The Proportional Value of a Cooperative Game

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ABSTRACT

The proportional value is the unique strictly consistent TU and NTU value which, in two-player TU games, gives players equal proportional gains from cooperation. Strict consistency means consistency with respect to the Hart and Mas-Colell (1989) reduced game. The proportional value is a nonlinear analog of the Shapley (1953) value in TU games and the egalitarian value (Kalai and Samet (1985)) in NTU games. It is derived from a ratio potential similar to the Hart and Mas-Colell (1989) difference potential. The proportional value is monotonic and is in the core of a log-convex game. It is also the unique equilibrium payoff configuration in a variation of the noncooperative bargaining game of Hart and Mas-Colell (1996) where players' probabilities of participation at any point in the game are proportional to their expected payoff at that time. Thus, it is a model of endogenous power in cooperative games. Application to cost allocation problems is considered.

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1 Introduction

This paper presents a theory of bargaining and value allocation in cooperative games based on the principle of equal proportional gain. This is in contrast to standard value theory, as represented by the Shapley (1953) value, which embodies the principle of equal gain. In a situation where two risk-neutral players are bargaining over the division of the proceeds of cooperation, a sum of money, the standard result is that they will split the surplus equally. The surplus is that amount over that which they could obtain acting alone. The proportional value, in contrast, would have them split the surplus so each gains in equal proportion to that which could be obtained by each alone. Further, it applies the principle of equal proportional gain to games where coalitions may form.

Equal proportional gain, or, more simply, proportional allocation, is not a new idea. Young (1994: 64) writes that "[p]roportionality is deeply rooted in law and custom as a norm of distributed justice." Moulin's (1999a) survey of the social choice perspective on allocation opens by quoting Aristotle: "Equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences." Adams (1965: 272) interprets Homans' (1961) social psychological theory of distributive justice to be that equity "in an exchange relationship . . . obtains when the profits of each are in proportion to their investments." Thompson's (1998: 197) text on negotiation puts proportionality at "the heart of equity theory." It is the standard of business practice: Profit is typically divided in proportion to investment; and cost is generally allocated on a pro rata basis. Moriarity (1975) extends cost allocation practice with the accounting theory proposal that the savings from jointly incurred costs be allocated in proportion to stand-alone costs, thus apparently proposing the first distinctly game theoretic model of proportional allocation.

There is only a small game theoretic literature related to proportionality (e.g., Raiffa (1953), Kalai and Smordinsky (1975), Kalai (1977), Roth (1979), Chun and Thomson (1992), Vorob'ev and Liapounov (1998), and Feldman (1998)). Most of the results on proportionality are, instead, found in the accounting and social choice literature. One reason for this omission is that proportional gain outcomes change with the choice of origin of a player's utility scale, and are thus not translation covariant. This has been thought to be an unacceptable property, as it is customary to consider that translation of a player's utility scale, changing its origin, should not change real outcomes. In the proportional approach, however, essential information is distorted by this process. The following example from Lemaire's (1991) survey of the application of cooperative theory to insurance problems shows that the proportional approach provides a rational and practical solution to an important class of problems which are poorly served by the Shapley value.

Example 1.1 *Three players have 1.8 million, 900 thousand, and 300 thousand Belgian Francs to invest, respectively. The interest rate on sums less than 1 million is 7.75%, then for sums less than 3 million the interest rate is 10.25%, and for sums of 3 million or greater it is 12%. If players pool their funds, they will receive the 12% rate. Lemaire* *suggests that the first player should "be entitled to a higher rate, on the grounds that she can achieve a yield of 10.25% on her own, and the others only 7.75%." (1991: 19)*

Lemaire then constructs the coalitional game (assuming 3 months of simple interest) for this problem, and illustrates different solution concepts. He shows the Shapley value of this game, expressed as a vector, is (51,750, 25,875, 12,375). Lemaire also reports these payoffs in annualized rates of interest, (11.5%, 11.5%, 16.5%), and comments that the "allocation is much too generous" to the third player, "who takes great advantage" because he is essential to achieve the highest interest rate. Lemaire shows that the nucleolus generates results close to that of the Shapley value and concludes that these solutions, "defined in an additive way, fail in this multiplicative problem." (1991: 37)

The proportional value of this game is (55,022, 26,552, 8,425), and it gives returns of (12.2%, 11.8%, 11.2%); an outcome consistent with the intuition that larger investors should not receive smaller rates of return in this problem.

This paper defines the proportional value and develops some of its key practical and theoretical properties. A brief review of the joint cost allocation literature addresses both the practical relevance of the proportional approach and some limitations of the standard theory in this application. A second paper, *A Dual Theory of Value* (referred to here as *ADTV*), develops further properties of the proportional value and considers its relationship to the existing cooperative theory of value.

Section 2 of this paper first presents basic definitions and notation. It then summarizes the relevant prior literature in accounting, social choice, and game theory. Finally, it describes the Hart and Mas-Colell (1989) difference potential for cooperative games.

Section 3 defines the ratio potential and defines the proportional value in TU and NTU games as its discrete derivative. The proportional value is defined only on positive games, those where no coalition has zero worth. (*ADTV* develops methods and results for games where coalitions may have zero worth.) It is proved to be unique, monotonic, and in the core of a log-convex game. It is conjectured to be in the core of almost all convex games. It is also shown to have an equal proportional game property analogous to the balanced contributions property (Myerson (1980)) of the Shapley value.

Section 4 addresses consistency, which requires the allocation any player receives in a suitably defined reduced game be the same as that received in the original game. The reduced game used here is the same as that used by Hart and Mas-Colell (1989) to prove the consistency of the Shapley and egalitarian (Kalai and Samet (1985)) values. I define consistency with respect to this reduced game as *strict consistency*. Theorem 4.4 proves the proportional value is the unique strictly consistent TU and NTU allocation rule which gives players equal proportional gains in two-player TU games.

Section 5 implements the proportional value based on a variation of the noncooperative bargaining game of Hart and Mas-Colell (1996). In the Hart and Mas-Colell game, when players' probabilities of participation are equal, the expected payoffs correspond to the Shapley value when the underlying cooperative game is TU and the Maschler-Owen value[1] if it is NTU. In the variation developed here, the *value-weighted participation game*, a player's probability of participation at any point in the game is proportional to her average proposed payoff at that time. The expected payoffs are proved to be given by the proportional value in TU games, in the limit as the probability of breakdown in negotiations goes to zero. The same is true in NTU games when participation is proportional to λ-weighted average proposed payoffs.

The proportional value provides a simple model of *endogenous power* in cooperative games. The weighted Shapley value (1953) (see Kalai and Samet (1987)) provides a model of exogenous bargaining power in coalitional games. Svejnar (1986) develops a similar type of pure-bargaining model. In these models there is an exogenous vector of weights, or bargaining powers, which must be specified. The nature of endogenous power in proportional value allocation can be seen in a two-player game such as the example which opens this paper. In the two-player version of the value-weighted participation game of Section 5.2, and in equilibrium, players' probabilities of selection to propose are proportional to their individual worths, the measure of their outside opportunities. In games with more players, a player's probability of participation reflects her contributions to other coalitions as well. The weighted proportional value may be used to further condition bargaining power on exogenous factors.

Section 6 considers the relevance of the proportional approach to joint cost allocation problems.

The conclusion finds that the discrete derivative of the ratio potential indeed deserves to be considered a cooperative value. A technical appendix follows.

2 Background

This section starts with basic definitions for TU cooperative games. Additional definitions for NTU games are be provided in Section 3.2. A systematic presentation of the basic elements of cooperative game theory can be found in Myerson (1991).

2.1 Basic Notation and Definitions

The *players* in a cooperative game are represented as the elements of a set. The set of all players, the *grand coalition*, is denoted by N. The number of players in the game is n, and $N = \{1, 2, \ldots, n\}$. A *coalition* is a subset of N. The relation $S \subset T$ implies S is a *proper subset* of T, i.e. $S \neq T$, while $S \subseteq T$ allows for equality. The *intersection* of

[1] The Maschler-Owen value was introduced as the consistent (Shapley) NTU value. The consistency of this value lies in the fact that all coalitions are treated symmetrically in a λ-weighting process. The Shapley NTU value is based on only one set of weights generated by the grand coalition (see, e.g., Myerson (1991: 468 or Hart and Mas-Colell (1996: 366)). Since the Maschler-Owen value is not consistent in the sense of Hart and Mas-Colell (1989) (see Owen (1994) and *ADTV*, Subsection 6.3), it is not referred to as the consistent NTU value in order to avoid unnecessary confusion.

two sets S and T is $S \cap T$ and their *union* is $S \cup T$. To simplify presentation, a small coalition will usually be identified with use of the overbar: i.e. $\bar{i} = \{i\}$ and $\overline{ij} = \{i, j\}$. The *set minus* operation is denoted by the backslash (\), so that $S \setminus T$ is the members of S with the members of $T \cap S$, if there are any, removed. The notation $S \ni i$ is read *S contains i*, and describes the same state as $i \in S$: i is a member of S. The *immediate subcoalitions* of S are those with one player removed.

A transferable utility (TU) game in *characteristic function form*, v, assigns a scalar *worth* $v(S)$ to every coalition $S \subseteq N$. The worth of a coalition is what it can guarantee for itself, regardless of the actions of other players. In TU games worth is utility which may be freely divided among the members of a coalition. The worth of Ø, the *empty set*, a formal subset of N, is zero. A game may be restricted to a subset of N, which is represented by the pair (S, v).

