

# Successful Uninformed Bidding\*

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January 19, 2000

Job Market Paper

## Abstract

This paper studies multiunit common value auctions with informed and less informed bidders. In these auctions, we show that bidders with less information can bid very aggressively and do surprisingly well. We also show that the degree of aggressiveness and success of bidders with less information is positively related to the number of units for sale. We explain these phenomena in terms of the balance of the winner's curse and the loser's curse and their differential effect on bidders with different quality of information.

JEL Classification Numbers: D44.

Keywords: common value, uninformed bidders, multiunit auctions, loser's curse.

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\*I would want to express my gratitude to my supervisor Tilman Börgers. I also want to acknowledge Jacques Cremer's guidance during my visit to the MPSE (Toulouse). Comments by Murali Agastya, Philippe Jehiel, Benny Moldovanu, Marco Ottaviani, Thomas Tröger and the participants in seminars at MPSE (Toulouse), Bristol University and at the ESRC Game Theory Conference 1999 in Kenilworth are gratefully acknowledged. I also thank Richard Norman for his help. This paper has been partially written while the author was visiting the MPSE (Toulouse) through the ENTER program. Its hospitality is gratefully acknowledged. Finally, I would want to recognise the financial support from Fundación Ramón Areces and Banco de España, and the hospitality of ELSE. The only responsibility of errors remains my own.

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# 1 Introduction

This paper studies multiunit, common value auctions in which some bidders have better information than others. In equilibria of such auctions the *worse* informed bidders can bid very aggressively and do surprisingly well. We display this effect in a sequence of models and discuss when it arises, and when it does not. We also argue that the correct intuitive explanation for these results relies on the balance of the *winner's curse* and the *loser's curse* effects.

The theoretical study of multiunit, common value auctions is important because a number of real life auctions have at least some similarity to such auctions. Examples are auctions of oil and gas leases, treasury bill auctions, and auctions of parts of the radio spectrum. It is important to understand how poorly informed or uninformed bidders behave in these auctions because their actions can influence the efficiency of the auction outcome as well as the expected revenue of the auctioneer. Their presence can then also affect the optimal auction design.

For the case that bidders have unit-demand, and that the number of units for sale is smaller than the number of well-informed bidders, Milgrom (1981) has displayed an equilibrium of a second price auction in which bidders without relevant private information lose out to better informed bidders with probability one. In this paper, we focus on the opposite case, that there are at least as many units for sale as there are well-informed bidders. In practice, for example in the auctions cited in the previous paragraph, it often seems realistic that well-informed bidders form only a small fraction of the total market.

We show that Milgrom's result is reversed, and that the uninformed or poorly informed bidders can win with positive probability. In fact, we find a kind of monotonicity: The more units are for sale, the more aggressively is the bidding behaviour of uninformed or poorly informed bidders, and the more likely it is that they win. In extreme cases, this probability can become one. We also show that the unconditional expected utility of the informed bidders may be less than that of the completely uninformed bidders.

It is important to emphasise that, although we consider multiunit auctions, like Milgrom we maintain the assumption that each bidder individually demands only one unit. Thus, our results are unrelated to the difficult problems arising in auctions in which bidders are allowed to submit multiunit-demands. Because we maintain the unit-demand assumption, it is also obvious how the second price auction needs to be defined in the multiunit case, say with  $k$  units for sale: the bidders with the  $k$  highest bids win and pay the  $k + 1$ -th highest bid.

The observation that uninformed bidders may win auctions is not original to this paper. In fact, Engelbrecht-Wiggans, Milgrom, and Weber (1983) showed that this may happen in the single unit case if the format is a first price auction.

Engelbrecht-Wiggans, Milgrom and Weber's result was extended by Daripa (1998) to a multiunit setup, using a generalisation of the first price auction. The auction format of Daripa is more difficult to analyse than ours. His analysis is also complicated by the fact that he allows for multiunit-demand. As a consequence, we obtain a more clear-cut analysis than Daripa. For example, we do not face as severe problems of multiplicity of equilibria as Daripa does.

Another reason for our interest in the second price format is that it allows us to develop particularly clearly the intuition for our findings. We explain the relatively good performance of poorly informed or uninformed bidders with respect to informed bidders in terms of the differential effect of the winner's curse and the loser's curse on the incentives to bid of bidders with different quality of information.

In the (generalised) second price auction a bidder will want to raise his bid by a small amount, say from  $b$  to  $b + \epsilon$ , if the expected value of a unit, conditional on its price being  $p \in (b, b + \epsilon)$ , is larger than  $p$ . The price is  $p$  if and only if the  $k$ -th highest bid of the other bidders is  $p$ . This event is the intersection of two events, one of which implies good news whereas the other implies bad news for the bidder. The good news is that at least  $k - 1$  other bidders have been willing to bid  $p$  or more, and that at least one other bidder has been willing to bid  $p$ . If these bidders had any private information at all, it must have been favourable. This is good news. This effect has been called the "loser's curse" as a bidder who neglects this effect will regret losing. The bad news is that at least  $m - k - 1$  bidders (where  $m$  denotes the total number of bidders) have bid  $p$  or less, and hence, if they had any private information at all, this must have been unfavourable. This effect has been called the "winner's curse" as a bidder who neglects this effect will regret winning.<sup>1</sup>

The winner's curse reduces the incentives to bid higher, whereas the loser's curse raises the incentives to bid higher. Both effects are stronger for less informed bidders. The reason is that the average quality of the information of the other bidders is higher from the point of view of a poorly informed bidder than from the point of view of a well-informed bidder. If the loser's curse is sufficiently strong in comparison to the winner's curse we can expect that in equilibrium bidders with less information win more often than bidders with more information. Moreover, we can also expect that the stronger the loser's curse in comparison to the winner's curse, the more often less informed bidders win. This explains the monotonicity of the behaviour of uninformed or poorly

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<sup>1</sup>The "winner's curse" is well-known in the auction literature, see for instance the survey by Milgrom (1989). The concept of "loser's curse" is less established. It was first used by Holt and Sherman (1994) in the context of a bargaining model. The concept was introduced in auction models by Pesendorfer and Swinkels (1997). They also presented a formal definition of the meaning of the winner's curse and the loser's curse in the spirit of that given in our paper.

informed bidders with respect to the number of units. The more units there are for sale, the more winners and the fewer losers there are in the auction, thus the loser's curse will be stronger and the winner's curse will be weaker.

Notice that the loser's curse can only arise when there is more than one unit for sale. At the opposite extreme is the case when the number of units for sale equals the number of bidders minus one. Then the winner's curse plays no role, and it can only be the loser's curse that affects the incentives to bid higher. Consequently, when the number of units for sale equals the number of bidders minus one and there is more than one unit for sale, less informed bidders bid more aggressively than better informed bidders.

One possible application of our results concerns the case in which the auctioneer can choose into how many "lots" to divide what he has for sale. For the case that the auctioneer is committed to a generalised second price auction, our results indicate that he should choose a large number of "lots" if, for some reason, the success chances of poorly informed bidders is important to him.

The most closely related papers are those of Milgrom (1981), Engelbrecht-Wiggans, Milgrom, and Weber (1983) and Daripa (1998) which were already discussed above. Another related study is that of Pesendorfer and Swinkels (1997). This paper, like ours, studies the generalisation of the second price auction to the multiunit case when bidders have unit-demand. Pesendorfer and Swinkels (1997) differs from our paper in two respects. Firstly, they assume that all bidders have signals of equal informativeness, whereas our focus is on the case that some bidders have more informative signals than others. Secondly, they focus on the case that the number of units for sale and the number of bidders are large. By contrast, our focus is on the case of a fixed, finite number.

This paper is structured as follows: In Section 2, we study a basic model in which there are one bidder with relevant, although potentially incomplete information, and several other, completely uninformed bidders. Section 3 extends the model and analyses a case in which there are several bidders who hold relevant information whereas other bidders are completely uninformed. In Section 4, we extend the model of Section 2 into a different direction, and allow the bidders who were uninformed in Section 2 to hold some pieces of information. We only assume that their information is less significant than that of the well-informed bidder. We show that the equilibria in this setup converge in an appropriate sense to the equilibrium in Section 2 as the significance of the less informed bidders' signals tends to zero.

