Why Are Beveridge-Nelson and Unobserved-Component Decompositions of GDP So Different?

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The decomposition of real GDP into trend and cycle remains a problem of considerable practical importance, but two widely used methods yield starkly different results. The unobserved component approach, introduced by Harvey (1985) and Clark (1987), implies a very smooth trend with a cycle that is large in amplitude and highly persistent. In contrast, the approach of Beveridge and Nelson (1981) implies that much of the variation in the series is attributable to variation in the trend while the cycle component is small and noisy. This conflict is apparent in Figures 1 and 2 in this paper where the two cycle components are plotted, and has been widely noted; see Watson (1986) Stock and Watson (1988) among others.

It should surprise us that the unobserved component (UC) and Beveridge-Nelson (BN) methods produce very different trend-cycle decompositions since both are model-based. Each implies an ARIMA representation. Neither imposes smoothness in the trend component *a priori* as does the smoother of Hodrick and Prescott (1997) or as in the polar case of a linear trend that forces all variation, save constant growth, into the cycle. The UC and BN both "let the data speak for itself" in this regard. While it is often stated that BN assumes a perfect negative correlation between trend and cycle innovations, that is a property of the estimated trend and cycle, not the unobserved components, and it is a property shared with the UC decomposition. This paper attempts to find out why we do not, after decades of research, have a consistent picture of how variation in a series like real GDP should be allocated between trend and cycle.

Briefly, section 1 demonstrates the theoretical equivalence between the approaches. Section 2 investigates the source of the difference observed in practice. Section 3 concludes.

1. Theoretical Equivalence of the Beveridge-Nelson and Unobserved Component Estimates of Trend and Cycle

The detrending problem is motivated by the idea that the log of aggregate output is usefully thought of as the sum of a nonstationary trend component that accounts for long-term growth and a stationary component that allows for transitory deviations of output from trend. We follow custom in refering to the latter as the “cycle” even if it is not periodic. The UC model takes the form:

(1a) $y_t = t_t + c_t$

(1b) $t_t = t_{t-1} + m + h_t; \quad h \sim i.i.d.\ N(0, s_h)$

(1c) c_t *is stationary and ergodic*

where $\{y_t\}$ is the observed series, $\{t_t\}$ is the unobserved trend assumed to be a random walk with average growth rate m, and $\{c_t\}$ is the unobserved stationary cycle component.[1] The UC-ARMA adds the condition that $\{c_t\}$ is a stationary and invertible ARMA(p,q) process with innovations that may be contemporaneously cross-correlated with trend innovations,

(1d) $f_p(L)\, c_t = q_q(L)\, e_t;\ e \sim i.i.d.\ N(0, s_e);\ Cov(h_t, e_{t+k}) = s_{he}$ *for k=0; 0 otherwise.*

In some implementations the rate of drift m is allowed to evolve as a random walk, and an additional irregular term may be added. Harvey (1985) and Clark (1987) suggest specifying $p=2$ which allows the cycle process to be periodic in the sense of having a

[1] As noted in Blanchard and Quah (1989), the structural trend of output does not necessarily follow a random walk. Therefore, any decomposition method which assumes a random walk trend runs the risk of lumping transitory movements in output due to supply shocks in with movements due to demand shocks, with only the latter movements directly connected to what economists traditionally refer to as the business cycle. However, removal of the permanent component is still useful in business cycle analysis since it should, in principal, render the output series stationary without removing any relevant information about the cycle. Furthermore, to the extent that transitory movements are dominated by demand shocks, the transitory component will provide a very accurate measure of the business cycle.

peak in its spectral density function. They and others also assume that the trend and cycle innovations are uncorrelated, setting

(1e) $s_{he} = 0$

thereby casting the UC model in state-space form by treating (1a) as the measurement equation and (1b) as the state transition equation. We denote this constrained zero-covariance UC-ARMA model as UC0.

