Precautionary Bidding:
First Price Auctions with Stochastic Values*

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Abstract
We consider a first-price auction where risk-averse bidders bid for an object whose value is risky, and provide the first analysis of the pure comparative statics of risk on bidding behavior. In the private values model we show that as risk increases, decreasingly risk-averse bidders will reduce their bids by more than the risk premium (we term the effect "precautionary bidding"). Ceteris paribus, bidders will be better off bidding for a more risky object. This effect arises because as risk increases, so does the expected marginal utility of income, so bidders are reluctant to bid so highly. Even in the presence of this effect, the expected revenue of a first price auction remains higher than that of a second price auction. We also show how this result extends to common values.

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1 Introduction

In many real world auctions, the value of the object for sale is subject to ex post risk. At the time of the sale, bidders can only estimate the value of the object, and they are well aware that the true value to them will be revealed only (some time) after the sale. The sale of oil tracts, art, antiques and wine provide obvious examples, as well as many procurement contracts. Despite the ubiquitousness of risk, there has been no analysis of the effects of ex post risk on bidding behavior. This paper provides the first such analysis.

We show that as risk increases, *decreasingly risk-averse bidders will reduce their bids by more than the risk premium.* Ceteris paribus, *bidders will be better off bidding for a more risky object.* Effectively, bidders engage in "precautionary bidding": as in precautionary saving problems, risk increases the marginal utility of income; bidders value each extra dollar of income more highly vis a vis increased probability of winning the object, and so bid less aggressively (Proposition 1). This is not trivial since (under some conditions) decreasingly risk-averse individuals become more risk-averse in facing one risk (i.e., losing the object) when forced to face an independent risk (i.e., object value).[1] And increasing risk aversion leads to more aggressive bidding in a first price auction.[2] However, the latter effect turns out to be of second order compared to the precautionary bidding effect.

This effect may reduce the advantage of the first price auction over the second price auction for the seller (in the case of decreasingly risk averse bidders). However, we show that the effect is always smaller than the expected revenue difference of the first and second price auctions; in other words, the expected revenue ranking cannot be reversed by the introduction of the ex post risk (Proposition 2).

The seller (facing bidders with DARA preferences) has incentive to reduce the riskiness of the valuations because that directly increases the bidders' willingness to pay, and also, because it intensifies competition. Even a seller who is as risk averse as the bidders are may provide insurance against the noise. This finding is similar in spirit to the linkage principle (although the context is quite different).

To the extent that risk has been incorporated into auction models, it is usually in the context of common (not purely private) values: one's valuation

[1] See Kimball (1993), Pratt and Zeckhauser (1987), Eeckhoudt et al (1996).

[2] See Maskin and Riley (1984), Milgrom and Weber (1982).

is uncertain because it depends on the other bidders' information. The most important consequence of the common values assumption is the (rational) winner's curse effect. However, the winner's curse does not arise because of the ex post riskiness of the object's value, but because winning provides information about the value of the object (even risk-neutral bidders should bid less aggressively in that case). Conversely, it is very possible for the private value of an object to an individual to be risky without any winners curse implications for bidding, since here winning provides no information about the object's value.

To avoid confusion of the winner's curse and the precautionary bidding effects, we first completely abstract from interdependencies in the valuations. We extend the analysis to common values in section 3. Our main finding is that decreasingly risk averse bidders are better off with a riskier object provided that the number of bidders is sufficiently large (Proposition 3).

Throughout the paper we use a symmetric model where valuations have a deterministic and an ex post stochastic component. In section 2 (private values) the deterministic component is private and independent, i.e., it equals the signal that the given buyer receives. In section 3 (common values) the deterministic part is a weighted average of the independent private signals of all buyers.

In both sections, the stochastic component is independent of the deterministic component. However, we do *not* require it to be independent or different across bidders. We can interpret the "noise term" as a result of common shocks (oil price or the amount of oil underground) or buyer-specific, but then symmetric, shocks (unforeseen production costs).

2 Private values

In this section we first describe the private values model and derive the first order condition characterizing the equilibrium bid function. Then we turn to the main results concerning the effects of precautionary bidding under private values. In the last subsection we show illustrative numerical examples.

2.1 The private values model

To keep things as simple as possible, in this section we use a symmetric private values framework with n risk-averse bidders. We will use as a benchmark the familiar riskless case where each bidder i values the object at v_i, with v_i distributed i.i.d. on $[\underline{v}, \overline{v}]$ with distribution function F. We will assume that F is strictly log-concave; i.e., $\frac{d}{dx}[F'(x)/F(x)] < 0$.[3] We assume that v_i is private information to bidder i.

In order to ascertain the effects of changes in the riskiness of the object as perceived by the bidders, we will consider subjecting each bidder i's value to a random mean zero shock, denoted $\tilde{z}_i$. Thus bidder i's actual valuation for the object is

$$\tilde{v}_i = v_i + \tilde{z}_i.$$

The bidders do not know the realization of their own shock $\tilde{z}_i$ at the time of bidding, and thus they regard the value of the object as being risky. The $\tilde{z}_i$-s may be correlated or independent across bidders, but their ex ante distribution must be symmetric. Even if the shock is purely common, i.e., $\tilde{z}_i = \tilde{z}$ for all i, there are no winner's curse effects because (the realization of) $\tilde{z}$ is not known at the time of bidding.