## 2 An Auction with One Informed and Many Uninformed Bidders

An auctioneer puts up for sale through auction  $k$  indivisible units of a good. There are  $n + 1$  bidders,  $n \geq 2$ .<sup>2</sup> Each bidder can bid for one or zero units of the good.<sup>3</sup> We assume that the number of bidders is greater than the number of units for sale,  $n + 1 > k$ .

Each bidder obtains a von Neumann Morgenstern utility of  $v - p$  if she obtains one unit of the good, and she obtains a von Neumann Morgenstern utility of zero if she obtains no unit. The value  $v$  is common to all bidders. One bidder, the *informed* bidder, receives privately a signal  $s$ , whereas the other bidders, the *uninformed* bidders, do not receive any signal. For simplicity we assume that  $v = s$ .<sup>4</sup> The signal  $s$  is drawn from the interval  $[\underline{s}, \bar{s}]$  (where  $0 \leq \underline{s} < \bar{s}$ ) with a continuous distribution function  $F(s)$ . This distribution is assumed to have support  $[\underline{s}, \bar{s}]$ .

The auction used is a uniform price auction. We assume that there are neither a reserve bid nor an entry fee. All bidders submit simultaneously non-negative bids. The bidders who make the  $k$  highest bids win one unit each. The price which they have to pay is the  $k + 1$ -th highest bid. If the  $k$ -th highest bid and the  $k + 1$ -th highest bid have the same value  $b$ , then the price in the auction is  $b$ , all bidders who make a bid strictly higher than  $b$  get one unit with probability 1, and the remaining winners are randomly selected among all bidders who have made bid  $b$ , whereby all such bidders have the same probability of being selected.

To analyse equilibrium bidding in this auction we begin with the following observation:

**Proposition 1.** *The informed bidder has a weakly dominant strategy:  $b_I^*(s) = s$  for all  $s \in [\underline{s}, \bar{s}]$ .*

*Proof.* This follows from the standard argument that is used to show that in single object, private value, second price auctions bidding one's true value is a dominant strategy. ■

Given Proposition 1 we can focus on the behaviour of the uninformed bidders. We shall assume that all uninformed bidders play the same pure or

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<sup>2</sup>In the case  $n = k = 1$  the auction game which we are considering has very many equilibria. Since an analysis of these equilibria would distract from the main point of this paper, we restrict attention to the case  $n \geq 2$ .

<sup>3</sup>Equivalently we could assume that a perfectly divisible good is for sale. All bidders have constant marginal utility. The auctioneer splits the good into  $k$  identical lots and allows each bidder to bid for at most one of these lots.

<sup>4</sup>Engelbrecht-Wiggans, Milgrom, and Weber (1983) show that this assumption is equivalent to assume that  $v$  and  $s$  are two affiliated random variables.

mixed strategy. We shall describe this mixed strategy by its distribution function  $G_U^* : [\underline{s}, \bar{s}] \rightarrow [0, 1]$ . Notice that we rule out bids which are not in the interval  $[\underline{s}, \bar{s}]$ . Such bids are weakly dominated. We shall call a strategy of the uninformed bidders an *equilibrium* strategy if together with the weakly dominant strategy of the informed bidder it constitutes a Bayesian Nash equilibrium of the auction game.

We consider first two cases that allow for an analysis specially clear-cut. The first of these is when the number of units for sale is only one. Then, the only effect on the incentives to bid is the winner's curse as it was already suggested in the introduction. In this case, due to the absence of the loser's curse, the incentives to win are always higher for the informed than for the uninformed bidders. As expected, the next proposition states that there is a unique equilibrium where the uninformed bidders bid lower than any type of the informed bidder.

**Proposition 2.** *If there is only one unit for sale,  $k = 1$ , there is only one equilibrium strategy for the uninformed bidders, to bid  $\underline{s}$  with probability one.*

*Proof that the proposed strategy is an equilibrium strategy:* In the proposed equilibrium the uninformed bidders get utility zero. The only possible deviation for uninformed bidders is to raise their bids. If all uninformed bidders except one bid  $\underline{s}$ , and one uninformed bidder raises her bid to some value  $b > \underline{s}$ , then this uninformed bidder wins if and only if the informed bidder's bid is between  $\underline{s}$  and  $b$ . Moreover, the price which the uninformed bidder has to pay is exactly the informed bidder's bid which equals the true value of one unit. Therefore, the expected utility from raising the bid is zero. Thus, there is no strict incentive for uninformed bidders to raise their bids.

*Proof that there are no other equilibrium strategies:* Suppose all uninformed bidders choose the same mixed strategy, and assume that this strategy assigns positive probability to bids above  $\underline{s}$ . Then each uninformed bidder can gain by changing her strategy, and bidding  $\underline{s}$  with probability 1. To see this distinguish the following two events: (i) the highest of all other uninformed bidders' bids is greater than the informed bidders' bid; and (ii) the highest of all other uninformed bidders' bids is less than or equal to the informed bidders' bid. Observe that both events occur with positive probability. In event (ii) all bids give expected utility zero, thus the change in bidding strategy has no effect. In event (i), however, there is a strict incentive to be among the losers of the auction, this is, there is a winner's curse. If the bidder adopts the same mixed strategy as all other uninformed bidders, there is a positive probability that she is among the winners. Thus, she can strictly gain by deviating to  $\underline{s}$ . ■

**Remark 1.** *If  $k = 1$  : (i) The price is completely uninformative, since it is always equal to  $\underline{s}$ . (ii) The informed bidder wins with probability 1 the unique*

unit for sale. (iii) The informed bidder has positive expected utility whereas the uninformed bidders have expected utility zero.

The other specially simple case is when the number of units for sale equals the number of uninformed bidders. As we suggested in the introduction, then there is no winner's curse, but loser's curse. In this case, the incentives to win of the uninformed bidders are always greater than those of the informed bidder. As a consequence the uninformed bidders bid higher than any type of the informed bidder:

**Proposition 3.** *If there are  $n$  units for sale,  $k = n$ , there is only one equilibrium strategy for the uninformed bidders, to bid  $\bar{s}$  with probability one.*

*Proof that the proposed strategy is an equilibrium strategy:* In the proposed equilibrium the uninformed bidders have utility zero. This is because they all win with probability one, but the price equals the bid of the informed bidder, i.e. the value of the good. If an uninformed bidder lowers her bid, she loses the auction whenever the informed bidder's bid is above her lower bid. Otherwise she wins, but at a price which equals the informed bidder's bid. Hence her expected utility is again zero. Thus, no uninformed bidder can gain by deviating.

*Proof that there are no other equilibrium strategies:* Suppose all uninformed bidders choose the same mixed strategy, and assume that this strategy assigns positive probability to bids below  $\bar{s}$ . Then each uninformed bidder can gain by changing her strategy, and bidding  $\bar{s}$  with probability 1. To see this distinguish the following two events: (i) the lowest of all other uninformed bidders' bids is greater than or equal to the informed bidders' bid; and (ii) the lowest of all other uninformed bidders' bids is less than the informed bidders' bid. Observe that both events occur with positive probability. In event (i) all bids give expected utility zero, thus the change in bidding strategy has no effect. In event (ii), however, there is a strict incentive to be among the winners of the auction, this is, there is a loser's curse. If the bidder adopts the same mixed strategy as all other uninformed bidders, there is a positive probability that she is not among the winners. Thus, she can strictly gain by deviating to  $\bar{s}$ . ■

**Remark 2.** *If  $k = n$  : (i) The price reveals the true value. (ii) With probability 1 all units are won by uninformed bidders. (iii) All bidders have expected utility zero.*

In other cases, namely when  $1 < k < n$ , both the winner's curse and the loser's curse effects can affect the equilibrium outcome. The study of the interaction of these two effects requires a slightly different analysis than that of the previous cases. This analysis is done in the next proposition:

**Proposition 4.** *If  $1 < k < n$ , then there exists a unique equilibrium strategy for the uninformed bidders.<sup>5</sup>*

$$G_U^*(b) = \frac{F(b)(n-k)(b - E[s | s \leq b])}{F(b)(n-k)(b - E[s | s \leq b]) + (1 - F(b))(k-1)(E[s | s \geq b] - b)},$$

for all  $b \in [\underline{s}, \bar{s}]$ .