In practice, the parameters are unknown and are estimated from the data series $(y_1, .., y_n)$ using the maximum likelihood method of Harvey (1981). Given the parameters, the Kalman filter is used to compute the expectation of the trend component conditional on data through time t:

$$\hat{t}_{t|t} = E\left[t_t \middle| Y_t\right], \text{ where } Y_t = (y_1,..,y_t)$$

Alternatively, the BN estimate of trend for an I(1) time series $\{y_t\}$ is defined to be the limiting forecast as horizon goes to infinity, adjusted for the mean rate of growth; so

$$BN_t = \lim_{M\to\infty} E\left[y_{t+M} - Mm \middle| Y_t\right].$$

BN showed that the time series $\{BN_t\}$ will be a random walk with drift, the deviation from trend is a stationary process, and that the innovations of $\{BN_t\}$ and $\{y_t - BN_t\}$ are perfectly correlated. The series $\{BN_t\}$ is calculated from the ARIMA representation of $\{y_t\}$, which in principle is unique after cancellation of any redundant AR and MA factors.

It is well known that the UC–ARMA model always implies a univariate ARIMA representation for $\{y_t\}$. This is what Nerlove, Grether, and Carvalho (1979) refer to as the canonical form of the UC model, and it may be useful to think of it as the reduced form. Substituting (1b) and (1d) into (1a), taking first differences, and rearranging we obtain

(2a) $f_p(L)\ (1\text{-}L)y_t = f_p(1)m + f_p(L)\ h_t + q_q(L)\ (1\text{-}L)e_t$.

Recognizing that the right hand side will have non-zero autocorrelations through lag max(p, q+1), Granger's Lemma implies that the univariate representation will be

(2b) $f_p(L)\ (1\text{-}L)y_t = m^* + q^*_{q*}(L)\ u_t;\ u \sim i.i.d.\ N(0, s_u);\ q^* = max(p, q+1)$

This ARIMA reduced form of fully describes the joint distribution of the $\{y_t\}$ and therefor the conditional distribution of future observations given the past and is unique.

Further, there is always a UC representation of any ARIMA process. As Cochrane (1988) pointed out, the existence of the BN decomposition guarantees it. However, there will not be a unique UC representation corresponding to a given ARIMA process. For example, a series that is autocorrelated at lag one only has a UC representation as a random walk plus random noise, but the variances of the two innovations and their covariance are not all separately identified; see the discussion in Nelson and Plosser (1981).

Given that a time series will not in general have a unique UC representation, it seems surprising to us that *the BN trend is the conditional expectation of the random walk component of an I(1) process*. As pointed out by Watson (1986), this is true regardless of the covariance structure of the unobserved components. To see this, consider the unconstrained UC model defined by (1a)-(1c), so cycle and trend innovations may be cross-correlated. The conditional expectation of the trend component at time t is

$$E\left[t_t \middle| Y_t\right] = E\left[t_t + c_{t+M} \middle| Y_t\right]$$

for large enough M, since the cycle, by its ergodicity, has expectation zero far enough in the future. Further, the expected value of any future innovation in the trend is zero, so we have

$$E[t_t|Y_t] = E\left[t_t + \sum_{j=1}^{M} h_{t+j} + c_{t+M}|Y_t\right].$$

Recognizing that the terms of the right include all the elements of y_{t+M} except the accumulated drift, we have

$$E[t_t|Y_t] = E\left[t_t + \sum_{j=1}^{M} h_{t+j} + c_{t+M}|Y_t\right] = E[y_{t+M} - Mm|Y_t] = BN_t .$$

Then the conditional expectation of the cycle at time *t* is simply

$$E[c_t|Y_t] = y_t - E[t_t|Y_t] = y_t - BN_t .$$

Thus, we can always compute conditional expectation estimates of trend and cycle at any point in time from the ARIMA representation of the observed series. The two assumptions, (a) the trend is a random walk, and (b) the cycle is ergodic, are sufficient to identify the components, and this does not depend on knowing the covariance between trend and cycle innovations. Intuitively, the forecast at a long enough horizon reflects only the permanence of the random walk trend. Stronger assumptions may be needed to identify the parameters of a UC representation, but they are irrelevant if the only objective is to estimate trend and cycle. In that case, only the conditional expectation of the future given the past is required, and the ARIMA reduced form provides the relevant conditional distribution.