The bidders evaluate the object in monetary terms, according to the von Neumann-Morgenstern utility function u, with $u(0) = 0$, $0 < u' < \infty$, $u'' < 0$. A buyer with valuation v_i who wins the object and pays price p has utility $U(v_i, p) = E_{\tilde{z}_i} u(v_i + \tilde{z}_i \Leftrightarrow p)$.[4] Zero initial wealth is an innocent normalization; in fact, $u'(0) < \infty$ suggests that bidders have some wealth. The utility of a buyer who does not get the object is 0, by normalization.

Suppose that in the equilibrium of a first price auction buyers use the (same, increasing) bid function $\beta(v_i)$.[5] A bidder with information v_i bids as if he had type $\hat{v}_i$ to maximize expected utility, $F^{n-1}(\hat{v}_i) E_{\tilde{z}_i} u(v_i + \tilde{z}_i \Leftrightarrow \beta(\hat{v}_i))$. The equilibrium is efficient, so that a bidder with (reported) type $\hat{v}_i$ will win with probability $F^{n-1}(\hat{v}_i)$.

Differentiating with respect to $\hat{v}_i$, substituting in the Nash-equilibrium condition $\hat{v}_i = v_i$, we obtain the following first order condition:

[3] With weak log-concavity, all inequalities in the paper hold weakly. In order to avoid tedious duplication of the statements we do not treat that case separately.

[4] We could work out our results with different classes of utility functions as well. For example, we could use the general form $U(v, p)$, where v, the object's value, and p, the transfer paid, enter as separate arguments.

[5] Maskin and Riley (1984, 1996a, 1996b) have shown the existence and uniqueness of the equilibrium of a first price auction. They also show that in a symmetric set-up, the unique equilibrium is symmetric and efficient, and therefore $\beta(v_i)$ is increasing.

$$\beta'(v_i) = (n \Leftrightarrow 1)\frac{F'(v_i)}{F(v_i)}\frac{E_{\tilde{z}_i} u(v_i + \tilde{z}_i \Leftrightarrow \beta(v_i))}{E_{\tilde{z}_i} u'(v_i + \tilde{z}_i \Leftrightarrow \beta(v_i))}. \tag{1}$$

This differential equation, together with the boundary condition $\beta(\underline{v}) = \underline{v} \Leftrightarrow \pi(\underline{v})$, determines the equilibrium bid function $\beta(v_i)$.

As a benchmark, we also define $b(.)$ the (unique, symmetric) equilibrium bid function when the object's value is deterministic, i.e., when $\tilde{z}_i \equiv 0$. It is well known (and follows from equation 1) that $b(.)$ solves

$$b'(v_i) = (n \Leftrightarrow 1)\frac{F'(v_i)}{F(v_i)}\frac{u(v_i \Leftrightarrow b(v_i))}{u'(v_i \Leftrightarrow b(v_i))}, \quad b(\underline{v}) = \underline{v}. \tag{2}$$

The interpretation of the first order conditions (1) or (2) is familiar. Bidders trade off the gains from winning the object with greater probability against the gains from paying less for the object when they win.

From (1), the intuition for our later results is straightforward. For all plausible utility functions, risk increases the marginal utility of income. When values are risky, bidders value the money they can save by shading their bids slightly relatively more. With decreasing absolute risk aversion, this marginal utility effect persists even after bids have been shaded by the whole amount of the risk premium, so bidders shade their bids by more than the risk premium. *This means that they will actually be better off in expectation if the object is risky.* We may call this the "precautionary bidding" effect.[6]

2.2 Results under private values

In order to prove the result that decreasingly risk-averse bidders can become better off when the auctioned object becomes more risky, a few intermediate steps are required.

We first note that – in the case of a riskless object – bidders with higher values v_i have higher surplus $u(v_i \Leftrightarrow b(v_i))$ conditional on winning the auction.

[6]It is the same effect that induces precautionary saving. When future income becomes risky, the marginal utility of future income goes up, and so income is shifted into the future by saving. With DARA preferences, the precautionary premium (the amount of money that must be given to compensate marginal utility) exceeds the risk premium (the amount of money that must be given to compensate total utililty). See Kimball (1990).

This is true because the bid function has slope less than one if the distribution function F is strictly log-concave (Maskin and Riley, 1996).[7]

We next formally define the (compensating) risk premium $\pi(v_i)$ associated with the risk $\tilde{z}_i$ as the solution to

$$u(v_i \Leftrightarrow b(v_i)) \equiv E_{\tilde{z}_i} u(v_i + \tilde{z}_i \Leftrightarrow b(v_i) + \pi(v_i)). \tag{3}$$

This risk premium would exactly compensate buyer i for $\tilde{z}_i$ in the equilibrium with $b(v_i)$. It is a function of wealth when the object is won, $v_i \Leftrightarrow b(v_i)$, which is monotone increasing in v_i by $b' < 1$. Decreasing absolute risk aversion of $u(.)$ implies that π is decreasing in v_i. (Weak DARA implies $\pi'(v_i) \leq 0$, while with strict DARA, $\pi'(v_i) < 0$.)

