# Efficient Sorting in a Dynamic Adverse Selection Model: The Hot Potato

Igal Hendel Alessandro Lizzeri Marciano Siniscalchi*

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**Abstract**

We study the possibility of achieving efficiency in a dynamic adverse selection market for durable goods. The idea is to use the number of times a car has been traded ("vintage") as a signal of its quality. Higher-valuation consumers experiment with younger vintages.

We first exhibit an impossibility result: no choice of (re)sale prices can induce consumers to follow this experimentation policy.

We then show that modified leasing contracts can be constructed so as to achieve efficiency if consumers are patient.

## 1 Introduction

This paper investigates the possibility of overcoming adverse selection in markets for durable goods.

We study the following environment. There is a mass of perfectly durable goods (cars for concreteness) of different qualities, and a mass of consumers who differ in their valuations for quality. Efficiency then requires matching qualities to consumers in such a way that consumers with higher valuations end up consuming higher-quality goods. Consumers cannot observe the qualities of cars they have not driven (say, for at least one period); therefore, the efficient allocation may not obtain, because of adverse selection.

Our main contribution is to exhibit a mechanism that achieves (*ex-post*) efficiency if consumers are patient, despite the presence of asymmetric information. The idea is to introduce new cars gradually over time, open markets for different "vintages," and induce consumers to "experiment" with the right vintage until they get the right quality. In equilibrium, the vintage of a car, i.e. the number of times it has been tried by different consumers, serves as a signal of its quality.

*Email: {igal, lizzeri, marciano}@princeton.edu. We would like to thank Patrick Bolton, Markus Brunnermeier, and Jose Scheinkman.

Specifically, consumers adopt the following policy. Assume that there are finitely many distinct qualities. Highest-valuation consumers only experiment with cars of vintage zero, i.e., new cars. They keep a car if and only if it is of the highest quality. Consumers in the group with second-highest valuations experiment only with cars of vintage one, and keep only the highest quality of this vintage, i.e. the second-highest quality; and so on.

The transactions for a particular unit thus continue until that unit finds the consumer that is the right match for that unit. Symmetrically, a consumer continues experimenting with a particular vintage until she gets the top quality of that vintage, and then exits the mechanism. Thus, cars "trickle down" from consumers with high valuation to consumers with lower valuations.

A natural starting point is to ask whether an appropriate choice of (re)sale prices can induce consumers to follow this policy. It turns out that this is impossible, except in the very special case in which a car has only two possible qualities.

We provide an interpretation of this impossibility in terms of the cost of experimentation. Loosely speaking, for efficiency to obtain, the costs of experimentation must become small (relative to its long-run benefits) when consumers become patient. Otherwise, consumers who are *ex-ante* indifferent between experimenting with cars of two different vintages will not be willing to continue experimenting *ex-post*, if they get the second-best draw for their vintage. We show that, in general, the costs of experimentation do not become small when consumers become patient. Thus, a resale mechanism cannot achieve efficiency.

The mechanism we propose employs a menu of modified leasing contracts to separate experimentation costs from buying costs. Consumers who lease a car of a particular vintage pay a rental cost for trying the car one period. In addition, they get the option to buy the good if they so wish, at a purchase (or strike) price which is set in advance, independently of the rental cost. If they do not keep the car, the unit becomes a lower-vintage car and is rented to consumers in the next group.

Under this mechanism, as consumers become patient, the costs of experimentation goes to zero, and the strike prices converge to the prices that would obtain if quality was observable. Thus, an efficient allocation can be implemented if consumers are patient. Of course, since efficient sorting is obtained through experimentation, it takes time for the right quality to find the right consumer. However, it turns out that the expected number of periods it takes to achieve sorting is independent of the discount factor. Hence this time cost is negligible for patient consumers.

The process whereby cars trickle down from high- to low-valuation consumers has one delicate feature. The flow of cars of vintage $n$ that is available at any given date need not match the number of consumers who are supposed to experiment with that vintage at that date. Thus, the mechanism has to delay the experimentation of a fraction of these consumers. Furthermore, in order to prevent consumers from experimenting with the "wrong" vintage, the algorithm must specify that this delay be monotonic in vintage, i.e., consumers of worse vintages face longer delays than consumers of better vintages. However, this delay is independent of the discount factor; as consumers become more patient, the cost it imposes on them vanishes.

We wish to emphasize that, in the mechanism that implements the efficient allocation, the purchase prices paid by consumers approximate those that would prevail if quality were observable.

This has two important implications. First, it is **not** the case that the planner is subsidizing the mechanism.