It follows that if a particular time series does have a representation as a UC0 process, then the Kalman filter and BN estimates of trend and cycle will be the same, as long as the parameters of the ARIMA model are those implied by the UC0 representation. In that case, BN is just another way to compute $\hat{\tau}_{t|t}$ and $\hat{c}_{t|t}$; the time series of those estimates will be the same and they will have the same properties. In particular, their innovations will be both be functions of the innovation in y_t since it is only new information that will cause the long range forecast, the trend, to change. Thus, UC0 and BN share the often-noted property of the BN decomposition, that the innovations of the *estimated* trend and cycle series are perfectly correlated. Further, none of these results depend on limiting UC representations that might be modeled to the constrained UC0 case; the corresponding ARIMA representation will always be an equivalent way of obtaining the information relevant to estimating the trend.

To sum up this section, we have shown that whether one uses the UC approach to trend-cycle decomposition based on a state-space representation and the Kalman filter, or the BN approach based on long-range forecasts from a univariate ARIMA model, the specific results should be the same. UC and BN are simply alternative ways of calculating the same conditional expectation of the unobserved trend and cycle at a point in time. The fact that the two have produced such different estimates of trend and cycle in practice implies, then, that they must be based on conflicting representations of the data. Identifying the source of the conflict is the subject of the next section.

2. In What Way Do UC and ARIMA Models of U.S. Real GDP Conflict?

The results of Section 1 imply that the differing results obtained in practice must be traceable to restrictions on the reduced form ARIMA implied by the UC approach that are in conflict with the unrestricted ARIMA model used in the BN approach. Those restrictions presumably arise from the restrictions that have been placed on the UC–ARMA representation used in implementing the Kalman filter, the particular ARMA(p, q) form of the cycle process and the zero correlation between trend and cycle innovations, what we called the UC0 model. Following Clark (1987) we set p=2, to allow for cyclical dynamics, and q=0 in the UC0 model and obtain for real GDP 1947:1 –

1998:2 the filtered estimate of $c_{t|t}$ shown in Figure 1.[2] Confirming results in the literature, the estimated cycle produced by the UC0 model is both large in amplitude and very persistent. It agrees reasonably well with the NBER dating of the business cycle, although it leads the NBER dating at peaks. Reflecting the smoothness of the trend, the cycle shown here is qualitatively similar to that obtained by simple linear detrending of log output by least squares. For example, both imply that the economy has been below trend throughout the 1990s.

Table 1 reports the maximum likelihood estimates of the parameters and their standard errors for the UC0 model. The roots of the estimated autoregressive polynomial are complex, implying that the business cycle has a period of almost 8 years with a standard deviation of about 3 percentage points around trend, confirming the visual impression of persistence, periodicity, and amplitude in Figure 1. In contrast, the trend process innovation has a standard deviation of only about 0.7 percentage points.

[2] Clark (1987) allowed the drift parameter to evolve as a random walk, but estimates of the variance are small. We have assumed that this parameter remains constant, implying that output is I(1).

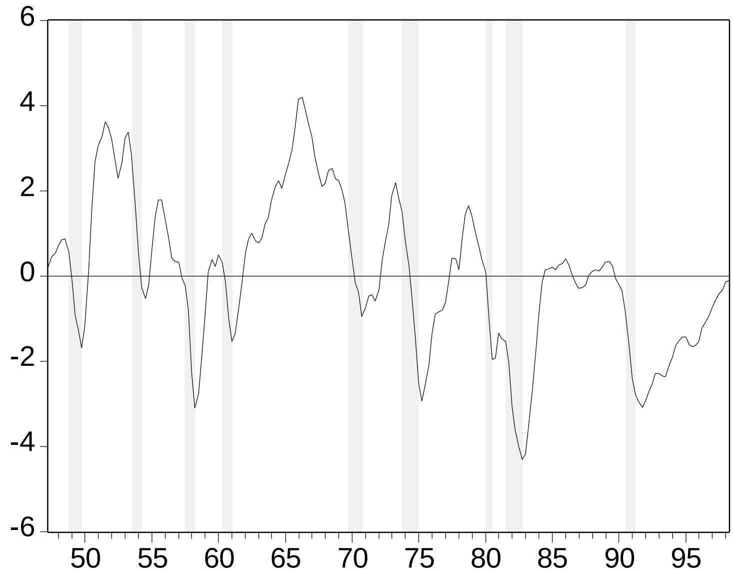

Fig. 1: UC0 Cycle, U.S. Real GDP.
Percent deviation from trend, NBER recessions shaded

Table 1: Maximum Likelihood Estimates of UC0 Parameters

	Estimate	Standard Error
Trend process		
Drift: μ	0.811914	(0.050050)
Innovation: s_h	0.689342	(0.103756)
Cycle process		
ϕ_1	1.530307	(0.101162)
ϕ_2	-0.609731	(0.114031)
Innovation: s_e	0.619867	(0.131859)
AR Roots (inverted)	0.765154 +/- 0.155792i	

Implied cycle: period 7.7 years, standard deviation .03.