*Proof.* This proof is broken down into two steps.

*Step 1.* In the first step we consider mixed strategies of the uninformed bidder that have a continuous distribution function. A necessary condition for such strategies to be an equilibrium is that each uninformed bidder is indifferent between all the bids in the support, if she takes as given that all the other uninformed bidders adopt the proposed strategy, and that the informed bidder plays her weakly dominant strategy. This is just the standard indifference condition characterising Nash equilibria in mixed strategies, extended to the case of infinite strategy spaces.

This indifference condition is satisfied only if each uninformed bidder gets zero expected utility. To see why notice that the number of units for sale is less than the number of uninformed bidders, thus the lowest bid in the support of uninformed bidders' strategy must lose with probability one. Thus, she gets zero expected utility.

To apply this condition we distinguish two events under which an uninformed bidder can win the auction: (i) the price in the auction equals the bid of the informed bidder, and (ii) the price in the auction equals the bid of another uninformed bidder. Under event (i), the expected utility of winning is trivially zero, the price equals the value of the good. Hence, the expected utility of winning must also be zero under event (ii) for all bids of the uninformed bidders that are in the support of  $G_U^*$ . Or equivalently, the expected utility of winning under event (ii) and conditional on the price equals  $b$  must be zero almost surely.

To formalise the last necessary condition, we introduce for an arbitrary  $b$  in the support of the equilibrium mixed strategy of the uninformed bidders, the notation  $\mathbb{P}(b)$ . This stands for the probability that the informed bidder's bid,  $s$ , is greater than  $b$ , conditional on the following event: there are exactly  $k-1$  bids above  $b$  among  $n-2$  uninformed bidders' bids and the informed bidder's bid. This is the probability that an uninformed bidder suffers a loser's curse at price  $b$ . Similarly,  $1 - \mathbb{P}(b)$  is the probability that an uninformed bidder suffers a winner's curse at price  $b$ . Using this notation, we can write our necessary condition as:

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<sup>5</sup>Here and in the following  $E[. | .]$  denotes the expected value of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line.



$$\mathbb{P}(b)E[s|s \geq b] + (1 - \mathbb{P}(b))E[s|s < b] - b = 0, \quad (1)$$

almost surely. Here  $\mathbb{P}(b)$  equals by definition:

$$\frac{\binom{n-2}{k-2}[1 - F(b)][1 - G_U^*(b)]^{k-2}G_U^*(b)^{n-k}}{\binom{n-2}{k-2}[1 - F(b)][1 - G_U^*(b)]^{k-2}G_U^*(b)^{n-k} + \binom{n-2}{k-1}F(b)[1 - G_U^*(b)]^{k-1}G_U^*(b)^{n-k-1}}.$$

The unique  $G_U^*$  that solves the above necessary condition for a given  $b$  must be as defined in the proposition. Since this function is continuous, strictly increasing and satisfies  $G_U^*(\underline{s}) = 0$  and  $G_U^*(\bar{s})=1$ , the unique candidate for an equilibrium continuous distribution function must be that in the proposition.

It only remains to be shown that this distribution function is in fact an equilibrium strategy. This follows since we have already shown that the expected utility of each uninformed bidder given that the other uninformed bidders play the proposed strategy, and that the informed bidder plays her weakly dominant strategy, is zero for all bids in  $[\underline{s}, \bar{s}]$ .

*Step 2.* In this second step we study mixed strategies that has a discontinuous distribution function. Assume that  $G_U^*$  is one of such strategies with an atom at  $\hat{b}$ . We focus on the incentives to deviate of an uninformed bidder, say bidder  $l$ . For the sake of simplicity we introduce the following notation. Let  $b_{(k)}$  be the  $k$ -th highest bid of all the bidders but  $l$ . Define the event " $\hat{b}$  wins" to be the event in which bidder  $l$  when making a bid  $\hat{b}$  wins one unit, and the event " $\hat{b}$  loses" the complement of " $\hat{b}$  wins", this is the event in which bidder  $l$  when making a bid  $\hat{b}$  loses the auction.

We begin by arguing that we must have:  $E[v|b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] \geq \hat{b}$ . Suppose instead  $E[v|b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] < \hat{b}$ . If this were the case, then bidder  $l$  could gain by shifting all probability mass that is placed on  $\hat{b}$  to some bid  $\hat{b} - \epsilon$  where  $\epsilon > 0$  is close to zero. This change would obviously make no difference to player  $l$ 's utility in the case that  $b_{(k)} > \hat{b}$ , nor would it affect  $l$ 's utility in the case that  $b_{(k)} = \hat{b}$  and  $\hat{b}$  loses. Finally, it would obviously also not make any difference in the case that  $b_{(k)} < \hat{b} - \epsilon$ . In the event that  $b_{(k)} = \hat{b}$  and  $\hat{b}$  wins, which has positive probability, the change in strategy would lead to a strict increase in player  $l$ 's utility. Finally, the probability of the event that  $\hat{b} - \epsilon \leq b_{(k)} < \hat{b}$  can be made arbitrarily small by choosing a sufficiently small  $\epsilon$ , so that it does not affect the advantageousness of the proposed deviation.

In a similar way it can be argued that we must have  $E[v|b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ loses}] \leq \hat{b}$ .

If  $\hat{b} = \bar{s}$ , the event  $b_{(k)} = \hat{b}$  means that the bid of the informed bidder is below  $\bar{s}$ . As a consequence the first of the conditions above cannot be satisfied.

Similarly, it can be shown that  $\hat{b} = \underline{s}$  violates the second of the conditions above.

We can complete our indirect proof by arguing that if  $\underline{s} < \hat{b} < \bar{s}$ , then  $E[v|b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] < E[v|b_{(k)} = \hat{b}] < E[v|b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ loses}]$ , this is, that there is a winner's and a loser's curse at price  $\hat{p}$ . This last inequality obviously contradicts the other two inequalities. Suppose you knew that  $b_{(k)} = \hat{b}$ , but you did not know whether the informed bidder is bidding above or below  $\hat{b}$ . If you learned that the informed bidder is bidding above  $\hat{b}$ , then the probability that  $\hat{b}$  wins would drop. Hence,  $\hat{b}$  wins has strictly negative correlation with the event that the informed bidder is bidding above  $\hat{b}$ , conditional on  $b_{(k)} = \hat{b}$ . This implies that whenever  $\hat{b}$  wins it is ex post more likely that the informed bidder is bidding below  $\hat{b}$ , and vice versa when  $\hat{b}$  loses. ■

**Remark 3.** *If  $1 < k < n$  : (i) The price contains information about the true value, but it is an imperfect signal. (ii) All bidders have positive probability of winning. (iii) The informed bidder has positive expected utility, but the uninformed bidders have expected utility zero.*

In order to state the next result of the paper, we introduce the following definition:

**Definition:** We say that the uninformed bidders bid *relatively more aggressively* than the informed bidder the more units there are for sale if and only if for every  $s$  in  $(\underline{s}, \bar{s})$  the probability that the bid of an uninformed bidder is above  $b_I^*(s)$  increases when the number of units for sale increases.