Second, it can be shown that the mechanism we construct would be approximately optimal for a monopolist. Indeed, suppose for a moment that the monopolist is operating in a world where quality is observable, and the monopolist chooses the distribution of quality to optimally screen consumers *à la* Mussa-Rosen (1978). Now suppose that we move back to a world where quality is initially unobservable. If the monopolist follows the mechanism that we propose and chooses the same menu of qualities, he makes approximately the same profits as in the optimal mechanism for the case in which quality is observable. Thus, optimal screening would obtain in spite of adverse selection.

The paper is organized as follows. Section 2 presents the model and the definition of (approximate) efficiency. Section 3.1 discusses the case of the resale mechanism and shows that efficiency cannot be obtained in such a mechanism. Section 4 introduces the mechanism that achieves efficiency. Section 5 concludes with some additional observations. Proofs of ancillary results are in the Appendix.

### 1.1 Related Literature

Akerlof (1970) provided the seminal analysis of adverse selection. He showed that markets may break down completely if there is asymmetric information. In Akerlof's model, goods are pre-assigned so that the world starts with the owner of a good being in possession of private information about some variable that matters to potential buyers. In this environment it is impossible to achieve ex-post efficiency. The essential difference with our model is that in our case, the world starts without the goods being pre-assigned. Thus, no consumer has superior information on the quality of the goods. This seems to be a more appropriate assumption for durable goods since no consumer is likely to possess superior information about the quality of the new goods.

Hendel and Lizzeri (1999) study a model of adverse selection that is closer to the one in the present paper. The main differences is that Hendel and Lizzeri (1999) study a world in which cars depreciate and only last two periods. Their main focus is not efficiency although they exhibit an example in which selling and re-trading can lead to efficient allocations.

Waldman (1999) and Hendel and Lizzeri (1998) study the role of leasing contracts under adverse selection in a similar model to the one in Hendel and Lizzeri (1999). Waldman provides an example in which leasing contracts lead to efficient allocations. However, Hendel and Lizzeri show that this is a special case, and that in general, in that environment, it is impossible to achieve efficiency through leasing contracts. Hendel and Lizzeri also provide a mechanism that does achieve efficiency. This mechanism is similar to the one described in Section 5, and suffers from the same lack of robustness that will be discussed below.

Janssen and Roy (1999) consider a dynamic version of Akerlof's problem in which used markets are open at every date. They show that goods of all qualities are traded in finite time. The main difference with our paper is that this trading does not lead to efficiency: this is unattainable when the good is pre-assigned. Indeed, in their model the number of periods it takes for transactions to be completed increases without bound as consumers become more patient.

# 2 Preliminaries

## 2.1 Model

Consider a discrete time, infinite horizon economy. There is a unit mass of infinitely lived consumers who differ in their valuation for quality, and a mass $y < 1$ of perfectly durable goods (cars) of several possible qualities. The environment also includes a monopolistic seller or planner, who initially owns the cars.

Technology determines a distribution of car qualities with finite support $\{q_0, q_1, \ldots, q_N\}$; we assume that $q_0 > q_1 > \ldots > q_N$[1]. For each $n = 0, \ldots, N$, the probability that the quality of a randomly selected car is $q_n$ is denoted by $\lambda_n$. It is also convenient to denote by $L_n$ the probability that the quality of a randomly selected car is $q_n$ or lower: that is, for every $n = 0, \ldots, N$, $L_n = \sum_{m=n}^{N} \lambda_n$. For notational ease, we shall let $L_{N+1} = 0$.

Consumers differ in their valuation of car qualities; moreover, per-period utility is quasi-linear. Specifically, we assume that each consumer is characterized by a "type" $\theta \in [\underline{\theta}, \overline{\theta}] \subset \mathbf{R}_+$, distributed according to the c.d.f. $F$, with strictly positive density. A consumer of type $\theta$ who drives a car of quality $q$ and effects a monetary transfer in the amount $p \geq 0$ (to another consumer, to a seller, or to the planner) for one period enjoys flow utility $\theta \cdot q - p$. Finally, consumers evaluate utility streams by discounting at the common rate $\delta \in (0, 1)$.

[1] In the experimentation equilibria we construct in Sections 3 and 4, the quality of a car that has been traded $n$ times is $q_n$ or worse. This makes it convenient to adopt the numbering convention in the text.

### 2.2 Efficiency

In the environment under consideration, the *ex-post* efficient allocation of cars to consumers ("efficient sorting" hereafter) can be described as follows.