A TU game is *positive* if $v(S) > 0$ for every $S \subseteq N$, $S \neq Ø$. The proportional value is defined only on positive TU and NTU games.

Pure bargaining games are those where gains are only possible when all players cooperate. The worth of a single player i, $v(\bar{i})$ is called i's *individual worth.* In a *coalitional* game, a proper subset of players forming a coalition may be able to obtain outcomes other than the sum of the individual worths of its players.

A TU game is *monotonic* if $S \subset T$ implies $v(T) \geq v(S)$. A TU game is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T : S \cap T = Ø$. This requires the worth of the union of two disjoint coalitions be at least the sum of worths of the coalitions. It is *subadditive* if $v(S \cup T) \leq v(S) + v(T)$ for all $S \cap T = Ø$. A TU game is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. And it is *concave* if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$ for all $S, T \subseteq N$. In all these cases, the strict definitions correspond to use of strict inequalities.

An *allocation rule* ϕ is a function from a domain of cooperative games to allocations for individual players. If $\sum_{i \in N} \phi_i(v) \leq v(N)$ then ϕ is feasible, if the relation is equality then ϕ is *efficient.* If $\phi_i(v) \geq v(\bar{i})$ for all $i \in N$ then ϕ is *individually rational.* A *value* is a distinguished allocation rule which represents, in some sense, the expected value of participation in a game. This term is first used by von Neumann to refer to the expected payoffs to players choosing minmax strategies in 2-player zero-sum games.

2.2 Prior Work Related to Proportional Allocation

2.2.1 Proportional approaches to cooperative theory in accounting

Moriarity (1975) makes the straightforward argument that joint costs are only rationally incurred because separable, *stand-alone* costs are greater, and that cost savings might fairly be allocated in proportion to the stand-alone costs. Thus, if total joint costs are $0 < v(N) < \sum_{i \in S} v(\bar{i})$, the share of each cost center i is $x_i = v(\bar{i}) - v(\bar{i}) / \sum_{j \in S} v(\bar{j}) \times (\sum_{j \in S} v(\bar{j}) - v(N))$. This is a model of equal proportional gain in a subadditive pure bargaining game. Louderback (1976) offers a refinement of this approach where cost savings are allocated in proportion to the difference between stand-alone costs and incremental internal costs. Balachandran and Ramakrishnan (1981) develop a continuum of solutions which encompasses both methods. Banker (1981) offers several characterizations of proportional allocation including one similar to O'Neill (1982), below. Gangolly (1981) suggests costs be allocated using the weighted Shapley value (see Kalai and Samet (1987)) with weights equal to cost centers' stand-alone costs. Under his Independent Cost Proportional Scheme (ICPS) the share of cost center i is $x_i = wSh(N, v, \{v(\bar{j})\}_{j\in N})$, where v represents the joint costs faced by different coalitions of cost centers. The ICPS provides equal proportional gain in two-player games. It is directly suggestive of Feldman (1998), below. The comments of Banker, Gangolly, and others in support of proportional allocation methods can be found in Section 6.

2.2.2 Proportionality in social choice theory

O'Neill (1982) identifies proportional allocation based on the equal proportional gain concept as one possible method of division in his study of the Talmudic methods of rights arbitration and characterizes it by efficiency, symmetry, continuity in at least one player's payoff, zero payoff to null players, and strategy proofness. The key last axiom requires that players cannot improve their allocation by merging or splitting their claims. Moulin (1987) shows that 5 pairwise combinations of 4 axioms identify separate 1-parameter families of values which include equal and proportional sharing of a surplus. He further finds that any 3 of these axioms identify a class with two elements: the equal and proportional solutions. Both O'Neill's and Moulin's results apply only to games with 3 or more players. Young (1987) shows that any consistent allocation method must be a member of a family of allocation functions which include both the equal and proportional sharing methods. Young (1988), in a paper on distributive justice in taxation, shows that the proportional allocation method is uniquely identified by continuity, self-duality and composition. Self-duality requires that losses and gains are treated identically and is introduced by Aumann and Maschler (1985). Composition requires that a bargaining problem can be solved in stages without affecting the outcome and is used in Kalai (1977) and Myerson (1977b). Young's result applies to games of 2 or players, but is defined only on positive subadditive games. Moulin (1999b) extends the proportional method to random allocation of indivisible units.

2.2.3 Principal game theoretic results on proportionality

Raiffa (1953) first introduced what is now called a claims model. In his model, each player claims the maximum he can attain consistent with the other player receiving at least what could be achieved in disagreement. These joint claims describe the "ideal point." Chun and Thomson (1992) allow the ideal point to be independent of the structure of the feasibility set. Call the gains over the individually rational outcomes represented by these claims the expected gains. The solution in both models is the maximal feasible allocation in which players' ratios of actual to expected gains are equal. Kalai and Smordinsky (1975) identify the Raiffa solution axiomatically in their search for a monotonic alternative to the Nash (1950) bargaining solution. Kalai (1977) introduces general proportional solutions where players' gains are in some fixed proportion which is determined exogenously. Roth (1979) provides further axiomatic analysis of this approach.

2.2.4 Recent and less known game theoretic results

In an insurance journal, Lemaire (1991) describes a proportional nucleolus, one where there excess of a coalition is defined in relative terms, as follows:

$$e(\alpha, S) = \left[v(S) - \sum_{i \in S} \alpha_i \right] \Big/ v(S), \tag{2.1}$$

where α is a feasible allocation. The proportional nucleolus results from the lexicographical minimization of these proportional excesses subject to the individual rationality constraint. It is derivative of the nucleolus (Schmeidler (1969)), which is the unique individually rational allocation resulting from the lexicographical minimization of coalitional absolute excesses. Lemaire reports that the proportional nucleolus gives all players in Example 1.1 a 12% return, which he writes, is the "common practice" (1991: 19) in such situations. It is easily verified that the proportional nucleolus provides equal proportional gain in two-player games.

Feldman (1998) defines the powerpoint which, in TU games, is based on the weighted Shapley value. The TU powerpoint is a fixed point in a map formed by the weighted Shapley value when the space of weights and values are the same. Player's weights are their values. If v is a positive game and x is a powerpoint, then $x = wSh(N, v, x)$. The powerpoint generates equal proportional gain bargaining in two-player games, but is not strictly consistent. For games of more than three players it does not appear to have an analytic form. Vorob'ev and Liapounov (1998) independently develop a similar result which they call the proper Shapley value.

2.3 The Difference Potential

Hart and Mas-Colell (1989) introduce a potential function for cooperative games. This potential, which will be called the *difference potential* here, satisfies the relation

$$v(S) = \sum_{i \in S} P^d(S, v) - P^d(S \setminus \bar{i}, v), \tag{2.2}$$

for any $S \subseteq N$, $S \neq \emptyset$. Clearly, $P^d(S)$ is well-defined for any $S \subseteq N$, once $P^d(\emptyset, v)$ is determined. Define the discrete derivative $D^i P^d(S, v) = P^d(S, v) - P^d(S \setminus \bar{i}, v)$. Hart and Mas-Colell show that $D^i P^d(S, v)$ is equal to the Shapley value of player i in the game (S, v) for any choice of $P^d(\emptyset, v)$ and use the difference potential to provide a series of results regarding the consistency of the Shapley and egalitarian values. It appears that almost all of these results have analogs for the proportional value. Only the most important are developed here.

3 The Proportional Value

3.1 TU Games

3.1.1 The ratio potential of a TU game

The *ratio potential* of a positive TU cooperative game is the function from the set of coalitions to the real numbers, $P : 2^N \to R$, defined by the recursive relation

$$(3.1) \qquad v(S) = \sum_{i \in S} \frac{P(S,v)}{P(S \setminus \bar{i}, v)} \qquad \text{or, equivalently,} \qquad P(S,v) = v(S) \left(\sum_{i \in S} \frac{1}{P(S \setminus \bar{i}, v)} \right)^{-1}.$$

Given any $P(\emptyset, v) \neq 0$, potentials for all other $S \subseteq N$ are uniquely determined. Unless otherwise noted, it will be assumed that $P(\emptyset, v) = 1$. Lemma 3.2 shows this is without loss of generality.