We can now state next corollary:

**Corollary 1.** *The uninformed bidders bid relatively more aggressively than the informed bidder the more units there are for sale.*

*Proof.* The corollary follows trivially from Proposition 2 and Proposition 3 when the starting number of units for sale is 1 or when the final number of units for sale is  $n$  respectively. In other cases since the bid of the informed bidder is  $b_I^*(s) = s$ , it is enough to show that  $1 - G_U^*(b)$  increases with  $k$ . This follows directly from the definition of  $G_U^*$  in Proposition 4. ■

This corollary can be explained in terms of the winner's and loser's curse. Increasing the number of units for sale increases the relative strength of the loser's curse with respect to the winner's curse. This implies that the uninformed bidders' incentives to win increase more than those of the informed bidder. As a consequence, each uninformed bidder bids relatively more aggressively than the informed bidder.

Figure 1 shows the plot of the density of the distribution of the equilibrium mixed strategy of the uninformed bidders for  $k = \{2, 3, 4, 5\}$  given that  $F$  is a

uniform distribution function on  $[0, 1]$  and  $n = 6$ . Since the bid of the informed bidder does not change with  $k$ , this graph illustrates Corollary 1.

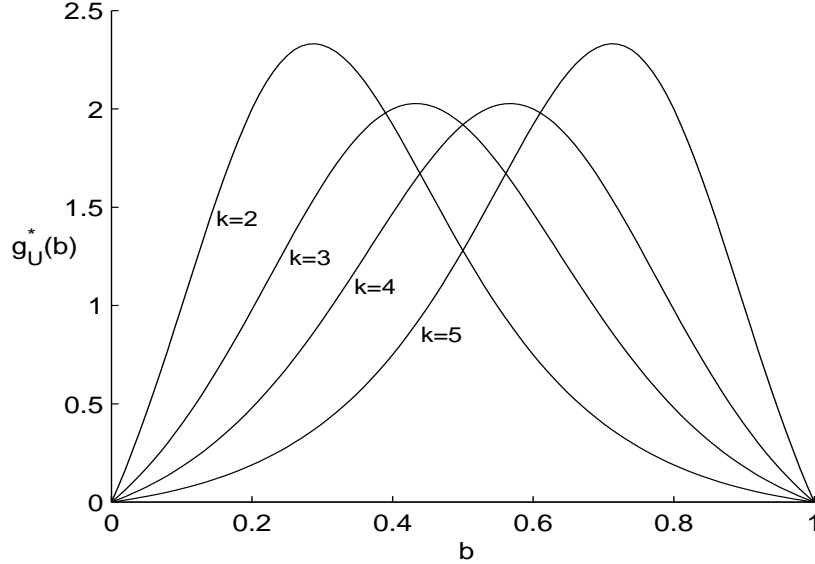


Figure 1: Plot of the density ( $g_U^*$ ) of  $G_U^*$  for  $n = 6$ .

### 3 An Auction in Which Uninformed Bidders have Positive Expected Utility

In the model of the previous section the uninformed bidders can win with positive probability but they will always receive zero expected utility. The purpose of this section is to construct a model in which the uninformed bidders can win, and their expected payoff is strictly positive.

As before, we assume that there are  $k$  units of the same good for sale. The number of bidders is now assumed to be  $n_I + n_U$ , where  $n_I + n_U > k$  and  $n_I < n_U$ . Among the bidders,  $n_I$  bidders are called *informed* bidders. Each of these bidders receives privately a signal  $s_i$ . The other  $n_U$  bidders are called *uninformed* bidders. They receive no signal. The value of the good,  $v$ , equals  $\frac{\sum_{i=1}^{n_I} s_i}{n_I}$ . We assume that the signals  $s_i$  are independently drawn from the set  $[\underline{s}, \bar{s}]$  ( $0 \leq \underline{s} < \bar{s}$ ) according to the same continuous distribution function  $F$  with support  $[\underline{s}, \bar{s}]$ . Bidders' preferences and the auction game are the same as in Section 2.

As in Section 2, we shall focus on symmetric equilibria. In this section, this will mean that all informed bidders play the same strategy, and all uninformed

bidders play the same strategy. For simplicity, we shall focus on equilibria in pure strategies instead of allowing for mixed strategies as in Section 2. We shall denote by  $b_I^* : [\underline{s}, \bar{s}] \rightarrow R^+$  the strategy of the informed bidders and by  $b_U^* \in R^+$  the bid of the uninformed bidders. We shall further simplify our arguments by assuming that the informed bidders play a continuous and strictly increasing strategy.

Some of the results of the previous section generalise in natural ways to the model of the current section. For example, in the case  $k \leq n_I$ , it can be proved that in the unique equilibrium outcome the uninformed bidders lose with probability one. Such equilibria generalise the equilibrium in Proposition 2. For the case  $n_I < k < n_U$  one can show that there is no equilibrium in pure strategies. In this respect this case is similar to the case of Proposition 4.

We shall not deal explicitly in this paper with the two cases mentioned in the previous paragraph. We shall also omit the rather special case  $k = n_U$ . Instead, we shall focus on the case that  $k > n_U$ . This case yields for our purposes the most interesting result. The result is similar to Proposition 3. We use the symbol  $s_{(r)}$  to refer to the  $r$ -th highest signal of the informed bidders.

**Proposition 5.** *Suppose  $k > n_U$ . Then the bidding strategies  $(b_I^*, b_U^*)$  constitute an equilibrium if and only if:*

$$\begin{aligned} b_I^*(s) &= E [v | s_{(q)} = s_{(q+1)} = s] \\ b_U^* &\geq E [v | s_{(q)} = s_{(q+1)} = \bar{s}]. \end{aligned}$$

Here, we define  $q \equiv k - n_U$ .

*These conditions are such that the uninformed bidders win with probability one in equilibrium.*

We first provide an example of equilibrium bidding functions. In this example we assume  $n_I = 6$ ,  $n_U = 8$ ,  $k = 10$  and  $F$  to be a uniform distribution function with support  $[0, 1]$ . Then  $b_I^* = 2/3s + 1/4$  and  $b_U^* = 11/12$  satisfy the conditions in Proposition 5. A plot of these equilibrium bidding functions appears in figure 2.

*Proof that the conditions in the Proposition are sufficient:* The strategy of the informed bidders is the same as in the standard symmetric equilibrium in an auction of  $q$  units and no uninformed bidders. The arguments used for the standard symmetric equilibrium by Milgrom (1981) to show that the bidders do not have incentives to deviate also explain why the informed bidders do not have incentives to deviate if the conditions in the proposition are satisfied.

To see that the uninformed bidders do not have incentives to deviate suppose that all the bidders stick to some strategies that satisfy the proposed conditions. Under this assumption we can state the following arguments. First,

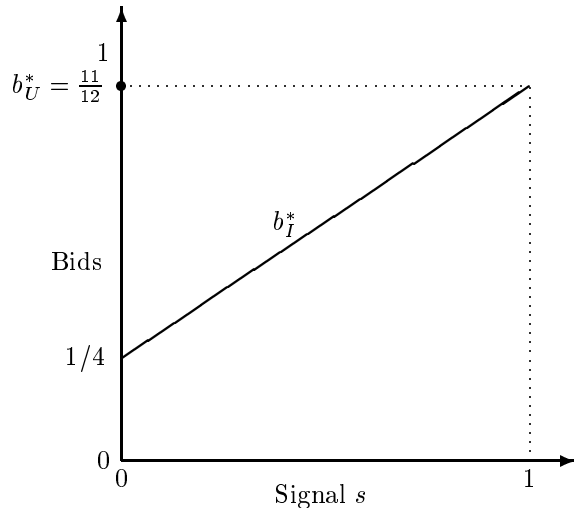


Figure 2: Equilibrium bidding functions with  $n_I = 6$ ,  $n_U = 8$  and  $k = 10$ .

each uninformed bidder does not have incentives to raise her bid since she is already winning with probability one. Second, if one uninformed bidder lowers her bid below  $b_U^*$  she does not improve her payoffs. This is because by deviating the bidder loses when the price in the auction is between her deviating bid and  $b_U^*$ , and the expected value of winning at a given price is positive for all the equilibrium prices. To see why, take an arbitrary price of the auction  $p$ . This price  $p$  must correspond to the equilibrium bid of a type  $s$  of the informed bidders. Then, the expected utility of winning at price  $p$  is the difference between the expected value of the good conditional on the  $q + 1$ -th highest signal equals  $s$ , and the price  $p$ . Since the price  $p$  equals the expected value of the good conditional on the  $q$ -th and  $q + 1$ -th highest signals equal  $s$  by definition of  $s$ , the difference is positive.

