Optimal Consumption and Investment with Lévy Processes

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Abstract

This paper study the intertemporal consumption and investment problem in a continuous time setting when the secutity prices follow a Geometric Lévy Process. Using stochastic calculus for semimartingals we obtain sufficient conditions for the existence of optimal consumption and investment policies.

Keywords: Lévy Processes; Hyperbolic Lévy Motion; Incomplete Markets.

1 Introduction

In a seminal paper Merton[1971], solved the intertemporal optimal consumption and investment problem in a continuous time setting using stochastic dynamic programming. He provided explicit solutions for an economy with incomplete markets in which security prices follow a geometric Brownian motion, the endowments follow a Poisson process and the investor has a negative exponential utility with an infinite horizon. Svensson and Werner[1993]

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obtained solutions for the same problem considering that the endowments follow an arithmetic Brownian motion. They used the fact that with this utility the investor's portfolio choices are independent of wealth. There exist other results in the literature that try to improve the above models adding income stream, transaction costs, borrowing constraints and other facts that appear in the real world, but all these models asume a Normal distribution for stock returns. Unfortunately, empirical results have shown that this hypothesis did not hold for the majority of stocks, since they present "fat tails". In order to circumvent this problem Fama[1965] and Mandelbrot and Taylor[1967], proposed a *Pareto-stable distribution*, but the Pareto-stable tails seem to be too heavy. Recently, many authors have developed models that try to describe correctly this phenomenum. Eberlein and Keller[1995] and Kulher et al.[1994] suggested Hyperbolic distributions for modelling German stock returns, they verify that these distributions fit well the data. Barndorff-Nielsen[1994] showed that Generalized hyperbolic distributions (GH) explain very well the behavior of (log) returns for Danish stocks[1]. Motivated by these ideas , we will give conditions for the existence of optimal consumption/investment policies when asset prices follow a Geometric Lévy Process, and finally we will analize the particular case of GH distributions.

2 Model

We will consider a financial market $\mathfrak{M}$ consisting of 2 assets. The first is called *bond* (the riskless asset) and the second will be called *stock* (the risky asset). We denote by $B(t)$ and $P(t)$ the bond's and stock's price at each time $t \in [0, T]$, respectively. The evolution of these prices are modelled by the following equations:

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1. \tag{1}$$

$$dP(t) = P(t^-)[\rho_t dt + \sigma_t dY(t)], \quad P(0) \in (0, \infty) \tag{2}$$

In this model the sources of risk are modelled by a Lévy process $Y(t), 0 \leq t \leq T$. i.e. a process with independent and stationary increments. Y is

[1]For more details on financial applications of these GH distributions see Eberlein,E. and K. Prause[1998]

defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and denote by $\mathbf{F} = \{\mathcal{F}(t), 0 \le t \le T\}$ the $\mathbf{P}-$ augmentation[2] of the natural filtration generated by Y:

$$\mathcal{F}_Y(t) = \sigma(Y(s), 0 \le s \le t), \quad 0 \le t \le T.$$

time horizon will be considered finite. The positiveness of the stock price will be analized in the next chapter. The *interest rate* $\{r(t) : 0 \le t \le T\}$, the *appreciation rate* $\{\rho(t), 0 \le t \le T\}$, and the of *volatilities* $\{\sigma(t), 0 \le t \le T\}$ will be referred as the *coefficients* of the financial market $\mathfrak{M}$. We will asume that these coefficients are deterministic continuous functions.
Now we will introduce a small investor (his decisions does not affect the market prices), who will decide at each moment $t \in [0, T]$:

1. How much money $\pi(t)$ he wants to invest in the stock.
2. His cumulative consumption $C(t)$.

Of course, these decisions must be made without foreknowledge of future events, so C and π must be adapted processes [3]. If we denote by $X(t)$ the wealth of this agent at time t , then the amount invested in the bond will be $X(t) - \pi(t)$, . From this,(1) and (2), we obtain the following equation for the wealth:

$$dX(t) = \pi(t)\frac{dP(t)}{P(t^-)} + (X(t) - \pi(t))\,\frac{dB(t)}{B(t)} - dC(t) \tag{3}$$

$$= \pi(t)\left[\rho(t)dt + \sigma(t)dY(t)\right] + (X(t) - \pi(t))\,r(t)dt - dC(t)$$
$$= r(t)X(t)dt + \pi(t)\left[(\rho(t) - r(t))dt + \sigma(t)dY(t)\right] - dC(t). \tag{4}$$

The solution of this linear stochastic differential equation with initial condition $x \in \mathbb{R}$ *i.e.* $X(0) = x$. is:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi(s)\left[\sigma(s)dY(s) + (\rho(s) - r(s))ds\right] \tag{5}$$

[2] The augmented filtration $\mathbf{F}$ is defined by $\mathcal{F}(t) = \sigma(\mathcal{F}_Y(t) \cup \mathcal{N})$, where $\mathcal{N} = \{E \subset \Omega : \exists G \in \mathcal{F}$ with $E \subseteq G, \mathbf{P}(\mathbf{G}) = \mathbf{0}\}$ denotes the set of $\mathbf{P}$-null events.

[3] We said that a process $\{X_t\}$ is adapted with respect to $\mathbf{F}$ if for all $t \in [0, T]$, X_t is an $\mathcal{F}(t)$-measurable random variable.

For all $0 \leq t \leq T$. Where

$$\gamma(t) \triangleq e^{-\int_0^t r(s)ds} \tag{6}$$

is the discount factor in $\mathfrak{M}$. Now we give analogous definitions as in the Brownian case [4]:

Definition 1 *(i) An* $\mathbf{F}-$ *adapted process* $C = \{C(t), 0 \leq t \leq T\}$ *with increasing, right-continuous paths and* $C(0) = 0, C(T) < \infty$ *a.s is called a cumulative consumption process.*

(ii) An $\mathbf{F}-$*progressively measurable,* $\mathbb{R}-$*valued process* $\pi = \{\pi(t), 0 \leq t \leq T\}$ *with*

$$\int_0^T |\pi(t)\sigma(t)|^2 \, dt + \int_0^T |\pi(t)(\rho(t) - r(t))| \, dt < \infty, a.s \tag{7}$$

is called a portfolio process.

(iii) For a given $x \in \mathbb{R}$ *and* (π, C) *as above, the process* $X(t) = X^{x,\pi,C}(t)$ *of (3), (5) is called the wealth process corresponding to initial capital* x*, portfolio* π*, and cumulative consumption process* C*.*

Notice that we are allowing $(X(t) - \pi(t))$ and $\pi(t)$ to take negative values, meaning that short-sales of stock and borrowing at interest rate $r(\cdot)$,, are permitted. So we need to impose some restriction on admissible portfolios.

