The Three Muskeeters. Old Solutions to Bankruptcy Problems.

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1. Introduction

1.1. The axiomatic approach to bankruptcy problems

When a firm goes bankrupt, how should its liquidation value be divided among its creditors? In this paper we deal with such *bankruptcy problems,* and search for well-behaved methods, or *rules*, of associating with each bankruptcy problem a division of the liquidation value of the firm.

This is a major practical issue and, as such, it has a long history. The best-known rule is the *proportional rule,* which recomends awards to be proportional to the claims. Some other rules were generated trying to acommodate examples appearing in the literature. This is the case of the *Talmud rule* that generates the numbers proposed in the Talmud as solutions to some examples of bankruptcy problems [see Rabinovitch (1973), O'Neill (1982), Aumann and Maschler (1985)].

Modern economic analysis has addressed this problem from two different perspectives. One is the game theoretical, in which bankruptcy problems are formulated either as TU coalitional games, or as bargaining problems, and rules are derived from solutions to coalitional games and from bargaining solutions, respectively [see O'Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1988), Dagan and Volij (1993)]. Most of the recent literature follows the axiomatic framework, in which appealing properties of rules are formulated, rules are compared on the basis of these properties, and the existence of rules satisfying various combinations of these properties together is investigated [see O'Neill (1982), Young (1987), Chun (1988a), Dagan (1996), Herrero, Maschler and Villar

(1998)]. The reader is referred to Thomson (1995) for a survey of this literature. Here we follow this approach.

Alternative rules typically represent different ways of applying a fairness criterion to the resolution of the bankruptcy problem. Which one should we choose? It seems that one would need a rule to select rules (and immediately comes the problem of which rule to select rules to choose). One way out of this dilemma is to identify the structural properties that each of these rules satisfy, so that choosing a rule means choosing a set of these properties. This venue becomes more fruitful the closer we get to the following recommendations:

(1) Each property is intuitive and represents a single and clear ethical principle.

(2) We can identify each rule as the only one satisfying a distintinctive set of properties (that is, a collecion of these properties *characterizes* the rule); moreover all these properties are logically independent.

(3) This set of distinctive properties is small whereas alternative rules share most of the properties (in order to clearly identify their ethical differences).

Structural properties express invariance of the solutions with respect to changes in the parameters, and are usually motivated by particular concerns. They are intented to ensure that the solution has some desirable features or to prevent some inconveniences. Hence it is not surprising that a particular rule can be characterized by different sets of independent axioms. Each characterization provides an insight on the type of problems for which a rule is satisfactory. The reader is referred to Thomson (1998) for a discussion of the axiomatic method.

1.2. Scope and outline of the paper

This paper concentrates on a comparative analysis of three basic rules to solve bankruptcy problems from an axiomatic viewpoint. These rules are:

(i) The *proportional rule*, that divides the estate proportionally to agents' claims.

(ii) The *constrained equal-awards rule*, that divides equally the estate among the agents under the condition that nobody gets more than her claim.

(iii) The *constrained equal-losses rule*, that divides equally the diference between the aggregate claims and the budget, provided no agent ends up with a negative award.

The *proportional rule* satisfies a number of appealing properties, and when compared with other rules, it has much to recommend itself. The idea of *equality* underlies another well-known rule: the *constrained equal-awards rule*. It makes awards as equal as possible to all creditors, subject to the condition that no creditor receives more than her claim. A dual formulation of equality, focusing on the losses creditors incur, as opposed to what they receive, underlies the *constrained equal-losses rule*. It proposes losses as equal as possible for all creditors, subject to the condition that no creditor ends up with a negative award.

Suppose that, when solving a problem, we start by temporarily awarding every agent his claim. Since it is not feasible, now we apply a particular rule to allocate losses. By this procedure we obtain a new rule, *the dual rule* of the initially used. When a rule coincides with its dual, it is called *self-dual*. The proportional rule is self-dual, and the constrained equal-losses and the constrained equal awards rules are dual from each other. This duality underlies the properties they fulfil, and help to better understand their different behavior.

The comparative analysis of the three aforementioned rules aims at clarifying the class of real life problems for which each of these solutions is better. With this purpose in mind and following the recommendations given above, we concentrate on those characterizations that permit an easy comparison of these three rules. In particular, we focus on a family of results that characterize each of these rules by three independent axioms, two of which are common to all of them.[1]

The choice of these three solutions is by no means arbitrary. First because they are the most common methods of solving practical problems. Second for their long tradition in history. And last but not least, because they are almost the only sensible ones within the family of solutions that treat equally equal claims. As Moulin (1997, p. 3) puts it: "One unambiguous conclusion emerges from the axiomatic analysis of rationing methods: ... [these three rules] stand out by virtue of their multifarious axiomatic properties".

As the Three Musketeers were four so are our three rules. The Talmud rule here will play the role of D'Artagnan. This is an appealing allocation rule that amounts to solve bankruptcy problems by combining the constrained equal awards rule and the constrained equal losses rule.

We start by formally introducing the family of bankruptcy problems and the three basic rules. Then we present several appealling properties for bankruptcy rules. We offer a joint characterization of the three bankruptcy rules in terms of some of those properties, as well as independent characterizations.

Previous properties help us to analyze also the *contested garment rule*, and its consistent extension, the *Talmud rule* (the rule playing here the role of D'Artag-

[1] The discusion that follows is largely based on the author former works of Herrero (1998), Herrero, Maschler & Villar (1998) and Herrero & Villar (1998a,b).

nan). Variants and extensions of the aforementioned rules are also analyzed.

The paper ends by providing noncooperative support of the *constrained equal-awards rule* and of the *constrained equal-losses rule*. Proofs are relegated to an Appendix.

2. The Model

Let N be a set of agents, with $|N| = n$. A **bankruptcy problem for** N is a pair (E, c), where $E \in \mathbb{R}_+$ represents the **net worth** of a firm, and $c \in \mathbb{R}_+^N$ is a **vector of claims:** c_i represents the **claim** of creditor $i \in N$. Moreover, $\sum_{i \in N} c_i > E$. We denote by $\mathbb{B}^N$ the family of all such problems.

The model describes the situation faced by a bankruptcy court. An alternative interpretation of the model is the division of an estate E among a group of heirs when the estate is insufficient to cover all the bequeathed amounts, c_i, $i \in N$. The same model can also be interpreted as a formalization of a class of tax assessment problems: there, the cost E of a project has to be divided among a group of taxpayers, where c_i stands for agent $i's$ income.

A **rule for** $\mathbb{B}^N$ is a mapping F that associates with every $(E, c) \in \mathbb{B}^N$ a unique point $F(E, c) \in \mathbb{R}^N$ such that: (i) $0 \leq F(E, c) \leq c$, and (ii) $\sum_{i \in N} F_i(E, c) = E$. The point $F(E, c)$ is to be interpreted as a desirable way of dividing E among the creditors in N. Requirement (i) is that each creditor receive an award that is non-negative and bounded above by his claim. Requirement (ii) is that the entire net worth of the firm be allocated. It is implicitly assumed that F is homogeneous of degree one in (E, c), meaning that E and c are measured in the same units.

A more general model refers to the case when we face a *variable population.* Let $\mathbb{N}$ stand for a (infnite) potential set of agents, and let $\mathbb{F}$ be the family of all finite subsets of $\mathbb{N}$. For any $N \in \mathbb{F}$, we denote by n the cardinal of N. Now a **bankruptcy problem** is a triple (N, E, c), where $N \in \mathbb{N}$ stands for the particular set of agents involved, $E \in \mathbb{R}_+$ is the net worth of the firm, and $c \in \mathbb{R}_+^N$ is the vector of claims, with $\sum_{i \in N} c_i \geq E$. We shall denote by $\mathbb{B} = \cup_{N \in \mathbb{F}} \mathbb{B}^N$ the family of all such bankruptcy problems with variable population. A **rule** is a mapping F that associates with every $(N, E, c) \in \mathbb{B}$ a unique point $F(N, E, c) \in \mathbb{R}^N$, such that: (i) $0 \leq F(N, E, c) \leq c$ and (ii) $\sum_{i \in N} F_i(N, E, c) = E$.

Both in the fixed population and in the variable population framework, given a rule F, we can consider another rule associated to it, by means of a **duality** procedure. This duality procedure can be easily explained in the following way: suppose that, when solving a problem, we start by temporarily awarding every agent his claim. Since it is not feasible, now we apply rule F to the problem of allocating losses. By this procedure we obtain a new rule, *the dual rule* of the initially used. Formally,

Dual rule of F, F^* (Aumann and Maschler, 1985): For all $N \in \mathbb{F}$ and all $(E,c) \in \mathbb{B}^N$, $F^*(E,c) = c - F(\sum_{i \in N} c_i - E, c)$.

The rules F and F^* are related in a simple way: F^* divides what is available in the same way as F divides what is missing. Note that for any problem $(E,c) \in \mathbb{B}^N$,we have that $\sum_{i \in N} c_i - E \in \mathbb{R}_+$ and $\sum_{i \in N} c_i > (\sum_{i \in N} c_i - E)$. Hence, the problem $(\sum_{i \in N} c_i - E, c)$ is also a problem in $\mathbb{B}^N$. Moreover, $0 \leq F(\sum_{i \in N} c_i - E, c) \leq c$ and $\sum_{i \in N} F_i(\sum c_i - E, c) = \sum c_i - E$, so that $0 \leq F^*(E,c) \leq c$ and $\sum_{i \in N} F_i^*(E,c) = E$, that is, F^* is well defined. A rule is **self-dual** whenever it coincides with its dual. Formally,

Self-duality (Aumann and Maschler, 1985)**:** F is called **self-dual** if $F^* = F$.

Note that duality is an idempotent operation, that is, $(F^*)^* = F$. The notion of duality is naturally extended to the properties a solution satisfies. Formally,

Dual properties: Given two properties $\mathcal{P}$, $\mathcal{P}^*$, we say that $\mathcal{P}^*$ is the **dual property of** $\mathcal{P}^*$ if for very rule F that satisfies $\mathcal{P}$, its dual rule F^* satisfies $\mathcal{P}^*$. Similarly, a property $\mathcal{P}$ is **self-dual** if $\mathcal{P}^* = \mathcal{P}$.