First, let $\theta_{-1} := \overline{\theta}$; next, proceeding iteratively for $n = 0, \ldots, N$, assuming that $\theta_{n-1}$ has been defined, choose $\theta_n$ such that

$$\forall n = 0, \ldots, N, \quad F(\theta_{n-1}) - F(\theta_n) = \lambda_n y;$$

observe that $\theta_{-1} > \theta_0 > \ldots > \theta_N$ by construction; also, $\theta_N > \underline{\theta}$, because $y < 1$.

Thus, for every $n = 0, \ldots, N$, the mass of consumers with types $\theta \in [\theta_n, \theta_{n-1}]$ is equal to the mass of cars of quality $q_n$. We then assign all cars of quality $q_n$ to consumer types $\theta \in [\theta_n, \theta_{n-1}]$.

In a multi-period setting, *ex-post* payoff efficiency requires that the above allocation be implemented in the first period. However, if efficient sorting obtains by a fixed, finite time $T > 1$ independent of the discount factor $\delta$, then the cost of delay vanishes as $\delta \to 1$; in other words, payoff efficiency obtains provided consumers are patient. We adopt this limiting notion as our main reference point.

We are interested in approximating efficient sorting and payoff efficiency. Specifically, an allocation achieves $\epsilon$-efficient sorting if all but a mass $\epsilon > 0$ of cars (hence, of consumers) are efficiently sorted. If $\epsilon$-efficient sorting obtains by a finite time $T(\epsilon)$ independent of $\delta$, the cost of delay vanishes as $\delta \to 1$; hence, we say that payoff $\epsilon$-efficiency obtains provided consumers are patient. Finally, we say that efficient sorting (resp. payoff efficiency) obtains asymptotically provided consumers are patient if, for every $\epsilon > 0$, $\epsilon$-efficient sorting (resp. $\epsilon$-payoff efficiency) obtains by a finite time $T(\epsilon)$ independent of $\delta$.

## 3 Experimentation and Resale

We noted in the Introduction that if consumers adopt appropriate, type-dependent experimentation policies, efficient sorting will result. This section provides a more detailed analysis of "efficient experimentation," and investigates whether a planner can induce consumers to adopt them by *selling* new cars and opening *resale* markets.

### 3.1 Efficient Experimentation

We begin by sketching the essential features of the *efficient experimentation* policy.

A consumer in the highest type bracket, $[\theta_0, \theta_{-1}]$, buys a new car in every period; she keeps it if its quality is $q_0$, and resells it otherwise.

Hence, high-type consumers supply *vintage-1* cars, i.e. cars that have been used for one period. Consumers in the second-highest type bracket, $[\theta_1, \theta_0]$, purchase them and keep only cars of quality $q_1$.

Consumers in the type bracket $[\theta_1, \theta_0]$ thus supply *vintage-2* cars, which are purchased by consumers in the type bracket $[\theta_2, \theta_1]$, and so on.

Hence, if consumers conform to this experimentation policy, the quality of a car of vintage $n$ is $q_n$ or worse. In particular, the quality of a vintage-$N$ car is certainly $q_N$, so consumers in the lowest type bracket never retrade.

Also observe that the supply of vintage-$n$ cars is positive only for $t \geq n$. Thus, in the first $N$ rounds of trading, some markets will be closed.

Moreover, at any time $t$, for arbitrary distributions of quality, the residual mass of consumers in the $n$-th type bracket may exceed the mass of available vintage-$n$ cars. In particular, this will *always* be the case for vintage-$N$ cars. However, for every integer $N$, there exists a generic set of distributions for which available supply exceed available demand in all but the highest-vintage market:

**Lemma 3.1** *For every $N > 0$, efficient experimentation will generate supply in excess of demand for all vintages except $N$ if and only if*

$$\frac{\lambda_0}{L_0} \geq \frac{\lambda_1}{L_1} \geq \cdots \geq \frac{\lambda_{N-1}}{L_{N-1}}$$

**Proof:** by induction, using Equation 20 in the Appendix.

Our results in Section 4 imply that the above experimentation policy *will* lead to asymptotically efficient sorting for any distribution of qualities and types.

### 3.2 Resale Mechanisms

Implementing efficient experimentation via a mechanism based on resale presents two distinct problems. First, as mentioned above, efficient experimentation may generate excess demand in the market for vintages other than $N$. The planner must then choose prices so as to clear all markets, but this creates incentives for some consumer types to experiment with the wrong vintages.

The second problem is more fundamental. In order to sustain efficient experimentation, prices must induce consumers in the $n$-th type bracket to experiment with vintage-$n$ cars; moreover, prices must induce these consumers to continue experimenting if they receive a car of quality $q_{n+1}$ or worse. It turns out that, in general, prices cannot achieve both objectives simultaneously.