Definition 2 *We say that a given portfolio process* $\pi(\cdot)$ *is tame, if the associated discounted gain process:*

$$M^{\pi}(t) \triangleq \int_0^t \gamma(s)\pi(s) \left[\sigma(s)dY(s) + (\rho(s) - r(s))ds\right] \tag{8}$$

$$is \ a.s. \ bounded \ from \ below \ by \ some \ real \ constant:$$

$$\mathbf{P}\left[M^{\pi}(t) \geq q_{\pi}, \forall 0 \leq t \leq T\right] = 1 \ for \ some \ q_{\pi} \in \mathbb{R} \tag{9}$$

[4]See Karatzas[1996].

In the absence of a condition like (9), one investor could construct doubling strategies, that is, portfolios that attain arbitrary large values of wealth with probability one at $t = T$, starting with zero initial capital at $t = 0$. (cf.Karatzas(1996) for an example in the case of Brownian motion)

Definition 3 *A tame portfolio that satisfies:*

$$\mathbf{P}\left[M^{\pi}(T) \geq 0\right] = 1, \quad \mathbf{P}\left[M^{\pi}(T) > 0\right] > 0 \tag{10}$$

is called an "arbitrage opportunity"(or free lunch). we say that a market $\mathfrak{M}$ *is arbitrage free if no such portfolio exist.*

So we need conditions for precluding these arbitrage opportunities. We know that the existence of an Equivalent Martingale Measure (EMM) in a general context rule out this opportunities. In our context Eberlein and Jacod[1997] show that when $r(t) = r,\ \forall t \in [0, T]$, there would be EMM and T. Chan[1999] shows analogous result with $r(\cdot)$ being a deterministic continuous function. I will show that in that case our market is arbitrage free, i.e. there will be only no arbitrage tame portfolios in the market. Unfortunately some models are incomplete, as we will see later the Hyperbolic process is purely discontinuous, then we have not an unique EMM. The criterion to choose one of these EMM will be also analized.

3 Lévy Processes and Equivalent Martingale Measures

In this section we will characterize all the equivalent martingale measures in the model introduced in section 2. Now as we know all the infinitely divisible distributions (Y_t) admit the following Lévy-Khintchine representation:

$$\phi(u) = exp\left(iau - \frac{c}{2}u^2 + \int \left(e^{iux} - 1 - iux\mathbf{1}_{[|x|\leq 1]}\right) G(dx)\right), \tag{11}$$

Where ϕ is the characteristic function of the infinitely divisible distribution, a is the drift, c is the quadratic variation coefficient and G is a positive measure with $\int(x^2 \wedge 1)G(dx) < \infty$. This measure is called the Lévy measure and describes the jumps of the process.

We also know that all Lévy Process must be a linear combination of a standard Brownian Motion (W_t) and a quadratic pure jump process[5] (N_t) which is independent of the Brownian Motion W_t, then

$$Y_t = cW_t + N_t$$

The process N_t has a Lévy decomposition: Let $L(dt, dx)$ be a Poisson measure on $\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$. with expectation (or compensator) measure $dt \times G$ [6], then[7]:

$$N_t = \int_{[|x|<1]} x(L((0,t], dx) - tG(dx)) + \int_{[|x|\geq 1]} xL((0,t], dx) \tag{12}$$
$$+tE\left[N_1 - \int_{|x|\geq 1} xG(dx)\right]$$

Now assume that

$$E\left[exp(-bY_1)\right] < \infty \quad \forall b \in (-b_1, b_2)$$

and

$$\int_{[|x|\geq 1]} e^{-bx} dG(x) < \infty \quad \forall b \in (-b_1, b_2)$$

Where $0 < b_1, b_2 \leq \infty$. The first assumption said that Y_t has all moments finites and the second is technical and will let us separate integrands. With this in mind we can return to the jumps and transform the equation (12) into:

$$N_t = \int_{\mathbb{R}} x\left(L((0,t], dx) - tG(dx)\right) + tEN_1$$

It is easy to see that the process

$$M_t = \int_{\mathbb{R}} x\left(L((0,t], dx) - tG(dx)\right)$$

is a martingale. Then $N_t = M_t + at$, with $a = EN_1$, as a consequence the original process can be written as

$$Y_t = M_t + cW_t + at \tag{13}$$

[5]A process X is said to be a quadratic pure jump process if $\langle N\rangle^c \equiv 0$, where $\langle N\rangle^c$ is the continuous part of its quadratic variation $\langle N\rangle$. Remember that $\langle N\rangle$ is the process such that $(N_t)^2 - \langle N\rangle_t$ is a martingale.

[6]$\forall B \in \mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$, $L(B)$ has Poisson distribution with parameter $(dt \times G)(B)$

[7]$E(\cdot)$ will denote the expectation with respect to $\mathbf{P}$

Before passing to characterize the absolutely continuous measures with respect to the original measure, let us introduce some elements of stochastic calculus: for any measurable function $f(t,x)$ we have

$$\sum_{0<s\leq t} f(s,\Delta N_s) = \int_0^t \int_{\mathbb{R}} f(s,x)L(ds,dx) \tag{14}$$

and for any C^2 function f, we have *the generalized Itô's formula* for cadlag semimartingals $X^1,...,X^n$:

$$df(X_t^1,..,X_t^n) = f_i(X_{t^-}^1,..,X_{t^-}^n)dX_t^i + \frac{1}{2}f_{ij}(X_{t^-}^1,..,X_{t^-}^n)d[X^i,X^j]_t^c$$
$$+f(X_t^1,..,X_t^n) - f(X_{t^-}^1,..,X_{t^-}^n) - f_i(X_{t^-}^1,..,X_{t^-}^n)\Delta X_t^i$$

With $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i x_j}$ and $[X^i,X^j]^c$ the continuous part of the mutual variation of X^i and X^j. Now with the above results we will study the solution of equation (2):

$$dP(t) = P(t^-)[\rho_t dt + \sigma_t dY(t)] = (a\sigma_t + \rho_t)P_{t^-}dt + \sigma_t P_{t^-}(cdW_t + dM_t)$$