We are now in a position to prove our first proposition:

Proposition 1. Consider the model of first price auction with ex post stochastic i.i.d. private values and risk averse bidders. Suppose the risk $\tilde{z}$ is sufficiently small that $\pi(v_i) \leq v_i$, $\forall v_i$. If preferences exhibit decreasing absolute risk aversion then the bidders are better off with noisy private values. With increasing absolute risk aversion, noise makes bidders worse off in the equilibrium.

Remarks. (**1**) The purpose of the assumption $\pi(v_i) \leq v_i$, $\forall v_i$ is to ensure that the distribution of potential bidders does not get truncated with the introduction of ex post risk. In the C/DARA case this condition is implied by $\pi(\underline{v}) \leq \underline{v}$.

(**2**) Under strict log-concavity of F, strictly DARA bidders are strictly better off with noisy values, strictly IARA bidders are strictly worse off.

(3) Our comparison is confined to noisy and deterministic private values. The noiser the values, the better off the buyers, under the assumptions of *decreasing strong risk aversion* introduced by Ross (1981),[8] or under the conditions discussed by Kihlstrom, Romer, and Williams (1981)[9]. The insufficiency of ordinary DARA in the case of the comparison of two noisy environments is not particular to our problem; for a general discussion, see the papers cited above.

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[7] If we used the utility function mentioned in footnote 4 above then we could just assume that $\pi(v_i)$ is decreasing, without any distributional assumption.

[8] Decreasing strong risk aversion requires that $\forall x, y > 0$, $\exists\lambda$ such that $u''(x+y)/u''(x) < \lambda < u'(x+y)/u'(x)$. This notion is stronger than ordinary strict DARA.

[9] That would require, besides ordinary DARA, that the "extra" noise be independent of the "original" noise.

Proof. We show that C/DARA players are better off with noise than without noise. The more general statements of the proposition follow similarly.

With stochastic private values, the equilibrium bid function solves (1) with initial condition $\pi(\underline{v}) = \underline{v}$. If $\pi(v_i) \leq v_i$ for all v_i and so all bidders participate in the auction, then the probability of winning with type v_i is the same as in the case without noise.

Consider $\hat{\beta}(v_i) \equiv b(v_i) \Leftrightarrow \pi(v_i)$ where $b(.)$ solves (2).

$$\begin{aligned}
\hat{\beta}'(v_i) &= b'(v_i) \Leftrightarrow \pi'(v_i) \\
&\geq (n \Leftrightarrow 1)\frac{F'(v_i)}{F(v_i)}\frac{u(v_i \Leftrightarrow b(v_i))}{u'(v_i \Leftrightarrow b(v_i))} \\
&\geq (n \Leftrightarrow 1)\frac{F'(v_i)}{F(v_i)}\frac{E_{\tilde{z}_i}u(v_i + \tilde{z}_i \Leftrightarrow b(v_i) + \pi(v_i))}{E_{\tilde{z}_i}u'(v_i + \tilde{z}_i \Leftrightarrow b(v_i) + \pi(v_i))} \\
&= (n \Leftrightarrow 1)\frac{F'(v_i)}{F(v_i)}\frac{E_{\tilde{z}_i}u(v_i + \tilde{z}_i \Leftrightarrow \hat{\beta}(v_i))}{E_{\tilde{z}_i}u'(v_i + \tilde{z}_i \Leftrightarrow \hat{\beta}(v_i))}.
\end{aligned}$$

The first inequality is true by (2) and $\pi'(v_i) \leq 0$. For the second inequality, note that by differentiating (3),

$$[1 \Leftrightarrow b'(v_i)]\; u'(v_i \Leftrightarrow b(v_i)) \equiv [1 \Leftrightarrow b'(v_i) + \pi'(v_i)]\; E_{\tilde{z}_i}u'(v_i + \tilde{z}_i \Leftrightarrow b(v_i) + \pi(v_i)).$$

By $b'(v_i) \leq 1$ and $\pi'(v_i) \leq 0$, $u'(v_i \Leftrightarrow b(v_i)) \leq E_{\tilde{z}_i}u(v_i + \tilde{z}_i \Leftrightarrow b(v_i) + \pi(v_i))$. This, together with (3) implies the second inequality.

Note that if $\pi(v_i)$ is constant (like in the case of CARA utilities) then the relations are equalities and therefore $\hat{\beta}$ is a solution to (1). From now on, consider strict log-concavity of F and strictly DARA preferences, i.e., $\pi'(v_i) < 0$. Then the inequalities are strict.

From (1) and the chain of inequalities it is clear that for all $v_i \in (\underline{v}, \overline{v}]$, $\beta(v_i) = \hat{\beta}(v_i)$ implies $\hat{\beta}'(v_i) > \beta'(v_i)$. Moreover, by concavity of u, $\beta(v_i) \geq \hat{\beta}(v_i)$ also implies $\hat{\beta}'(v_i) > \beta'(v_i)$. It is then clear that $\hat{\beta}(v_i) > \beta(v_i)$ for all $v_i \in (\underline{v}, \overline{v}]$.[10] By definition of the risk premium, this means that a bidder with valuation $v_i \in (\underline{v}, \overline{v}]$ prefers the equilibrium of an auction with noise than the equlibrium of an auction with no noise.■

[10] Note that in general, the implication is not that obvious because the condition $\beta(v_i) \geq \hat{\beta}(v_i) \Rightarrow \hat{\beta}'(v_i) > \beta'(v_i)$ holds only for $v_i > \underline{v}$. However, the statement is still true.