3. The three musketeers

Next we introduce three well-known rules. The **proportional rule** is the most widely used rule. It makes awards proportional to claims.[2] Formally:

Proportional rule, P: For all $N \in \mathbb{F}$, all $(E,c) \in \mathbb{B}^N$, and all $i \in N$, $P_i(E,c) = \lambda c_i$, where $\lambda \geq 0$ solves $\sum_{i \in N} \lambda \ c_i = E$.

The **constrained equal-awards rule** makes awards as equal as possible, subject to the condition that no agent receives more than his claim. As it is made explicit in Aumann and Maschler (1985), "this rule has been adopted as law by

[2]Hence, provided that $\sum_i \ c_i \neq 0$, it equalizes the ratios between claims and awards.

most major codifiers, including Maimonides (in his *Laws for Lending and Borrowing)"*. From a geometrical viewpoint this way of solving the problem amounts to selecting that point in the feasible set which is closest to the origin. Formally,

Constrained equal-awards rule, CEA:For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $i \in N$, $CEA_i(E, c) = \min\{c_i, \lambda\}$, where λ solves $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

The **constrained equal-losses rule** makes awards such that losses are as close to equal as possible, subject to the condition that no creditor ends up with a negative award. According to Aumann and Maschler (1985), this rule also appears in Maimonides, in dealing with auctions, and looking at the losses the seller may experience when bidders renege (in his *Laws of Appraisal).* From a geometric viewpoint this way of solving the problem amounts to selecting that point in the feasible set which is closest to the vector of claims. Formally,

Constrained equal-losses rule, CEL: For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $i \in N$, $CEL_i(E, c) = \max\{0, c_i - \lambda\}$,where λ solves $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.The

constrained equal-awards rule corresponds to the *uniform rule* in the context of distribution problems with single-peaked preferences, when the task is smaller than the supply of effort. In the context of taxation this rule is known as the "head tax". The principle underlying the *constrained equal-losses rule*, the *equal-loss* principle, has been applied to other distribution problems, such as cost-sharing, taxation or axiomatic bargaining [see Young (1987), (1988), Chun (1988b), Herrero and Marco (1993)]. In the context of taxation it is known as the "leveling tax".

- **Claim 1.** It is easy to see that $CEL = CEA^*$ whereas $P^* = P$, that is, the *constrained equal-awards rule* and the *constrained equal-losses rule* are dual from each other, whereas the *proportional rule* is self-dual.

4. Common Properties

Let us now consider some properties that represent value judgements that a solution might be asked to satisfy. Each property is usually motivated by a particular concern and thus directed to prevent this inconvenience to occur.

4.1. Two basic properties: equal treatment and consistency.

The first property is a basic equity requirement: agents with identical claims should be treated identically. Hence, we exclude differentiating between agents on the basis of their names, gender, religion, political ideas, etc. Formally:

Equal treatment of equals: For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $i, j \in N$, if $c_i = c_j$, then $F_i(b) = F_j(b)$.

Equal treatment of equals is an instance of impartiality. It establishes that all agents with the same claims will receive the same amount. A stronger requirement is that of **anonymity** (given two sets $N, N' \in \mathbb{F}$, with $|N| = |N'|$, for all $(N, E, c), (N', E, c') \in \mathbb{B}$, and for all one-to-one mapping $\Pi : N \to N'$ such that $c'_{\Pi(i)} = c_i$, then $F_{\Pi(i)}(N', E, c) = F_i(N, E, c)$ for all $i \in N$); a weaker version of the impartiality principle is that of **symmetry** (for all $N \in \mathbb{F}$ and for all $(E, c) \in \mathbb{B}^N$ such that for all $i, j \in N$, $c_i = c_j$, then $F_i(E, c) = F_j(E, c)$, for all $i, j \in N$).

The second property refers to the case of a *variable population.* **Consistency** is a well-known and powerful property that links the solution of a problem for a given society N with the solutions of the problems corresponding to its subsocieties. To formally define this property, let S be a proper subset of N and suppose that, after solving a problem (N, E, c) by means of the rule F, the members of group S reconsider the allocation of what they got, $\sum_{i \in S} F_i(N, E, c)$. Let $[S, \sum_{i \in S} F_i(N, E, c), c_S]$ be the associated **reduced problem**, where $c_S = (c_i)_{i \in S}$. The rule F is *consistent* if applied to any of its reduced problems it gives the incumbent agents the same shares as they got in the original problem. Formally:

Consistency: For all $N \in \mathbb{F}$, all $S \subset N$, all $(N, E, c) \in \mathbb{B}$, and all $i \in S$, we have: $F_i(N, E, c) = F_i[S, \sum_{i \in S} F_i(N, E, c), c_S]$.

Consistency is a procedural requirement with two relevant implications:

(i) Once an allocation has been agreed upon, no group of agents is willing to re-aply the rule in the reduced problem that appears when the other agents leave bringing with them their alloted shares. Hence, what is good for the large group is also good for the small ones.

(ii) If the agreement on how to distribute an estate between two agents can be consistenly extended to any number of them, then that extension is unique.[3] Hence, what is good for the smallest group is good for larger ones.

[3]It may happen that no consistent extension of a particular solution exists [see Dagan, Ser-

The first implication provides us with a stability feature: consistency prevents subgroups of agents to renegociate once there is a solution proposed for the society. The second one helps in assesing the value judgements of alternative solutions, as what is "fair" is easier to check and to understand in the two-person case.

4.2. Composition and path-independence

To motivate the next two properties, suppose that a tentative distribution is made by first forecasting the value of the estate. Assume that, once the tentative division is done, the actual value of the estate is greater than initially thought. Then, two options are open: either the tentative division is cancelled altogether and the actual problem is solved, or the rule is applied to the problem of dividing the incremental value of the estate, after adjusting the claims down by the amounts already assigned. **Composition** requires the rule to be invariant with respect to the chosen option. Alternatively, assume that, once the tentative division is done, it turns out that the actual value of the estate falls short of what was assumed. **Path-independence** requires the solution of the actual problem to agree with the solution of the problem in which the initial claims are substituted by the (unfeasible) allocation initially proposed. Formally:

Composition (Young, 1988): For all $N \in \mathbb{F}$, all $b = (E, c) \in \mathbb{B}^N$, and all $E_1, E_2 \in \mathbb{R}_+$ such that $E_1 + E_2 = E$, if $b_1 = (E_1, c)$ and $b_2 = [E_2, c - F(b_1)]$, then $F(b) = F(b_1) + F(b_2)$.

This property says that the problem (E, c) can also be solved as the sum of two partial problems. The first one corresponds to a problem with the initial claims c and a fraction E_1 of the estate; the second one is that problem made out of the outstanding claims $c' = c - F(E_1, c)$ and the reminder estate, $E - \sum_{i \in N} F_i(E_1, c)$. When a solution satisfies *composition*, solving a problem in stages does not change agents' final awards.

rano and Volij (1997)]. The uniqueness in the procedure of consistenly extending bankruptcy problems was first noticed by Aumann and Maschler, for the consistent extension of the *contested garment rule.*

The requirement of agreement on the shares from small groups to large groups is usually refered to as "converse consistency". In general, consistency and its converse are independent properties, but in the case of bankruptcy problems, if a rule satisfies consistency, then it also satisfies converse consistency. See Chun (1998).

Consider now the case in which, after solving a problem, it turns out that the actual worth of the firm falls short of what was expected. Consider a group of creditors with claims c, and a worth firm's forecast E. If we solve the problem (E,c), the vector of awards is z. Assume now that the net worth turns out to be smaller than expected, $\widetilde{E} < E$. **Path-independence** requires that the solution of the problem $(\widetilde{E}, c)$ be the same as that of the problem (E, z), namely, if we adjust claims down to z, the final awards do not change. Formally:

Path Independence (Moulin, 1987): For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $E' > E$, we have $F(E, c) = F(E, F(E', c))$

- **Claim 2**. If a rule satisfies either *composition* or *path-independence* it is *monotonic with respect to the estate*, and also it is *continuous with respect to the estate.*

- **Claim 3.** The properties of *equal treatment of equals* and *consistency* are self-dual whereas *composition* and *path-independence* are dual properties.

- **Claim 4.** The three rules, P, CEA, and CEL satisfy *equal treatment of equals, consistency, composition,* and *path-independence.*

5. Separating Properties, or How Important Claims are.

Suppose now that we face a bankruptcy problem in which agents' claims are very assymetric. How should we ask a solution to treat this situation? To make the point clearer think of a two-person problem in which one agent has a relatively small claim, $c_1 < \frac{1}{2}E$, say, whereas the other one has a claim larger than the estate, e.g. $c_2 > E + c_1$. One can approach this question from two opposite points of view:

(i) The large claim c_2 is meaningless as it asks for more than what it is available, so it has to be scaled down to reality in one way or another. Or, put in a different way, the agent with a realistic claim is going to have a relatively higher satisfaction.

(ii) One should give priority to agent two in the distribution because, even if we give everything to her, she will have a loss larger than that of agent one. Or, stated differently, the agent 1 cannot expect to get something until agent 2's net claim has been scaled down to a magnitud "comparable" with that agent 1's claim.

These alternative notions of claims enforceability express clear cut values on how allocation rules should perform in extreme situations.[4] They will help us choosing among different rules, depending upon the problem at hand. We formalize these ideas by using four different properties: **sustainability**, **preeminence**, **independence of claims truncation**, and **composition from minimal rights**. These properties happen to be dual by pairs.

5.1. Sustainability and preeminence

Let $(E, c) \in \mathbb{B}^N$ and $i \in N$. Now, consider the problem $(E, c^i(b))$, where, for all $j \in N$, $c^i_j(b) = \min\{c_i, c_j\}$. That is, we truncate all claims by agent $i's$ claim. Agent i's claim is **sustainable** in (E, c) if $\sum_{j\in N} c^i_j \leq E$. Thus, an agent's claim is sustainable in a problem if, by truncating the claims of all agents by c_i, the problem becomes feasible. The following property states that sustainable claims should be fully honored.