When the coefficients ρ_t and σ_t are deterministic continuous function the solution of this equation is given by the Doléans-Dade exponential[8]:

$$P_t = P_0 exp\left\{\int_0^t \sigma_s dY_s + \int_0^t \left(\rho_s - \frac{c^2\sigma_s^2}{2}\right)ds\right\}\prod_{0<s\leq t}(1+\sigma_s\Delta Y_s)e^{-\sigma_s\Delta Y_s}$$

with (13) we obtain:

$$P_t = P_0 exp\left\{\int_0^t c\sigma_s dW_s + \int_0^t c\sigma_s dM_s + \int_0^t \left(a\sigma_s\rho_s - \frac{c^2\sigma_s^2}{2}\right)ds\right\}\prod_{0<s\leq t}(1+\sigma_s\Delta M_s)e^{-\sigma_s\Delta M_s} \tag{15}$$

To ensure that $P_t \geq 0,\ a.s.\ \forall t\in[0,T]$ we need that

$$1+\sigma_t\Delta M_t \geq 0,\ \ \forall t\in[0,T]$$

If we assume the convention '$\sigma > 0$', we only need that the jumps of N_t be bounded from below, i.e. $\Delta N_t \geq -\frac{1}{\sigma_t}$[9], it means that we will consider only

[8]See Jacod and Shirjaev[1987]

[9]Observe that from $N_t = M_t + at$ we have $\Delta M_t = \Delta N_t$.

"semi-fat tailed" distributions as Poisson, Gamma, Hyperbolic and Normal Inverse Gaussian and we will eliminate processes with heavy tails, it is worth noting that the stable distributions (without including the Gaussian Case) were eliminated when we supposed that Y has all moments finites.
The follow step will be characterize all the measures that are absolutely continuous with respect to $\mathbf{P}$, to this end let:

$$\mathcal{M}(dt, dx) = L(dt, dx) - dtG(dx)$$

then

$$M_t = \int_0^t \int_{\mathbb{R}} x\mathcal{M}(ds, dx)$$

Now two useful results[10]:

Lemma 1 *Let R_t and $K(t,x)$ be a previsible and a Borel previsible processes*[11] *respectively. Suppose that*

$$E(\int_0^t R_s^2 ds) < \infty$$

and $K \geq 0$, $K(t,0) = 1 \quad \forall t \in \mathbb{R}^+$. *Let $k(t,x)$ be another Borel previsible process such that*

$$\int_{\mathbb{R}} \left[K(t,x) - 1 - k(t,x)\right] G(dx) < \infty$$

Define a process $\mathcal{Z}_t$ by

$$\mathcal{Z}_t = exp\left\{\int_0^t R_s dW_s - \frac{1}{2}\int_0^t R_s^2 ds + \int_0^t \int_{\mathbb{R}} k(s,x)\mathcal{M}(ds,dx) - \int_{[0,t)\times\mathbb{R}} [K(s,x) - 1 - k(s,x)]G(dx)ds\right\} \prod_{0<s\leq t} K(s,\Delta N_s)e^{-k(s,\Delta N_s)}$$

Then $\mathcal{Z}$ is a local martingale with $\mathcal{Z}_0 = 1$ and $\mathcal{Z}$ is positive if and only if $K > 0$.

[10] See T. Chan[1999]

[11] a Process $K_\omega(t,x)$ is said to be a Borel previsible function or process if the process $t \mapsto K_\omega(t,x)$ is a previsible function for fixed x and the function $x \mapsto K_\omega(t,x)$ is Borel-measurable for fixed t.

Proof.- From the fact

$$\mathcal{Z}_t - \mathcal{Z}_{t^-} = \mathcal{Z}_{t^-}\left(K(t, \Delta N_t) - 1\right)$$

and applying Itô's formula and (14), we can obtain:

$$\mathcal{Z}_t = 1 + \int_0^t R_s \mathcal{Z}_{s^-} dW s + \int_0^t \int_{\mathbb{R}} \mathcal{Z}_{s^-}\left[K(s,x) - 1\right] \mathcal{M}(ds, dx)$$

This expresion is a local martingale.□

Theorem 1 *Let* $\mathbf{Q}$ *be a measure which is absolutely continuous with respect to* $\mathbf{P}$ *on* $\{\mathcal{F}_T\}$*. Then*

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_T} = \mathcal{Z}_T$$

where $\mathcal{Z}$ *is as in the lemma 1, for some* R, K *and* k *for which* $EZ_T = 1$*. Moreover under* $\mathbf{Q}$*, the process*

$$\hat{W}_t = W_t - \int_0^t R_s ds \tag{16}$$

is a Brownian Motion and the proces N_t *is a quadratic pure jump process with compensator measure given by* $dt\hat{G}_t(dx)$ *with*

$$\hat{G}_t(dx) = K(t,x)G(dx)$$

and previsible part given by

$$\hat{a}_t = E^{\mathbf{Q}} N_t = at + \int_0^t \int_{\mathbb{R}} x(K(s,t) - 1)G(dx)ds$$

Then under $\mathbf{Q}$ the process N_t can be represented as

$$N_t = \hat{M}_t + at + \int_0^t \int_{\mathbb{R}} x(K(s,t) - 1)G(dx)ds$$

with

$$\hat{M}_t = M_t - \int_0^t \int_{\mathbb{R}} x(K(s,t) - 1)G(dx)ds \tag{17}$$

This process is a $\mathbf{Q}-$martingale and it is easy to see that $\Delta\hat{M}_t = \Delta M_t$.
Now let

$$\hat{P}_t = exp\left(-\int_0^t r_s ds\right) P_t$$

be the discounted price process. Replacing the processes W_t and M_t in the equation (15) by their respective $\mathbf{Q}-$versions, we obtain

$$\hat{P}_t = P_0 exp\left\{\int_0^t c\sigma_s d\hat{W}_s + \int_0^t c\sigma_s d\hat{M}_s + \int_0^t \left(a\sigma_s + c\sigma_s R_s + \rho_s - r_s - \frac{c^2\sigma_s^2}{2}\right) ds\right.$$

$$\left. + \int_0^t \sigma_s \int_{\mathbb{R}} x(K(s,x)-1)G(dx)ds\right\} \prod_{0<s\leq t}(1+\sigma_s\Delta\hat{M}_s)e^{-\sigma_s\Delta\hat{M}_s}$$

A necessary and sufficient condition for $\hat{P}_t$ be a $\mathbf{Q}-$martingale is the existence of R and $K > 0$ for which :

$$cR_s + a + \frac{\rho_s - r_s}{\sigma_s} + \int_{\mathbb{R}} x(K(s,x)-1)G(dx)ds = 0 \quad \forall s \tag{18}$$

And $E\mathcal{Z}_t = 1, \ \ \forall t > 0$. Since the process

$$exp\left\{\int_0^t c\sigma_s d\hat{W}_s + \int_0^t \sigma_s d\hat{M}_s - \int_0^t \frac{c^2\sigma_s^2}{2} ds\right\} \prod_{0<s\leq t}(1+\sigma_s\Delta\hat{M}_s)e^{-\sigma_s\Delta\hat{M}_s}$$

is a $\mathbf{Q}-$martingale. Now we can stay the following

Theorem 2 *If there exist R and $K > 0$ for which $E\mathcal{Z}_t = 1 \ \ \forall t$ and the market price of risk is given by*

$$\eta(s) = \frac{\rho_s - r_s}{\sigma_s} = \int_{\mathbb{R}} x(1-K(s,x))G(dx) - cR_s - a \ \ \forall s \tag{19}$$

Then the market $\mathfrak{M}$ is arbitrage free.