The intuition behind the proof is clear from the previous section. When risk is added and bids are shaded by exactly the risk premium, the gain to bidding more and increasing the probability of winning is left unchanged, equal to $F^{n-1}(v_i)u(v_i \Leftrightarrow b(v_i)) = F^{n-1}(v_i)E_{\tilde{z}}u\,(v_i + \tilde{z}_i \Leftrightarrow [b(v_i) \Leftrightarrow \pi(v_i)])$. But with strictly decreasing absolute risk aversion, the gain to bidding less and getting more when one wins – equal to $F^n(v_i)E_{\tilde{z}}u'\,(v_i + \tilde{z}_i \Leftrightarrow [b(v_i) \Leftrightarrow \pi(v_i)]) > F^n(v_i)u'(v_i \Leftrightarrow b(v_i))$ – is higher than before, so bids must be lower. This in turn means that decreasingly (constant, increasingly) risk-averse bidders will be better off (just as well off, worse off) when the value of the auctioned object becomes risky.

It is well known (Maskin and Riley, 1984) that with risk averse bidders and deterministic private values, a first price auction yields higher expected revenue than an English or second price auction. The intuition behind that result is that the inherent riskiness of the first price auction makes the buyers submit higher bids when they are more risk averse (think of the first price auction as a Dutch auction). However, their behavior in a second price auction is not affected by risk aversion, and therefore the seller's expected revenue in a second price auction equals the revenue of any efficient auction with risk neutral bidders.

Under the assumptions of stochastic valuations and DARA preferences, the "precautionary bidding" effect may reduce the bids in the first price auction by substantially more than the risk premium. In a second price auction, there is no precautionary bidding effect. As we see below, bidders reduce their bids by the amount of the risk premium at $\underline{v}$ (which is also a greater reduction than the "fair" risk premium at v_i). We may wonder if the expected revenue order of the first and second price auctions can be reversed by the precautionary effect. The answer is no; the classical result remains true.

Proposition 2. In our model of stochastic private values and decreasingly risk averse bidders, a first price auction provides higher expected revenue than a second price auction.

Proof. First note that in a second price auction bidders submit bids equal to $\sigma(v) = v \Leftrightarrow \pi$ where $\pi = \pi(\underline{v})$ solves $Eu(\tilde{z} + \pi) = u(0) = 0$. (By submitting lower bids they can only decrease their chances of winning a positive surplus; their surplus doesn't depend on their bid conditional on winning.) The expected revenue of the auction equals the expectation of the second highest valuation less π.

Consider the bid function $\eta(v_i) = E[\max_{j \neq i} v_j \mid \max_{j \neq i} v_j < v_i,\ v_i] \Leftrightarrow \pi$ in the first price auction. Clearly, if bidders use this bid function then the expected revenue coincides with that of the second price auction. (Without risk aversion, this is indeed the equilibrium bid function with $\pi = 0$.)

Note that $\eta(\underline{v}) = \underline{v} \Leftrightarrow \pi = \beta(\underline{v})$ where β is the true equilibrium bid function in the first price auction. So $\eta(.)$ and $\beta(.)$ start from the same level at $\underline{v}$.

$$\eta(v) = E[\max_{j \neq i} v_j \mid \max_{j \neq i} v_j < v] \Leftrightarrow \pi = \frac{(n \Leftrightarrow 1) \int_{\underline{v}}^{v} x F(x)^{n-2} F'(x) dx}{F(v)^{n-1}} \Leftrightarrow \pi,$$

$$\eta'(v) = (n \Leftrightarrow 1) \frac{F'(v)}{F(v)} \left[v \Leftrightarrow \frac{(n \Leftrightarrow 1) \int_{\underline{v}}^{v} x F(x)^{n-2} F'(x) dx}{F(v)^{n-1}} \right],$$

and from now on let $\hat{v} = v \Leftrightarrow (n \Leftrightarrow 1) \int_{\underline{v}}^{v} x F(x)^{n-2} F'(x) dx \,/\, F(v)^{n-1}$. Note that $v + \tilde{z} \Leftrightarrow \eta(v) \equiv \hat{v} + \tilde{z} + \pi$.