Sustainability (Herrero and Villar, 1998b)**:** For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $i \in N$, if c_i is sustainable in (E, c), then $F_i(E, c) = c_i$.

Sustainability says that when the resources to be divided are large enough, only individuals with the highest claims are to be rationed. This amounts to saying that agents with smaller claims are given priority in the distribution. It is worth mentioning that this principle is applied by Law in some real-life bankruptcy problems (particularly in the case of banks or other financial intermediaries, where the debts of clients with small savings are honored first).

Let us now introduce the notion of claims domination, in order to define the alternative property. Let $(E, c) \in \mathbb{B}^N$ and let $j \in N$. We say that creditor j's claim **is dominated** in (E, c) if there is some $i \in N$ such that $c_i - E \geq c_j$. An agent's claim c_j is dominated in a problem (E, c) if there is another claim c_i so large that even if we give the whole budget to her, i's loss is still greater than j's demand. The next property states that agents whose claims are dominated should not be allocated anything. Formally:

[4] John Stuart Mill (1859) argued that the strength of our moral values is to be judged in extreme situations.

Preeminence (Herrero and Villar, 1998b) **:** For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $j \in N$, if j's claim is dominated in (E, c), then $F_j(E, c) = 0$.

Preeminence says that those agents with dominated claims do not get anything. To understand better the extent of this requirement, take the extreme case in which there is an agent, say agent 1, whose claim dominates every other agent (namely, $c_1 - E \geq c_j$, for all $j > 1$). If a solution F satisfies preeminence it follows that $F_1(E, c) = E$. Note that the number $c_1 - E$ measures the *greatest minimal loss* agent 1 can experience. Namely, the rationing suffered by that agent with the highest claim in the most favourable case (that in which the whole estate were allotted to her). In such a case c_j measures the rationing experienced by the jth agent, as she is allotted zero. Preeminence says that when this minimal loss of individual 1 is greater than or equal to the claim of every other agent, it seems fair to allocate the estate only to her (note that any other assignment will increase the rationing suffered by agent 1, which is always greater than that experienced by any other agent).

- **Claim 5.** *Sustainability* and *preeminence* are dual properties. CEA satisfies sustainability and fails to satisfy preeminence. CEL, on the contrary, satisfies preeminence and fails to satisfy sustainability. P fails to satisfy both properties.

5.2. Independence of claims truncation and composition from minimal rights.

Consider again a problem in which some claims are larger than the estate, $c_i > E$, and how a rule should treat these demands. Here again there are two opposite views that yield two dual properties. These two properties appear in the Talmud, providing with a natural way of solving two-person problems.[5]

The first one states that a rule should not consider any claim that is greater than the estate: replacing c_i by E if $c_i > E$ should not affect the recommendation. Formally,

Independence of claims truncation (Dagan, 1996)**:** For all $N \in \mathbb{F}$ and all $(E, c) \in \mathbb{B}^N$, $F(E, c) = F(E, c^T)$, where $c_i^T = \min\{E, c_i\}$.

[5] The *contested garment rule* (see Section 6) is presented as the most sensible way of solving two-person bankruptcy problems, by applying these two principles.

This property establishes that if an individual claim exceeds the total to be allocated, the excess claim should be considered irrelevant. The rationale behind is that "one cannot claim more than there is; thus the excess of a claim above the estate is irrelevant. A rule is independent of claims truncation if it allocates the estate taking into account only the relevant claims." [Cf. Dagan (1996, p.53)].

In order to present the second property let us start by introducing the notion of *minimal rights*. For a given problem $(E, c) \in \mathbb{B}^N$ define the ith agent's minimal right as:

$$m_i(E, c) = \max\{0,\ E - \sum_{j \neq i} c_j\}$$

The number $m_i(E, c)$ represents the amount of the budget which is left to agent i when all other agents' claims are honored, provided this amount is nonnegative; and is taken to be zero otherwise. Let $m(E, c)$ denote the vector in $\mathbb{R}^N_+$ whose components are the minimal rights, $m_i(E, c)$, $i \in N$.

The next property says that a rule should honor agents' minimal rights before any further step is taken. Hence it asks the rule to allocate first the amounts corresponding to these minimal rights and then solving the remaining problem. Formally,

Composition from minimal rights: For all $N \in \mathbb{F}$ and all $(E, c) \in \mathbb{B}^N$, $F(E, c) = m(E, c) + F[E - \sum_{i \in N} m_i(E, c), c{-}m(E, c)]$.

Composition from minimal rights is a particular form of *composition* that says the following: the solution of any problem $(E, c) \in \mathbb{B}^N$ coincides with the outcome of a process in which minimal rights $m(E, c)$ are allocated first, and the rule is applied to the problem consisting of the remaining estate $E - \sum_{i \in N} m_i(E, c)$ and the outstanding claims $c{-}m(E, c)$.

- **Claim 6.** *Independence of claims truncation* and *composition from minimal rights* are dual properties [see Herrero (1998)]. CEA satisfies independence of claims truncation and fails to satisfy composition from minimal rights. CEL satisfies composition from minimal rights and fails to satisfy independence of claims truncation. P fails to satisfy both.

- **Claim 7.** There is no relationship in between the former properties, namely, in between *independence of claims truncation* and *sustainability*, on the one hand, and in between *composition from minimal rights* and *preeminence*, on the other hand.

6. The Three Musketteers in Focus.

This section is devoted to the characterization of the three rules in terms of the properties presented in former sections. Two types of results will be presented. The first one is a joint characterization of the three rules that both emphasizes their common features and gives support to their choice as the leading candidates to the resolution of bankruptcy problems. Then we shall present some individual characterizations of these rules underlining the duality relations between them. A relevant feature of these characterizations is that they allow us to compare the three rules in terms of a single differential property. This facilitates the selection among these rules depending on the nature of the bankruptcy problem considered.

6.1. A joint characterization

As it was mentioned in Claim 4, it can be easily verified that the *proportional rule*, the *constrained equal-awards rule,* and the *constrained equal-losses rule* satisfy the properties of equal treatment of equals, consistency, composition and path-independence. The following result tells us that these rules are actually the only ones satisfying all these requirements.

Theorem 6.1. *(Moulin, 1997, Corol. to Th.2). There are three and only three rules on $\mathbb{B}$ satisfying simoultaneously equal treatment of equals, composition, path-independence, and consistency: The proportional rule, the constrained equal-awards rule, and the constrained equal-losses rule.*

This impressive result provides additional support to the choice of the three bankruptcy rules discussed so far. One way of checking the strengh of this theorem is by asking ourselves what property to drop in order to buy other solutions. On the one hand, *equal treatment of equals* seems difficult to object, unless we consider a wider family of problems where agents have other relevant differences, to be included in the information that describes the problem. *Composition*, *path-independence* and *consistency* can be regarded as procedural requirements. The first two ensure coherence with respect to subdivisions of the estate; the third one ensures coherence with respect to all reduced problems, when some of the agents leave, taking with them their alloted shares. If either composition or path-independence fails, then the outcome of the resolution becomes dependent on the agenda; namely, it varies according to the way in which the problem is subdivided into partial problems. The lack of consistency implies that the resolution is sensitive to the way in which the division of the problem in population subgroups is organized.

6.2. Independent Characterizations

Let us consider now the characterization of each of these rules. The following results are obtained:

Theorem 6.2. *(Herrero and Villar, 1998b) The constrained equal-awards rule is the only rule in $\mathbb{B}$ satisfying equal treatment of equals, path-independence, and sustainability.*

Theorem 6.3. *(Herrero and Villar, 1998b) The constrained equal-losses rule is the only rule in $\mathbb{B}$ satisfying equal treatment of equals, composition, and preeminence.*

Theorem 6.4. *(Dagan, 1996) The constrained equal-awards rule is the only rule in $\mathbb{B}$ satisfying equal treatment of equals, composition, and independence of claims truncation.*

Theorem 6.5. *(Herrero, 1998b) The constrained equal-losses rule is the only rule in $\mathbb{B}$ satisfying equal treatment of equals, path independence, and composition from minimal rights.*

- **Claim 8.** Theorems 2 and 3 are dual results. Theorems 4 and 5 also are dual results.

In order to present a characterization of the proportional solution, consider now the following property:

Continuity: For all $N \in \mathbb{F}$, all $(E, c) \in \mathbb{B}^N$, and all $(E_q, c_q) \in \mathbb{B}^N$ such that $\lim_{q\to\infty} E_q = E$, and $\lim_{q\to\infty} c_q = c$, then $\lim_{q\to\infty} F(E_q, c_q) = F(E, c)$.

The following characterization, involving *self-duality*, is obtained:

Theorem 6.6. *(Young, 1988) The proportional rule is the only rule in $\mathbb{B}$ satisfying continuity, self-duality and composition.*

- **Claim 9.** *Continuity* is redundant in previous result. Consequently, Young's theorem can be refined by dropping continuity.

By using the duality relation, the following result is also obtained:

Theorem 6.7. *The proportional rule is the only rule in $\mathbb{B}$ satisfying self-duality and path-independence.*

Remark 1. *Note that even though the three rules satisfy consistency, it is not required in the characterizations presented here. It follows from the other requirements.*

These results illuminate on the kind of problems for which each solution is better. The constrained equal-losses rule is a sensible rationing scheme for those problems in which claims represent real entities of an absolute nature (e.g. unalienable rights or vital needs, to take two extreme cases). The constrained equal awards rule instead, seems more appropriate for those problems in which individuals are the primary concern, whereas their claims only represent maximal aspirations (inheritance, say). The proportional rule is a natural distribution rule when we think of bankruptcy as a subfamily of distribution problems in which E can exceed of fall short of C, as self-dual rules allocate awards and losses in the same manner.