Proof Take a tame portfolio π and suppose that $\mathbf{P}(M^\pi(T) \geq 0) = 1$, by definition we have

$$M^\pi(t) = \int_0^t \gamma(s)\pi(s)\left[\sigma(s)dY(s) + (\rho(s) - r(s))ds\right]$$

With equation (13) and (19) we obtain

$$M^{\pi}(t) = \int_0^t \gamma(s)\pi(s)\sigma(s) \left[cdW(s) + dM(s) + \left(\int_{\mathbb{R}} x(1 - K(s,x))G(dx) - cR_s) ds \right) \right]$$

Now using (16) and (17)

$$M^{\pi}(t) = \int_0^t \gamma(s)\pi(s)\sigma(s)[cd\hat{W}(s) + d\hat{M}(s)]$$

We have that M^{π} is a $\mathbf{Q}-$ local martingale bounded from below, since π is tame, then a supermartingale. Hence

$$E^{\mathbf{Q}} M^{\pi}(T) \leq 0$$

Then $\mathbf{Q}(M^{\pi}(T) > 0) = 0$, so there are not arbitrage opportunities.□
Now we introduce our

4 Optimization Problem

In this section we will formalize the individual problem of the investor and give some definitions, and finally we will present the main result.

Definition 4 *A pair (π, C) of portfolio consumption process is called admissible for the initial capital $x \geq 0$. If*

$$X(T) = X^{x,\pi,C}(T) \geq 0, \quad a.s. \tag{20}$$

The class of all such pairs will be denoted by $\mathcal{A}(x)$.

Now take any equivalent martingale measure $\mathbf{Q}$ and define the following state price density:

$$H^{\mathcal{Z}}(t) = \gamma(t)\mathcal{Z}(t), \quad \forall t \in [0,T] \tag{21}$$

where

$$\frac{d\mathbf{Q}}{d\mathbf{P}}\Big|_{\mathcal{F}_t} = \mathcal{Z}_t$$

using (5) and the generalized Itô's lemma for $F(\mathcal{Z}(t), \gamma(t)X(t)) = H^{\mathcal{Z}}(t)X(t)$, we obtain:

$$d\left(H^{\mathcal{Z}}(t)X(t)\right) = -H^{\mathcal{Z}}(t^-)dC(t) + H^{\mathcal{Z}}(t^-)\pi(t)(\rho(t) - r(t))dt + H^{\mathcal{Z}}(t^-)\pi(t)\sigma(t)dY(t)$$
$$+\gamma(t)X(t^-)d\mathcal{Z}(t) + H^{\mathcal{Z}}(t)\Delta X(t) \tag{22}$$

Now with the decomposition (13), we obtain

$$d\left(H^{\mathcal{Z}}(t)X(t)\right) = -H^{\mathcal{Z}}(t^-)dC(t) + H^{\mathcal{Z}}(t^-)\pi(t)\sigma(t)[cdW(t) + dM(t)]$$
$$+\gamma(t)X(t^-)d\mathcal{Z}(t) + dD(t)$$

where $D(t)$ is given by

$$D(t) = D(0) + \int_0^t H^{\mathcal{Z}}(t^-)\pi(t)(\sigma(t)a + \rho(t) - r(t))dt + \sum_{0<s\leq t} H^{\mathcal{Z}}(s)\Delta X(s)$$

In order to formulate our optimization problem we need the concept of utility function.

Definition 5 *We say that a function $u : (0, \infty) \to R$ is a utility function if it is strictly increasing, strictly concave and continuously differentiable and*

$$u'(\infty) \triangleq \lim_{x\to\infty} u'(x) = 0 \quad and \quad u'(0_+) \triangleq \lim_{x\downarrow 0} u'(x) = \infty \tag{23}$$

Examples of utility function are $u(x) = \log x$ and $u(x) = \frac{x^\delta}{\delta}, \delta \in (-\infty, 1)\backslash\{0\}$.

We will denote by $I(\cdot)$ the inverse of the derivative $u'(\cdot)$, both these functions are continuous, strictly decreasing, and map (0,∞) onto itself with $I(0_+) = u'(0_+) = \infty, I(\infty) = u'(\infty) = 0$. We shall consider also de convex dual

$$\widehat{u}(y) \triangleq \max_{0<x<\infty}[u(x) - xy] = u(I(y)) - yI(y), 0 < y < \infty. \tag{24}$$

Of $u(\cdot)$: a convex decreasing function, continuously differentiable on (0,∞) and satisfies

$$\widehat{u}'(y) = -I(y),\ 0 < y < \infty \tag{25}$$

$$u(x) \triangleq \min_{0<y<\infty}[\widehat{u}(y) + xy] = \widehat{u}(u'(x)) + xu'(x) \tag{26}$$

$$\widehat{u}(\infty) = u(0_+),\ \widehat{u}(0_+) = u(\infty) \tag{27}$$

4.1 Optimization without income stream

Consider an small investor who has an initial capital $x > 0$, and he wants to choose a portfolio $\pi(\cdot)$ and consumption processes $\{c(t),\ 0 \le t \le T\}$ in order to maximize his expected utility from the terminal wealth $X^{x,\pi,C}(T)$ and from consumption.
Given the utility functions g and $u(t,\cdot)$ as in the above definition, we will define the following classes:

$$\mathcal{A}_g(x) \triangleq \left\{(\pi, C) \in \mathcal{A}(x) \big/ Eg^-\left(X^{x,\pi,C}(T)\right) < \infty\right\}$$

$$\mathcal{A}_u(x) \triangleq \left\{(\pi, C) \in \mathcal{A}(x) \big/ E\int_0^T u^-(t, c(t))dt < \infty\right\}$$

remember $f^-(x) = \max\{-f(x), 0\}$.
Then our small investor will have to maximize the expected utility from consumption and terminal wealth over the following class:

$$\mathcal{A}_0(x) \triangleq \mathcal{A}_u(x) \cap \mathcal{A}_g(x)$$

The value function will be

$$V_{\mathcal{Z}}(x) = \sup_{(\pi,C)\in\mathcal{A}_0(x)} E\left[\int_0^T u(t, c(t))dt + g\left(X^{x,\pi,C}(T)\right)\right] \tag{28}$$