*Proof that the conditions in the Proposition are necessary:* Assume that the bid of the uninformed bidders is above all the bids of the informed bidders, then the informed bidders' equilibrium strategy must be an equilibrium strategy of the same auction game with  $q$  units for sale,  $n_I$  informed bidders and no uninformed bidders. In this last case, we can use a proof similar to that by Harstad and Levin (1986) to show that there is a unique equilibrium strategy. This equilibrium strategy is such that each informed bidder bids the expected value of the good given that the  $q$  and the  $q + 1$  highest signal of the informed bidders equal her own private signal. This is the equilibrium bidding function that appears in the proposition.

In order to complete the proof it only remains to be shown that the unin-

formed bidder cannot lose the auction with positive probability in equilibrium. This proof follows a similar structure and notation to *Step 2* of the proof of Proposition 4.

Assume that the uninformed bidders' bid,  $b_U^*$ , is below the maximum bid of the bidding function of the informed bidders. We focus on the incentives to deviate of an uninformed bidder, say bidder  $l$ . We reintroduce the notation  $b_{(k)}$  for the  $k$ -th highest bid of all the bidders but  $l$ . Define the event " $l$  wins" to be the event in which bidder  $l$  wins when making the bid  $b_U^*$  and the event " $l$  loses" as its complement, this is, the event in which bidder  $l$  loses when making the bid  $b_U^*$ .

We can use the same arguments as in *Step 2* of the proof of Proposition 4 to show that two necessary conditions of equilibrium are:  $E[v|b_{(k)} = b_U^* \text{ and } l \text{ wins}] \geq b_U^*$  and  $E[v|b_{(k)} = b_U^* \text{ and } l \text{ loses}] \leq b_U^*$ .

We start showing that these inequalities cannot be met simultaneously if  $b_U^*$  is between the minimum and the maximum bid of the informed bidders. We prove this using a random variable  $\tilde{I}$ <sup>6</sup> that stands for the number of informed bidders that bid above  $b_U^*$ . Our indirect proof is to show that the conditional distribution of  $\tilde{I}$  shifts in the sense of strictly first order stochastic dominance, upwards when  $l$  loses and downwards when  $l$  wins. Since we restrict attention to equilibria in which the informed bidders' bidding function is strictly increasing, this is sufficient for our claim.

By definition:<sup>7</sup>

$$\begin{aligned} \Pr\left(l \text{ wins} \mid \tilde{I} = I, k - n_U < \tilde{I} < k\right) &= \frac{k-I}{n_U} \\ \Pr\left(l \text{ loses} \mid \tilde{I} = I, k - n_U < \tilde{I} < k\right) &= \frac{n_U - k + I}{n_U}, \end{aligned}$$

where  $k - n_U < \tilde{I} < k$  is the same event than  $b_{(k)} = b_U^*$ . If  $\tilde{I}$  is less than  $k$  but more than  $k - n_U$ , the  $k$ -th highest bid of the other bidders is the bid of one of the uninformed bidders, all of which bid  $b_U^*$ .

Hence,

$$\frac{\Pr\left(l \text{ wins} \mid \tilde{I} = I, b_{(k)} = b_U^*\right)}{\Pr\left(l \text{ loses} \mid \tilde{I} = I, b_{(k)} = b_U^*\right)}$$

decreases strictly with  $I$ . Therefore,  $l$  wins and  $l$  loses, conditional on  $b_{(k)} = b_U^*$  can be interpreted as a pair of signals that satisfy the Monotone Likelihood Ratio Property. Consequently, the distribution of  $\tilde{I}$  conditional on  $l$  loses and

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<sup>6</sup>We use the standard notation  $\tilde{I}$  for the random variable and  $I$  for its realisation.

<sup>7</sup>Here and in the following  $\Pr(\cdot|\cdot)$  denotes the expected probability of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line.

$b_{(k)} = b_U^*$  strictly first order stochastically dominates the distribution of  $I$  conditional on  $l$  wins and  $b_{(k)} = b_U^*$ .

We complete our proof with the case in which  $b_U^*$  is equal or below the minimum bid of the informed bidders. In this case, since the events  $l$  wins,  $l$  loses and  $b_{(k)} = b_U^*$  are uninformative of  $v$ , the two inequalities above simplify to  $b_U^* = E[v]$ . Moreover, since the number of informed bidders is less than the number of units for sale, this means that the price in the auction is  $E[v]$  with probability one. Hence, an informed bidder with a signal low enough has incentives to deviate and bid below  $b_U^*$ . Bidding above means winning with probability one at price  $E[v]$  and bidding below losing with probability one. ■

**Proposition 6.** *If  $k > n_U$ , then the expected utility of each uninformed bidder is strictly positive and strictly greater than the unconditional expected utility of each informed bidder in equilibrium.*

*Proof.* Each uninformed bidder wins with probability one and pays the  $q + 1$  highest bid of the informed bidders. This difference is strictly positive because the expected value of the good given that the  $q + 1$  highest bid of the informed bidders equals the price is higher than the price for all potential prices. On the other hand, from an ex ante point of view, an informed bidder wins if and only if her signal is among the  $q$  highest signals of the informed bidders. This is with probability  $q/n_I$ . Conditional on winning, the informed bidder gets one unit of the good and pays the  $q + 1$  highest bid of the informed bidders, this is, she gets the same utility as each uninformed bidder. Since each uninformed bidder wins with probability one and each informed bidder only with probability  $q/n_I$  the proposition follows. ■

It is remarkable that, contrary to the previous section, competition among the uninformed bidders does not dissipate all the uninformed bidders' rents. The reason for this is that their demand ( $n_U$  units) is less than their supply ( $k$  units).

## 4 An Auction with One Informed and Many Poorly Informed Bidders

In this section we extend the analysis of Section 2 to a model where there is one informed bidder and some "poorly" informed bidders. The purpose of this extension is double. First, to show that similar results to those in Section 2 also hold in this more general set-up. Second, to prove that the equilibria in this extended model converge in an appropriate sense to the equilibria in the

model of Section 2 when the informativeness of the poorly informed bidders' signal goes to zero.

In this section we keep all the assumptions of Section 2 except the information structure. This is modified to allow for less informative signals.

More precisely, we assume that the value of the good  $v$  is a weighted average of one signal  $s$  and  $n$  signals  $s_i^P$  ( $i = 1, 2, \dots, n$ ). Each signal  $s_i^P$  is less informative about  $v$  than  $s$  in the sense of a smaller weight in the former average. Formally,  $v = \frac{s + \lambda \sum_{i=1}^n s_i^P}{1 + n\lambda}$ , with  $0 < \lambda < 1$ . We assume that the signal  $s$  and all the signals  $s_i^P$  are independently drawn from the set  $[\underline{s}, \bar{s}]$  ( $0 \leq \underline{s} < \bar{s}$ ) according to the same continuous distribution function  $F$  with support  $[\underline{s}, \bar{s}]$ . We shall assume that one bidder (the *informed* bidder) receives privately the signal  $s$ , whereas each of the other bidders (the *poorly informed* bidders) receives privately a different signal  $s_i^P$ .

An equilibrium of the game is a bidding function  $b_I^* : [\underline{s}, \bar{s}] \rightarrow \mathbb{R}^+$  for the informed bidder and a bidding function  $b_P^* : [\underline{s}, \bar{s}] \rightarrow \mathbb{R}^+$  for the poorly informed bidders, such that  $(b_I^*, b_P^*)$  is a Bayesian Nash equilibrium of the game.