7. And D'Artagnan

The *contested garment rule* is another old bankruptcy rule. It appears int the Talmud as a way of solving two-person bankruptcy problems. The idea behind this rule is to concede first to each agent her minimal right and then to distribute equally the remaining estate. Formally,

Contested garment rule, G: For all $N \in \mathbb{F}$, with $|N| = 2$, all $(E, c) \in \mathbb{B}^N$, and all $i \in N$, $G_i(E, c) = m_i(E, c) + \frac{1}{2}\left[E - \sum_{j \in N} m_j(E, c)\right]$

The *contested garment rule* satisfies some interesting properties: it is *symmetric, continuous, monotone with respect to the estate, self-dual,* and satisfies simultaneously *independence of claims truncation* and *composition from minimal* *rights.* It fails to satisfy composition, path-independence, preeminence and sustainability.

The following characterization result is obtained:

Theorem 7.1. *The contested garment rule is the only two person rule satisfying self-duality and composition from minimal rights.*

Thus, the *contested garment rule* appears as a way of reaching an agreement in between the principles of *composition from minimal rights* and *independence of claims truncation.* It also can be viewed as a compromise in between the CEA and the CEL rules, as we can write

$$G(E,c) = \begin{cases} CEA(E, \frac{1}{2}c) & 0 \leq E \leq \frac{1}{2}\sum_i c_i \\ \frac{1}{2}c + CEL(E - \frac{1}{2}\sum_i c_i, c) & \frac{1}{2}\sum_i c_i \leq E \leq \sum_i c_i \end{cases}$$

Hence, the *contested garment rule* behaves as the *constrained equal awards rule* for values of the estate not exceeding the minimum claim of the agents, and it behaves as the *constrained equal-losses rule* for values of the estate above the maximum claim.

Aumann and Maschler (1985) introduced the **Talmud rule** as the consistent extension of the *contested garment rule.*

Talmud rule, T (Aumann and Maschler, 1985)**:** For all $N \in \mathbb{F}$, all $(E,c) \in \mathbb{B}^N$, and all $i \in N$, $T_i(E,c) = \begin{cases} \min\{\frac{1}{2}c_i, \lambda\} & \text{if } E \leq \frac{1}{2}\sum_{i\in N} c_i \\ \max\{\frac{1}{2}c_i, c_i - \mu\} & \text{if } E \geq \frac{1}{2}\sum_{i\in N} c_i \end{cases}$
where λ and μ are chosen so that $\sum_{i\in N} T_i(E,c) = E$.

Apart from its justification as the consistent extension of the *constested garment rule,* the rationale of the *Talmud rule* is based in the psychological principle of "more than half is like the whole, whereas less than a half is like nothing". Thus, it seems natural to look at the size of the *awards* when they are below half of the claim, and to look at the size of the *losses* above half of the claim. This, together with a principle of equal treatment, in which all agents are at the same side of the *half-way psychological watershed* amounts to construct the Talmud rule.

As a consequence, we have the following result:

Theorem 7.2. *The Talmud rule is the only rule in $\mathbb{B}$ satisfying consistency, self-duality and composition from minimal rights.*

Because of self-duality, and the dual relationship between composition from minimal rights and independence of claims truncation, the following alternative characterization of the Talmud solution is obtained:

Theorem 7.3. *The Talmud rule is the only rule in* $\mathbb{B}$ *satisfying consistency, self-duality and independence of claims truncation.*

8. Variants

The principles of independence of claims truncation and composition from minimal rights permit to obtain some additional rules, when combined with the solutions already discussed.

By applying independence of claims truncation we may consider solving bankruptcy problems in two steps: First, we substitute the original claims by the truncated claims, and then we apply the corresponding solution to the *truncated problem.* In this way, the following variants appear:

Truncated Proportional rule, TP: For all $N \in \mathbb{F}$ and all $(E,c) \in \mathbb{B}^N$, $TP(E,c) = P(E,c^T)$

Truncated Constrained Equal-Losses rule, TCEL: For all $N \in \mathbb{F}$ and all $(E,c) \in \mathbb{B}^N$, $TCEL(E,c) = CEL(E,c^T)$.

Similarly, if we insist on *composition from minimal rights* to be satisfied, we may also consider a second variant of previous solutions by solving the problem into two steps, but now, starting by giving each agent first her minimal right. In this way, the *adjusted* variants of previous solutions appear. Formally,

Adjusted Proportional rule, AP: For all $N \in \mathbb{F}$ and all $(E,c) \in \mathbb{B}^N$, $AP(E,c) = m(E,c) + P\left[E - \sum_{i\in N} m_i(E,c), c - m(E,c)\right]$.

Adjusted Constrained Equal Awards rule, ACEA: For all $N \in \mathbb{F}$ and all $(E,c) \in \mathbb{B}^N$, $ACEA(E,c) = m(E,c) + CEA\left[E - \sum_{i\in N} m_i(E,c), c - m(E,c)\right]$.

These solutions satisfy some interesting properties: by construction, TP and $TCEL$ satisfy *independence of claims truncation.* Furthermore, $TCEL$ also satisfies *composition from minimal rights.* AP and $ACEA$ satisfy *composition from minimal rights.* $ACEA$ also satisfies *independence of claims truncation.* Additionally, all four solutions satisfy *equal treatment of equals, estate monotonicity,* and *continuity with respect to the estate.* AP and TP are dual solutions, and $ACEA$ and $TCEL$ are also dual solutions.

- **Claim 10.** For the two-person case, $TCEL$ and $ACEA$ coincide with the contested garment solution.
- **Claim 11.** $TCEL$ and $ACEA$ fail to satisfy consistency
- **Claim 12.** AP and TP fail to satisfy consistency. Furthermore, there is no consistent extension of the two-person version of AP and TP.

The interest of these extensions, thus, is seriously affected by the lack of consistency.

9. Noncooperative support of rules

In previous Sections, we defended different bankruptcy rules from an axiomatic perspective. If agents agree on a particular set of properties, they also agree in choosing a particular rule to solve their problem. The planner's job consists of convincing the agents on which procedure is better, taking into account the characteristics of the particular problem at hand.

In this Section, we consider a different approach. For each problem, we shall isolate a multi-valued solution concept, the *set of fair allocations.* Given a problem (E, c), $\mathbb{F}(E, c)$ contains all distributions of E such that no agent gets more than her claim, and agents with higher claims receive higher awards and suffer higher losses.

To isolate a single outcome in $\mathbb{F}(E, c)$, we consider a noncooperative approach. Different agents may have different opinions on how the liquidation value of the firm should be distributed, and then, we allow agents to propose different shares, within the set of fair allocations. This can be understood as that agents choose *rules* in a certain family, and apply the chosen rule to the problem at hand.

Two *natural procedures* to solve the differences are analyzed. The *diminishing claims procedure,* proposed by Chun (1989) for surplus sharing problems, and the *unanimous concessions procedure* (Herrero (1998c)).

In the *diminishing claims procedure,* if at the first step agents disagree on the proposed shares, the initial claims are truncated by the maximal amount proposed to each agent, and the new problem is then presented to the agents.

They continue proposing shares, up to the moment they agree, or otherwise, the limit of the procedure is proposed as the solution of the conflict.

In the *unanimous concessions procedure*, if at the first step agents disagree on the proposed shares, the minimal amounts proposed to each agent are assigned to them. Then, the agents face the residual problem, made out of the remaining value and where the claims are adjusted down by the amounts just given. They continue proposing shares for the residual problem, up to the moment they agree, or otherwise, the limit of the procedure is proposed as the solution of the conflict.

Each of previous procedure induces a game, where agents choose strategically fair allocations. Our main results are the following: In the game induced by the *diminishing claims* procedure, in any Nash equilibrium of the game, the agents receive the share recommended by the *constrained equal-awards rule.* In the game induced by the *unanimous concessions* procedure, in any Nash equilibrium of the game, the agents receive the share recommended by the *constrained equal-losses rule.*

9.1. Fair allocations

Let N be a set of agents, and $(E, c) \in \mathbb{B}^N$,and let $x \in \mathbb{R}^N$ such that $\sum x_i = E$ an allocation of E. x is **fair** if (i) for all $i \in N$, $0 \leq x_i \leq c_i$, and (ii) for all $i, j \in N$, if $c_i \geq c_j$ then $x_i \geq x_j$ and $c_i - c_j \geq x_i - x_j$.

In a fair allocation, we allocate a feasible amount of money, with the condition that no agent receives more than her claim, and no agent is alloted a negative award. Furthermore, we respect the order of the claims: agents with higher claims receive higher awards and face higher losses.

Given a problem $(E, c) \in \mathbb{B}^N$, let $\mathbb{F}(E, c)$ denote the set of *fair allocations.*

- **Claim 13.** For all $(E, c) \in \mathbb{B}^N$, the set $\mathbb{F}(E, c)$ has a particular structure: it is a *convex polihedral* with at most $|N|$ vertices.

Recall that a **rule** is a mapping $r : \mathbb{B} \to \mathbb{R}^N$ that associates with every $(E, c) \in \mathbb{B}$ a unique feasible allocation $r(E, c)$, to be interpreted as a desirable way of solving the problem. We shall say that a rule r is **fair** if for all $(E, c) \in \mathbb{B}$, $r(E, c)$ is fair.

- **Claim 14.** CEA, CEL and P are fair rules. Furthermore, for all $(E, c) \in \mathbb{B}$, $CEA(E, c)$ and $CEL(E, c)$ are vertices of $\mathbb{F}(E, c)$.

- **Claim 15.** $CEA(E,c)$ is the best preferred element in $\mathbb{F}(E,c)$ for the agent with the smallest claim, whereas $CEL(E,c)$ is the best preferred element in $\mathbb{F}(E,c)$ for the agent with the highest claim. The best preferred element in $\mathbb{F}(E,c)$ for any other agent always corresponds to some vertix of $\mathbb{F}(E,c)$.

9.2. Procedures to solve discrepancies

Different agents may have different ideas on the way the available amount of money in a problem should be allocated. Thus, when facing a problem, they may propose different allocations, and possibly, they do not find a unanimous compromise. In that case, some *natural procedures* come to mind in order to solve the discrepancies. In these procedures, given a problem, each agent proposes a division of the estate, and the procedure selects a particular division of the estate. Thus, once a procedure is chosen, it induces a game: agents may choose "rules" strategically to obtain a division of the estate as favorable to them as possible.