Now to solve this optimization problem consider in the context of the market model $\mathfrak{M}$ described above a contingent claim ξ[12] and a consumption process C that satisfy

$$E\left[H^{\mathcal{Z}}(T)\xi + \int_0^T H^{\mathcal{Z}}(t^-)dC(t)\right] = x > 0 \tag{29}$$

Then if there exist a portfolio process $\pi(\cdot)$, such that $(\pi, C) \in \mathcal{A}(x)$ and $X^{x,\pi,C}(T) = \xi$; we could conclude that the optimal problem is in some sense equivalent to the problem of maximize $E\left[\int_0^T u(t, c(t))dt + g(\xi)\right]$ over all pairs

[12]A contingent claim ξ is a random variable $\mathcal{F}_T-$measurable.

(ξ, c) of contingent claims and consumption rate process that satisfy the constraint (29) .
Now with $y > 0$(*Lagrange multiplier*) and with (24)

$$E\left[\int_0^T u(t,c(t))dt + g(\xi)\right] + y\left[x - E\left[H^{\mathcal{Z}}(T)\xi + \int_0^T H^{\mathcal{Z}}(t^-)c(t)dt\right]\right] =$$

$$= E\left[\int_0^T \left[u(t,c(t))dt - yH^{\mathcal{Z}}(t^-)c(t)\right] dt\right] + E\left[g(\xi) - yH^{\mathcal{Z}}(T)\xi\right] + xy \quad (30)$$

$$\leq E\left[\int_0^T \widehat{u}(t, yH^{\mathcal{Z}}(t^-))dt\right] + E\left[\widehat{g}\left(yH^{\mathcal{Z}}(T)\right)\right] + xy$$

And the equality hold if and only if

$$\xi_{\mathcal{Z}} = I_g\left(yH^{\mathcal{Z}}(T)\right) \quad and \quad c_{\mathcal{Z}}(t) = I_u\left(t, yH^{\mathcal{Z}}(t^-)\right) \quad (31)$$

Then in the constraint (29), we define

$$\mathcal{X}_{\mathcal{Z}}(y) \triangleq E\left[H^{\mathcal{Z}}(T)I_g\left(yH^{\mathcal{Z}}(T)\right) + \int_0^T H^{\mathcal{Z}}(t^-)I_u\left(t, yH^{\mathcal{Z}}(t^-)\right)dt\right] = x$$

and if we consider $x \in (0, \infty)$ that is $\mathcal{X}_{\mathcal{Z}}(y) < \infty, \quad \forall 0 < y < \infty$. This function maps (0,$\infty$) onto itself and is continuous, strictly decreasing with

$$\mathcal{X}_{\mathcal{Z}}(0_+) \triangleq \lim_{y \downarrow 0} \mathcal{X}_{\mathcal{Z}}(y) = \infty, \mathcal{X}_{\mathcal{Z}}(\infty) \triangleq \lim_{y \to \infty} \mathcal{X}_{\mathcal{Z}}(y) = 0.$$

If we denote by $\mathcal{Y}^{\mathcal{Z}}(\cdot) = \mathcal{X}_{\mathcal{Z}}^{-1}(\cdot)$. Then the lagrange multiplier $y > 0$ is uniquely determined by

$$y = \mathcal{Y}^{\mathcal{Z}}(x)$$

Now we can formalize the above intuition with the following

Theorem 3 *Suppose $x \in (0, \infty)$ and $V_{\mathcal{Z}}(x) < \infty, \forall x \in (0, \infty)$ For any $x > 0$, consider the optimization problem with value function $V_{\mathcal{Z}}(x)$ as in (28) and define $\xi_{\mathcal{Z}}$ and $c_{\mathcal{Z}}(\cdot)$ as in (31). Then if* *a) there is a portfolio process $\pi_{\mathcal{Z}}(\cdot)$ such that $(\pi_{\mathcal{Z}}, C_{\mathcal{Z}}) \in \mathcal{A}(x)$ and*

$$X^{x,\pi_{\mathcal{Z}},C_{\mathcal{Z}}}(T) = \xi_{\mathcal{Z}};$$

b) the process D_t is a Local martingale and $D_0 = 0$.

Then $(\pi_{\mathcal{Z}}, C_{\mathcal{Z}})$ are the solutions of the optimal problem and the the value function is given by

$$V_{\mathcal{Z}}(x) = \mathcal{G}(\mathcal{Y}_{\mathcal{Z}}(x))$$

Where

$$\mathcal{G}(y) \triangleq E\left[\int_0^T u(t, I_u(t, yH^{\mathcal{Z}}(t^-))dt + g\left(I_g\left(yH^{\mathcal{Z}}(T)\right)\right)\right], \forall y \in (0,\infty) \quad (32)$$

And the convex dual of $V_{\mathcal{Z}}(\cdot)$ is

$$\widehat{V_{\mathcal{Z}}}(y) = \mathcal{G}(y) - y\mathcal{X}_{\mathcal{Z}}(y) = E\left[\int_0^T \widehat{u}(t, yH^{\mathcal{Z}}(t^-))dt\right] + E\left[\widehat{g}\left(yH^{\mathcal{Z}}(T)\right)\right]$$

Proof.- By construction $\xi_{\mathcal{Z}}$ and $c_{\mathcal{Z}}$ satisfy (29) and using the following inequality

$$f(I_f(y)) \geq f(x) + y\left[I_f(y) - x\right]$$

for every utility function f, we obtain

$$u(t, c_{\mathcal{Z}}(t)) \geq u(t,1) + \mathcal{Y}_{\mathcal{Z}}(x)H^{\mathcal{Z}}(t^-)(c_{\mathcal{Z}}(t) - 1),; \; 0 \leq t \leq T$$

$$g(\xi_{\mathcal{Z}}) \geq g(1) + \mathcal{Y}_{\mathcal{Z}}(x)H^{\mathcal{Z}}(T)(\xi_{\mathcal{Z}} - 1), \;\; a.s.$$

