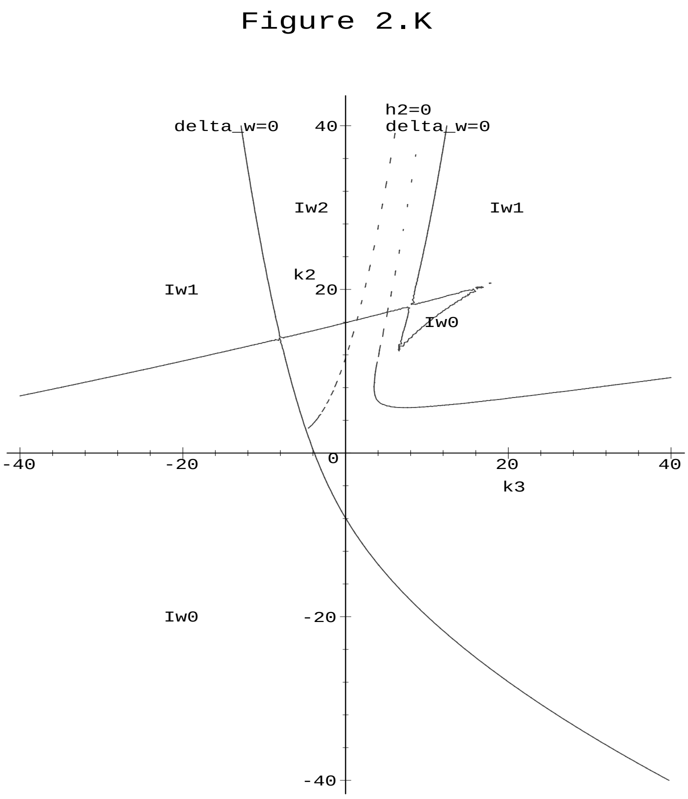


Figure 13: k_2 and k_3 for $k_0 = 1$ and $k_1 = -4$

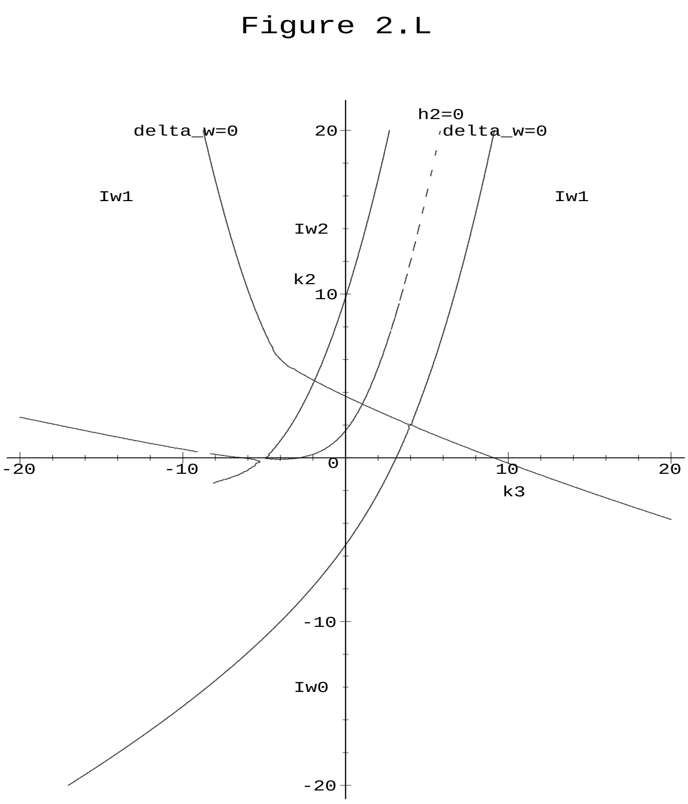


Figure 14: k_2 and k_3 for $k_0 = 1$ and $k_1 = -8$

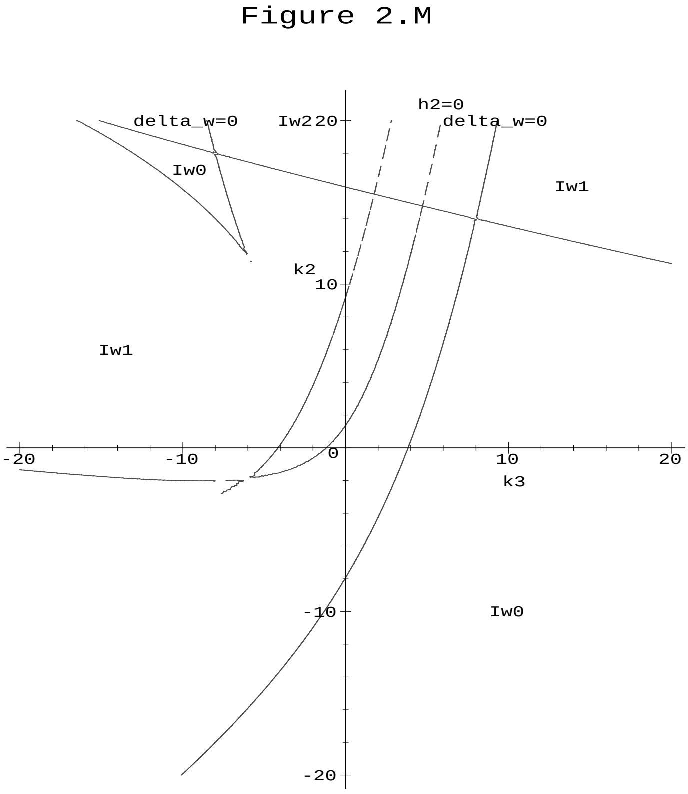


Figure 15: k_2 and k_3 for $k_0 = 2.5$ and $k_1 = 2$