We analyze the latter problem first; in order to focus solely on the incentive issues, we assume that the condition appearing in Lemma 3.1 applies, so that all but the oldest vintage are in nonnegative excess supply.

### 3.2.1 Ex-ante and ex-post incentives are incompatible

A *resale mechanism* then functions as follows. At each time $t = 0, 1, \ldots$, the planner:

1. Fixes prices $p_0^t, \ldots, p_N^t$ for each vintage; if the supply of some vintage is zero, the corresponding price may be taken to be infinity (or greater than $(1-\delta)^{-1} q_0 \overline{\theta}$);

2. Determines the supply of new (vintage-0) cars, and, if there is excess supply in the market for vintage $n < N$, clears that market by buying out all cars in excess at the current price.

On the other side of the mechanism, at each time $t = 0, 1, \ldots$, consumers can:

1. do nothing, i.e. keep their car, if they own one, or remain without a car, if they do not own one;

2. possibly buy a new or used car, or trade their current car for another (of equal or different vintage).

Finally, consumers enjoy per-period utility.

As a preliminary observation, note that implementing this mechanism will be costly for the planner, if $\delta < 1$. Prices will decline with time (see below), so that clearing markets in excess supply is costly for the planner. In any case, we shall propose an alternative resolution of the market clearing problem shortly.

We now analyze the problem faced by consumers in some detail. For simplicity, we focus on periods in which all vintages are in positive supply; the arguments may be adapted to the first $N$ trading rounds.

For efficient experimentation to obtain, at any time $t \geq N$ and for every $n = 0, \ldots, N$, an agent of type $\theta \in [\theta_n, \theta_{n-1}]$ must be willing to buy vintage $n$ if she does not own a car; moreover, she must be willing to retrade her vintage-$n$ car for another car of the same vintage if its quality is less than $q_n$.

Denote by $V^t(n;\theta)$ the value at date $t$ of following this policy to a consumer of type $\theta$ who owns no car; then

$$V^t(n;\theta) = -p_n^t + \frac{\lambda_n}{L_n} \frac{\theta q_n}{1-\delta} + \frac{L_{n+1}}{L_n} \Big( \mathbf{E}[q | q \leq q_{n+1}] \theta + \delta p_{n+1}^{t+1} + \delta V^{t+1}(n;\theta) \Big). \tag{1}$$

That is, the consumer pays a price $p_n^t$; with probability $\frac{\lambda_n}{L_n}$, she receives a car of quality $q_n$, and keeps it forever; with complementary probability, she receives a car of worse quality, so only enjoys per-period consumption and continues to experiment. More precisely, at time $t+1$ she sells her car (whose vintage is now $n+1$) and adopts the same policy she adopted at time $t$.

Observe that $V^t(N;\theta) = \frac{\theta q_N}{1-\delta} - p_N^t$ for all $t$, because there is no uncertainty as to the quality of vintage-$N$ cars and, by assumption, consumers in the lowest type bracket never retrade.

**Remark 3.1** *For every $\theta \in [\underline{\theta}, \overline{\theta}]$, $n = 0, \ldots, N-1$ and $t \geq N$:*

$$V^t(n;\theta) = B_n\theta - p_n^t - \delta\frac{L_{n+1}}{L_n}C_n^{t+1},$$

*where, for every $n = 0, \ldots, N-1$, $B_n$ is a measure of the benefits from experimentation:*

$$B_n = \left(1 - \delta\frac{L_{n+1}}{L_n}\right)^{-1}\left(\frac{\lambda_n}{L_n}\frac{q_n}{1-\delta} + \frac{L_{n+1}}{L_n}\mathbf{E}[q|q \leq q_{n+1}]\right)$$

*and, for every $n = 0, \ldots, N-1$ and $t \geq N$, $C_n^t$ measures the cost of experimentation:*

$$C_n^t = \sum_{s=t}^{\infty}\left(\delta\frac{L_{n+1}}{L_n}\right)^{s-t}(p_n^s - p_{n+1}^s).$$

*Note that $(1-\delta)B_n \uparrow q_n$ as $\delta \uparrow 1$, and $C_n^t = (p_n^t - p_{n+1}^t) + \delta\frac{L_{n+1}}{L_n}C_n^{t+1}$.*

All remarks in this section are proved in the Appendix (see Subsection 6.1).