For the sake of simplicity we restrict attention to equilibria in continuous and strictly increasing strategies. Moreover, only equilibria in which all the bidders have an unconditional positive probability of winning the auction are analysed. This constraint rules out some strange equilibria that exist in the case  $k = 1$  and  $k = n$ .<sup>8</sup>

Next, we introduce an assumption that simplifies the analysis of equilibria.

**Assumption 1.**  $\sigma - E[s | s \leq \sigma]$  and  $E[s | s \geq \sigma] - \sigma$  are respectively strictly increasing and strictly decreasing in  $\sigma$ .

Assumption 1 is satisfied by many distribution functions. If  $F$  has a continuously differentiable density, Bikhchandani and Riley (1991) show (Lemma 3) that a sufficient condition for the first part of the assumption is that  $F(s)$  is strictly log-concave. Similarly, if  $F$  has a continuously differentiable density, a sufficient condition for the second part is that  $1 - F(s)$  is strictly log-concave, this is that  $F(s)$  has an increasing hazard rate.

For the analysis of this section we use a function  $\phi : [\underline{s}, \bar{s}] \rightarrow [\underline{s}, \bar{s}]$ . This function assigns to each type  $s^P$  of the poorly informed bidders the type  $\phi(s^P)$  of the informed bidder who, in equilibrium, makes the same bid  $b$  as  $s^P$  does.

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<sup>8</sup>For instance, if  $k = 1$ , there exists a set of equilibria where the informed bidder bids high enough and the poorly informed bidders bid low enough. In this case, the informed bidder does not have incentives to deviate because she wins with probability one at a very low price. On the other hand, a poorly informed bidder could win only if she would bid above the very high bid of the informed bidder. Since in this case winning would mean paying the very high bid of the informed bidder, the poorly informed bidders do not have incentives to deviate. Equilibria of this type have been called in a different set-up *degenerate* by Bikhchandani and Riley (1991).



This function will be implicitly defined by an equation  $\Psi(\phi(s^P), s^P) = 0$ , where  $\Psi : [\underline{s}, \bar{s}]^2 \rightarrow \mathbb{R}$  is the difference between the expected value of the good given that there are  $k - 1$  bidders bidding above  $b$ , two poorly informed bidders bidding  $b$  and all the other bidders bidding below  $b$ ; and the expected value of the good given that there are  $k - 1$  poorly informed bidders bidding  $b$ , the informed bidder and one poorly informed bidder bidding  $b$  and all the other bidders bidding below  $b$ . In order to express this condition formally we project the first expected value on the events  $s \geq \phi$  and  $s < \phi$ :

$$\begin{aligned} \Psi(\phi, s^P) = & \mathbb{P}^*(\phi, s^P)E[v|s \geq \phi, s_{(k-1)}^P = s_{(k)}^P = s^P] \\ & + (1 - \mathbb{P}^*(\phi, s^P))E[v|s < \phi, s_{(k)}^P = s_{(k+1)}^P = s^P] \\ & - E[v|s = \phi, s_{(k)}^P = s^P], \quad (2) \end{aligned}$$

where  $\mathbb{P}^*(\phi, s^P)$  is the probability that the bid of the informed bidder is above  $b$  given that there are  $k - 1$  bidders bidding above  $b$  among the bid of the informed bidder and the bids of  $n - 2$  poorly informed bidders. This is  $\mathbb{P}^*(\phi, s^P) = 0$  if  $k = 1$ ,  $\mathbb{P}^*(\phi, s^P) = 1$  if  $k = n$  and if  $1 < k < n$ :

$$\mathbb{P}^*(\phi, s^P) = \frac{\binom{n-2}{k-2}[1 - F(\phi)][1 - F(s^P)]^{k-2}F(s^P)^{n-k}}{\binom{n-2}{k-2}[1 - F(\phi)][1 - F(s^P)]^{k-2}F(s^P)^{n-k} + \binom{n-2}{k-1}F(\phi)[1 - F(s^P)]^{k-1}F(s^P)^{n-k-1}}.$$

**Lemma 1.** *There exists a unique  $\phi(s^P)$ . This function  $\phi$  is continuous and strictly increasing in  $s^P$ .*

*See the proof in the Appendix.*

The next proposition makes use of the function  $\phi$  to characterise the set of equilibrium bidding functions.

**Proposition 7.** *The pair of bidding strategies  $(b_I^*, b_P^*)$  is an equilibrium, if and only if:*

$$b_P^*(s^P) = b_I^*(\phi(s^P)) = E[v|s = \phi(s^P), s_{(k)}^P = s^P], \quad (3)$$

for all  $s^P \in [\underline{s}, \bar{s}]$ , and:

- If  $k = 1$ :

$$\int_{\phi(\bar{s})}^{\bar{s}} (E[v|\tilde{s} = \nu, s_{(1)}^P = \bar{s}] - b_I^*(\nu))f(\nu)d\nu \leq 0, \quad (4)$$

for all  $s$  in  $[\phi(\bar{s}), \bar{s}]$ .

- If  $k = n$ :

$$\int_s^{\phi(\underline{s})} (E[v|\tilde{s} = \nu, s_{(n)}^P = \underline{s}] - b_I^*(\nu)) f(\nu) d\nu \geq 0, \quad (5)$$

for all  $s$  in  $[\underline{s}, \phi(\underline{s})]$ .

See the proof in the Appendix.

Next lemma shows that if  $1 < k < n$ , equation (3) determines uniquely not only  $b_P^*$  but also  $b_I^*$ . In the other cases, Proposition 7 does not determine a unique  $b_I^*$ .

**Lemma 2.**

- (i) If  $k = 1$ , then  $\phi(\underline{s}) = \underline{s}$  and  $\phi(\bar{s}) < \bar{s}$ .
- (ii) If  $k = n$ , then  $\phi(\underline{s}) > \underline{s}$  and  $\phi(\bar{s}) = \bar{s}$ .
- (iii) If  $1 < k < n$ , then  $\phi(\underline{s}) = \underline{s}$  and  $\phi(\bar{s}) = \bar{s}$ .

See the proof in the Appendix.

In Figures 3, 4 and 5 there appear some examples of equilibrium bidding functions that illustrate Lemma 2. All these examples are done assuming that  $F$  is a uniform distribution function with support  $[0, 1]$ .

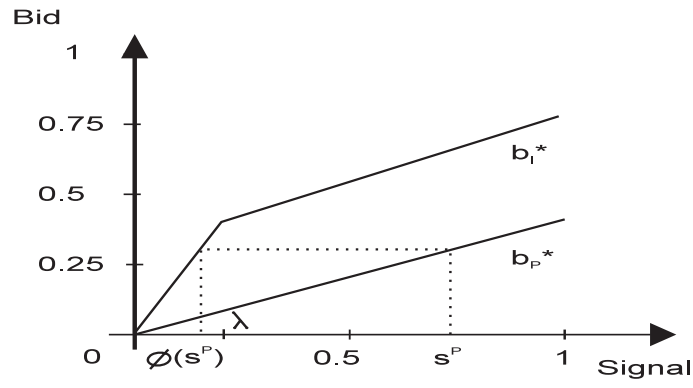


Figure 3: Equilibrium bidding functions with  $k = 1$ ,  $n = 5$  and  $\lambda = 0.2$ .

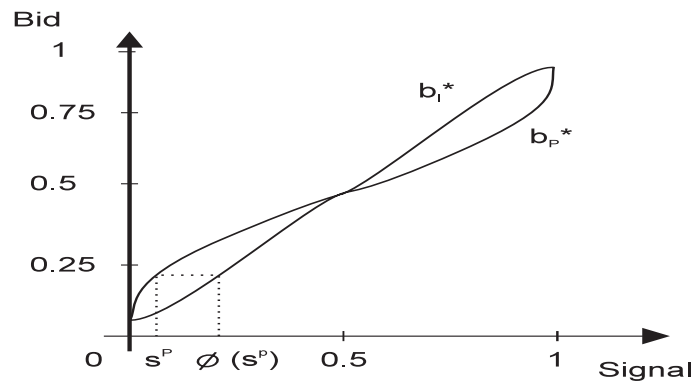


Figure 4: Equilibrium bidding functions with  $k = 3$ ,  $n = 5$  and  $\lambda = 0.2$ .