Therefore

$$E\left[\int_0^T u^-(t, c_{\mathcal{Z}}(t))dt + g^-(\xi_{\mathcal{Z}})\right] \leq |g(1)| + \int_0^T |u(t,1)|\, dt + \mathcal{Y}_{\mathcal{Z}}(x)\left[H^{\mathcal{Z}}(T) + \int_0^T H^{\mathcal{Z}}(t^-)dt\right] < \infty \quad (33)$$

Since $EH^{\mathcal{Z}}(t) \leq e^{rT}, 0 \leq t \leq T$, where $r = \max_{0 \leq t \leq T} r(t)$ and by (a) we have that there exist a portfolio process $\pi_{\mathcal{Z}}$ with $(\pi_{\mathcal{Z}}, C_{\mathcal{Z}}) \in \mathcal{A}(x)$ (also in $\mathcal{A}_0(x)$ thanks to (33)) and $X^{x,\pi_0,C_{\mathcal{Z}}}(T) = \xi_{\mathcal{Z}}$ *a.s.*).

Now take an arbitrary $x>0$, $(\pi,C)\in\mathcal{A}_0(x)$ and $y>0$ from (b) we have that the following process is a bounded (from below) local martingale :

$$H^{\mathcal{Z}}(t)X(t)+\int_0^t H^{\mathcal{Z}}(s^-)dC(s)$$

then a supermartingale and with (30) we have

$$\begin{aligned}E\left[\int_0^T u(t,c(t))dt+g\left(X^{x,\pi,C}(T)\right)\right]&\le E\left[\int_0^T u(t,c(t))dt+g\left(X^{x,\pi,C}(T)\right)\right]+\\&+y\left[x-E\left[H^{\mathcal{Z}}(T)X^{x,\pi,C}(T)+\int_0^T H^{\mathcal{Z}}(t^-)dC(t)\right]\right]\\&\le Q(y)+xy\end{aligned} \tag{34}$$

where

$$Q(y)\triangleq E\left[\int_0^T \widehat{u}(t,yH^{\mathcal{Z}}(t^-))dt+\widehat{g}\left(yH^{\mathcal{Z}}(T)\right)\right]=\mathcal{G}(y)-y\mathcal{X}_{\mathcal{Z}}(y)$$

In particular follows

$$V_{\mathcal{Z}}(x)\le Q(y)+xy,\quad \forall x>0$$

Whence

$$\widehat{V}_0(y)\le Q(y),\quad \forall y>0 \tag{35}$$

On the other hand the inequality (34) holds as equality if and only if $y=\mathcal{Y}_{\mathcal{Z}}(x)$ and $(\pi,C)=(\pi_{\mathcal{Z}},c_{\mathcal{Z}})$ then

$$\begin{aligned}E\left[\int_0^T u(t,c_{\mathcal{Z}}(t))dt+g\left(X^{x,\pi_0,c_{\mathcal{Z}}}(T)\right)\right]&=Q(\mathcal{Y}_{\mathcal{Z}}(x))+x\mathcal{Y}_{\mathcal{Z}}(x)\\&=\mathcal{G}(\mathcal{Y}_{\mathcal{Z}}(x))\end{aligned}$$

Now

$$V_{\mathcal{Z}}(x)=\mathcal{G}(\mathcal{Y}_{\mathcal{Z}}(x))$$

And also

$$Q(y) = V_{\mathcal{Z}}(\mathcal{X}_{\mathcal{Z}}(y)) - y\mathcal{X}_{\mathcal{Z}}(y) \leq \sup_{x>0} [V_{\mathcal{Z}}(x) - xy]$$

For every $y > 0$, and in conjuction with (35) we obtain $Q(y) = \widehat{V}_{\mathcal{Z}}(y), \ \ \forall y > 0$.□

5 Choosing a Measure

There exists some approachs to choose one EMM, we will concentrate our atention in the approach introduced by Gerber and Shiu[1994]: Define the new probability

$$\frac{d\mathbf{P}_t^\theta}{d\mathbf{P}_t} = \mathcal{Z}^\theta(t) = e^{\{\theta Y_t - t log\varphi(\theta)\}} \tag{36}$$

Where $\varphi(\theta) = -Ee^{\theta Y_1}$. When the stock price process has constant coefficients Gerber and Shiu [1994] prove that for a given r (constant), is possible to find a solution θ, of the following:

$$r = log\left(\frac{\varphi(\theta+1)}{\varphi(\theta)}\right) \tag{37}$$

Then we can verify that the process $\hat{P}_t = e^{-rt}P_t$ will be a martingale under $\mathbf{P}^\theta$,i.e. $\mathbf{P}^\theta \in \mathcal{Q}$. Moreover the process will be a Lévy Process under this probability and will be called the Esscher transform of the original process. In our model we consider time dependent functions, then we consider the generalized Esscher Transform:

$$\frac{d\mathbf{P}_t^\theta}{d\mathbf{P}_t} = \mathcal{Z}^\theta(t) = e^{\{\int_0^t \theta_s dY_s - \int_0^t log\varphi(\theta_s)ds\}}$$

Now we can choose θ_s in order to satisfy equation (18), Since this is the case of $K(s,x) = exp(-\theta_s x)$, $k(s,x) = -\theta_s x$ and $R_s = -c\theta_s$. In fact with these expresions we obtain

$$-c^2\sigma_s\theta_s + a\sigma_s + \rho_s - r_s + \sigma_s \int_{\mathbb{R}} x(e^{-\theta_s x} - 1)G(dx) = 0$$

It easy to verify that this equation has an unique solution for which $\varphi(\theta_s) < \infty$ and $\theta_s \in (-b_1, b_2)$ $\forall s$. Then we can get a EMM, and important fact of this measure is that this is the measure of minimum relative entropy with respect to $\mathbf{P}$. to see this remember the definition of entropy:

$$I_{\mathbf{P}}(\mathbf{Q}) = E^{\mathbf{Q}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}\right]$$

Where $\mathbf{Q}$ is any absolutely continuous measure with respect to $\mathbf{P}$, with lemma 1 we have

$$I_{\mathbf{P}}(\mathbf{Q}) = E^{\mathbf{Q}}\left[\frac{1}{2}\int_0^T R_s^2 + \int_0^T\int_{\mathbb{R}}[K(s,x)(logK(s,x)-1)+1]G(dx)\right]$$

Where $\mathbf{Q}$ depends on the choice of K and R, and these functions have to satisfy equation (18). We can show [13] that this minimum is obtained when $K = exp(-c\sigma\lambda)$ and $R = -c\sigma\lambda$, where λ is the lagrange multiplier associated to the constraint (18), this can justify the choice of the measure associated to $\theta = \sigma\lambda$. Now we use this measure to characterize optimal consumption in the case of GH distributions.