Log Likelihood	-286.605332	

The reduced form ARIMA representation for this UC0 model corresponding to (2b) is obtained as follows:

$$\begin{aligned}\Delta y_t &= (1-L)t_t + (1-L)c_t \\ &= m + h_t + (1-L)(1-f_1L - f_2L^2)^{-1}e_t\end{aligned}.$$

Multiplying both sides by $(1-f_1L - f_2L^2)$ gives:

(3) $(1-f_1L - f_2L^2)\Delta y_t = m^* + h_t - f_1h_{t-1} - f_2h_{t-2} + e_t - e_{t-1} = m^* + u_t + q_1^*u_{t-1} + q_2^*u_{t-2}$.

using the fact that the right-hand side has a representation as an MA(2) by Granger's lemma, the univariate innovations u_t being i.i.d. N(0, σ_u), and m^* is $m(1- f_1- f_2)$.

While the reduced form of the UC0 model is ARIMA(2,1,2), when we estimate that model and compute the BN cycle component from it we get the very different results seen in Figure 2. As reported in the literature, the estimated BN cycle is small in amplitude compared to the UC0 cycle and much less persistent.

Table 2 reports the maximum likelihood estimates of the parameters for the reduced-form ARIMA(2,1,2) model. Confirming the visual impression from Figure 2, the period of the cycle implied by the AR parameters here is much shorter, only 2.4 years instead of 8. The fact that the value of the log likelihood is greater by roughly 2 for the unrestricted ARIMA must reflect restrictions in UC0 model not imposed in the reduced form, in particular zero correlation between trend and cycle innovations. To see what correlation is implied by the ARIMA parameters, we next solve for the parameters of the unrestricted UC model of equations (1a)-(1d) that correspond to the estimated unrestricted ARIMA parameters.

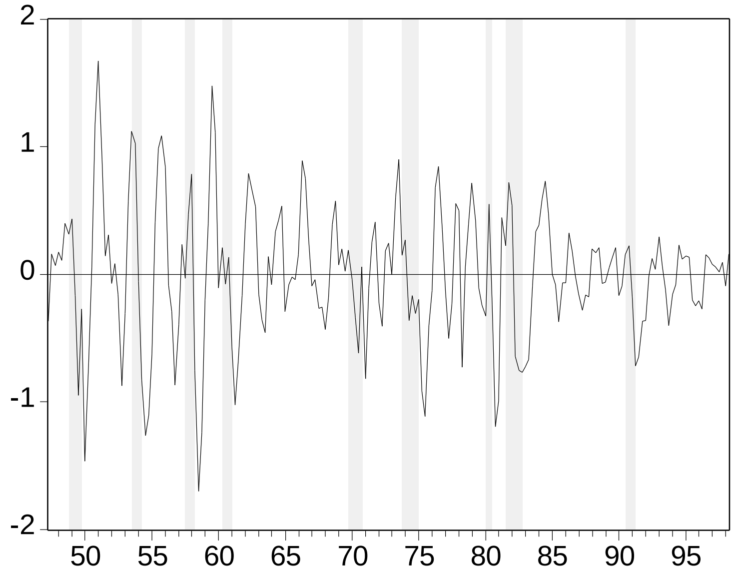

Fig. 2: Beveridge-Nelson Cycle, U.S. Real GDP.
Percent deviation from trend, NBER recessions shaded.