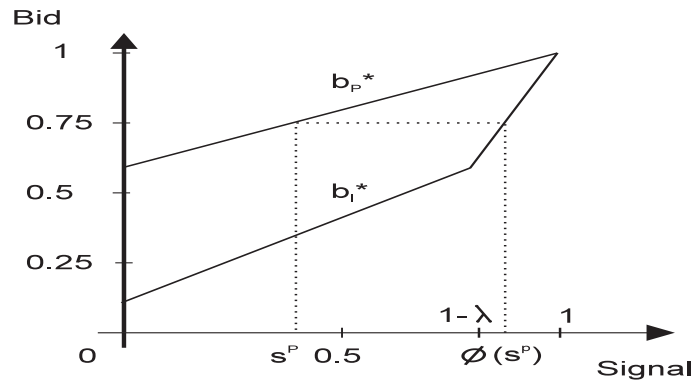


Figure 5: Equilibrium bidding functions with  $k = 5$ ,  $n = 5$  and  $\lambda = 0.2$ .

We apply Proposition 7 to show that we can state a similar result to Corollary 1 in Section 2 in this new framework. We start with the following definition:

**Definition:** We say that the poorly informed bidders bid *relative more aggressively* than the informed bidder the more units are for sale if and only if for every  $s$  in  $(\underline{s}, \bar{s})$  the probability that the bid of a poorly informed bidder is above  $b_I^*(s)$  increases when the number of units for sale increases.

We can now state the next corollary:

**Corollary 2.** *Each poorly informed bidder bids relatively more aggressively than the informed bidder the more units there are for sale.*

*See the proof in the Appendix.*

The last point of this section is to provide a robustness test for the equilibrium outcomes given in Section 2. The next proposition accomplishes this task.

**Proposition 8.** *When  $\lambda$  goes to zero:*

- (i) *If  $k = 1$ , then the equilibrium bidding function of the poorly informed bidders converges point-wise to  $\underline{s}$ . Moreover, in the limit each poorly informed bidder loses with probability one.*
- (ii) *If  $k = n$ , then the equilibrium bidding function of the poorly informed bidders converges point-wise to  $\bar{s}$ . Moreover, in the limit each poorly informed bidder wins with probability one.*
- (iii) *If  $1 < k < n$ , then the equilibrium bidding function of the informed bidder converges point-wise to  $b_I^*(s) = s$ , the equilibrium bidding function of the informed bidder in Section 2. The equilibrium distribution of bids of the poorly informed bidders converges to the equilibrium distribution of bids of the uninformed bidders in Proposition 4, Section 2.*

*See the proof in the Appendix.*

## 5 Conclusions

In this paper we have studied several models in which there were one or more well informed and some other bidders with either no information or worse information. We have shown in this set-up that the relative performance of

the more informed-less informed bidders depends on the interaction of two effects, the winner's curse and the loser's curse, and its differential effect on bidders with different quality of information. Basically, we showed that the leading effect is the loser's curse if the number of units for sale is large and the winner's curse if it is small. We based on these arguments to explain the surprising result that when there are several units for sale, bidders with less information can do better than bidders with better information in terms of probability of winning and expected revenue.

Our analysis leaves several extensions for further research. We distinguish two profitable ways of complementing our analysis. The first one is the study of other auction procedures, especially the open ascending auction. This popular auction is similar to the (generalised) second price auction studied in this paper. The main difference to our concern is that the losers' bids are revealed along the auction and so the winner's curse is very much alleviated. Thus, we could expect that the loser's curse dominates (if there is more than one unit for sale) and consequently, that less informed bidders do better than more informed bidders. The second branch of extensions is the study of the consequences for auction design of our analysis on the performance of less informed bidding. For instance, this should be a major worry if the auctioneer concern is to promote the entry of poorly informed or uninformed bidders, or if he is interested in providing incentives to acquire information.

## Appendix

In this appendix we provide the proofs of the lemmas and propositions in Section 4. To follow our arguments more easily, it is useful to notice that  $\Psi$  simplifies to:

$$\begin{aligned} \Psi(\phi, s^P) = & \\ & \frac{(k-1)(1-F(\phi))F(s^P)}{A} [E[s|s \geq \phi] - \phi - \lambda(E[s|s \geq s^P] - s^P)] - \\ & \frac{(n-k)F(\phi)(1-F(s^P))}{A} [\phi - E[s|s \leq \phi] - \lambda(s^P - E[s|s \leq s^P])], \quad (6) \end{aligned}$$

where  $A \equiv (k-1)(1-F(\phi))F(s^P) + (n-k)F(\phi)(1-F(s^P))$ .

*Proof of Lemma 1.* We start the proof showing by contradiction that  $E[s|s \geq \phi] - \phi - \lambda(E[s|s \geq s^P] - s^P)$  and  $\phi - E[s|s \leq \phi] - \lambda(s^P - E[s|s \leq s^P])$  must be non negative when  $\Psi(\phi, s^P) = 0$ . Assume that one is negative. Then  $\Psi(\phi, s^P) = 0$  implies that they are both negative. But under Assumption 1, if the first expression is negative,  $\phi > s^P$ , and if the second expression is negative,  $\phi < s^P$ .

As a consequence, under Assumption 1,  $\Psi$  must be strictly decreasing in its first argument and strictly increasing in its second when  $\Psi(\phi, s^P) = 0$ . Hence, the lemma follows as  $\Psi$  is continuous. ■

*Proof of Proposition 7.*

*Necessary proof.* A local necessary condition that must be satisfied by the equilibrium bidding functions almost everywhere is the following: if marginal changes in a bidder's bid change marginally her probability of winning, then her conditional bid must be equal to the expected value of the good conditional on her private information and on the  $k$ -th highest bid of the other bidders equals her bid.

Our proof starts with the study of a set of bids  $B$  where the condition above binds. This set  $B$  is defined as the intersection of the interior of the range of the bidding function of the informed bidder and the interior of the range of the bidding function of the poorly informed bidders.

The first step is to show that  $B$  is not empty. Since the functions  $b_I^*$  and  $b_P^*$  are by assumption continuous and strictly increasing,  $B$  can be empty if and only if either  $b_I^*(\underline{s}) \geq b_P^*(\bar{s})$  or  $b_P^*(\underline{s}) \geq b_I^*(\bar{s})$ . We only need to check that none of these two possibilities can happen in equilibrium.

If  $k = 1$  then it can be neither  $b_I^*(\underline{s}) \geq b_P^*(\bar{s})$  nor  $b_P^*(\underline{s}) \geq b_I^*(\bar{s})$ . The reason is that in these cases there are bidders that lose the auction with probability one, and by assumption we rule out this possibility in equilibrium. If  $k = n$  the case  $b_P^*(\underline{s}) \geq b_I^*(\bar{s})$  is ruled out because of identical reasons.

If  $k > 1$  it cannot be that  $b_I^*(\underline{s}) \geq b_P^*(\bar{s})$ . To see why, assume that this is the case. Then, the informed bidder gets one unit with probability one and the poorly informed bidders compete for the  $k - 1$  units left. As a consequence, the equilibrium strategies of the poorly informed bidder must be equilibrium strategies of an auction with  $n$  poorly informed bidders,  $k - 1$  units for sale and without informed bidder. In this last case, we can use a proof similar to that by Harstad and Levin (1986) to show that there is a unique equilibrium strategy such that:  $b_P^*(s^P) = E[v|s_{(k-1)}^P = s_{(k)}^P = s^P]$ . But given this bidding function, types of the informed bidder with a very low signal have incentives to bid below  $b_P^*(\bar{s})$ . This is because the expected utility of winning the auction of a type of the informed bidder  $s'$  conditional on the price close enough to  $b_P^*(\bar{s})$  is arbitrary close to the difference between the expected value conditional on  $b_{(k)} = b_P^*(\bar{s})$  ( $E[v|s = s', s_{(k)}^P = \bar{s}]$ ) and  $b_P^*(\bar{s})$  ( $E[v|s = s', s_{(k)}^P = s_{(k+1)}^P = \bar{s}]$ ), that is strictly negative for  $s'$  close enough to  $\underline{s}$ .