6 Generalized Hyperbolic Distributions

The GH distribution was introduced by Barndorff-Nielsen[1977] to study the distribution of sand particles, as we mentioned earlier. Many authors have used these distributions and their subclasses to model the returns of some stocks. These distributions have interesting properties: they are invariant under margining, conditioning and affine transformations. Many important distributions belong to this class or are the limit of one of its members, for example the Gaussian distribution, Laplace distribution, Student's t- distribution, Gamma distribution, hyperbolic distribution and the Normal inverse gaussian distributions (NIG)[14]. The density of the GH distribution is given by

$$gh(x;\lambda,\alpha,\beta,\delta,\mu) = a_\lambda(\delta^2+(x-\mu)^2)^{(\lambda-1/2)/2}K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)exp(\beta(x-\mu)), \tag{38}$$

[13]See T. Chan[1999]

[14]See Barndorff-Nielsen and Blaesild[1981]

$$a_\lambda = a_\lambda(\alpha, \beta, \delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}; \quad x, \mu \in \mathbb{R}$$

$$\begin{aligned} &\delta \geq 0, \mid \beta \mid < \alpha \;\; if \;\; \lambda > 0 \\ &\delta > 0, \mid \beta \mid < \alpha \;\; if \;\; \lambda = 0 \\ &\delta > 0, \mid \beta \mid \leq \alpha \;\; if \;\; \lambda < 0 \end{aligned} \tag{39}$$

Where K_λ is a modified Bessel function. The parameters μ and δ describe the location and the scale of the distribution. We can represent this distribution as a Normal Variance-Mean Mixture (NVMM) distribution.[15]. To verify it, use as a mixing distribution a generalized inverse Gaussian distribution $N^-(\lambda, \nu, \psi)$ which has the following density :

$$f(x) = \frac{(\psi/\nu)^{\lambda/2}}{2K_\lambda(\sqrt{\psi\nu})} x^{\lambda-1} exp\{\frac{1}{2}(\nu x^{-1} + \psi x)\}. \;\; x > 0$$

The parameter domain is $\lambda \in \mathbb{R},\;\; \nu > 0,\;\; \psi > 0$. And $\nu = 0$ is allowed for $\lambda > 0$, $\psi = 0$ is allowed for $\lambda < 0$. Then

$$GH(\lambda, \alpha, \beta, \delta, \mu, \Delta) = NVMM(\mu, \beta\Delta, \Delta, N^-(\lambda, \delta^2, \epsilon^2)) \tag{40}$$

$$\epsilon^2 = \alpha^2 - <\beta, \Delta\beta>$$

Remember that in (38) $n = 1$ then $\Delta = 1$. Also Barndorff-Nielsen[1978] showed that the normal distribution is obtained as a limiting case for $\delta \to \infty$ and $\delta/\alpha \to \sigma^2$.

Now as Barndorff-Nielsen and Halgreen[1977] showed N^- distributions are infinitely divisible. They used (40) to prove that GH distributions are infinitely divisible. Then they generate a Lévy processes. Now using properties

[15] A random variable $X \in \mathbb{R}^n$ is said to be distributed with a normal variance-mean mixture with location μ, drift β, structure matrix Δ and mixing distribution F, if there is a random variable z with distribution F on $[0, \infty)$ and the conditional distribution of X under z is normal:

$$P^{X|z} = N_n(\mu + z\beta, z\Delta)$$

$\mu, \beta \in \mathbb{R}^n$, Δ is symmetric, positive definite and det(Δ)=1. This last condition exclude the case $\Delta = 0$ and makes the choice of parameters unique. This distribution will be denoted by $NVMM(\mu, \beta, \Delta, F)$. Notice that any distribution F on $[0, \infty)$ could be written as $NVMM(0, 1, 0, F)$.

of the Bessel functions, we have that for $\lambda = 1$ we get the Hyperbolic distribution (H):

$$h(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\delta\alpha K_1\left(\sqrt{\alpha^2-\beta^2}\right)} exp\left(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right) \tag{41}$$

$$x,\mu \in \mathbb{R},\ 0 \le \delta,\ \mid \beta \mid < \alpha.$$

One fact in favour of using this GH distributions is that they fit very well the data, since they have semi-fat tails:

$$gh(x;\lambda,\alpha,\beta,0,1) \sim \mid x \mid^{\lambda-1} e^{-(\pm\alpha-\beta)x} \ \ as \ \ x \to \pm\infty \ and \ \lambda > 0$$

But one argument against its use is the fact that its parameters do not have a clear meaning. To circumvent this problem many parametrizations have been suggested, following Barndorff-Nielsen et al.[1985]:

$$\xi = (1+\delta\sqrt{\alpha^2-\beta^2})^{-\frac{1}{2}}, \quad \chi = \frac{\xi\beta}{\alpha} \tag{42}$$

$$\zeta = \delta\sqrt{\alpha^2-\beta^2}, \qquad \varrho = \frac{\beta}{\alpha}$$

They showed that in the case of hyperbolic distribution (χ, ξ) may be plotted in a shape triangle, which reflects asymptotically the shape, i.e. skewness and kurtosis of the distribution.
As in Eberlein and Keller[1995] empirical results on stocks returns motivates the choice of a centered symmetric hyperbolic distribution i.e. $\beta = \mu = 0$. In terms of the parametrization (42), we have that (41) is given by:

$$h(x;\zeta,\delta) = \frac{1}{2\delta K_1(\zeta)} exp\left(-\zeta\sqrt{1+(\frac{x}{\delta})^2}\right) \tag{43}$$

We will denote by $(Z_t^{\zeta,\delta})_{t\ge 0}$ the Lévy process generated by the Hyperbolic distribution with density h as in (43) i.e. the process with independent and stationary increments such that $Z_0^{\zeta,\delta} = 0$ and the distribution of $Z_1^{\zeta,\delta}$ has density h. Eberlein and Keller[1995] called this $(Z_t^{\zeta,\delta})_{t\ge 0}$ Hyperbolic Lévy motion.
It is easy to see that this process is a martingale, since it has centered independent increments. And using the stationarity we have:

$$E\left[(Z_t^{\zeta,\delta})^2\right] = tE\left[(Z_1^{\zeta,\delta})^2\right] < \infty$$

since

$$E\left[(Z_1^{\zeta,\delta})^2\right] = \delta^2 \frac{K_2(\zeta)}{\zeta K_1(\zeta)} \tag{44}$$