Table 2: Maximum Likelihood Estimates for ARIMA(2,1,2)

	Estimate	Standard Error
Drift μ	0.815603	(0.086490)
ϕ_1	1.341846	(0.151854)
ϕ_2	-0.705894	(0.173021)
θ_1	-1.054277	(0.195914)
θ_2	0.518756	(0.225004)
SE of Regression	0.969392	(0.047822)

AR roots (inverted) 0.670923 +/- 0.505724i
Implied cycle: period 2.4 years

Log Likelihood -284.650664

First note that the AR parameters are the same in both the UC and ARIMA reduced form since the AR polynomial on the left side of (2) is the AR polynomial of the UC cycle. Now the observable moments on the MA side of (2) are the mean, which identifies μ, and the autocovariances:

$$\begin{aligned}
\gamma_0 &= (1+\phi_1^2+\phi_2^2)\sigma_\eta^2 + 2\sigma_\varepsilon^2 + 2(1+\phi_1)\sigma_{\eta\varepsilon} \\
\gamma_1 &= -\phi_1(1-\phi_2)\sigma_\eta^2 - \sigma_\varepsilon^2 - (1-\phi_2+\phi_1)\sigma_{\eta\varepsilon} \\
\gamma_2 &= -\phi_2\sigma_\eta^2 - \phi_2\sigma_{\eta\varepsilon} \\
\gamma_j &= 0,\ j \geq 3
\end{aligned} \tag{4}$$

The three non-zero autocovariance for the MA(2) are just sufficient to identify the three remaining parameters of the UC representation, namely σ_η^2, σ_ε^2, and $\sigma_{\eta\varepsilon}$. We note that in a particular case the solution to (4) might not imply a positive definite covariance matrix for the trend and cycle innovations, in which case there would not exist a corresponding UC-ARMA(2,0) representation.

Table 3 compares the estimates from Table 1 for the UC0 model with the implied estimates from the unrestricted ARIMA(2,1,2) reduced form. While the parameters for the cycle component are somewhat similar, the implied standard deviation of the trend innovations is almost twice as large for the unrestricted reduced form, and the implied correlation between trend and cycle innovations is large and negative. To avoid misunderstanding, we note that the latter is the estimated correlation between unobserved innovations, not the correlation between the innovations in the observed series $\hat{\tau}_{t|t}$ and $\hat{c}_{t|t}$. The former is a function of parameters of the ARIMA reduced form representation which could imply any correlation value, while the latter is always –1 when the two representations imply the same ARIMA representation, as discussed above.

Table 3: Parameters of UC0 Model and Those Implied by Unrestricted ARIMA(2,1,2) Reduced Form

	UC0 Model	Implied by ARIMA
Trend process		
Drift: μ	0.811914	0.815603
Innovation: s_h	0.689342	1.2368
Cycle process		
ϕ_1	1.530307	1.341846
ϕ_2	-0.609731	-0.705894
Innovation: s_e	0.619867	0.74867
Covariance s_{he}	zero (constrained)	-0.83913
Correlation r_{he}	zero (constrained)	-0.90621

The fact that s_{he} is identified in this case implies that we can relax the constraint that it is zero in the UC model and estimate it directly by maximum likelihood.[3] The unconstrained UC-ARMA(2,0) model is recast in state-space form simply by including the cycle component with the trend in the state equation; see appendix for details. Figure 3 displays the filtered estimate $c_{t|t}$ of the transitory component for this model. The estimated cycle is essentially identical to the estimated cycle from the BN decomposition.[4] This verifies that the filtered estimates from the UC model and the BN estimates are equivalent.

More generally, the order condition for identification of the cross-correlation between trend and cycle innovations is satisfied, in the sense of having as many moment equations as parameters, when p=q+2 as it is in this case with p=2, q=0.

[3] By matching moments of the MA part of the ARIMA and UC representations, it is readily shown that a condition for identification of the covariance between innovations is that $p \geq q+2$.

[4] The only difference is for the first observation, which is different due to the need to provide an initial guess for the value of the random walk trend in estimation via the Kalman filter.