The benefits from experimentation may be seen as a weighted average[2] of the net present value of the quality the consumer will ultimately obtain by experimenting with vintage $n$, $q_n$, and the net present value of the average quality she enjoys whenever she receives a car of inferior quality, $\mathbf{E}[q|q \leq q_{n+1}]$. Note that, consistently with this observation, we can define $C_N = 0$ and $B_N = (1-\delta)^{-1}q_N$.

We now list two necessary conditions for efficient experimentation to be optimal at and after time $t = N$, given the price sequence $\{p_0^t, \ldots, p_N^t\}_{t\geq 0}$.

First, whenever a consumer does not own a car, she find experimentation with the "right" vintage at least as attractive as any alternative policy she could adopt; that is, efficient experimentation must be *ex-ante* incentive-compatible.

In particular, a consumer in the $n$-th type bracket must (weakly) prefer to experiment with vintage $n$ rather than with any other vintage. As long as $B_n > B_{n+1}$ for all vintages $n = 0, \ldots, N-1$

[2]The weights are $(1-\delta\frac{L_{n+1}}{L_n})^{-1}\frac{\lambda_n}{L_n}$ and $(1-\delta\frac{L_{n+1}}{L_n})^{-1}\frac{L_{n+1}}{L_n}(1-\delta)$.

(which can be guaranteed by choosing $\delta$ sufficiently close to 1), by standard arguments the following condition is necessary (and sufficient) to ensure that this will be the case: for every $t \geq N$,

$$V^t(N;\theta_N^t) = 0 \quad \text{and} \quad \forall n = 0,\ldots,N-1, \;\; V^t(n;\theta_n) = V^t(n+1;\theta_n). \tag{2}$$

The cutoff type $\theta_N^t$ is chosen so that $F(\theta_{N-1}) - F(\theta_N^t)$ equals the available supply of vintage-$N$ cars at time $t$. Notice that $\theta_N^t \downarrow \theta_N$ as $t \to \infty$.

Equation 2 pins down the entire sequence of price vectors. Also, it implies that the cost of experimentation $C_n^t$ is *time-independent*.

**Remark 3.2** *If Equation 2 holds for every $t \geq N$, then:*
*(i) $p_N^t = (1-\delta)^{-1} q_N \theta_N^t$.*
*Moreover, for all vintages $n = 0,\ldots,N-1$:*
*(ii) $C_n^t = (B_n - B_{n+1})\theta_n + \delta \frac{L_{n+2}}{L_{n+1}} C_{n+1} \equiv C_n$.*
*(iii) $p_n^t = p_{n+1}^t + \left(1 - \delta \frac{L_{n+1}}{L_n}\right) C_n$.*

Efficient experimentation must also be incentive-compatible *ex-post*, i.e. after a consumer has observed the quality of the car she has bought. Specifically, whenever a consumer in the $n$-th type bracket buys a car of vintage $n$ and learns that its quality is $q_{n+1}$, she must be willing to continue experimenting. Thus, for every $t \geq N$ and $n = 0,\ldots,N-1$,

$$\frac{\theta_n q_{n+1}}{1-\delta} \leq p_{n+1}^t + V^t(n;\theta_n).^3 \tag{3}$$

We emphasize that Equations 2 and 3 do *not* exhaust all necessary conditions for optimality.[4] We now show that Equations 2 and 3 cannot hold simultaneously for all vintages and periods if there are more than three qualities. This suffices to conclude that experimentation cannot be supported in a resale mechanism; of course, the prices defined in Remark 3.2 may also fail additional necessary conditions.

We can use Remark 3.1 to substitute for the value function in the right-hand side of Equation 3; this yields

$$\frac{\theta_n q_{n+1}}{1-\delta} \leq p_{n+1}^t - p_n^t + B_n\theta_n - \delta\frac{L_{n+1}}{L_n} C_n = B_n\theta_n - C_n \tag{4}$$

where we have used the recursive decomposition of $C_n^t$ and the fact that by Remark 3.2, $C_n^t$ is independent of $t$.

[3]Suppose that, at the end of time period $t-1$, the consumer owns a car of vintage $n$ and quality $n+1$. Then she can either leave the market and keep the car forever, for a discounted payoff of $\delta\frac{\theta_n q_{n+1}}{1-\delta}$, or else she can reenter the market and continue experimenting, for a discounted payoff of $\delta p_{n+1}^t + \delta V^t(n;\theta_n)$.

[4]For instance, consumers must not have an incentive to delay experimentation. This implies that prices cannot drop too quickly relative to the discount factor $\delta$. See Subsection 4.2 for a discussion of this issue.

Equation 4 emphasizes the trade-offs between the costs and benefits of experimentation. However, for $\delta$ large, the costs will outweigh the benefits.