By the concavity of $u(.)$, we have $u(\hat{v} + \tilde{z} + \pi) \Leftrightarrow u(\tilde{z} + \pi) > \hat{v}\, u'(\hat{v} + \tilde{z} + \pi)$ (for every realization of $\tilde{z}$). Taking expectation and noting $E_{\tilde{z}} u(\tilde{z} + \pi) = 0$, we obtain

$$E_{\tilde{z}} u(\hat{v} + \tilde{z} + \pi) > \hat{v}\ E_{\tilde{z}} u'(\hat{v} + \tilde{z} + \pi).$$

Then at any $v > \underline{v}$ where $\beta(v) = \eta(v)$,

$$\eta'(v) = \frac{F'(v)}{F(v)} \hat{v} < \frac{F'(v)}{F(v)} \frac{E_{\tilde{z}} u(\hat{v} + \tilde{z} + \pi)}{E_{\tilde{z}} u'(\hat{v} + \tilde{z} + \pi)} = \beta'(v)$$

which implies that $\eta(v) < \beta(v)$ for all $v \in (\underline{v}, \overline{v}]$.■

3 Numerical examples

Here we briefly describe some numerical results we obtained by computer calculations for the stochastic private values model with risk averse bidders. In the numerical computations we specifically assume

- two bidders ($n = 2$);
- the deterministic part of the valuation is an i.i.d. draw from a uniform distribution on $[0, 1]$, i.e., $F(v) = v$ for $v \in [0, 1]$;
- CRRA(ρ), $\rho < 1$, preferences with initial wealth w, i.e.,

$$u(x) = (w + x)^{1-\rho} / (1 \Leftrightarrow \rho);$$

• the "noise" is $\pm\varepsilon$ added to the object's value with 50-50 percent chance. As a result, the first order condition becomes

$$\beta'(v) = \frac{1}{(1-\rho)v}\frac{(w+v-\beta(v)+\varepsilon)^{1-\rho}+(w+v-\beta(v)-\varepsilon)^{1-\rho}}{(w+v-\beta(v)+\varepsilon)^{-\rho}+(w+v-\beta(v)-\varepsilon)^{-\rho}}.$$

The boundary condition is $\beta(0) = -\pi_\varepsilon(w)$, where $\pi_\varepsilon(w)$ stands for the risk premium at wealth level w. Assume the seller has sufficiently negative reservation value (e.g., she incurs a cost to keep the object). We solve the above differential equation numerically for different parameter values (w, ρ, ε).[11]

Figure 1 demonstrates the effect we describe in the paper (the "precautionary bidding" or DARA-effect). The parameters are set to $w = \varepsilon = 1/2, \rho = 3/4$; we see three graphs.

[insert Figure 1 about here]

The upper, continuous line shows the bid without noise less the fair risk premium (at the wealth level determined by the bid function).

The middle, dotted line shows the actual bid function $\beta(v)$ in the presence of noise. We see that it lies below $b - \pi$, confirming Proposition 1.

The lower graph, consisting of $+$ signs, shows the bids in a hypothetical first price auction that would yield the same expected revenue as a second price auction. It is the expectation of the other's valuation given that is less than v, minus the risk premium at w. We see that it lies below the true bid function β, as Proposition 2 claims.

[insert Figure 2 about here]

Now let us look at Figure 2. It depicts a graph in a different space. On the horizontal axis, ε is running from 0 to $1/3$, while on the vertical axis we measure the difference between the expected revenue from a first price and a second price auction. (The remaining parameters are set $w = 1/2, \rho = 3/4$.)

[11] Note that with no ex post uncertainty and zero initial wealth, i.e., for $\varepsilon = w = 0$, the equation can be written as

$$b'(v) = \frac{1}{(1-\rho)v}(v - b(v)), \quad b(0) = 0.$$

The solution is linear, $b(v) = v/(2-\rho)$. With either positive initial wealth or noise the solution cannot be given in closed form.

We see that the difference is always positive (according to Proposition 2), but apparently, as ε grows, the advantage of the first price auction over the second price auction first increases, then decreases.

[insert Figure 3 about here]

In Figure 3, we illustrate the possibility that with stochastic private values, increasing risk aversion doesn't necessarily increases the seller's revenue. In the figure we show the expected revenue of the first price auction as a function of the bidders' coefficient of relative risk aversion. The other parameters are set to $w = .25, \varepsilon = .197$.

We see that for low values of ρ, the revenue decreases, which is in contrast with the well-known result (valid under deterministic values), namely, that increasing risk aversion of the bidders implies higher revenue in a first price auction. The intuition is clear behind this result: risk-averse bidders bid more highly because they are afraid of losing the object, but less highly because they dislike the riskiness of the object, and even less highly because of the precautionary effect.

4 Common values

Here we briefly outline the extension of the model to common values. The model is essentially the same as the "general symmetric model" of Milgrom and Weber (1982), with independent signals and the specific noise structure. For notational simplicity, we take the valuations to be weighted arithmetic averages of the signals. In the first subsection we develop the model and its equilibrium. Then a short digression follows concerning the differences between the winner's curse and the precautionary bidding effect. In the third subsection we turn to the main result under common values.

4.1 The common values model

Suppose that bidder i's private information is t_i, distributed i.i.d. on $[0,1]$ according to a cumulative distribution function F. The deterministic valuations are given by $v_i(t_1, ..., t_n) = \alpha t_i + \frac{1-\alpha}{n-1} \sum_{j \neq i} t_j$, with $\alpha \in (0,1)$; i.e., all pieces of information matter for all bidders. Sometimes we will use the shorter notation $v_i(t_i, t_{-i})$ for i's valuation.