The ratio potential $P(\overline{12}, v)$ is easily determined by (3.1) to be

$$(3.2) \qquad P(\overline{12}, v) = \frac{v(\overline{12})}{\dfrac{1}{v(\bar{1})} + \dfrac{1}{v(\bar{2})}} = \frac{v(\bar{1})v(\bar{2})v(\overline{12})}{v(\bar{1}) + v(\bar{2})},$$

similarly, $P(\overline{123}, v)$ has the following recursive structure:

$$P(\overline{123}, v) = \frac{v(\overline{123})}{\dfrac{\frac{1}{v(\bar{1})} + \frac{1}{v(\bar{2})}}{v(\overline{12})} + \dfrac{\frac{1}{v(\bar{1})} + \frac{1}{v(\bar{3})}}{v(\overline{13})} + \dfrac{\frac{1}{v(\bar{2})} + \frac{1}{v(\bar{3})}}{v(\overline{23})}}.$$

Let $\mathcal{R}(S)$ be the set of all orderings of the players in S and $r = (r_1, r_2, \dots, r_s) \in \mathcal{R}$ be any such ordering. Set $s = \mid S \mid$ and denote by $T_i = \{r_j : j \leq i\}$ the coalition composed of r_i and all players before r_i in the ordering r. If $S = \{1, 2, 3\}$ and $r = (2, 3, 1)$, then $T_1 = \bar{2}$, $T_2 = \overline{23}$, and $T_3 = \overline{123} = S$. Consider any ordering of players r, call the product of worths $\prod_{i=1}^{n} v(T_i)$ the *ordered worth product* of S according to r. The potential of a coalition S is then the harmonic mean of its ordered worth products:

Lemma 3.1 *For any positive TU game v, and for all $S \subset N$, $S \neq \emptyset$, the ratio potential of S, may be represented as:*

$$P(S,v) = P(\emptyset, v) \left(\sum_{r \in \mathcal{R}(S)} \frac{1}{\prod_{i=1}^{s} v(T_i)} \right)^{-1} = P(\emptyset, v)\, v(S) \left(\sum_{r \in \mathcal{R}(S)} \frac{1}{\prod_{i=1}^{s-1} v(T_i)} \right)^{-1}.$$

Proof: Observe that the two formulations are clearly equivalent because $v(S)$ is common to all of the products on the left-hand side and factors out. If all potentials for coalitions of cardinality $s-1$ satisfy the relation, then it is easy to see that potentials of cardinality s must as well: Use the right-hand version of (3.1) and substitute the potentials of the immediate subcoalitions into the first form of the lemma and the right-hand version of the lemma results. To complete the proof, observe that it is clearly true for singleton coalitions: $P(\bar{\imath}, v) = P(\emptyset, v)\, v(\bar{\imath})$. □

3.1.2 The proportional value in TU games

The *discrete derivative* of the ratio potential of a coalition S with respect to a player $i \in S$ in a game v is defined as the ratio of the potential of S to the potential of the immediate subcoalition of S without player i, coalition $S \setminus \bar{\imath}$:

$$D^i P(S, v) = P(S, v)/P(S \setminus \bar{\imath}, v). \tag{3.3}$$

The *proportional value* of a positive, monotonic TU game is the allocation rule which assigns each player the discrete derivative of the potential of the coalition of all players with respect to that player:

$$\varphi_i(N, v) = D^i P(N, v) = \frac{\displaystyle\sum_{r \in \mathcal{R}(N \setminus \bar{\imath})} \prod_{i=1}^{n-1} v(T_i)^{-1}}{\displaystyle\sum_{r \in \mathcal{R}(N)} \prod_{i=1}^{n} v(T_i)^{-1}}, \tag{3.4}$$

where the T_i are defined as in Section 3.1.1.

Lemma 3.2 *The proportional value is efficient, symmetric, and unique in positive TU games.*

Proof: Efficiency follows directly from the definition of the ratio potential (3.1) and the definition of φ (3.3): The sum of values of all players must be equal to the worth $v(N)$. Symmetry follows from the observation that any permutation of players' labels must lead to the same permutation of their value allocations: e.g., if any two players change names so that they appear in each other's place in all coalitions, then their assigned values will be exchanged as well. The proportional value is unique because it is the ratio of two potentials and Lemma 3.1 shows that potentials scale linearly with $P(\emptyset, v)$. □

Consider the proportional value for a two-player game. Application of (3.2) and (3.3) shows that the proportional value assumes the following form in two-player games

$$\varphi_i(v) \;=\; \frac{v(\bar{\imath})}{v(\bar{\imath}) + v(\bar{\jmath})}\, v(\overline{ij}) \;=\; v(\bar{\imath}) + \frac{v(\bar{\imath})}{v(\bar{\imath}) + v(\bar{\jmath})} \left(v(\overline{ij}) - v(\bar{\imath}) - v(\bar{\jmath}) \right). \tag{3.5}$$

Clearly, the gain from cooperation is shared in proportion to each player's individual worth. This outcome is *equal proportional gain* in two-player games. The proportional value for a player in 3-player game has the following form:

$$\varphi_1(v) = v(\overline{123})\,\frac{v(\overline{1})v(\overline{12})v(\overline{13})\big(v(\overline{2})+v(\overline{3})\big)}{v(\overline{1})v(\overline{12})v(\overline{13})\big(v(\overline{2})+v(\overline{3})\big)+v(\overline{2})v(\overline{12})v(\overline{23})\big(v(\overline{1})+v(\overline{3})\big)+v(\overline{3})v(\overline{13})v(\overline{23})\big(v(\overline{1})+v(\overline{2})\big)}.$$

Remark 3.1 *Observe that each player's share is independent of the worth of the whole. Examination of (3.4) shows that this is always true in proportional value allocation.*

Remark 3.2 *Lemma 3.1 shows that the ratio potential is a harmonic mean, a kind of expectation. The value of a player relative to a coalition is this expectation divided by the equivalent expectation when the player is not present. In this sense, the proportional value is a player's expected* marginal proportional contribution.

3.1.3 The weighted proportional value

The Shapley and egalitarian values have weighted variants. Weights are a vector $\omega \in R^N_{++}$, each weight corresponding to a player, which may be used to represent exogenous factors such as variations in players' bargaining power. The proportional value also has a weighted variant, which is defined with the weighted ratio potential. The *weighted ratio potential* P^ω is defined by the following relation:

$$(3.6) \qquad v(S) = \sum_{i \in S} \omega_i \frac{P^\omega(S, v)}{P^\omega(S \setminus \bar{i}, v)}.$$

The *weighted proportional value* is the discrete derivative of the weighted ratio potential: $\varphi^\omega_i(N, v) = \omega_i(P^\omega(N, v)/P^\omega(N \setminus \bar{i}, v)$.

These relations are in direct analogy to the potential representation of weighted Shapley values in Hart and Mas-Colell (1989). It is easy to see that the weighted proportional value is efficient and uniquely defined. The proof of consistency is a straightforward extension of the unweighted case, as is the extension to NTU games. These results will not be presented here.

3.2 The NTU Ratio Potential and Proportional Value

In an NTU game utility is not transferable on a 1-to-1 basis between the players in a coalition, and, as a consequence, the worth of a coalition cannot be summarized by a single number. Instead, the worth is represented by a *feasible set* which represents the feasible allocations which may be achieved by the members of a coalition.

Feasible sets are represented as subsets of R^S, the $|S|$-dimensional Euclidean space whose dimensions are indexed by the members of S. These sets are required to be positive, comprehensive, closed, and bounded. A feasible set $V(S)$ is *positive* if there is an $x \in V(S)$ such that $x \in R_{++}$. A set is *comprehensive* if, when x is feasible for S and $y \geq x$, y is feasible as well. A set is *closed* if every convergent sequence in that set converges to a point in the set. A coalitional worth $V(S) \subset R^S$ is *bounded* if there is some $y \in R^S$ such that for any $x \in V(S)$, $x < y$. Except for positivity, these are the same conditions used by Kalai and Samet (1985).

An NTU characteristic function game is a collection of such feasible sets, coalitional worths, one for each coalition in the game. An NTU game V is *monotonic* if $V(S) \times \{0^{T \setminus S}\} \subseteq V(T)$ for all $S \subset T$: i.e., any allocation for the players of S in $V(S)$ is also feasible in $V(T)$. A game V is *superadditive* if, for any disjoint coalitions S and T, and for every $x \in V(S)$ and every $y \in V(T)$, the joint vector (x, y) is in the feasible set of the union: $(x, y) \in V(S \cup T)$.

As in the TU game, the proportional value for a player i in an NTU game is defined as the ratio of the potential of the coalition of all players to the potential of the immediate subcoalition that does not contain i. The NTU ratio potential is a straightforward generalization of the TU ratio potential and is defined in an exactly parallel manner to the Hart and Mas-Colell (1989) definition of the NTU difference potential.

The TU condition (3.1) is modified to require that the discrete derivatives of the players in any coalition S identify an allocation on the efficient surface of $V(S)$. Thus, given $(P(S \setminus \bar{i}, V))_{i \in S}$, the *NTU ratio potential* $P(S, V)$ is the unique scalar which satisfies

$$(3.7) \qquad \left(\frac{P(S,V)}{P(S \setminus \bar{i}, V)} \right)_{i \in S} = \left(D^i P(S,V) \right)_{i \in S} \in \partial V(S).$$

If V represents a TU game, then (3.7) clearly reduces to (3.1). The *NTU proportional value* for i in the game (N, V) is: $\varphi_i(N, V) = D^i P(N, V) = P(N, V)/P(N \setminus \bar{i}, V)$.

Lemma 3.3 *The proportional value is unique in positive, monotonic NTU games.*

Proof: Once $P(\emptyset, V) \neq 0$ is determined, $P(S, V)$ for all $S \subseteq N$ are uniquely determined. It is easily seen that $P(S, V)$ scales linearly with $P(\emptyset, V)$, as in the TU case. Thus the discrete derivative and the value are independent of the choice of $P(\emptyset, V)$. □

As in the TU case, it will generally be assumed that $P(\emptyset, V) = 1$.

3.3 Some Properties of the Proportional Value

Further properties related to individual rationality and values when the worths of some coalitions are zero are developed in *ADTV*.

3.3.1 Pareto efficiency

An allocation is weakly *Pareto efficient* if there is no feasible allocation which can strongly improve upon it. If there is no allocation which can weakly improve it, then it is strongly Pareto efficient.