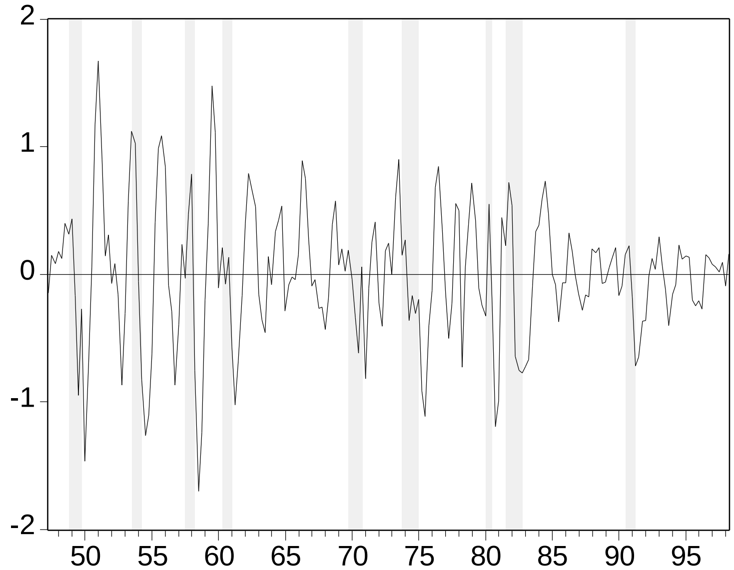

Figure 3 – UC1 Cycle (NBER dated recessions shaded)

Table 4 reports the maximum likelihood estimates of the parameters for the unconstrained UC model. The most striking feature of these estimates is that they are all essentially the same as the implied estimates from the unrestricted ARIMA model reported in Table 3. Confirming identification of the covariance between trend and cycle innovations, the standard error of the estimate of $\sigma_{\varepsilon\eta}$ is small, and a .95 confidence interval does not include zero. The log likelihood value is also the same as for the ARIMA model, and significantly larger than for the restricted UC0 model. The likelihood ratio statistic for testing the restriction $\sigma_{\varepsilon\eta} = 0$ is 3.909, with a corresponding *p*-value of 0.048. Thus we can strongly reject the restriction of a zero correlation between permanent and transitory shocks by comparing the results for the UC0 model with either the results for the reduced-form ARIMA model or the unrestricted UC model.

Finally, the actual estimated correlation is –0.906. This finding implies that a positive permanent shock is very closely related to a negative transitory shock, and vice versa.

Table 4: Maximum Likelihood Estimates for Unconstrained UC Model

	Estimate	Standard Error
Trend process		
Drift: μ	0.815608	(0.086518)
Innovation: σ_η	1.236757	(0.151798)
Cycle process		
ϕ_1	1.341909	(0.145616)
ϕ_2	-0.705974	(0.082245)
Innovation: σ_ε	0.748524	(0.161431)
Roots of AR process	0.670955 + 0.505761i	
	0.670955 - 0.505761i	
Covariance: $\sigma_{\varepsilon\eta}$	-0.838944	(0.109599)
Log Likelihood Value	-284.650664	

3. Summary and Conclusions

We have shown that trend-cycle decompositions based on unobserved component models cast in state-space form and on the long run forecast implied by an ARIMA model differ not because they differ in principle but because the empirical models that have been used differ. In particular, the restriction that trend and cycle innovations are uncorrelated has been imposed in the former while it is not imposed in the latter. We note that when this restriction is relaxed in the state-space model, the two approaches lead to identical trend-cycle decompositions and identical univariate representations. Further, this restriction is strongly rejected by the data for U.S. real GDP, quarterly 1947-1998.

If we accept the implication that innovations to trend are strongly negatively correlated with innovations to the cycle, then case for the importance of real shocks in the macro economy is strengthened. For example, a positive productivity shock will immediately shift the level of potential output upward, leaving actual output below trend until it catches up with potential. In contrast, a positive nominal shock, say a shift in Fed policy towards stimulus, will be an innovation to the cycle without any immediate impact on potential output.

Closing with a caveat, we note that the decompositions considered here share a common restriction, that the cycle process is symmetric. Recent business cycle research suggests that postwar recessions have exhibited important asymmetry; see Neftci (1984), Sichel (1993, 1994), Beaudry and Koop (1993), and Kim and Nelson (1999). If asymmetry is important, then the dominating variation of the trend component that we have seen in this paper may reflect the dominating influence of expansions in postwar GDP, periods of time when actual output is relatively close to potential and the cycle component is short-lived and small in amplitude.

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Appendix

This appendix contains notes on how to calculate the BN decomposition and how to set up the state-space model in order to estimate the UC1 model via the Kalman filter.