Next, we present two *natural procedures:* the **diminishing claims procedure** and the **unanimous concessions procedure:**

Diminishing claims procedure (dc) (Chun, 1989): Let $(c,E) \in \mathbb{B}$ be given. At the first stage, each agent chooses a *rule* $f^i \in \mathbb{F}$. Let $f = (f^i)$ be the profile of rules reported. The division proposed by the diminishing claims procedure, $dc[f,(c,E)]$ is obtained as follows:

Step 1. Let $c^1 = c$. For all $i \in N$, calculate $f^i(c^1,E) \in FE(c^1,E)$. If all coincide, then, $dc[f,(c,E)] = f^i(c^1,E)$. Otherwise, go to the next step.

Step 2. Let $c_i^2 = \max_{j \in N} f_i^j(c^1,E)$. For all $i \in N$, calculate $f^i(c^2,E) \in FE(c^2,E)$. If all coincide, then $dc[f,(c,E)] = f^i(c^2,E)$. Otherwise, go to the next step.

Step k+1. Let $c_i^{k+1} = \max_{j \in N} f_i^j(c^k,E)$. For all $i \in N$, calculate $f^i(c^{k+1},E) \in FE(c^{k+1},E)$. If all coincide, then $dc[f,(c,E)] = f^i(c^{k+1},E)$. Otherwise, go to the next step.

If previous process does not terminate in a finite number of steps, then

Limit case. Compute $\lim_{t\to\infty} f(c^t,E)$. If it converges to an allocation x such that $\sum x_i \leq E$, the allocation $x = dc[f,(c,E)]$. Otherwise, $dc[f,(c,E)] = 0$.

Unanimous concessions procedure (u) (Herrero, 1998): Let $(c,E) \in \mathbb{B}$ be given. At the first stage, each agent chooses a rule $f^i \in \mathbb{F}$. Let $f = (f^i)$ be the profile of rules reported. $u[f,(c,E)]$ is obtained as follows:

Step 1. Let $c^1 = c$. For all $i \in N$, calculate $f^i(c^1, E)$. If all them coincide, then, $u[f, (c, E)] = f^i(c^1, E)$. Otherwise, go to the next step.

Step 2. Let $m_i^1 = \min f_i^j(c^1, E)$, $c^2 = c^1 - m^1$, and $E^2 = E - \sum m_i^1$. For all $i \in N$, calculate $f^i(c^2, E^2)$. If all them coincide, then $u[f, (c, E)] = m^1 + f^i(c^2, E^2)$. Otherwise, go to the next step.

Step k+1. Let $m_i^k = \min f_i^j(c^k, E^k)$, $c^{k+1} = c^k - m^k$, and $E^{k+1} = E^k - \sum m_i^{k-1}$. For all $i \in N$, calculate $f^i(c^{k+1}, E^{k+1})$. If all them coincide, then $u[f, (c, E)] = m^1 + \cdots + m^k + f^i(c^{k+1}, E^{k+1})$. Otherwise, go to the next step.

If previous process does not terminate in a finite number of steps, then

Limit case. Compute $\lim_{t\to\infty}(m^1 + \cdots + m^k + \cdots)$. If it converges to an allocation x such that $\sum x_i \leq E$, the allocation $x = u[f, (c, E)]$. Otherwise, $u[f, (c, E)] = 0$.

In the *diminishing claims procedure,* once agents choose their preferred rules, we sequentially reduce agents' claims by substituting them by the highest amount assigned to every agents by the chosen rules. If the process ends in a feasible allocation in a finite number of steps, this allocation is chosen as the solution of the conflict. If not, but this process has a feasible limit, then this limit is the solution of the problem. Otherwise, nobody gets anything.

In the *unanimous concessions procedure,* once agents choose they preferred rules, we sequentially allocate parts of the estate to the different agents by assigning each of them the minimum amount assigned to him. If the process ends in a finite number of steps, this allocation is chosen as the solution of the conflict. If not, but this process has a feasible limit, then this limit is the solution of the problem. Otherwise, nobody gets anything.

Let us now analyze the behavior of the previous procedures when agents act strategically. Each procedure induces a game, where the set of players is N, and the strategies for all players are rules in $\mathbb{F}$. Let Γ^{dc} be the game induced by the *diminishing claims* procedure and let Γ^u be the game induced by the *unanimous concessions* procedure.

Now, we obtain the following results:

Theorem 9.1. *(Chun, 1989) In the game Γ^{dc}, the constrained equal-awards rule is a dominant strategy for the agent with the smallest claim. In any Nash equilibrium of Γ^{dc}, the outcome corresponds to the recommendation of the constrained equal-awards rule.*

Theorem 9.2. *(Herrero, 1998) In the game Γ^u, the constrained equal-loss rule is a dominant strategy for the agent with the highest claim. In any Nash equilibrium of Γ^u, the outcome corresponds to the recommendation of the constrained equal-losses rule.*

10. Final Remarks

10.1. Additional comments and open problems

1. Thinking of the psychological principle behind the *Talmud rule* "more than half is like the whole, whereas less than a half is like nothing", and in the case we are mainly worried about losses, it may be sensible to take the oppossite view, namely, to look at the size of the *losses* from half of the claim, and to look at the size of the *awards* above half of the claim. This, together with a principle of equal treatment, in which all agents are at the same side of the *half-way psychological watershed* amounts to construct a different rule, which may be interpreted as the *mirror image of the Talmud rule.* This rule can be defined in the following way:

$$MT(E,c) = \begin{cases} CEL(E, \frac{1}{2}c) & 0 \leq E \leq \frac{1}{2}\sum_i c_i \\ \frac{1}{2}c + CEA(E - \frac{1}{2}\sum_i c_i, \frac{1}{2}c) & \frac{1}{2}\sum_i c_i \leq E \leq \sum_i c_i \end{cases}$$

This rule satisfies *equal treatment of equals, consistency, self-duality,* and weaker versions of both *preeminence* and *sustainability.* A combination of this rule and the Talmud rule, was introduced by Chun and Thomson (1998), in the following way:

$$CT(E,c) = \begin{cases} CEA(E, \frac{1}{2}c) & 0 \leq E \leq \frac{1}{2}\sum_i c_i \\ \frac{1}{2}c + CEA(E - \frac{1}{2}\sum_i c_i, \frac{1}{2}c) & \frac{1}{2}\sum_i c_i \leq E \leq \sum_i c_i \end{cases}$$

2. The *diminishing claims procedure* and the *unanimous concessions procedure* provide non-cooperative support to the *constrained equal awards* and the *constrained equal-losses rules,* respectively. They can also be interpreted as *natural* non-cooperative ways of making selections of the *fair and efficient* multivalued solution.

As was previously mentioned, the allocation recommended by the *constrained equal-awards rule* is the best preferred by the agent with the smallest claim in the set of *fair and efficient* allocations. Similarly, the recommendation of the *constrained equal-losses rule* is the best preferred by the agents with the highest claim. For each agent i, the best preferred option of agent i corresponds to one of the vertices of the set of *fair and efficient* allocations. In this allocation, awards of agents with claims higer than $i's$ are as equal as possible, and losses of agents with claims below $i's$ are as equal as possible. In the same way as the *constrained equal-awards* and the *constrained equal-losses* can be interpreted as *dictatorial selections* of the *fair and efficient* solution, the rule given to agent in *position* k its preferred share, could be interpreted as a *positional dictatorial* selection of the *fair and efficient* solution. Properties and characterizations of those rules are open problems, as well as the possibility of designing procedures supporting them from a noncooperative point of view.

Once we look at our main results, other question comes to mind. In a society with more than two agents, which of the two previous procedures would receive more support? The answer is clearly related with the share of the median claimant in the recommendations made by the *constrained equal awards* and the *constrained equal-losses,* respectively. In the 3-person case, for example, the *diminishing claims procedure* would be preferred by the median claimant either if the estate is very small or very close to the sum of the claims, whereas the *diminishing claims procedure* is more likely to be preferred by the median claimant for intermediate values of the estate.

3. An interesting problem is that of manipulating the outcome of a particular rule by merging or splitting agents, namely when agents pretend to be different, by putting together their claims or else an agent appearing as several agents by splitting her claim into two pieces. The *proportional rule* can be characterized by Non advantageous Merging and Non Advantageous Splitting, simultaneously [see de Frutos, 1994). The *constrained equal-awards rule* satisfies *non advantageous merging* and it fails to satisfy *non advantageous splitting,* whereas for the *constrained equal losses rule* the opposite is true. These two properties are dual to each other. The question of finding further characterization results by using this properties is an openproblem.

4. We chose the axiomatic approach as a way of selecting clear-cut procedures to solve these type of situations. There are several reasons why we addressed the problem this way.

In the *game-theoretic approach* the traditional way of associating a TU cooperative game to a bankruptcy problem is by defining the characteristic function in the following way: For all $(E, c) \in \mathbb{B}^N$,

$$v_{(E,c)}(T) = \max\{0, E - \sum_{i \in N-T} c_i\}$$

Previous characteristic function defines a TU-game, and then cooperative solution concepts are applied in order to solve the bankruptcy problem (E, c).

Note that if we consider for all $i \in N$, $c_i^T = \min\{c_i, E\}$, it follows that $v_{(E,c)} = v_{(E,c^T)}$. As a consequence, the game theoretic approach imposes the condition of *independence of claims truncation*. This is not satisfactory in general, since in some cases, as we explained above, claims must be considered in full.

10.2. Related literature

An excellent survey of the literature on bankruptcy problems, solution concepts, applications and properties is Thomson (1995). A new version of this paper is Thomson (1998).

Two papers deserve to be mentioned in dealing with the introduction of the axiomatic method in analyzing bankruptcy problems: O'Neill (1982) can be quoted as the first systematic analysis. Aumann and Maschler (1985) initiated the increasing literature on the subject and for the first time thought of considering a variable number of agents and clarified the consistency principle. They also introduced the idea of *duality*, that we exploit here substantially.