Lemma 3.4 *The proportional value is weakly Pareto efficient in comprehensive, closed, and bounded positive NTU games. It is strongly Pareto efficient if the game is also required to be nonlevel.*

Proof: Let $V(\overline{1}) = V(\overline{2}) = \{x \in R : x \leq 1\}$, and $V(\overline{12}) = \{(x_1, x_2) \in (2,3) - R^2_+\}$. Then $\varphi(V) = (2,2)$, while the unique strongly Pareto optimal outcome is $(2,3)$. However, if V is nonlevel, then the efficient surface $\partial V(S)$ is necessarily a set of strongly Pareto efficient outcomes. Since $\varphi(S,v) \in \partial V(S)$ it must then be strongly Pareto efficient. □

3.3.2 Value monotonicity

A value is *monotonic* if an increase in the worth of a coalition never reduces the value allocated to any of its players. For TU games, ϕ is weakly monotonic if $\partial\phi_i(v)/\partial v(S) \geq 0$ for $S \ni i$, and strongly so if the inequality is strict. This definition is equivalent to that of Kalai and Samet (1985), but weaker than that of Young (1985b).

Lemma 3.5 *The proportional value is strongly monotonic in TU and weakly monotonic in NTU games.*

Proof: In the TU case this follows directly from the definition of the potential given by (3.1). Clearly, an increase of $v(S)$ will increase $P(S,v)$ for any $S \subseteq N$. Further, an increase in $V(S)$ will also increase $P(T,v)$ for any $T \supset S$ because of the following induction argument. Choose any $R \supseteq S$ and assume that $P(R,v)$ increases with an increase in $v(S)$. Consider $T = R \cup \overline{j}$. Then (3.1) shows that $P(T,v)$ must also increase. On the other hand, if $S \not\subseteq T$ then $P(T,v)$ is clearly independent of $v(S)$. Therefore φ is strongly monotonic in positive TU games.

In NTU games a similar argument is based on Definition (3.7). In this case, however, an "increase" in $V(S)$, an outward expansion of the efficient surface, may have no effect on $P(T,V)$, $S \subseteq T \subseteq N$. To see this, let $V'(S) \supset V(S)$ be an enlargement of $V(S)$ such that there is no $y \in V'(S)$ such that for all $i \in S$, $y_i > \varphi_i(S,V)$. Then (3.7) requires that $P(S,V)$ be unchanged, and therefore, no potentials will change and the value allocation will not change. However, there is no way that an increase in $V(S)$ can lead to a decrease in $P(T,V)$ for $S \subseteq T \subseteq N$ or effect any $P(T,V)$ where $S \not\subseteq T$. □

3.3.3 Equal proportional gain

A value is characterized by *equal proportional gain* if and only if the proportional gain to a player j from a second player k joining a coalition is equal to the proportional gain of k when j joins.

Lemma 3.6 *The proportional value has the equal proportional gain property in TU and NTU games:*

$$\frac{\varphi_j(S,V)}{\varphi_j(S \setminus \overline{k}, V)} = \frac{\varphi_k(S,V)}{\varphi_k(S \setminus \overline{j}, V)}.$$

Proof: The proof follows immediately from the restatement of the relationship in terms of potentials. In both cases, the ratios reduce to $P(S,V)/P(S \setminus \overline{jk}, V)$. □

This result is directly analogous to the "balanced contributions" property of Myerson (1980) which is called "preservation of differences" in Hart and Mas-Colell (1989). In weighted proportional values, this result would generalize to parallel Hart and Mas-Colell Equation 5.4.

3.3.4 The proportional value and the TU core

The proportional value is in the core of a game where the proportional marginal contribution of a player never decreases from the addition of players to a coalition. A *core allocation* is one that cannot be (weakly) improved upon for every player in some coalition that seeks to implement an alternative allocation on its own. The *core* is the set of all core allocations in a game. For a TU game, $core(v) = \{x \in R^N : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$.

The potential of a game v is *log-convex* if and only if for all coalitions S and $T \subseteq N$, $P(S \cup T, v)P(S \cap T, v) \geq P(S,v)P(T,v)$. A positive TU game v is *log-convex* if and only if (1) for coalitions S and T such that $S \cap T \neq \emptyset$, $v(S \cup T)v(S \cap T) \geq v(S)v(T)$, and (2) v is superadditive otherwise. Let $S = R \cup \overline{j}$ and $T = R \cup \overline{k}$, then log-convexity implies $v((R \cup \overline{j}) \cup \overline{k})/v(R \cup \overline{k}) \geq v(R \cup \overline{j})/v(R)$. Thus, if v is log-convex it must be convex as well. The proof that φ is in the core of a log-convex game is in several steps.

Lemma 3.7 *If the potential v is log-convex for all $R \subset S$, with $S \subseteq N$, then $\varphi_i(S,v) \geq \varphi_i(R,v)$ for all $i \in R \subset S$.*

Proof: For any $T \subset S \subseteq N$ choose any $i \in S$ and $j \in S \setminus T$. Then, since $P(S,v)P(S \setminus \overline{ij}, v) \geq P(S \setminus \overline{i}, v)P(S \setminus \overline{j}, v)$, $P(S,v)/P(S \setminus i, v) \geq P(S \setminus \overline{i}, v)/P(S \setminus ij, v)$, which, by the definition of the proportional value implies that $\varphi_i(S,v) \geq \varphi_i(S \setminus \overline{j})$. This series of steps may be repeated sequentially for all $k \in S \setminus (T \setminus \overline{i})$ to show first that $\varphi_i(S \setminus \overline{j}) \geq \varphi_i(S \setminus \overline{jk})$, and then continue to build a chain, $\varphi^i_S \geq \varphi_i(S \setminus \overline{j}) \geq \varphi_i(S \setminus \overline{jk}) \geq ... \geq \varphi_i(T)$, proving the result. □

The proof that log-convex games have log-convex potentials utilizes the following technical lemma, the proof of which is provided in the appendix.

Lemma 3.8 *If v is log-convex then for all $S \subseteq N$ with $\mid S \mid \geq 3$:*

$$\sum_{k \in S \setminus \overline{i}} \frac{1}{P(S \setminus \overline{ik}, v)} \sum_{k \in S \setminus \overline{j}} \frac{1}{P(S \setminus \overline{jk}, v)} \geq \sum_{k \in S} \frac{1}{P(S \setminus \overline{k}, v)} \sum_{k \in S \setminus \overline{ij}} \frac{1}{P(S \setminus \overline{ijk}, v)}. \tag{3.8}$$

Lemma 3.9 *If v is log-convex then its potential is log-convex as well.*

Proof: Start with Lemma 3.8 and use $P(T,v) = v(T)/(\sum_{k\in T} P(T\setminus \overline{k}, v))$ to remove the sums in $P(S\setminus\overline{ijk}, v)$, $P(S\setminus\overline{ik}, v)$, and $P(S\setminus\overline{jk}, v)$ and then reorganize to obtain:

$$\left(\sum_{k\in S}\frac{1}{P(S\setminus\overline{k},v)}\right)^{-1}\frac{P(S\setminus\overline{ij},v)}{v(S\setminus\overline{ij})}\geq\frac{P(S\setminus\overline{i},v)}{v(S\setminus\overline{i})}\frac{P(S\setminus\overline{j},v)}{v(S\setminus\overline{j})}$$

Since v is log-convex, use the relation $v(S)v(S\setminus\overline{ij}) \geq v(S\setminus\overline{i})v(S\setminus\overline{j})$ to remove $v(S\setminus\overline{i})$, $v(S\setminus\overline{j})$, and $v(S\setminus\overline{ij})$ and obtain:

$$\left[v(S)\left(\sum_{k\in S}\frac{1}{P(S\setminus\overline{k},v)}\right)^{-1}\right]P(S\setminus\overline{ij},v)\geq P(S\setminus\overline{i},v)P(S\setminus\overline{j},v).$$

The result follows directly because the terms in brackets are equal to $P(S,v)$. □

Theorem 3.1 *If v is log-convex then $\varphi(S,v)\in core(S,v)$ for all $S\subseteq N$.*

Proof: Since v is log-convex, by Lemma 3.9 its potential is log-convex as well. Choose any $T\subset S$. Then $\sum_{i\in T}\varphi_i(S,v)\geq\sum_{i\in T}\varphi_i(T,v) = v(T)$, by Lemma 3.7. □

Is the proportional value in the core of a convex TU game? It appears to be difficult to answer this question one way or the other. The problem does not appear amenable to simple analytical methods. Monte Carlo simulation with random convex games suggests the answer is "Yes." Simulations were conducted on games of up to 7 players, where 400 random convex games were generated. Much larger samples were generated for smaller games. In all cases, the proportional value is in the convex core. This process, however, clearly cannot not rule out the existence of a very small subset of convex games, perhaps of measure zero, where φ is not in the core. In fact, the proportional value is frequently found close to the boundary of the core of a convex game.

Conjecture 3.1 *The proportional value is in the core of almost every convex TU game.*

4 Strict Consistency

Consistency is an important characteristic of a rational allocation method and a hallmark of any claim to its fairness. O'Neill (1982) and Aumann and Maschler (1985) document the consistent nature of rulings in the Babylonian Talmud, a collection of ancient Jewish religious and legal texts. Young (1994) notes the role of consistency in many rules including the allocation of seats in representative bodies. Hart and Mas-Colell (1989: 601) trace the development of consistency concepts in cooperative game theory and provide further references and discussion. A *reduced game* is based on a formula for constructing a new game in which any chosen set of players are removed from the original game. Consistency of an allocation rule with respect to a type of reduced game means that, in any reduced game of this type, the allocations of the remaining players are always the same as in the original game.