**Remark 3.3** *Suppose that Equation 2 holds. Then*

$$\lim_{\delta \to 1}(1-\delta)C_{N-1} = (q_{N-1} - q_N)\theta_{N-1} \text{ and } \forall n = 0, \ldots, N-2, \ \lim_{\delta \to 1}(1-\delta)C_n > (q_n - q_{n+1})\theta_n.$$

Thus, we obtain the main result of this section.

**Proposition 3.2** *For every $N > 2$, for any distribution of qualities which satisfy the conditions in Lemma 3.1, and for $\delta$ sufficiently close to 1, no sequence of prices can induce efficient experimentation.*

**Proof:** From Remarks 3.1 and 3.3, $\lim_{\delta \to 1}(1-\delta)(B_n\theta_n - C_n) = q_n\theta_n - \lim_{\delta \to 1}(1-\delta)C_n < q_n\theta_n - (q_n - q_{n+1})\theta_n = q_{n+1}\theta_n$. Thus, for $\delta$ sufficiently close to 1, Equation 4 must be violated. ■

The intuition for this result can be gleaned from Remark 3.3. Consider a consumer of type $\theta_n$ who receives a car of vintage $n$ and quality $q_{n+1}$. For $\delta$ sufficiently close to 1, the benefit from continuing experimentation is approximately $(q_n - q_{n+1})\theta_n$ per period—the difference between her payoff from the quality she will ultimately receive and the payoff she can secure now. On the other hand, for $n < N - 1$, the flow cost of experimentation is higher than $(q_n - q_{n+1})\theta_n$; hence, the consumer will prefer to stop experimenting.

In other words, experimentation costs (which are determined by prices) are such that, *ex-ante*, consumers are willing to try out the appropriate vintages. However, it turns out that they are too high to induce consumers *ex-post* to continue experimenting when efficiency dictates that they do. Loosely speaking, the mechanism attempts to rely on "too few prices" to meet "too many constraints."

Proposition 3.2 states that, for a set of parameters of the model (determined by the quality distribution and the discount factor) having positive Lebesgue measure, efficient experimentation cannot be sustained in a resale mechanism. Thus, it provides an impossibility statement which mirrors the standard results for static environments (e.g. Akerlof (1970)).

Alternatively, Proposition 3.2 may be interpreted as stating that, for a set of quality distributions of positive Lebesgue measure, efficient sorting may be achieved, but consumers have to be sufficiently *im*patient.[5] This implies that efficient sorting has a cost in terms of consumer payoffs, and hence it does *not* imply payoff efficiency.

[5] On the other hand, consumers must not be too impatient: otherwise, in general, Equation 2 is not sufficent to ensure that ex-ante incentive compatibility will hold for *all* consumer types (not just marginal types). In fact, we conjecture that Proposition 3.2 can be strengthened by dropping the condition on the discount factor.

### 3.2.2 Excess demand for intermediate vintages

The possibility that vintages other than the highest might be in short supply poses additional difficulties. We view Proposition 3.2 as our main impossibility result; hence, our aim here is merely to point to some consequences of demand-supply imbalances, which must be taken care of by imposing additional constraints on prices.

Assume that there are at least four qualities, and choose $n < N-1$. Suppose that, at date $t$, the supply of vintage-$n$ cars generated by efficient experimentation is less than the residual mass of consumers in the $n$-th type bracket. For simplicity, suppose that vintages $n$ and $N$ are the only ones in short supply.

Prices at dates $s \geq t$ should therefore be chosen so as to prevent consumers whose type $\theta$ is in at the lower end of the interval $[\theta_n, \theta_{n-1}]$ from experimenting with vintage $n$. Proceeding as we did above to accommodate excess demand in the vintage-$N$ market, we can find a type $\theta_n^t \in [\theta_n, \theta_{n-1}]$ such that $F(\theta_{n-1}) - F(\theta_n^t)$ equals the available supply of vintage-$n$ cars, and choose prices so that $V^t(n; \theta_n^t) = V^t(n+1; \theta_n^t)$.

Ideally, consumers with types $\theta \in [\theta_n, \theta_n^t)$ should be induced to "do nothing", i.e. defer experimentation (and therefore per-period consumption), until enough vintage-$n$ cars are available.