Similarly, it can be shown that if  $1 < k < n$ , it cannot be that  $b_P^*(\underline{s}) \geq b_I^*(\bar{s})$ . In this case, types of the informed bidder with signals close enough to  $\bar{s}$  have incentives to rise their bids.

We continue our argument considering a generic bid  $b$  in  $B$ . The local necessary condition implies that the type of the informed bidder that bids

$b$  must be such that  $b$  equals the expected value of the good conditional on the private information of this type and on the  $k$ -th highest bid of the poorly informed bidders equals the bid  $b$ . If we define  $\sigma(b)$  and  $\sigma^P(b)$  as the inverse bidding functions of the informed bidder and of the poorly informed bidder respectively, we can formalise this condition as:

$$b = E [v | s = \sigma(b), s_{(k)}^P = \sigma^P(b)]. \quad (7)$$

Similarly, the local necessary condition implies that the type of the poorly informed bidders that bids  $b$  must be such that  $b$  equals the expected value of the good conditional on the information of this type and on the  $k$ -th highest bid of the other bidders equals the bid  $b$ . In order to simplify this condition we distinguish two different events: either (i) the  $k$ -th highest bid of the other bidders is the bid of the informed bidder, or (ii) the  $k$ -th highest bid of the other bidders is the bid of another poorly informed bidder. Both events happen with strictly positive probability. The local necessary condition of the poorly informed bidders is identical to the local necessary condition of the informed bidder under event (i). This implies that the local necessary condition of the poorly informed bidders must also be satisfied under event (ii). We formalise this last condition projecting on two events: the event the informed bidder's bid is above  $b$  and the event the informed bidder's bid is below  $b$ :

$$b = \mathbb{P}^*(\sigma, \sigma^P) E [v | s \geq \sigma(b), s_{(k-1)}^P = s_{(k)}^P = \sigma^P(b)] + (1 - \mathbb{P}^*(\sigma, \sigma^P)) E [v | s \leq \sigma(b), s_{(k)}^P = s_{(k+1)}^P = \sigma^P(b)]. \quad (8)$$

Using simultaneously equations (7) and (8) we get the condition:  $\Psi(\sigma(b), \sigma^P(b)) = 0$ . Thus,  $\sigma(b) = \phi(\sigma^P(b))$  for all  $b$  in  $B$ .

We next show that  $B = (b_P^*(\underline{s}), b_P^*(\bar{s}))$ . Suppose that the infimum of  $B$  is not  $b_P^*(\underline{s})$ . Since  $b_I^*$  and  $b_P^*$  are continuous and strictly increasing,  $b_I^*(\underline{s})$  must be strictly greater than  $b_P^*(\underline{s})$ , but this contradicts that  $\phi$  is strictly increasing and that  $\phi(\underline{s}) \geq \underline{s}$  according to its definition. Similarly, we can show that the supremum of  $B$  must be  $b_P^*(\bar{s})$ .

To complete this part of the proof it only remains to be shown that the inequality (4) is necessary if  $k = 1$  and the inequality (5) is necessary if  $k = n$ . We only consider the case  $k = 1$ , the other case can be proved in a symmetric way. Suppose that there is an  $s$  in  $[\phi(\bar{s}), \bar{s}]$  such that the inequality (4) is not satisfied. Then, it can be shown that the types of the poorly informed bidders arbitrarily close to  $\bar{s}$  are better off bidding  $b_I^*(s)$  than their equilibrium bid.

*Sufficient Proof.* We start with two remarks that we use to show that no type of the bidders has incentives to lower her bid. The first remark is that a type of one bidder does not have incentives to lower her bid if lower types do not have

incentives to do so. The reason is that higher types of a bidder put higher value on winning than lower types, they expect to pay the same price but they give a higher expected value to the good. The second remark is that the necessary proof conducted above showed that the conditions of the proposition assure that the type  $\underline{s}$  of the informed bidder and of the poorly informed bidders do not have incentives to lower her bid and that no type of the bidders has incentives to reduce her bid locally. Since we restrict attention to continuous and increasing bidding functions, the two remarks above are enough to prove that no type of no bidder has incentives to lower her bid.

Similarly, we can show that no type of no bidder has incentives to rise her bid. ■

*Proof of Lemma 2.*

- (i) Define  $\eta(s) = s - E[\tilde{s} | \tilde{s} \leq s]$ . Assumption 1 assures  $\eta$  is strictly increasing for all  $s \in [\underline{s}, \bar{s}]$ . If  $k = 1$ , then  $\phi(s^P) = \eta^{-1}(\lambda\eta(s^P))$ . Hence,  $\phi(\bar{s}) = \eta^{-1}(\lambda\eta(\bar{s})) < \eta^{-1}(\eta(\bar{s})) = \bar{s}$ .
- (ii) Define  $\mu(s) = E[s | s \geq \underline{s}] - s$ . Assumption 1 assures  $\mu$  is strictly decreasing for all  $s \in [\underline{s}, \bar{s}]$ . If  $k = n$ , then  $\phi(s^P) = \mu^{-1}(\lambda\mu(s^P))$ . Hence,  $\phi(\underline{s}) = \mu^{-1}(\lambda\mu(\underline{s})) > \mu^{-1}(\mu(\underline{s})) = \underline{s}$ .

The other claims are trivial since  $\Psi(\phi, s^P)$  has a straight forward unique solution in those cases. ■

*Proof of Corollary 2.* To prove the corollary it is enough to show that the type of the informed bidder that bids the same bid as a given type of the poorly informed bidders in equilibrium increases when the number of units increases. Since by Proposition 3  $b_P^*(s^P) = b_I^*(\phi(s^P))$ , the statement before follows if  $\phi(s^P)$  shifts upwards when we increase  $k$ . We can use the same arguments than in the proof of Lemma 1 to show that  $\Psi$  is strictly decreasing in its first argument and shifts upwards when we increase  $k$  around points such that  $\Psi(\phi, s^P) = 0$ . This completes the proof. ■

*Proof of Proposition 8.*

- (i) We use the function  $\eta$  defined in the proof of Lemma 2. Since  $\eta(\underline{s}) = 0$ ,  $\lim_{\lambda \rightarrow 0} \phi(s) = \lim_{\lambda \rightarrow 0} \eta^{-1}(\lambda\eta(s)) = \eta^{-1}(0) = \underline{s}$ .
- (ii) We use the function  $\mu$  defined in the proof of Lemma 2. Since  $\mu(\bar{s}) = 0$ ,  $\lim_{\lambda \rightarrow 0} \phi(s) = \lim_{\lambda \rightarrow 0} \mu^{-1}(\lambda\mu(s)) = \mu^{-1}(0) = \bar{s}$ .
- (iii) The first part follows directly from  $\lim_{\lambda \rightarrow 0} b_I^*(s) = s$  because of Proposition 3. For the second part, notice that  $\Psi$  is continuous in  $\lambda$  and has unique solution to  $\Psi(\phi, s^P) = 0$  for all  $s^P \in [\underline{s}, \bar{s}]$  when  $\lambda = 0$ . Then



$\phi(s^P)$  must satisfy in the limit  $\Psi(\phi(s^P), s^P) = 0$ . Given that in the limit  $b_I^*(s) = s$ , we can write the condition before as  $\Psi(b, \phi^{-1}(b)) = 0$ . This implies that in the limit:  $F(\phi^{-1}(b)) = G_U^*(b)$  for all  $s \in [\underline{s}, \bar{s}]$ , where  $G_U^*(b)$  is as defined in Proposition 4.

■

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