Hart and Mas-Colell (1989) define a new reduced game and show that the Shapley value is consistent with it. I show the proportional value is consistent with the same reduced game. I call consistency with respect to the Hart and Mas-Colell game *strict consistency* in order to distinguish it from other types of consistency in cooperative theory. The name is apt in comparing the structure of this reduced game with others and because strictly consistent values adhere so closely to the basic principles of distributive justice that they are not strongly Pareto optimal in (level) NTU games. Strict consistency can also be called Hart and Mas-Colell consistency. The distinguishing aspect of strict consistency is this: The worth of a coalition in a reduced game is what remains of the worth of their union with all reduced players, after the reduced players are given the allocation which the allocation rule specifies for them in the game based on this union. Other types of consistency differ in the reduced players that cooperate with S or the way that the reduced players' allocations are determined.

4.1 Strict Consistency in TU Games

Let ϕ be an allocation rule and T be the players remaining in the game after players $T^c = N \setminus T$ are reduced. The worth of a coalition $S \subseteq T$ in the *strictly reduced game* v_T^ϕ is equal to the worth of the coalition with the players in T^C minus the total due to the players in T^C in the game $(S \cup T^c, v)$ according to ϕ:

$$(4.1) \qquad v_T^\phi(S) = v(S \cup T^C) - \sum_{i \in T^C} \phi_i(S \cup T^C, v), \qquad \text{for all } S \subseteq T.$$

An allocation rule ϕ is *strictly consistent* if and only if

$$(4.2) \qquad \phi_i(T, v_T^\phi) = \phi_i(N, v), \qquad \text{for all } i \in T \text{ and } T \subseteq N.$$

In contrast, the Davis and Maschler (1965) reduced game, for example, requires S to give reduced players their allocation in (N, v). Further, the reduced players who cooperate with S and are given their allocation are the $Q \subseteq T^C$ which maximize $v^\phi(S)$. The nucleolus is consistent with respect to the Davis and Maschler reduced game.

The notation v_{-i} refers to a game where i has been reduced.

Theorem 4.1 *The proportional value is strictly consistent in TU games.*

Proof: First observe that φ is (strictly) consistent in two-player games since $\varphi_j(\overline{j}, v_{-i}^\varphi) = v_{-i}^\varphi(\overline{j}) = v(\overline{ij}) - \varphi_i(\overline{ij}, v) = \varphi_j(\overline{ij}, v)$. Now choose a player i to reduce from the game and assume that φ is consistent for games of $m > 2$ players or less. Then for any $S \not\ni i$, $\mid S \mid < m$, and any $j \in S$, $P(S, v^{\varphi}_{-i})/P(S \setminus \bar{j}, v^{\varphi}_{-i}) = P(S \cup \bar{i}, v)/P((S \cup \bar{i}) \setminus \bar{j}, v)$. Since this is true for all players $j \neq i$ and all coalitions $S \not\ni i$, $\mid S \mid < m$, there must be a constant c such that $P(S, v^{\varphi}_{-i}) = cP(S \cup \bar{i}, v)$ for the potentials of all these coalitions. Now consider the reduced-game potential of a coalition R with m players. Since $\sum_{j \in R}(P(R, v^{\varphi}_{-i})/P(R \setminus \bar{j}, v^{\varphi}_{-i}) = v(R \cup \bar{i}) - \varphi_i(S, v)$, then $P(S, v^{\varphi}_{-i})$ must scale with c as well. Thus, φ is consistent in $m+1$-player games, and the result follows by induction. □

Corollary 4.1 *The proportional value is the unique strictly consistent TU value which gives equal proportional gain in two-player TU games.*

This is an immediate consequence of Theorem 4.4 in the following subsection. The same result could be reached directly though application of the steps taken in Theorem 4.3 to TU games.

4.2 Strict Consistency in NTU Games

The NTU definition of the strictly reduced game is, again, as in Hart and Mas-Colell (1989), and is the straightforward extension of the TU definition:

$$(4.3) \qquad V^{\phi}_{T}(S) = \left\{ x \in R^S : \left(x, \left(\phi_i(S \cup T^C, V) \right)_{i \in T^C} \right) \in V(S \cup T^C) \right\}.$$

Theorem 4.2 *The proportional value is strictly consistent in NTU games.*

Proof: We parallel the steps of the TU case. When i is reduced from a two-player game we see from (4.3) that $V^{\varphi}_{-i}(\bar{j}) = \{x \leq \varphi_j(\overline{ij}, V)\}$. Therefore $\varphi_j(\bar{j}, V^{\varphi}_{-i}) = P(\bar{j}, v^{\varphi}_{-i}) = P(\overline{ij}, V)/P(\bar{i}, V) = \varphi_j(\overline{ij}, V)$. Assume φ is (strictly) consistent in m-player games and create V^{φ}_{-i} by reducing i from V. By the same argument as in the TU case, the potentials of all coalitions with less than m players must scale by the same factor c. Now consider a coalition R with m players. By the definition of the reduced game the values of the players in R in the reduced game together with $\varphi_i(R \cup \bar{i}, V)$ must lie in $\partial V(R \cup \bar{i})$:

$$(4.4) \qquad \left(\left(\frac{P(R, V^{\varphi}_{-i})}{cP((R \cup \bar{i}) \setminus \bar{j}, V)} \right)_{j \in S}, \varphi_i(R \cup \bar{i}) \right) \in V(R \cup \bar{i}).$$

Clearly then, $P(R, V^{\varphi}_{-i}) = cP(R \cup \bar{i}, V)$ as well and φ must be consistent in $m+1$ player games as well. □

Theorem 4.3 *The proportional value is the unique strictly consistent NTU allocation rule which gives equal proportional gain two-player NTU games.*

Proof: Assume a second NTU allocation rule ϕ which is also strictly consistent and has equal proportional gain outcomes in two-player NTU games. Thus, φ and ϕ agree in two-player games. This implies that their singleton reduced games must agree. Consistency requires that both values in the singleton reduced games are equal to the values in the two-player games. Thus, φ and ϕ must agree for one-player games as well.

Now, for any coalition S, assume that φ and ϕ agree for games of $s-1$ players or less and choose any two distinct players in $i, j \in N$ and construct the reduced games consisting of only this pair of players, V_{ij}^{φ} and V_{ij}^{ϕ}, one for each value, according to (4.3). The players $N \setminus \overline{ij}$ are reduced from the game. Since φ and ϕ agree for games of $s-1$ players the individual worths of i and j must be the same in both reduced games.

Since both rules give equal proportional gain outcomes in 2-player games and the individual worths in both games are the same, each player will gain in the same proportion in both games. Thus both players' allocations will be equal or larger according to one of the values than the other. By consistency, these values are equal to player values in the game (S, V). This outcome applies to any pair of players in S. Thus, all players' allocations according to one of the rules must be at least as great as according to the other. If φ and ϕ are both efficient in s player games then they must be equal.

By assumption, ϕ is efficient in $s-1$ player games. But, by the construction of the strictly reduced game, this immediately implies

$$\left(\left(\phi_j(S \setminus \bar{i}, V_{-i}^{\phi})\right)_{j \in S \setminus \bar{i}}, \phi_i(S, V)\right) \in \partial V(S).$$

Therefore, ϕ is efficient in s-player games as well and $\phi = \varphi$ in (S, V). The conclusion follows by induction. □

Again parallel to Hart and Mas-Colell (1989), this result can be strengthened in that consistency with equal proportional gain in two-player TU games is sufficient to guarantee the uniqueness of the proportional value.

Theorem 4.4 *The proportional value is the unique strictly consistent NTU value which gives equal proportional gain in TU two-player games.*

Proof: Given Theorem 4.3 all that must be proved is that strict consistency with equal proportional gain bargaining in TU two-player games implies it in NTU two-player games as well. The proof strategy is the same as in Hart and Mas-Colell (1989) Lemma 6.9: A two-player NTU game $(\overline{ij}, V)$ is embedded in a three-player game $(\overline{ijk}, \widehat{V})$ whose other coalitional worths are all transferable. Because $\widehat{V}(\overline{ijk})$ is equivalent to a transferable worth, all games reduced by one player are equivalent to TU games. Through consistency, the value of the original two-player NTU game is identified.