The approach in Section 9 is closely related with that adopted by van Damme (1986) in bargaining. The *diminishing claims procedure* can be looked at as a modification of van Damme's suitable for the type of problems at hand.

Marco, Peris and Subiza (1996) present a modification of van Damme's procedure. The *unanimous concessions procedure* can be looked at as a modification of Marco et al's suitable for bankruptcy problems. Criticisms and modifications of Van Damme and Marco Peris and Subiza are in Naeve-Steinweg (1997). A major difference between them, however is that in our case, the procedures can be used for any number of agents, while the procedures for bargaining problems are only suitable for the 2-person case.

Non-cooperative foundations of bankruptcy rules also appear in O'Neill (1982), and Dagan, Serrano and Volij (1997).

11. Appendix

Proof of Claim 7.

The Talmud rule satisfies *independence of claims truncation* and *composition from minimal rigths*, and it does not fulfil the other two properties. Consider now the following two rules: F assigns identical quantities to all agents with the minimum claim, up to the moment in which this claim is fully honored; then, any remaining amount is equally shared among the agents with the second minimal claim, up to when their claims are honored, etc. G starts by assigning any amount to the agents with the highest claim, up to when this claim is fully honored; then, any additional amount goes to the agents with the second highest claim, up to when this claim is fully honored, etc. F and G are dual rules, F satisfies *sustainability,* and does not satisfy *independence of claims truncation,* and G satisfies *preeminence* and does not satisfy *composition from minimal rights.* ■

Proof of Theorem 3:

Given $b = (E, c) \in \mathbb{B}$, let $\delta^1(c) = \max_{j \in N} c_j$, $N_1(c) = \{i \in N \mid c_i = \delta^1(c)\}$, and $n_1(c) = |N_1(c)|$. Consider now the following lemma:

Lemma. *Let F be a rule satisfying equal treatment of equals, composition, and preeminence, and let $b = (E, c) \in \mathbb{B}$ be such that for all $j \in N \backslash N_1(c)$ $\delta^1(c) \geq c_j + \frac{E}{n_1(c)}$. Then, for all $i \in N \backslash N_1(c)$, $F_i(b) = 0$.*

Proof: 1. Let $\delta^2(c) = \max_{j \in N \backslash N_1(c)} c_j$. Obviously, if for all $j \in N \backslash N_1(c)$, $\delta^1(c) \geq c_j + \frac{E}{n_1(c)}$, then $\delta^1(c) \geq \delta^2(c) + \frac{E}{n_1(c)}$.

Now, let $c^1 = c$, $E_1 = E$, and $b_1 = (E_1, c^1)$. Also, let $E_2 = \frac{1}{n_1(c)} E_1$ and $b_2 = (E_2, c^1)$. Since $\delta^1(c^1) \geq \delta^2(c^1) + \frac{E_1}{n_1(c^1)}$, by *preeminence* and *equal treatment of equals*, for all $i \in N_1(c^1)$, we have $F_i(b_2) = \frac{1}{n_1(c^1)} E_2$.

Let $\widetilde{E}_2 = E_2 - E_1$, $c^2 = c^1 - F(b_2)$, and $\widetilde{b}_2 = (\widetilde{E}_2, c^2)$. By *composition,* $F(b_1) = F(b_2) + F(\widetilde{b}_2)$.

2. Note that $N_1(c^2) = N_1(c^1)$ and $\delta^1(c^2) = \delta^1(c^1) - \frac{1}{n_1(c^1)} E_1 \geq \delta^2(c^1) + \frac{1}{n_1(c^1)} E_1 - \frac{1}{n_1(c^1)} E_2 = \delta^2(c^1) + \frac{1}{n_1(c^1)} \widetilde{E}_2$.

Let $E_3 = \frac{1}{n_1(c^1)} \widetilde{E}_2$ and $b_3 = (E_3, c^2)$. By *preminence* and *equal treatment of equals*, for all $i \in N_1(c^2)$, we have $F_i(b_3) = \frac{1}{n_1(c^1)} E_3$.

Let $\widetilde{E}_3 = \widetilde{E}_2 - E_3$, $c^3 = c^2 - F(b_3)$, and $\widetilde{b}_3 = (\widetilde{E}_3, c^3)$. By *composition,* $F(\widetilde{b}_2) = F(b_3) + F(\widetilde{b}_3)$.

k. Suppose that $\widetilde{b}_k = (\widetilde{E}_k, c_k)$ has been defined. Note that $N_1(c^k) = N_1(c^{k-1})$ and $\delta^1(c^k) = \delta^1(c^{k-1}) - \frac{1}{n_1(c^1)} E_k \geq \delta^2(c^1) + \frac{1}{n_1(c^1)} E_{k-1} - \frac{1}{n_1(c^1)} E_k = \delta^2(c^1) + \frac{1}{n_1(c^1)} \widetilde{E}_k$.

Let $E_{k+1} = \frac{1}{n_1(c)} \widetilde{E}_k$ and $b_{k+1} = (E_{k+1}, c^k)$. By *preeminence* and *equal treatment of equals*, for all $i \in N_1(c^k)$, we have $F_i(b_{k+1}) = \frac{1}{n_1(c^1)} E_{k+1}$.

Let $\widetilde{E}_{k+1} = \widetilde{E}_k - E_{k+1}$, $c^{k+1} = c^k - F(b_{k+1})$, and $\widetilde{b}_{k+1} = (\widetilde{E}_{k+1}, c^{k+1})$. By *composition,* $F(\widetilde{b}_k) = F(b_{k+1}) + F(\widetilde{b}_{k+1})$.

Observe that $E_{k+1} = \left(\frac{n_1(c^1)-1}{n_1(c^1)}\right)^{k-1} E_2$. Consequently, $\lim_{k\to\infty}(E_2 + \cdots + E_k) = E$. Therefore, by *composition*, and since, for all $i \in N_1(c^1)$, F_i is *continuous with respect to* E,

$F_i(b) = \lim_{k\to\infty}[F_i(b_2) + \cdots + F_i(b_k)] =$

$= \lim_{k\to\infty} \left[1 + \frac{n_1(c^1)-1}{n_1(c^1)} + \left(\frac{n_1(c^1)-1}{n_1(c^1)}\right)^2 + \cdots + \left(\frac{n_1(c^1)-1}{n_1(c^1)}\right)^{k-1}\right] = E_2$

$= \frac{E}{n_1(c)}$. □

Proof of Theorem 3: The *constrained equal-losses rule* satisfies all the properties. Let us prove the converse implication.

Let $\delta^1(c) = \max_{j\in N} c_j$, $N_1(c) = \left\{i \in N \mid c_i = \delta^1(c)\right\}$, and $n_1(c) = |N_1(c)|$. Similarly, let $\delta^2(c) = \max_{j\in N\setminus N_1} c_j$, $N_2(c) = \left\{i \in N \mid c_i = \delta^2(c)\right\}$, and $n_2(c) = |N_2(c)|$, and so forth.

(i) Let $0 \leq E \leq n_1(c)[\delta^1(c) - \delta^2(c)]$. Then, for all $j \in N\setminus N_1(c)$, we have $\delta^1(c) - \frac{E}{n_1(c)} \geq c_j$. By Lemma 1, for all $i \in N_1(c)$, we have $F_i(E,c) = \frac{E}{n_1(c)}$. Therefore, $F(b) = CEL(b)$.

(ii) Let $n_1(c)[\delta^1(c) - \delta^2(c)] < E \leq n_1(c)\delta^1(c) - n_2(c)\delta^2(c) - [n_1(c) + n_2(c)]\delta^3(c)$.

Let $E_1 = n_1(c)[\delta^1(c) - \delta^2(c)]$, $b_1 = (E_1, c)$, and $b_2 = [c - F(b_1), E - E_1]$. By *composition*, $F(b) = F(b_1) + F(b_2)$.

By (i), $F(b_1) = CEL(b_1)$, namely, for all $i \in N_1(c)$, $F_i(b_1) = \frac{E_1}{n_1(c)}$, and otherwise $F_i(b_1) = 0$. Consequently, for all $i \in N_1(c)$, $c_i - F_i(b_1) = \delta^1(c) - \frac{E_1}{n_1(c)} = \delta^2(c)$, and otherwise, $c_i - F_i(b_1) = c_i$.

Let $c' = c - F(b_1)$. That is, for all $i \in N_1(c)$, $\delta^1(c') = \delta^1(c) - F_i(b_1)$, for all $j \in N_2(c)$, $\delta^2(c') = \delta^2(c) - F_j(b_1)$, etc. By construction, $\delta^1(c') = \delta^2(c') > \delta^3(c') \geq \cdots \geq \delta^n(c')$.

Moreover, for all $j \in N\setminus[N_1(c') \cup N_2(c')]$,

$$c_j - F_j(b_1) \leq \delta^1(c') - \frac{E - E_1}{n_1(c) + n_2(c)}.$$

Again, by (i), for all $j \in N\setminus[N_1(c') \cup N_2(c')]$, we have $F_j(b_2) = 0$, and for all $i \in N_1(c') \cup N_2(c')$, we have $F_i(b_2) = \frac{E-E_1}{n_1(c)+n_2(c)}$ namely, for all $i \in N_1(c')$,

$F_i(b) = \frac{E_1}{n_1(c)} + \frac{E-E_1}{n_1(c)+n_2(c)}$; for all $i \in N_2(c')$, $F_i(b) = \frac{E-E_1}{n_1(c)+n_2(c)}$, and for all other $i \in N$, $F_i(b) = 0$. Consequently, $F(b) = CEL(b)$.

We repeat the previous procedure until all possible values of the estate smaller than or equal to $\sum_i c_i$ are covered. □

Proof of Theorem 5:

The CEL rule satisfies all the properties. Conversely, let F be a rule that satisfies all the properties. Let us show that $F = CEL$.

Let $(c, E) \in \mathbb{C}$. Let $C(c) = \sum_i c_i$, $\delta(c) = \min_i c_i$, and $D(c) = C(c) - \delta(c) = \max_{i \in N}\{\sum_{j \in N\setminus\{i\}} c_j\}$. Also, let $m_i(c, E) = \max\{0, E - \sum_{j \neq i} c_j\}$

Case 1. $C(c) - \delta(c) \leq E$.