BN decomposition

The simplest way to calculate the BN decomposition for any ARIMA model is to first convert the model into its companion VAR(1) form. For the ARIMA(2,1,2) model, this is given as follows:

$$\begin{bmatrix} \Delta y_t - m \\ \Delta y_{t-1} - m \\ e_t \\ e_{t-1} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & q_1 & q_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} - m \\ \Delta y_{t-2} - m \\ e_{t-1} \\ e_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} e_t , \qquad \text{(A.1)}$$

or, more compactly,

$$b_t = F b_{t-1} + v_t . \qquad \text{(A.1')}$$

Then, the BN cycle can be calculated at any given point of time as

$$c_{t|t} = -\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} F (I - F)^{-1} b_t . \qquad \text{(A.2)}$$

Note that this is just the vector generalization of the BN cycle for a simple AR(1). Specifically, the BN cycle represents the accumulation of the forecastable momentum inherent in a series, given its present relationship to a long-run equilibrium.

The State-Space Model

One possible explanation for prevalence of the unnecessary zero correlation assumption in empirical work is the way the state-space model for a UC model is traditionally set up and estimated. In particular, the random walk trend is often treated as the state variable, while the AR(2) cycle is treated as a residual in the observation equation. Given this setup, estimation via the Kalman filter requires an assumption of independence between trend and cycle innovations.

However, there are other ways of setting up the state-space model for a UC model. One possibility is to make the observation equation an identity, with both the trend and cycle treated as state variables. This is the approach we take:

<u>Observation Equation</u>:

$$y_t \equiv \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \end{bmatrix}, \tag{A.3}$$

or, more compactly,

$$y_t = H\beta_t, \tag{A.3'}$$

<u>State Equation</u>:

$$\begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_{t-1} \\ c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix}, \tag{A.4}$$

or, more compactly,

$$\beta_t = \tilde{\mu} + F\beta_{t-1} + v_t. \tag{A.4'}$$

Then, we can use the Kalman filter to estimate the model, even if we allow the trend and cycle innovations to be correlated. That is,

$$Q \equiv E[v_t v_t'] = \begin{bmatrix} s_h^2 & s_{he} & 0 \\ s_{he} & s_e^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \text{(A.5)}$$

The Kalman filter is given by the following six equations:

$$b_{t|t-1} = \tilde{m} + F b_{t-1|t-1}, \qquad \text{(A.6)}$$

$$P_{t|t-1} = F P_{t-1|t-1} F' + Q, \qquad \text{(A.7)}$$

$$y_t - y_{t|t-1} = y_t - H b_{t|t-1}, \qquad \text{(A.8)}$$

$$f_{t|t-1} = H P_{t|t-1} H', \qquad \text{(A.9)}$$

$$b_{t|t} = b_{t|t-1} + K_t (y_t - y_{t|t-1}), \qquad \text{(A.10)}$$

$$P_{t|t} = P_{t|t-1} - K_t H P_{t|t-1}, \qquad \text{(A.11)}$$

where $b_{t|t-1} \equiv E[b_t | y_{t-1}]$, for example, is the expectation of b_t conditional on information up to time $t-1$; $P_{t|t-1}$ is the variance of $b_{t|t-1}$; $f_{t|t-1}$ is the variance of $(y_t - y_{t|t-1})$; and $K_t \equiv P_{t|t-1} H' f_{t|t-1}^{-1}$ is the Kalman gain.[5]

Given some initial values $b_{0|0}$ and $P_{0|0}$, we can iterate through (A.6)-(A.11) for $t = 1, \ldots, T$ to obtain filtered inferences about b_t conditional on information up to time *t*. Also, as a by-product of this procedure, we obtain $(y_t - y_{t|t-1})$ and $f_{t|t-1}$, which we can use to find maximum likelihood estimates of the hyper-parameters based on the prediction error decomposition (Harvey, 1990):

[5] For a more general discussion of the Kalman filter and state-space models, as well as details on the derivation of the Kalman gain, refer to Hamilton (1994a,b) and Kim and Nelson (1998).

$$\max_{q} l(q) = -\frac{1}{2}\sum_{t=t+1}^{T}\ln(2pf_{t|t-1}) - \frac{1}{2}\sum_{t=t+1}^{T}(y_t - y_{t|t-1})\, f_{t|t-1}^{-1}(y_t - y_{t|t-1}), \qquad \text{(A.12)}$$

where $q = (m, f_1, f_2, s_h, s_e, s_{he})$.

In terms of the initial values $b_{0|0}$ and $P_{0|0}$, we assume an arbitrary estimate for the random walk component, but assign it an extremely large variance. For the transitory component, we use the unconditional mean and variance of the AR(2) process.