We will consider an ex ante symmetric, additive and signal-independent ex post uncertainty, $\tilde{z}_i$, for bidder i. We strengthen the assumption of signal-independence (relative to the one used in the private values model) by requiring that $\tilde{z}_i$ be independent of t_j for $j \neq i$ as well. The uncertain valuation is given by $\tilde{v}_i = v_i + \tilde{z}_i$, as before. Assume bidders are risk averse with utility function u, satisfying $u(0) = 0$, $0 < u' < \infty$, $u'' < 0$.

We will denote the (symmetric) equilibrium bid function by $\gamma(.)$, in the benchmark case without noise by $c(.)$. Similarly what we did in the private values model, let us define the appropriate compensating risk premium for the income noise $\tilde{z}_i$ by $\pi_n(t_i)$. In this section $\pi_n(t_i)$ must solve

$$\begin{aligned} E_{\tilde{z}}\left[\, E[u(\, v(t_i, t_{-i}) \Leftrightarrow c(t) + \tilde{z} + \pi_n(t)\,) \,|\, t_j < t_i,\, \forall j \neq i]\,\right] &= \\ E[u(\, v(t, t_{-i}) \Leftrightarrow c(t)\,) \mid t_j &< t_i,\, \forall j \neq i]. \end{aligned}$$

Note that the bidder's utility in the common values auction equals his *expected* utility conditional on all other bidder's signal being lower than his.

Denote the inverse of the equilibrium bid function by $\phi(.)$. In equilibrium, bidder 1 solves

$$\begin{aligned} &\max_b \; \Pr[t_j < \phi(b),\, \forall j > 1] \cdot E_{\tilde{z}}\left[\, E\left[\, u(\tilde{v}_1 \Leftrightarrow b) \mid t_j < \phi(b),\, j > 1\right]\right] \\ \Leftrightarrow \; &\max_b \int_0^{\phi(b)} \ldots \int_0^{\phi(b)} E_{\tilde{z}} u\left(\alpha t_1 + \tfrac{1-\alpha}{n-1} \textstyle\sum_{j=2}^n t_j + \tilde{z}_1 \Leftrightarrow b\right) dF(t_2) \ldots dF(t_n). \end{aligned}$$

The first order condition is

$$\begin{aligned} &(n \Leftrightarrow 1)\, \phi'(b)\, F'(\phi(b)) \cdot \\ &\int_0^{\phi(b)} \ldots \int_0^{\phi(b)} E_{\tilde{z}} u\left(\alpha t_1 + \tfrac{1-\alpha}{n-1}\phi(b) + \tfrac{1-\alpha}{n-1} \textstyle\sum_{j=3}^n t_j + \tilde{z}_1 \Leftrightarrow b\right) dF(t_3) \ldots dF(t_n) \\ = \; &\int_0^{\phi(b)} \ldots \int_0^{\phi(b)} E_{\tilde{z}} u'\left(\alpha t_1 + \tfrac{1-\alpha}{n-1} \textstyle\sum_{j=2}^n t_j + \tilde{z}_1 \Leftrightarrow b\right) dF(t_2) \ldots dF(t_n). \end{aligned}$$

In Nash equilibrium $b = \gamma(t_1)$, i.e., $t_1 = \phi(b)$; and $\phi'(b) = 1/\gamma'(t_1)$. We obtain

$$\gamma'(t_1) = \frac{(n \Leftrightarrow 1) F'(t_1)}{F(t_1)} \cdot \frac{E\left[E_{\tilde{z}} u\left(\alpha t_1 + \tfrac{1-\alpha}{n-1} t_1 + \tfrac{1-\alpha}{n-1} \sum_{j=3}^n t_j + \tilde{z}_1 \Leftrightarrow \gamma(t_1)\right) \mid t_j < t_1,\, j > 2\right]}{E\left[E_{\tilde{z}} u'\left(\alpha t_1 + \tfrac{1-\alpha}{n-1} \sum_{j=2}^n t_j + \tilde{z}_1 \Leftrightarrow \gamma(t_1)\right) \mid t_j < t_1,\, j > 1\right]}. \tag{4}$$

With no ex post risk, i.e., $\tilde{z} \equiv 0$, this equation becomes

$$c'(t_1) = \frac{(n \Leftrightarrow 1)F'(t_1)}{F(t_1)} \frac{E\left[u\left(\alpha t_1 + \frac{1-\alpha}{n-1} t_1 + \frac{1-\alpha}{n-1}\sum_{j=3}^{n} t_j \Leftrightarrow c(t_1)\right) \mid t_j < t_1,\, j > 2\right]}{E\left[u'\left(\alpha t_1 + \frac{1-\alpha}{n-1}\sum_{j=2}^{n} t_j \Leftrightarrow c(t_1)\right) \mid t_j < t_1,\, j > 1\right]}. \tag{5}$$

The intuition behind these expressions is simple. In (5), when bidder 1 contemplates about raising his bid by dc, he faces a trade-off. He will gain if he just outbids the current winner (with probability $(n \Leftrightarrow 1)F'(t_1)/c'(t_1)$ his expected gain is $E\left[u\left(\alpha t_1 + \frac{1-\alpha}{n-1} t_1 + \frac{1-\alpha}{n-1}\sum_{j=3}^{n} t_j \Leftrightarrow c(t_1)\right) \mid t_j < t_1,\, j > 2\right]$). However, he surely loses utility in situations where he would have won anyway (his loss equals $E\left[u'\left(\alpha t_1 + \frac{1-\alpha}{n-1}\sum_{j=2}^{n} t_j \Leftrightarrow c(t_1)\right) \mid t_j < t_1,\, j > 1\right]$). He sets his bid in order to equalize the two.