By *composition from minimal rights,* $F(c, E) = m(c, E) + F[c - m(c, E), E - \sum_i m_i(c, E)]$.

Note that for all $i \in N$, $m_i(c, E) = E - C(c) + c_i$ and $c_i - m_i(c, E) = C(c) - E$. Thus, by *equal treatment of equals,* for all $k \in N$, we have $F_k[c - m(c, E), E - \sum_i m_i(c, E)] = \frac{n-1}{n}[C(c) - E]$. Thus, for all $i \in N$, $F_i(c, E) = c_i + E - C(c) + \frac{n-1}{n}[C(c) - E] = c_i - \frac{C(c)-E}{n} = CEL_i(c, E)$.

Case 2. $C(c) - n\delta(c) < E \leq C(c) - \delta(c)$.

Without loss of generality, assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. Thus, $\delta(c) = c_1$, and $D(c) = c_2 + \cdots + c_n$.

Step 1. Note that $(c, D(c)) \in \mathbb{C}$ and by construction, it is covered by Case 1. Thus, for all $i \in N$, $F_i(c, D(c)) = CEL_i(c, D(c)) = c_i - \frac{\delta(c)}{n} = c^1_i$.

By *path independence,* $F(c, E) = F(c^1, E)$.

Now, note that $c^1_1 \leq c^1_2 \leq \cdots \leq c^1_n$. Thus, $\delta(c^1) = c^1_1 = \frac{n-1}{n}\delta(c)$, and $D(c^1) = C(c^1) - \delta(c^1) = C(c) - \delta(c) - \frac{n-1}{n}\delta(c)$.

Now, two possibilities are open: either $D(c^1) \leq E$, or $D(c^1) > E$.

If $D(c^1) \leq E$,then (c^1, E) is covered by Case 1. Thus, $F(c^1, E) = CEL_i(c^1, E)$. Since CEL satisfies *path independence,* $CEL(c^1, E) = CEL(c, E)$, and thus, $F(c, E) = CEL(c, E)$.

If $D(c^1) > E$, go to step 2.

Step k. Note that $(c^{k-1}, D(c^{k-1})) \in \mathbb{C}$ and by construction, it is covered by Case 1. Thus, for all $i \in N$, $F_i(c^{k-1}, D(c^{k-1})) = CEL_i(c^{k-1}, D(c^{k-1})) = c^{k-1}_i - \frac{\delta(c^{k-1})}{n} =$
$c_i - \frac{\delta(c)}{n}\left[1 + \frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^{k-1}\right] = c^k_i$.

By *path independence,* $F(c^{k-1}, E) = F(c^k, E)$. Note that $c^k_1 \leq c^k_2 \leq \cdots \leq c^k_n$. Thus, $\delta(c^k) = c^k_1 = \left(\frac{n-1}{n}\right)^k \delta(c)$ and $D(c^k) = C(c^k) - \delta(c^k) = C(c) - \delta(c)\left[1 + \frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots\right.$

Now, two possibilities are open: either $D(c^k) \leq E$, or $D(c^k) > E$.

If $D(c^k) \leq E$, then (c^k, E) is covered by Case 1. Thus, $F(c^k, E) = CEL_i(c^k, E)$. Since CEL satisfies *path independence*, $F(c, E) = CEL(c, E)$.

If $D(c^k) > E$, go to step k+1...

We claim that for some $k \in \mathbb{N}$, $D(c^k) \leq E$. Suppose not. Then, for all $k \in \mathbb{N}$, $D(c^k) > E$. Thus, $\lim_{k\to\infty} D(c^k) \geq E$. That is,

$$E \leq C(c) - \delta(c) \lim_{k\to\infty} \left[1 + \frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^k\right] = C(c) - n\delta(c),$$

which contradicts the fact that (c, E) is covered by Case 2. Thus, $F(c, E) = CEL(c, E)$.

Case 3. $E = C(c) - n\delta(c)$.

Let $\{E_k\}$ be a sequence such that $E_k > E_{k+1}$, and $\{E_k\} \to E$. Thus, the sequence of problems $\{(c, E^k)\}$ converges to (c, E). All problems (c, E^k) in the sequence are covered by Case 2. Thus, for all $k \in \mathbb{N}$, $F(c, E^k) = CEL(c, E^k)$. By *path independence*, F is *continuous with respect to the estate.* Thus, $F(c, E) = \lim_{k\to\infty} F(c, E^k) = CEL(c, E)$.

Case 4. $E < C(c) - n\delta(c)$.

Let $N_1(c) = \{i \in N \mid c_i = \delta(c)\}$, and let $n_1 = |N_1(c)|$. By *path independence*, $F(c, E) = F(F(c, C(c) - n\delta(c)), E)$. For all $i \in N_1(c)$, $F_i(c, C(c) - n\delta(c)) = 0$, and for any other $i \in N \backslash N_1(c)$, $F_i(c, C(c) - n\delta(c)) = c_i - \delta(c)$. Let $d = F(c, C(c) - n\delta(c))$. Then, $F(c, E) = F(d, E)$. Let $\delta_2(d) = \min\{d_i \mid d_i > 0\}$. We consider several subcases:

4.a. $C(d) - \delta_2(d) \leq E < C(c) - n\delta(c) = C(d)$.

For all $i \in N \backslash N_1(c)$, $m_i(d, E) = d_i + E - C(d)$, and thus, $E - \sum_{i\in N} m_i(d, E) = (n - n_1 - 1)[C(d) - E]$. By *equal treatment of equals*, for all $i \in N \backslash N_1$, $F_i(d, E) = d_i - C(d) + E + \frac{n-n_1-1}{n_1}[C(d) - E] = CEL_i(d, E)$.

4.b. $C(d) - (n - n_1)\delta_2(d) < E < C(d) - \delta_2(d)$.

Let $D_2(d) = C(d) - \delta_2(d)$, and consider the problem $(d, D_2(d))$. Note that $F(d, D_2(d)) = CEL(d, D_2(d)) = d^1$. By *path independence*, $F(d, E) = F(d^1, E)$. Now, two options are open: either $C(d^1) - \delta_2(d^1) \leq E$, or the opposite. Then, we may repeat the procedure of Case 2, by only considering the agents in $N \backslash N_1$.

4.c. $E = C(d) - (n - n_1)\delta_2(d)$. Repeat the procedure in Case 3 only considering the agents in $N \backslash N_1$.

4.d. From then on, repeat the procedure, considering at any step only the agents in $N \backslash N_1 \cup \cdots \cup N_k$, until all possible values of E are covered. $\square$

Proof of Claim 9:

Obviously, P satisfies both properties. Let F be a rule that satisfies *self-duality* and *composition.* Then, it also satisfies *path-independence.*

Let $(E, c) \in \mathbb{B}^N$, and let $C = \sum_{i \in N} c_i$. By *self-duality,* $F(\frac{1}{2}C, c) = \frac{1}{2}c$. By *path-independence,* $F(\frac{1}{4}C, c) = F(\frac{1}{4}C, \frac{1}{2}c) = \frac{1}{4}c$. By *self-duality,* $F(\frac{3}{4}C, c) = \frac{3}{4}c$. Similarly, we obtain that for all $m, n \in \mathbb{N}$, with $m < n$, $F(\frac{m}{n}C, c) = \frac{m}{n}c$. Finally, and since *composition* implies *continuity with respect to the estate,* for all $0 \leq \lambda \leq 1$, $F(\lambda C, c) = \lambda c$. ■

Proof of Theorem 8:

First, consider the following lemma:

Lemma 11.1. *The contested garment rule is the only two-person rule satisfying symmetry, estate monotonicity, independence of claims truncation and composition from minimal rights.*

Proof :

Let $N \in \mathbb{F}$ be such that $N = \{i, j\}$, and consider a problem $(E, c) \in \mathbb{B}^N$. If $c_i = c_j$, by *symmetry,* $F(E, c) = G(E, c)$. Let us consider the case $c_i \neq c_j$. Without loss of generality, assume that $c_i < c_j$.

(1) $0 \leq E \leq c_i$. By *independence of claims truncation* and *symmetry,* $F_i(E, c) = F_j(E, c) = \frac{E}{2}$. Consequently, $F(E, c) = CEA(E, c) = G(E, c)$.

(2) $c_j \leq E \leq C = c_i + c_j$. By *composition from minimal rigths,*

$$\begin{aligned} F(E, c) &= m(E, c) + F\left[E - \sum_k m_k(c, E), c - m(E, c)\right] = \\ &(E - c_j, E - c_i) + F\left[C - E, (C - E, C - E)\right]. \end{aligned}$$

By symmetry,

$$F\left[C - E, (C - E, C - E)\right] = \left(\frac{1}{2}(C - E), \frac{1}{2}(C - E)\right).$$

and consequently,

$$F(E, c) = (E - c_2, E - c_1) + \left(\frac{1}{2}(C - E), \frac{1}{2}(C - E)\right) = CEL(E, c) = G(E, c).$$

(3) $c_i \leq E \leq c_j$. Note that $F_i(c_1, c) = F_i(c_2, c) = \frac{1}{2}c_i$. By *estate monotonicity*, for all E such that $c_i \leq E \leq c_j$, it happens that $F_i(E, c) = \frac{1}{2}c_i$. Consequently, $F_j(c, E) = E - \frac{1}{2}c_i$, and thus, $F(E, c) = \left(\frac{1}{2}c_i, E - \frac{1}{2}c_i\right) = G(E, c)$. ■

Proof of Theorem 8:

The *contested garment rule* satisfies both properties. Let F be a solution satisfying *self-duality* and *composition from minimal rights*. By *self-duality*, it also satisfies *independence of claims truncation*. Let us see that also *symmetry* and *estate monotonicity* are fulfilled.

Suppose that F is not symmetric. Then, there is a set $N = \{i, j\}$, and a symmetric problem $(E, c) \in \mathbb{B}^N$, with $c_i = c_j$, such that $F_i(E, c) \neq F_j(E, c)$.