Moreover we can verify that $(Z_t^{\zeta,\delta})_{t\geq 0}$ is a $L^p - martingale$, $(p \geq 1)$.
For the hyperbolic Lévy motion we can compute its characteristic function and obtain

$$\psi(u) = \frac{\alpha}{K_1(\delta\alpha)} \frac{K_1(\delta\sqrt{\alpha^2 + u^2})}{\sqrt{\alpha^2 + u^2}}$$

Using this expresion Eberlein and Keller[1995] obtained the following representation

$$\phi(u) = exp\left(\int \left(e^{iux} - 1 - iux\right) g(x)dx\right), \tag{45}$$

where g is the density of the Lévy measure:

$$g(x) = \frac{1}{|x|}\left(\int_0^\infty \frac{e^{-\sqrt{2y+(\zeta/\delta)^2}|x|}}{\pi^2 y \left(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y})\right)} dy + e^{-(\zeta/\delta)|x|}\right) \tag{46}$$

Where Y_1 and J_1 are Bessel functions of first and second kind. We can see from this representation that there is no continuous part $(c = 0)$, then this process is purely discontinuous.
Now a good candidate to model the stock prices will be a hyperbolic Lévy motion plus a drift:

$$dP_t = \rho P_{t^-} dt + P_{t^-} dZ_t^{\zeta,\delta}. \tag{47}$$

we can rewrite as $dP_t = P_{t^-} dX_t$ with $X_t = \rho t + Z_t^{\zeta,\delta}$ the solution of this equation is given by the Doléans-Dade exponential:

$$P_t = P_0 exp(Z_t^{\zeta,\delta} + \rho t) \prod_{0<s\leq t} (1 + \Delta Z_s^{\zeta,\delta}) e^{-\Delta Z_s^{\zeta,\delta}} \tag{48}$$

Where $\Delta Z_s^{\zeta,\delta} = Z_s^{\zeta,\delta} - Z_{s^-}^{\zeta,\delta}$. To see the volatility as an explicit parameter do the following change of variable:

$$\delta =: \delta_\zeta = \sqrt{\zeta \frac{K_1(\zeta)}{K_2(\zeta)}}$$

Since by (44) we have

$$\sigma^2 = \delta^2 \frac{K_2(\zeta)}{\zeta K_1(\zeta)}$$

Then we obtain the process $(Z_t^{\zeta})_{t\geq 0} := (Z_t^{\zeta,\delta_\zeta})_{t\geq 0}$ and $E[(Z_1^{\zeta})^2] = 1$. Now introducing the process $(\sigma Z_t^{\zeta})_{t\geq 0}$ we have:

$$dP_t = \rho P_{t^-} dt + \sigma P_{t^-} dZ_t^{\zeta}. \tag{49}$$

Here σ is the daily volatility.[16]

Now we will solve the optimal consumption/investment problem, considering the Essher Transform of the process introduced in the last chapter. Take the following state price density, where $\mathcal{Z}^{\theta}(t)$ is given by (36):

$$H^{\theta}(t) = \gamma(t)\mathcal{Z}^{\theta}(t), \quad \forall t \in [0,T] \tag{50}$$

using (5) and the generalized Itô's lemma for $F(\mathcal{Z}^{\theta}(t), \gamma(t)X(t)) = H^{\theta}(t)X(t)$, we obtain:

$$d\left(H^{\theta}(t)X(t)\right) = -H^{\theta}(t^-)dC(t) + H^{\theta}(t^-)\pi(t)(\rho(t)-r)dt + H^{\theta}(t^-)\pi(t)\sigma(t)dZ^{\xi}(t) \\ +\gamma(t)X(t^-)d\mathcal{Z}^{\theta}(t) + H^{\theta}(t)\Delta X(t) \tag{51}$$

Now applying the generalized Itô's Lemma to (36), we obtain:

$$d\mathcal{Z}^{\theta}(t) = \theta\mathcal{Z}^{\theta}(t^-)dZ^{\xi}(t) + \mathcal{Z}^{\theta}(t^-)\left[e^{\theta\Delta Z^{\xi}(t)} - 1 - \theta\Delta Z^{\xi}(t)\right] \tag{52}$$

Then replacing (52) in (51), we have

$$d\left(H^{\theta}(t)X(t)\right) = -H^{\theta}(t^-)dC(t) + H^{\theta}(t^-)\left[\pi(t)\sigma(t) + \theta X(t^-)\right]dZ^{\xi}(t) + dD(t) \tag{53}$$

where $dD(t)$ is given by

$$dD(t) = H^{\theta}(t^-)\pi(t)(\rho(t)-r)dt + H^{\theta}(t^-)X(t^-)\left[e^{\theta\Delta Z^{\xi}(t)} - 1 - \theta\Delta Z^{\xi}(t)\right]$$

Then if the he process D_t and X_t satisfied the asumptions of the theorem 3, we would have the optimal policies given by

$$\xi_{\theta} = I_g\left(yH^{\theta}(T)\right) \quad and \quad c_{\theta}(t) = I_u\left(t, yH^{\theta}(t^-)\right)$$

[16] One problem with this formulation is that the price could be negative, to avoid this we have to imposse a restriction to the jumps: $\sigma Z_t^{\zeta} > -1$ or make the folllowing reformulation:

$$dP_t = P_{t^-}\left[\rho dt + \sigma dZ_t^{\zeta} + \left(e^{\sigma\Delta Z_t^{\zeta}} - 1 - \sigma\Delta Z_t^{\zeta}\right)\right]$$

Then we obtain $P_t = P_0 exp(\rho t + \sigma Z_t^{\zeta})$.

7 Conclusions

In this paper we have obtained optimality results for more realistics distribution than the Normal distribution. It would be important to extend these results to other processes, as processes with dependent increments. Since it could allow us to improve the model, because these processes can be used to model other stylized facts of financial data, as the observed persistence in the correlations of the absolute and squared log returns, fact that can not be performed with Lévy processes, see for instance Rydberg, T.[1999]. Another important fact in this paper is the use of the characterization of the EMM in a context given by a Lévy process, to characterize arbitrage free economies. As we know, recently many empirical results have supported the use of alternative models for stock prices, such as the Generalized hyperbolic distributions. But without any equilibrium analysis,so we could use this characterization and optimality conditions to study the existence of equilibrium and give a theoretical support to these alternative models.

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