4.2 The winner's curse vs. precautionary bidding

In order to make the private and common values models comparable we denote $v_i \equiv t_i$ in the private values model and have the types or values distributed on $[0, 1]$. This implies $c(t) \equiv b(t)$ for $\alpha = 1$.

As it is well known, the (rational) winner's curse effect arises because winning the object provides additional information about the distribution of the other bidders' signals. The bidders take this into account and bid less, shifting the curse onto the auctioneer – even if they are not risk averse.[12] To make this point precise, consider the bid functions under risk neutrality, i.e., $u(w) \equiv w$. Denote the risk neutral equilibrium bid functions by the appropriate capital letters.

$$B'(t_1) = (n \Leftrightarrow 1)\frac{F'(t_1)}{F(t_1)}[t_1 \Leftrightarrow B(t_1)] \tag{6}$$

$$\begin{aligned} C'(t_1) &= (n \Leftrightarrow 1)\frac{F'(t_1)}{F(t_1)} E\left[\alpha t_1 + \tfrac{1-\alpha}{n-1} t_1 + \tfrac{1-\alpha}{n-1}\textstyle\sum_{j=3}^{n} t_j \Leftrightarrow C(t_1) \mid t_j < t_1,\, j > 2\right] \\ &= (n \Leftrightarrow 1)\frac{F'(t_1)}{F(t_1)} \left[\tfrac{1}{n-1} t_1 + \tfrac{n-2}{n-1}\{\alpha t_1 + (1 \Leftrightarrow \alpha)E[t_2 \mid t_2 < t_1]\} \Leftrightarrow C(t_1)\right] \end{aligned} \tag{7}$$

[12]The winner's curse is sometimes referred to as a non-equilibrium (irrational) phenomenon: the bidders losing money by failing to realize that the object's value conditional on winning is less than its unconditional expectation. This is why we use the term rational winner's curse for the effect considered here.

The initial conditions are $B(0) = C(0) = 0$.

Note that for $n = 2$, there is no difference between the two differential equations, and so the bidders bid the same for any value of $\alpha \in (0, 1]$. If the object has common valuation then the only effect is that their expected profits will be lower, i.e., the rational winner's curse stays with the bidders. However, for $n > 2$, $B(t_1) > C(t_1)$ for all $t_1 > 0$. **Proof:** at any $t_1 > 0$, $B(t_1) \leq C(t_1)$ implies $B'(t_1) > C'(t_1)$ because $t_1 > E[t_2 \,|\, t_2 < t_1]$. But $B(0) = C(0)$, therefore $t_1 > 0$ and $B(t_1) \leq C(t_1)$ is impossible.■

The precautionary effect only arises when bidders are risk averse.

4.3 Results under common values

The main result of this section is that decreasingly risk averse bidders may end up being better off with an object of noisier value, exactly as in the private values model.

Proposition 3. Consider the symmetric common values model with stochastic valuations and n risk averse bidders. Suppose the risk $\tilde{z}$ is sufficiently small. If preferences exhibit decreasing absolute risk aversion then the bidders are better off with noisy valuations in a first price auction provided that the number of bidders is sufficiently large.

Proof. Consider bidder 1 submitting $\hat{\gamma}(t_1) \equiv c(t_1) - \pi_n(t_1)$ under noisy values, where $c(.)$ solves (5). Assume that DARA implies $\pi_n'(t_1) < 0$ so that $\hat{\gamma}'(t_1) > c'(t_1)$. Now we show that

$$c'(t_1) > \frac{(n-1)F'(t_1)}{F(t_1)} \frac{E\left[u\left(\alpha t_1 + \frac{1-\alpha}{n-1}t_1 + \frac{1-\alpha}{n-1}\sum_{j=3}^{n} t_j - \hat{\gamma}(t_1)\right) \mid t_j < t_1,\, j > 2\right]}{E\left[u'\left(\alpha t_1 + \frac{1-\alpha}{n-1}\sum_{j=2}^{n} t_j - \hat{\gamma}(t_1)\right) \mid t_j < t_1,\, j > 1\right]} \tag{8}$$

By the law of large numbers,

$$\begin{aligned}\lim_{n\to\infty} E\left[u\left(\alpha t_1 + \tfrac{1-\alpha}{n-1}t_1 + \tfrac{1-\alpha}{n-1}\textstyle\sum_{j=3}^{n} t_j - c(t_1)\right) \mid t_j < t_1,\, j > 2\right] &= \\ \lim_{n\to\infty} E\left[u\left(\alpha t_1 + \tfrac{1-\alpha}{n-1}\textstyle\sum_{j=2}^{n} t_j - c(t_1)\right) \mid t_j < t_1,\, j > 1\right] &= \\ u(\alpha t_1 + (1-\alpha)E[t_2 \,|\, t_2 < t_1] - c(t_1)).&\end{aligned}$$

Therefore $\pi_n(t_1)$, which compensates the second expression in the above chain of equations, also compensates the first expression in the limit as $n \to \infty$.