Suppose that $E < \frac{1}{2}C. = c_i$. By *independence of claims truncation,* $F(E, c) = F(E, c^T) = F[E, (E, E)] = \frac{1}{2}(E, E)$, by *self-duality,* against the hypothesis of $F_i(E, c) \neq F_j(E, c)$.

If $E > \frac{1}{2}C$, by self-duality, also there will be a violation of symmetry for $E' = (C - E) < \frac{1}{2}C$, and we just proved that it cannot be the case. Consequently, F is symmetric.

Let us now prove that F satisfies estate monotonicity. If $0 \leq E \leq \min\{c_i, c_j\}$, by *independence of claims truncation* and *symmetry,* $F(E, c) = \frac{1}{2}(E, E)$. By *self-duality,* if $\max\{c_i, c_j\} \leq E \leq c_i + c_j$, $F(E, c) = CEL(E, c)$. Consider then the possibility of having some vector of claims, $c = (c_i, c_j)$, with $c_i < c_j$, and a estate E, such that $c_i < E < c_j$, and $F_i(E, c) \neq \frac{1}{2}c_i$. Furthermore, by *self-duality,* $E \neq \frac{1}{2}[c_i + c_j]$.

Suppose that $E < \frac{1}{2}[c_i + c_j]$. By *independence of claims truncation,* $F(E, c) = F(E, c^T)$, where $c^T = (c_i, E)$. But then, $\max\{c_i^T, c_j^T\} = E$, and consequently, $F_i(E, c^T) = \frac{1}{2}c_i$, against the hypothesis. By *self-duality,* $F_i(E, c) = \frac{1}{2}c_i$, for $\frac{1}{2}[c_i + c_j] < E < c_j$. Thus, F satisfies estate monotonicity.

Consequently, by Lemma 1, we get the desired result. ■

Properties in Theorem 9 are independent:

In order to prove that independence, we provide with examples of solutions fulfilling all but one properties at any time. We mention the property that is violated:

Consistency: The truncated and adjusted proportional solution

Self-duality: The constrained equal-loss solution

Composition from minimal rights: The proportional solution. ■

Proof of Claim 10:

It follows from the fact that they satisfy simultaneouslsy *composition from minimal rights* and *independence of claims truncation,* and Lemma 1. ■

Proof of Claim 11:

It is enough to see that they do not coincide with the Talmud solution. It can be done by way of example. Consider a three agents problem, with claims $c = (100, 200, 300)$, and let the estate be $E = 300$. Then, $ACEA(E, c) = CEA(E, c) = (100, 100, 100)$, $TCEL(E, c) = CEL(E, c) = (0, 100, 200)$, and finally, $T(E, c) = (50, 100, 150)$.

Since the only consistent extension of the contested garment solution is the Talmud solution, then $TCEL$ and $ACEA$ are not consistent.■

Proof of Claim 12:

Let us see it by way of example. Let $c = (100, 200, 300)$, and $E = 400$. Then, $AP(E, c) = (60, 120, 220)$. Consider now the group $S = \{1, 3\}$, and the reduced problem for this group. Their claims are $c_S = (100, 300)$, and what they jointly receive is $E' = 280$. Now, $AP(E', c_S) = (0, 180) + P[100, (100, 120)]$. Clearly, the first agent is going to receive less than 50, whereas initially he received 60. Consequently, AP is not consistent.

πThe failure of consistency for TP can be obtained via the duality relation: if TP were consistent, then $AP = (TP)^*$, also would be consistent. But we just proved that it is not. Consequently, also TP fails to be consistent.■

Proof of Theorem 12:

First, consider the following lemma:

Lemma 11.2. *Let $(c, E) \in \mathbb{B}$ be given. If for all $i \in N$, $f^i \in \mathbb{F}$, and for some $j \in N$ $f^j = cel$, then $u[f, (c, E)] = cel(c, E)$.*

Proof :

Without loss of generality, assume that $c_1 \geq c_2 \geq \cdots \geq c_n$. Let $N_1 = \{i \in N \mid cel_i(c, E) = 0\}$, $n_1 = |N_1|$, $N_2 = N \backslash N_1$, and $n_2 = |N_2|$. Let $C_2 = \sum_{N_2} c_i$

Thus, for all $i \in N_1$, $cel_i(c, E) = 0$ and for all $i \in N_2$ $cel_i(c, E) = c_i - \frac{C_2 - E}{n_2}$.

At the first stage, agent j proposes cel. Agent $i \in N \backslash \{j\}$ proposes $f^i \in \mathbb{F}$.

Step 1.- Compute $cel(c, E)$, and for all $i \in N \backslash \{1\}$, compute $f^i(c, E)$. If all them coincide, the chosen allocation is $cel(c, E)$, and we are done.,Otherwise, go to step 2.

Step 2.- For all $i \in N_1$, $m_i^1 = 0$. Thus, for all $i \in N_1$, $c_i^2 = c_i$. Now, by *fairness,* for all $i \in N$, $c_1 - f_1^i(c,E) \geq c_2 - f_2^i(c,E) \geq \cdots \geq c_n - f_{n.}^i(c,E)$

Since, at $cel(c,E)$ individual losses from the respective claims are as equal as possible, for all $k \in N\backslash\{1\}$and all $F \in \mathbb{F}$, $cel_k(c,E) \leq f_k(c,E)$. In particular, for all i, $k \in N\backslash\{1\}$, $cel_k(c,E) \leq f_k^i(c,E)$.Thus, for all $k \in N\backslash\{1\}$, $m_k^j = cel_k(c,E)$ and $m_1^1 \geq \frac{E}{n}$.

Thus, we have the vector m^1, where for all $k \in N_1$, $m_k^1 = 0$, for all $k \in N_2\backslash\{1\}$, $m_k^1 = c_k - \frac{C_2 - E}{n_2}$, and for $k = 1$, $\frac{E}{n} \leq m_1^1 \leq cel_1(c,E) = c_1 - \frac{C_2 - E}{n_2} = A$. Let $E^1 = E - \sum_N m_i^1 = A - m_1^1$ and $c^1 = c - m^1$.

For all $i \in N$, compute, $f^i(c^1, E^1)$. If all them coincide, and since $f^j = cel$, the selected allocation is $x = m^1 + cel(c^1, E^1)$. Note that, for all $j \in N_2\backslash\{1\}$, $c_j^1 = c_j - m_j^1 = \frac{C_2 - E}{n_2}$. Furthermore, by *fairness,* $c_1^1 \geq \cdots \geq c_n^1$. Now, note that by *fairness,* $c_1^1 - cel_1(c^1, E^1) \geq \cdots \geq c_n^1 - cel_n(c^1, E^1)$. Consequently, since $c_1^1 - m_1^1 \geq c_2^1$, for all $k \in N\backslash\{1\}$, $cel_k(c^1, E^1) = 0$. Consequently, $m + cel(c^1, E^1) = cel(c,E)$, and we are done. Otherwise, go to step 3.

Step t. For all $t \geq 1$, and for all $i \in N\backslash\{1\}$, we have that $m_i^t = 0$. Thus, at any step, for all $i \in N\backslash\{1\}$, $cel_i(c^t, E^t) = 0$. Furthermore, by *fairness,* for all $i \in N$, $f_1^i(c^t, E^t) \geq \frac{E^t}{n}$. Thus, in case the process terminates after a finite number of steps, we have that the final allocation is $x = m^1 + m^2 + \cdots + m^t + cel(c^t, E^t) = cel(c,E)$, and we are done.

Limit case.- If previous process does not terminate in a finite number of steps, compute $\lim_{t\to\infty}(m^1 + \cdots + m^t)$.

Note that, for all $i \in N\backslash\{1\}$, for all $t \in \mathbb{N}$, $(m_i^1 + \cdots + m_i^t) = m_{ij}^1 = cel_i(c,E)$. Thus, we only need to consider convergence for agent 1.

First, note that for all $t \in \mathbb{N}$, $m_1^1 + \cdots + m_1^t \leq A$. Furthermore, $m_1^1 \geq \frac{E}{n} \geq \frac{A}{n}$, and $m_2^2 \geq \frac{A - m_1^1}{n}$. Consequently, $m_1^1 + m_1^2 \geq m_1^1 + \frac{A - m_1^1}{n} \geq \frac{A}{n} + \frac{n-1}{n^2}A$.

In general, $m_1^1 + \cdots + m_1^t \geq \frac{A}{n}\left[1 + +\frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots \left(\frac{n-1}{n}\right)^{t-1}\right]$.

Thus, $\lim_{t\to\infty}(m_1^1 + \cdots + m_1^t) \geq A$. It then follows that $\lim_{t\to\infty}(m_1^1 + \cdots + m_1^t) = A$, and therefore, $\lim_{t\to\infty}(m^1 + \cdots + m^t) = cel(c,E)$. $\square$

Proof of Theorem 12:

Let $i \in N$ be such that $c_i = \max_N c_j$. First, note that for all $(c,E) \in \mathbb{B}$, and for all $x \in FP(c,E)$, $cel_i(c,E) \geq x_i$. Furthermore, for all profile of reported rules f either $dc[f,(c,E)] \in FP(c,E)$, or $dc[f,(c,E)]$ is inefficient. If $dc[f,(c,E)] \in FP(c,E)$, then $cel_i(c,E) \geq dc_i[f,(c,E)]$. If $dc[f,(c,E)]$ is inefficient, there are two posibilities, either $dc[f,(c,E)] = 0$, or $dc[f,(c,E)] = \lim_{t\to\infty} f(c^t, E)$, and since for all $k \in N$ and all $t \in N$, $f^k(c^t, E) \in FP(c^t, E)$, then for all $k \in N$, $f_i^k(c^t, E) \leq cel_i(c, E)$. Thus, if $c_i = \max_N c_j$, then $u_i[f, (c, E)] \leq cel_i(c, E)$. Furthermore, by Lemma 2, if $f^i = cel$, then for any preferences chosen by the other agents, $u[f, (c, E)] = cel(c, E)$. Therefore, cel is a dominant strategy for agent i.

Additionally, once the agent with the maximum claim chooses cel, the outcome of the procedure is determined. □

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