Let $\alpha_i = \sup\{x_i \in V(\bar{i})\}$, define α_j similarly, and set $\alpha_k > 0$. Define $\widehat{V}$ as follows:

$$\widehat{V}(S) = \begin{cases} V(S), & S = \bar{i},\, \bar{j}, \text{ or } \overline{ij}, \\ \left\{x \in R^S : \sum_{i \in S} x_i \leq \sum_{i \in S} \alpha_i\right\}, & \text{otherwise.} \end{cases}$$

Let ϕ be an allocation rule which satisfies the conditions of the theorem and let $(y_i, y_2, y_3) = \phi(\overline{ijk}, \widehat{V})$. Define the TU versions of the two-player reduced games: $\widehat{v}_{-i}^{\phi}, \widehat{v}_{-j}^{\phi}$, and $\widehat{v}_{-k}^{\phi}$. In $\widehat{v}_{-i}^{\phi}$

and $\widehat{v}^{\phi}_{-j}$ the individual worths of i and j must be their values under $\phi(\overline{ij}, V)$, which is what is left over after the other player is given his value under $\phi(\overline{ij}, V)$, by efficiency and the definition of the reduced game. All other two-player games are additive so players get their individual worth:

$$\begin{array}{lll} \widehat{v}^{\phi}_{-i}(\overline{j}) = \phi_j(\overline{ij}, V), & \widehat{v}^{\phi}_{-i}(\overline{k}) = \alpha_k, & \widehat{v}^{\phi}_{-i}(\overline{ik}) = y_j + y_k, \\ \widehat{v}^{\phi}_{-j}(\overline{i}) = \phi_i(\overline{ij}, V), & \widehat{v}^{\phi}_{-j}(\overline{k}) = \alpha_k, & \widehat{v}^{\phi}_{-j}(\overline{jk}) = y_i + y_k, \\ \widehat{v}^{\phi}_{-k}(\overline{i}) = \alpha_i, & \widehat{v}^{\phi}_{-k}(\overline{j}) = \alpha_j, & \widehat{v}^{\phi}_{-k}(\overline{ij}) = y_i + y_j. \end{array}$$

The TU values for these games are easily calculated from (3.5). Since ϕ is consistent, these games generate a system of equations equating allocations in two of the reduced games, one for each player. For example, for player i, $\phi_i(\widehat{v}^{\phi}_{-j}) = \phi_i(\overline{ijk}, \widehat{V}) = \phi_i(\widehat{v}^{\phi}_{-k})$. From the game $\widehat{v}^{\phi}_{-k}$ we can determine that $\phi_i(\overline{ijk}, \widehat{V})/\phi_j(\overline{ijk}, \widehat{V}) = \alpha_i/\alpha_j$. Solving for $\phi_k(\overline{ijk}, \widehat{V})$ in the first two games gives $(\phi_i(\widehat{v}^{\phi}_{-j}) + \alpha_k)/(\phi_j(\widehat{v}^{\phi}_{-i}) + \alpha_k) = (y_i + y_k)/(y_j + y_k)$. Dividing $\phi_i(\widehat{v}^{\phi}_{-j})$ by $\phi_j(\widehat{v}^{\phi}_{-i})$ gives

$$\frac{\phi_i(\overline{ijk}, \widehat{V})}{\phi_j(\overline{ijk}, \widehat{V})} = \frac{\phi_i(\widehat{v}^{\phi}_{-j})}{\phi_j(\widehat{v}^{\phi}_{-i})} \; \frac{\phi_j(\widehat{v}^{\phi}_{-i}) + \alpha_k}{\phi_i(\widehat{v}^{\phi}_{-j}) + \alpha_k} \; \frac{y_i + y_k}{y_j + y_k}.$$

Combining these results and recalling that $\phi_i(\overline{ij}, V) = \phi_i(\widehat{v}^{\phi}_{-j})$ and $\phi_j(\overline{ij}, V) = \phi_j(\widehat{v}^{\phi}_{-i})$ we find that $\phi_i(\overline{ij}, V)/\alpha_i = \phi_j(\overline{ij}, V)/\alpha_j$. Strict consistency and equal proportional gain in two-player TU games thus imply equal proportional gain in two-player NTU games as well. Then, by Theorem 4.3, $\phi = \varphi$. □

5 Noncooperative Implementation

A noncooperative game with equilibria whose payoffs correspond to a cooperative solution is said to implement the solution. This section implements the TU and NTU proportional value. The principal insight it offers is that value allocation according to the proportional value corresponds to simple bargaining environments where a player's probability of having the opportunity to make a proposal is proportional to her average proposed payoff at that point in the game. This is in contrast to models of the Shapley value, the Maschler-Owen value, and, as shown in *ADTV*, the egalitarian value; which all result from games where players' probabilities of selection to propose are independent of their average proposed payoffs. Here, the ability to make a proposal is taken to be synonymous with the ability to participate in the bargaining process.

Gul (1989) implements the Shapley value based on a straightforward (TU) extension of the Rubinstein (1982) game where players controlling coalitions of resources have random pairwise meetings during which one, chosen with equal probability, may bid for the resources of the other. A player that is bought out leaves the game. The worth of any combination of resources is described by a cooperative game. The Gul approach is not amenable to implementing NTU values and Hart and Mas-Colell (1996) introduce a new type of game. In this game a player is selected in every round to propose a division of the worth of the coalition of players present. If no player rejects the proposal, the division is effected and the game ends. If any player rejects, the proposal is rejected. Before the next round begins, and with probability $1-\rho$, $0 \leq \rho < 1$, where ρ is probability of continuation, a breakdown occurs and the proposer is ejected from the game and receives zero final payoff.

When all players then present participate equally, the stationary subgame perfect (SSP) equilibria of the game have payoffs which correspond to the Shapley value when the underlying cooperative game is TU, to the Nash bargaining solution in two-player NTU games, and to the Maschler-Owen NTU value otherwise. The last two results hold in the limit as $\rho \to 1$. Hart and Mas-Colell generalize these basic results to allow for unequal probabilities of selection to propose (as well as different consequences following the rejection of a proposal) in TU and NTU games. These results are utilized here to implement the proportional value.

Hart and Mas-Colell define a *payoff configuration* as a set of payoff vectors, $a = \{a_S\}_{S \subseteq N}$, one for each $S \subseteq N$. Each a_S specifies payoffs for all $i \in S$. Following Hart and Mas-Colell, $a^j_{S,k}$ identifies the proposed payoff to j by k when the remaining players are S and a^i_S is player i's ex ante continuation payoff when the players are S. Given an opportunity to accept or reject a proposal, we will say that a player we will say that a player follows a *simple rational reply strategy* if he always accepts a proposal which is at least as much as he could expect to obtain following rejection, but rejects otherwise. Assume that all players follow simple rational reply. A complete (pure) strategy for a player is thus specified by a (proposed) payoff configuration. I use boldface type to indicate a profile of proposal strategies, so that **a** represents a complete strategy profile for all players and $\mathbf{a}_S = \{a_{S,j}\}_{j \in S}$ is the profile of proposals for when the remaining bargainers are S.

5.1 Value-Weighted Participation Games

The variation described will be called a *value-weighted participation game*. The following mechanism is used to operationalize the notion that participation, as reflected in the probability of selection to make a proposal, is proportional to a player's average proposed payoff. In every round, all players still in the game submit proposals for the division of the worth of the coalition of the remaining players. The probability of a player i being selected to make a proposal is then made proportional to the average of i's payoffs under the different proposals. If i is selected to make a proposal she must make the proposal which she submitted. This is to prevent manipulation of the mechanism. Note that all players in the game at that time still have equal opportunity to reject the proposal.[2]

Feldman (1998) shows payoffs converge to the powerpoint in TU games when this

[2] The ability to reject a proposal is also an aspect of participation in the bargaining process. Further, participation would be affected by differing probabilities of proposers' ejection given rejection of a proposal. Both of these might be modeled as conditioned by average proposed payoffs. The model presented here is thus very simple representation of value-weighted participation.

procedure is conducted only once during the first round of the game and a player's probability of selection in subsequent rounds of the game, should they occur, is proportional to their probability of selection in the first round.

5.2 TU Implementation

Let $a_S = \sum_{j \in S} p_S^j a_{S,j}$ be the expected payoff vector for S given proposal profile $\mathbf{a}_S$, where p_S^j is the probability of j being selected to propose when the players are S. Hart and Mas-Colell (1996) Proposition 1 develops the basic conditions of the SSP equilibrium for both TU and NTU games. These include that all proposals are positive and efficient and that

$$a_{S,k}^j = \rho a_S^j + (1-\rho) a_{S \setminus \bar{k}}^j, \tag{5.1}$$

for all $j, k \in S$, $j \neq k$, and all $S \subseteq N$. That is, each player is offered the weighted average of her continuation value and the expected payoff conditional on ejection of the proposer, where the weights are the probabilities of continuation and breakdown.

Hart and Mas-Colell's Proposition 9 shows that expected equilibrium payoffs for TU bargaining games under the circumstances described here satisfy the following recursive relation:

$$a_S^j = \sum_{k \in S \setminus \bar{j}} p_S^k a_{S \setminus \bar{k}}^j + p_S^j \left(v(S) - v(S \setminus \bar{j}) \right). \tag{5.2}$$

To formally describe the TU value-weighted participation mechanism, first define player j's average proposed payoff as $\bar{a}_S^j = (1/s) \sum_{k \in S} a_{S,k}^j$, where s is the number of players in S. Let the notation $p_S^j(\mathbf{a}_S)$ explicitly reflect the dependence of p_S on the proposal profile $\mathbf{a}_S$. The mechanism sets

$$p_S^k(\mathbf{a}_S) = \begin{cases} \bar{a}_S^k / \sum_{j \in S} \bar{a}_S^j & : \quad a_{S,j} \in R_+^S \text{ for all } j \in S, \\ 1/s & : \quad \text{otherwise.} \end{cases} \tag{5.3}$$

If any player proposes a negative payoff or all players propose zero payoffs for all players, all players have equal probability of selection. This will never happen in equilibrium.