This implies that the numerator of the fraction on the right hand side of (8) equals the numerator in (5) as $n \to \infty$. However, by the strict DARA hypothesis, the denominator in (8) exceeds the denominator in (5). This establishes (8) for some large n.

The rest follows as in the proof of Proposition 1. We have found that for all $t_1 > 0$, $\gamma(t_1) = \hat{\gamma}(t_1)$ implies $\gamma'(t_1) < \hat{\gamma}'(t_1)$. The two functions start from the same value at $t_1 = 0$, namely, $\gamma(0) = \hat{\gamma}(0) = \Leftrightarrow\pi(0)$. This implies that $\gamma(t_1) < \hat{\gamma}(t_1)$ for all $t_1 > 0$.■

The intuition for this result is very simple. As n becomes large, by the law of large numbers, the common values auction essentially becomes a private value auction with valuations equal to $\alpha t_1 + (1 \Leftrightarrow \alpha)E[t_2 \,|\, t_2 < t_1]$. Then Proposition 1 applies.

Actual numerical calculations suggest that n doesn't need to be large in calibrated models. For example, with log utility, unit initial wealth, uniform type distributions, and $\alpha = 1/2$, two bidders are sufficient for the result to hold.

5 Conclusions

We have shown that in first price auctions with stochastic valuations, *decreasingly risk-averse bidders are better off when the value of the object auctioned becomes more risky.* This is because they bid less aggressively, reducing their bids by more than the amount of the risk premium. They do so because – as in the precautionary saving literature – risk increases the marginal utility of income; the trade-off between raising one's bid to win more often and lowering it to win with a larger surplus is thus shifted in favor of the latter.

Our result (the "precautionary bidding" effect) is not trivial. Risk increases the risk aversion of decreasingly risk-averse bidders, and conventional wisdom states that an auctioneer is better off if his bidders are more risk-averse (see Maskin and Riley, 1984). However, the effect of risk on risk aversion turns out to be of second-order importance compared to the direct effect on marginal utility.

This result stands in contrast to the effects of risk in a second price auction (as we have shown, for private values), where bidders have a dominant strategy of bidding their value less the risk premium of the noise (calculated at 0 initial wealth level). It is well-known that with risk-averse bidders, revenue is higher in a first price than a second price auction, and this remains true under stochastic private values. However, *the advantage of the first over the second price auction might be diminished when risk is added*, because decreasingly risk-averse bidders bid less aggressively than before. Also, *more risk averse bidders do not imply higher seller's revenue* in a first price auction with stochastic private values.

Our paper can be interpreted as an investigation into the justifications of using the simpler and more familiar model with *deterministic values.* We find that the custom of abstracting away from the ex post stochasticity of valuations is completely justified only in the case of CARA preferences. Another qualitative claim, namely, that a first price auction is more advantageous for the seller than a second price auction remains valid under any utility function.

The challenge is now for empirical work to separate out the effects of risk per se in reducing the aggressiveness of bidding in risky settings, from the more well-documented effects of the winners curse; we believe that up until now these two effects have often been confounded.

An additional implication is that bidders may have very little incentive to collectively reduce the uncertainty that they face, since the benefits of this will be lost in more aggressive bidding. One example might be the case of procurement auctions: according to our model, contractors have an incentive to influence the buyer to procur risky objects. Also, they may be willing to commit not to acquire information on common shocks (for instance, not to drill on the tract for sale) if that action is verifiable by competitors.

6 Bibliography

Eeckhoudt, L., C. Gollier, and H. Schlesinger (1996): Changes in Background Risk and Risk-Taking Behavior. Econometrica 64, 683-689.

Gollier, C. and Kimball (1995): Toward a Systematic Approach to the Economic Effects of Uncertainty II: Characterising Utility Functions. Mimeo, IDEI Toulouse

Kihlstrom, R., D. Romer, and S. Williams (1981): Risk Aversion with Random Initial Wealth. Econometrica 49, 911-920.

Kimball, M. (1990): Precautionary Saving in the Small and in the Large. Econometrica 58, 53-73.

Kimball, M. (1993): Standard Risk Aversion. Econometrica 61, 589-611.

Maskin, E., and J. Riley (1984): Optimal Auctions with Risk Averse Buyers. Econometrica 52, 1473-1518.

Maskin, E., and J. Riley (1996a): Existence of the Equilibrium in Sealed High Bid Auctions. Mimeo

Maskin, E., and J. Riley (1996b): Uniqueness of the Equilibrium in Sealed High Bid Auctions. Mimeo

Pratt, J. W., and R. J. Zeckhauser (1987): Proper Risk Aversion. Econometrica 55, 143-154.

Ross, S. A. (1981): Some Stronger Measures of Risk Aversion in the Small and in the Large with Applications. Econometrica 49, 621-638.