However, in general it will not be possible to force these consumers to simply wait. Note that, as an alternative, they can buy vintage-$(n+1)$ cars instead (or indeed any higher-vintage car), and sell them as soon as enough vintage-$n$ cars are available. That is, they may wish to buy the "wrong" cars solely for *temporary consumption.*

If they are allowed to do so, some of these consumers will end up buying, say, vintage-$(n+1)$ cars of quality $q_{n+1}$, and eventually reselling them in the vintage-$(n+2)$ market; we call these cars *tainted.* It follows that some consumers in the $(n+1)$-th type bracket will *not* eventually receive a car of quality $q_{n+1}$, whereas some consumers in the $(n+2)$-th type bracket will receive a (tainted) car of quality $q_{n+1}$.[6]

To deter temporary consumption and tainting, the prices of downstream vintages must be raised. But this prevents some consumer types from experimenting with the respective "right" vintages; as a consequence, the same sort of consumption-motivated deviations we are trying to eliminate might appear down the vintage hierarchy.

[6]It also follows that the (equilibrium) inference that a vintage-$(n+1)$ car is of quality $q_{n+1}$ is now unwarranted, and this undermines the logic behind efficient experimentation as defined above. One might perhaps devise a different, more complicated experimentation scheme, which takes this possibility into account; we prefer to take a different approach. Also observe that, with only three qualities, tainting is not an issue. There is always excess supply of new cars; thus, only consumers in the intermediate type bracket (other than lowest-type consumers) may face a shortage of their designated cars. Their only profitable consumption-motivated deviation is to buy vintage-2 cars; but these are known to be of quality $q_2$.

Thus, in general, temporary consumption and tainting cannot be deterred in a resale mechanism. This introduces an additional source of inefficiency.

# 4 Experimentation in a Modified Leasing Mechanism

We now attempt to resolve the problems we have identified in the previous sections and construct a mechanism which achieves asymptotic efficiency.

In preparation for the formal analysis, we first argue that, by adopting a leasing mechanism, the incentive issues highlighted in Subsection 3.2.1 can be resolved. Next, we describe how we handle excess demand in markets for intermediate vintages; the key idea is to allow the planner to serve demand for different vintages at different rates, while keeping prices constant (after the initial $N$ periods).

We then state and prove our main result: the mechanism we construct achieves asymptotic efficiency for all distributions of qualities—even those violating the conditions in Lemma 3.1. We also emphasize that the revenues to the planner from the mechanism approximate, for $\delta$ sufficiently close to one, the revenues that would accrue to him if qualities were observable.

## 4.1 Reconciling ex-ante and ex-post incentive compatibility

In resale mechanisms, prices have two distinct roles: they obviously represent the cost of keeping a car forever, but they also determine the cost of experimentation. As we noted in our comments following Proposition 3.2, this may be viewed as the main reason why resale mechanisms cannot achieve efficiency.

Thus, it seems natural to address the problem by *decoupling* these costs. Specifically, we envision a mechanism whereby consumers *rent* (or lease) a car from the planner for one period, and have the right, but not the obligation, to *keep* it forever.

The *rental price* is paid at the beginning of the period; the consumer chooses whether or not to exercise her option to keep the car she has rented at the end of the period, i.e. after learning its quality.

If she does exercise it, she must pay a *buying* (or exercise) *price* to the planner. It is notationally convenient to assume that the buying price is actually paid at the beginning of the subsequent period.

If she does not exercise it, she must return the car to the planner. The vintage of a car equals the number of times it has been rented.

Thus, the cost of experimenting with a given vintage equals the respective rental price. This is distinct from the cost of keeping a car of that vintage forever, i.e. the buying price.

The timing of the mechanism is as follows. At each date $t$, the planner fixes rental and buying prices. Notice that no direct intervention in secondary markets is required: the planner only has to make her stock of cars, subdivided according to vintage, available for consumers to choose from.

On the opposite side of the mechanism, consumers who have not yet purchased a car:

1. Choose which vintage to rent.

2. Learn the quality of the car they have rented and enjoy per-period utility.

3. Decide whether to exercise the option to keep the car, or return it.

It should be intuitively clear that, by introducing separate prices for experimentation and eventual purchase, we can avoid the problems described in Subsection 3.2.1. This will be formally established in the course of the proof of our main result, but we shall provide an informal explanation at this stage.

An equation similar to 2 will ensure that consumers prefer to experiment with the "right" vintage rather than with any other vintage. Note that, in a resale mechanism, Equation 2 determines all prices, and hence the *total* cost of implementing the experimentation policies.

In a leasing mechanism, the counterpart to Equation 2 will still determine the total cost of these policies, but not the *split* between experimentation and purchase costs.

On the other hand, an equation similar to 3 will ensure that consumers be willing to continue experimenting if they do not receive the best car for their designated vintage. The idea is then to choose the split between experimentation and purchase costs so as to achieve *ex-post* incentive compatibility.