Lemma 5.1 *For any $0 \leq \rho < 1$ the value-weighted participation game has an equilibrium outcome when the underlying cooperative game is positive, monotonic, and TU.*

Proof: Since v is positive the mechanism (5.3) ensures that $p_S(\mathbf{a}_S)$ is well-defined for any proposal profile and for all $S \subseteq N$, thus the game is well-defined. Now consider the existence of a pure strategy SSP equilibrium based on the mechanism and the Hart and Mas-Colell game. We allow players' actual proposals to differ from those submitted to the mechanism and show that a proposal profile always exists such that the submitted proposals and resulting "partial" equilibrium proposals are the same. Denote by $\mathbf{d}_S = \{d_{S,j}\}_{j\in S}$ the profile of proposals submitted to the mechanism, and by $\mathbf{e}_S = \{e_{S,j}\}_{j\in S}$ the resulting partial equilibrium proposal profile according to the probabilities generated by the mechanism (5.3), (5.2), and (5.1). Assume that an equilibrium exists for all $R \subset S$ such that $\mathbf{a}_R = \mathbf{d}_R = \mathbf{e}_R$ and that $a^j_{S\setminus\bar{k}} \geq 0$ for all $j, k \in S$. Then, for any equilibrium proposal profile $\mathbf{e}_S$, $e^j_{S,k} \geq 0$ for every $j, k \in S$ and for every $k \in S$, $e_{S,k}$ must be efficient, by Hart and Mas-Colell Proposition 1 and the monotonicity of v. Thus $e_{S,k}$ must lie within the $s-1$-dimensional simplex $A^k(S) = \{x \in R^S_+ : \sum_{i\in S} x_i = v(S)\}$. Let $A^S(S) = \times_{k\in S} A(S)$. Observe that $A^S(S)$ is closed, bounded, and convex.

Now consider that the value-weighted participation mechanism is continuous with respect to the proposal strategy profile when proposals are efficient, and that (5.2) shows that a_S is continuous with respect to p_S. Further, (5.1) determines all equilibrium proposals as a continuous function of a_S, given ρ and $\{a_R\}_{R\subset S}$. The composition of these functions is thus a continuous function $J : A^S(S) \to A^S(S)$. The Kakutani fixed point theorem then guarantees the existence of a proposal profile $\mathbf{a}_S$ such that $\mathbf{a}_S = J(\mathbf{a}_S)$, i.e., that $\mathbf{d}_S = \mathbf{e}_S$. Note that equilibrium outcomes trivially exist for the singleton coalitions. This then completes the induction argument. □

Theorem 5.1 *The proportional value is the unique equilibrium payoff configuration of the stationary subgame perfect equilibria of the value-weighted participation game in the limit as $\rho \to 1$, when the underlying cooperative game is positive, monotonic, and TU.*

Proof: By Lemma 5.1 a value-weighted participation equilibrium exists for every $0 \leq \rho < 1$. In the limit as $\rho \to 1$, $a_{S,j} \to a_S$ for every $j \in S$ and all $S \subseteq N$. Also $\bar{a}_{S,j} \to a_S$. Now assume that the proportional value is the equilibrium proposal profile for all coalitions $R \subset S$. If $a^i_S = \varphi_i(S, v)$, then p^i_S can be written $p^i_S = 1/v(S) \times (P(S, v)/P(S \setminus \bar{i}, v))$ because v is positive. Clearly, these probabilities sum to unity. Now substitute these into the equilibrium payoff characterization along with payoffs in subgames into (5.2) and we see that

$$a^i_S = \left(\sum_{\substack{k\in S\\ k\neq i}} \frac{P(S,v)}{v(S)P(S\setminus\bar{k},v)} \frac{P(S\setminus\bar{k},v)}{P(S\setminus\overline{ik},v)} \right) + \frac{P(S,v)}{v(S)P(S\setminus\bar{i},v)} \left(v(S) - v(S\setminus\bar{i})\right).$$

The $P(S \setminus \bar{k}, v)$ terms in the sum cancel and the result may be rearranged to get

$$a^i_S = \frac{P(S,v)}{P(S\setminus\bar{i},v)} - \frac{P(S)}{v(S)} \left(\frac{v(S\setminus\bar{i})}{P(S\setminus\bar{i},v)} - \sum_{\substack{k\in S\\ k\neq i}} \frac{1}{P(S\setminus\overline{ik},v)} \right).$$

The first term on the right is the proportional value for player i in the game (S, v). The term in parenthesis must equal zero by the definition of the potential. Thus, if all subgames of S yield the proportional value, using the proportional value to determine participation probabilities in the subgame based on S is an equilibrium. Since the relation is trivially true for singleton coalitions, the conclusion follows by induction. Uniqueness can be proved by the same approach as taken in Theorem 5.2 ((5.6) and (5.7), here with all $\lambda^i_S = 1$). □

5.3 NTU Implementation

In order to guarantee the existence of equilibria in general NTU games the following additional restrictions on the structure of the NTU game are added to those set forward in Section 3.2. In addition to being (A.1) positive, (A.2) comprehensive, (A.3) closed, and (A.4) bounded; feasible sets must also be (A.5) smooth, (A.6) nonlevel, and (A.7) convex. *Smoothness* requires that the tangent hyperplane at any point $x \in \partial V(S)$ is well-defined. A surface is *nonlevel* if the outward normal vector at any point in the surface is positive in all directions. *Convexity* requires that for any $x, y \in V(S)$, and any $\alpha : 0 \leq \alpha \leq 1$, $\alpha x + (1-\alpha)y \in V(S)$ as well. The game V must also be (A.8) monotonic (see definition in Section 3.2). Assumptions A.2-A.8 are the Hart and Mas-Colell (1996) assumptions.

Hart and Mas-Colell show that the subgame equilibrium conditions of their Proposition 9 for the TU case (5.2 here) generalize to NTU games when conditions A.2-A.8 are met and the expected marginal contribution of i is λ-weighted. The λ-weights are the marginal rates of substitution of the players' payoffs, as determined by the tangent hyperplane to the equilibrium payoff. (The argument is contained in the paragraph which follows the proof of Hart and Mas-Colell (1996) Proposition 9.) Thus, in the limit at $\rho \to 1$, any equilibrium must satisfy the following relation:

$$(5.4) \qquad a_S^j = \sum_{k \in S \setminus \bar{j}} p_S^k a_{S \setminus \bar{k}}^j + p_S^j \frac{1}{\lambda_S^j} \left(\sum_{k \in S} \lambda_S^k a_S^k - \sum_{k \in S \setminus \bar{j}} \lambda_S^k a_{S \setminus \bar{j}}^k \right).$$

In order to implement the proportional value the value-weighted participation mechanism must be modified so that players' participation is proportional to their λ-weighted average proposed payoff. In a general NTU game, the average of players' proposals will not usually lie on the efficient surface. Therefore, we will select an appropriate point in $\partial V(S)$ in order to generate the λ weights. Let a_S^* be the intersection of the ray passing from the origin through the average proposal $\bar{a}_S$ and $\partial V(S)$, and let $\lambda(a_S^*)$ be the weights associated with the hyperplane tangent to $V(S)$ at a_S^*. Now define player i's λ-weighted average proposed payoff as: $b_S^i = (1/s)\lambda_i(a_S^*) \sum_{j \in S} a_{S,j}^i$. This, then, is the λ-value weighted participation mechanism:

$$(5.5) \qquad p_S^k(\mathbf{a}_S) = \begin{cases} b_S^i / \sum_{j \in S} b_S^j. & : \quad a_{S,j} \in R_+^S \text{ for all } j \in S, \\ 1/s & : \quad \text{otherwise.} \end{cases}$$

Extension of the methods for the TU case can be used to prove the existence of an equilibrium in the λ-value weighted participation game for hyperplane games under the same conditions as in the TU case. It is doubtful, however, that conventional methods can prove the existence of an equilibrium in general NTU games, even in the limit as $\rho \to 1$. First, the equilibrium proposal profile as a function of p_S may be multivalued (see Owen (1994) and Hart and Mas-Colell (1996: 366)). It is not clear that this correspondence is upper-hemicontinuous, except in the limit as $\rho \to 1$. Further, if multivalued, it will not be convex, even in the limit. Nevertheless, we can prove the following theorem.

Theorem 5.2 *The proportional value is the unique equilibrium payoff configuration, in the limit as $\rho \to 1$, of the stationary subgame perfect equilibria of the λ-value-weighted bargaining game when the underlying NTU cooperative game V meets conditions A.1-A.8.*

Proof: In the limit, $a_{S,j} \to a_S \in \partial V(S)$ for any p_S and all $j \in S$ and $S \subseteq N$ because conditions A.2-A.8 allow us to apply (5.4). Assume that $a^i_R = \varphi_i(R, V)$ for all $i \in R \subset S$. Define $V_S = \sum_{i \in S} \lambda^i_S(a_S) a^i_S$. Assume that we can set $p^j_S = \lambda^j_S(a_S) a^j_S / V_S$. Condition A.1 guarantees p_S will always be well-defined. We will prove that these conditions are always consistent with a unique proposal profile. Substitute into (5.4) to obtain

$$a^i_S = \sum_{k \in S \setminus \bar{i}} \frac{\lambda^k_S}{V_S} a^k_S \ \varphi_i(S \setminus \bar{k}, V) + \frac{a^i_S}{V_S} \left(\sum_{k \in S} \lambda^k_S a^k_S - \sum_{k \in S \setminus \bar{i}} \lambda^k_S \ \varphi_k(S \setminus \bar{i}, V) \right). \tag{5.6}$$

Replace φ by discrete derivatives of the potential and simplify to get

$$a^i_S P(S \setminus \bar{i}, V) = \frac{1}{\sum_{k \in S \setminus \bar{i}} \beta^k_S} \sum_{k \in S \setminus \bar{i}} \beta^k_S a^k_S P(S \setminus \bar{k}, V), \tag{5.7}$$

where $\beta^k_S = \lambda^k_S / P(S \setminus ik, V)$, for all $k \in S \setminus \bar{i}$. Clearly, all products $a^k_S P(S \setminus \bar{k}, V)$ must have the same magnitude since each is equal to the weighted average of all the others. Thus $a^i_S = \varphi_i(S, V)$ and $a^k_S P(S \setminus \bar{k}, V) = P(S, V)$. Since the theorem is trivially true for singleton coalitions the conclusion follows by induction. □

Remark 5.1 *The λ-value weighted participation game has a unique equilibrium outcome in general NTU games in the limit as $\rho \to 1$ even though the Hart and Mas-Colell game, on which it is based, does not. When the players are S, there may be other partial equilibrium proposal profiles supported by $p_S(\mathbf{a}_S)$, but players have already committed to $\mathbf{a}_S$. The participation mechanism thus plays the role of a sort of coordination device.*

6 Application to Problems of Cost Allocation

Shubik (1962) first makes the case for the use of cooperative game theory to allocate costs by use of the Shapley value of a cost allocation game. Since then, a considerable literature has developed around this method (see Young, 1985a), but it appears to be seldom used. One reason for this lack of adoption may be that cost allocation appears to be an inherently proportional procedure. Standard activity methods allocate costs in proportional to some measure of activity (see, e.g., Belkaoui (1991)). Further, as noted in Section 2.2.1, accounting theorists have gravitated to methods with proportional features. Review of the literature also shows significant resistance to the linearity of the Shapley value.

Gangolly (1981) notes that the choice between his ICPS and the Shapley value involves "trade-offs between the attributes of invariance and proportional equity" and argues that "cost centers with higher outputs (and hence lower averages costs) should receive a higher proportion of the cost savings, if for no other reason than the fact that their lower average costs give them a bargaining advantage in negotiating a share in the cost savings." (1981: 300) Boatsman, Hansen, and Kimbrell (1981) argue that while the Shapley value "has embodied in it a notion of equity, it appears to be a poor surrogate for the bargaining process." (1981: 73) Banker (1981) develops a partial characterization of pure proportional allocation and finds the Shapley value is "not consistent with our axioms, and hence with traditional cost allocation methods." (1981: 127) Further reservations regarding the use of the Shapley value to do cost allocation may be found in Thomas (1977) and Hamlen, Hamlen, and Tschirhart (1977).

Barton (1988) conducts an experiment where accounting students are asked to allocate costs when three farmers agree to buy feed together and the information is presented as a cooperative game. The results are classified by their closeness (measured by minimum squared error) to 6 allocation rules (which include the Shapley value and the nucleolus). In the unrestricted trial 75% of the allocations are classified as being closest to the Moriarity method. None are classified as closest to the Shapley value.

The following corollary to Theorem 4.4 formalizes the claim that there is no necessary conflict between accounting practice and the theory of cooperative value. The corollary is clearly true by virtue Moriarity's method being the allocation of cost savings in equal proportion to stand-alone costs.

Corollary 6.1 *The proportional value is the unique strictly consistent TU and NTU coalitional cost allocation method which agrees with the Moriarity (1975) cost allocation method for two cost centers.*

It can further be shown that the proportional value is in the core of a cost allocation game when proportional marginal costs are nonincreasing. This condition is a property of log-concave games, which stand in the same relation to the concave game as the log-convex game stands to the convex game (see Section 3.3.4). The proof is exactly the same as the steps that lead to Theorem 3.1 except that the inequalities are reversed. This works out well, because in a cost allocation game, a core allocation is one such that no coalition can, on its own, guarantee all members a lower cost.

Recent work has explored the application of cooperative game theory to other allocation problems. For example Suijs, de Waegenaere, and Borm (1998) show how to construct a TU game representing the reinsurance problem where some players are risk averse and Bilodeau (1999) shows how the problem of determining the distribution of surpluses in a defined-contribution plan may be represented as a cooperative game. In both cases, the proportional value appears to be a viable method for determining an effective allocation.

7 Discussion and Conclusion

The discrete derivative of the ratio potential has been shown to have many of the essential qualities commonly associated with a cooperative game value: With respect to the classical properties of a value, it is efficient, symmetric, and monotonic. It is a type of expectation, and it is an element of the core of a sufficiently convex game. With respect to more recent developments, it has been shown to be both strictly consistent and the expected payoff configuration of a value-style noncooperative bargaining game. I believe these results justify the conclusion that this discrete derivative should be considered a cooperative value.

The essential distinguishing property of this value is proportionality. Three aspects of this proportionality are established here. Lemma 3.6 shows it has the equal proportional gain property. Theorem 4.1 shows it is strictly consistent with equal proportional gain in two-player games. And Theorem 5.1 shows that it is implemented in the Hart and Mas-Colell (1996) noncooperative game when players' probability of selection to propose is proportional to their expected payoff at that point in the game.

The proportional value is also defined on and results are developed for NTU games. The results, except for the noncooperative implementation, are straightforward extensions of those for the TU case. Just as the proportional value is a nonlinear analog of the Shapley value in TU games, it is the analog of the egalitarian value in NTU games. The systematic parallels between linear and proportional values in both TU and NTU games are a salient aspect of this work.

The noncooperative implementation of the NTU value requires λ-weighting of participation probabilities, which appears to be the only wrinkle in an otherwise tidy picture. If we accept the reasonableness of this requirement then we achieve the striking result of unifying, in the realm of proportional value, the approach of Hart and Mas-Colell (1989) with that of Hart and Mas-Colell (1996). It is shown, however, in $ADTV$ that the egalitarian value may also be implemented with λ-weighted participation probabilities. This forces us to consider λ-weighting more carefully as Hart and Mas-Colell (1996) conclude, on the basis of equal participation in the same game, that the Maschler-Owen value is the appropriate NTU generalization of the Shapley value. This topic is taken up in $ADTV$.

With respect to application, it appears that an important practical use of the proportional value may be cost allocation. Section 6 shows that there is significant resistance to the use of the Shapley value and partiality toward proportional methods among accountants and that the proportional value addresses many of these concerns. There are, of course, still significant computational, informational, and skill-transfer barriers to the use of cooperative game theory as a standard method of cost allocation.

Another use of the proportional value may be models of endogenous power in cooperative games. Models of power based on the Shapley value clearly embody an equality of sharing and democratic quality of participation which may be inappropriate in many situations. Many features of the economy, and of our social and political institutions as well, appear to correspond more closely correspond to a proportional model of power.

8 Appendix

This is the proof of Lemma (3.8).

Proof: The proof is by induction. Assume the result is true for all coalitions $T \subset S$ and select any two players $i, j \in S$ and multiply out the terms in sums on each side.

Consider first the terms which arise from the restrict that $k \in S \setminus \overline{ij}$. The left-hand side contains $(n-2)^2$ such terms of the form $1/P(S \setminus \overline{iq}, v)1/P(S \setminus \overline{jr}, v)$, and the right-hand side has the same number of terms of the form $1/P(S \setminus \overline{q}, v)1/P(S \setminus \overline{ijr}, v)$, where $q, r \in S \setminus \overline{ij}$. Now identify a direct correspondence between terms on both sides based having the same omitted players q and r.

(a) When $q = r = k$, compare $1/P(S \setminus \overline{ik}, v)1/P(S \setminus \overline{jk}, v)$ on the left with $1/P(S \setminus \overline{k}, v)1/P(S \setminus \overline{ijk}, v)$ on the right. Multiplying the left side by the inverse of the right side should yield a term equal or greater than unity. This term is equal to $\varphi_i(S \setminus \overline{k}, v)/\varphi_i(S \setminus \overline{jk}, v)$, which, because of the assumption of the log-convexity of P for proper subsets of S and Lemma 3.7, must be true.

(b) When $q \neq r$, match the terms $1/P(S \setminus \overline{iq}, v)1/P(S \setminus \overline{jr}, v)$ on the left with $1/P(S \setminus \overline{q}, v)1/P(S \setminus \overline{ijr}, v)$ on the right, and the same argument as in (a) concludes that the left side term is at least as large as the right.

Thus the sum these terms on the left-hand side must be equal or greater than the sum of the corresponding terms on the right-hand side.

Now, the remaining terms will be treated. The desired result can be represented as:

$$\left[A + \frac{1}{P(S \setminus \overline{ij}, v)}\right]\left[B + \frac{1}{P(S \setminus \overline{ij}, v)}\right] \geq \left[C + \frac{1}{P(S \setminus \overline{i}, v)} + \frac{1}{P(S \setminus \overline{j}, v)}\right][D],$$

where $AB \geq CD$. It is not difficult to determine that this inequality is true when $P(S \setminus \overline{i}, v) \geq P(S \setminus \overline{ij}, v)$ and $P(S \setminus \overline{j}, v) \geq P(S \setminus \overline{ij}, v)$. This will always be true when $\varphi_i(S \setminus \overline{j}, v) \geq 1$ and $\varphi_j(S \setminus \overline{i}, v) \geq 1$. Games with values less than unity can be rescaled. All the ratios here can be seen to be invariant with respect to scaling, therefore relationships that hold for the scaled game will hold for the original game as well. This proves the induction step. It can be verified that the argument is valid for all S with $\mid S \mid \geq 3$.

To complete the proof, note that $v(\overline{ij}) \geq v(\overline{i}) + v(\overline{j})$ implies $P(\overline{ij}, v) \geq P(\overline{i}, v)P(\overline{j}, v)$. This follows immediately from the relations $P(\overline{i}, v) = v(\overline{i})$, $P(\overline{j}, v) = v(\overline{j})$, $P(\overline{ij}, v) = v(\overline{i})v(\overline{j})v(\overline{ij}) / (v(\overline{i}) + v(\overline{j}))$. □

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