We conclude this subsection by noting that prices will need to ensure that no other policy is a profitable deviation from efficient experimentation. This includes deviations we did not need to consider in order to establish the impossibility result in Section 3. In particular, we wish to draw the reader's attention to two possible deviations.

First, consumers may adopt a "pure consumption policy," whereby, at each date, they rent a car of some relatively inexpensive vintage, and never exercise their option.[7]

Second, if prices are non-stationary, consumers may delay experimentation and perhaps rent the "wrong" vintage for temporary consumption purposes, as discussed in Subsection 3.2.2.

In order to deter the first deviation, we choose rental prices so as to ensure that, among all possible "pure-consumption" policies, a consumer in the $n$-th type bracket prefers to rent vintage-$n$ cars indefinitely. Loosely speaking, since efficient experimentation ensures that this consumer will

[7]For large discount factors, this policy is not profitable in a resale mechanism, because the implied "rental cost" of a vintage-$n$ car, $p_n^t - \delta p_{n+1}^{t+1}$, diverges to infinity as $\delta \to 1$.

eventually receive a car of quality $q_n > \mathbf{E}[q|q \leq q_n]$, for large discount factors no pure-consumption deviation will be profitable.

Moreover, it turns out that the buying prices implicitly determined by the choice of rental prices also satisfy the *ex-post* incentive constraints.

The second kind of deviation warrants a more extensive discussion, and motivates a key ingredient of our modified leasing mechanism.

## 4.2 Market-Clearing and Stationarity with Decreasing Servicing Rates

Both resale and leasing mechanisms run into difficulties if prices are nonstationary. Temporary consumption of "wrong" vintages is but one variant of the problems which stem from the opportunity to delay experimentation.

These problems are not directly related to adverse selection. To see this, consider the market for vintage-$N$ cars in a resale mechanism. The rate at which the price of the worst vintage $p_N^t$ drops is driven by the distribution of qualities (which determines the supply of vintage-$N$ cars) and the c.d.f. $F(\cdot)$ (which, loosely speaking, determines residual demand). The discount factor $\delta$ does not play any role.

Hence, for any distribution of qualities and consumer types, sufficiently patient consumers will prefer to delay buying vintage-$N$ cars, and this will of course break the equilibrium in the resale market. The problem is of course not confined to the market for vintage-$N$ cars, and afflicts leasing mechanisms as well.[8]

We are thus led to consider alternative ways to tackle excess demand. Incidentally, if we assume that the conditions in Lemma 3.1 hold, then we only need to take care of the market for vintage-$N$ cars. However, the mechanism we propose allows us to handle arbitrary distributions of quality.

The basic intuition is to restrict the planner to choose stationary prices after time $N$, but allow her to *serve demand for different vintages at different rates.*

More specifically, at each date $t \geq 0$, and for each vintage $n = 0, \ldots, N$, the planner announces a *service rate* for vintage $n$ at time $t$. The interpretation is that, out of the total mass of consumers who request a car of vintage $n$, only a (randomly selected) fraction equal to the then-prevailing service rate for that vintage actually receives one. The planner chooses the service rate so as to ensure that the fraction of demand served equals available supply.

[8]This implies that, apart from the incentive issues discussed in Subsection 3.2.1, excess demand in the lowest-quality market will surely break a resale mechanism, if consumers are sufficiently patient. However, the methods described in this section can be easily adapted to resale mechanisms. On the other hand, it is easy to show that the same incentive problems will afflict modified resale mechanisms as well, which suggests that the analysis in Subsection 3.2.1 emphasizes robust consequences of adverse selection for trade in secondary markets.

The key feature of this mechanism is that each consumer in any type bracket is equally likely to be served. Hence, prices can be chosen so as to ensure that the marginal consumer type in each type bracket is willing to rent the appropriate vintage. Excess demand is tackled not by excluding low types from current consumption, but by serving an appropriate fraction of consumers, randomly chosen irrespective of their type.

We allow the planner to choose different vectors of "service rates" at each date; however, we shall prove that, if consumers adopt the efficient experimentation policy and service rates are chosen appropriately as soon as markets open, these rates will be stationary. This implies that, in the mechanism we construct, *after the initial $N$ periods, prices will be constant.*

We emphasize that, while at any date $t$ a fraction of consumers will not be served, it is still the case that, eventually, *every* consumer will receive a car (of the appropriate quality).

Moreover, the service rate, hence the expected time before a consumer is served, are chosen independently of the discount factor. Thus, loosely speaking, patient consumers will not mind waiting. More formally, the inefficiency caused by delay vanishes in the limit as $\delta \to 1$.

One last issue must be discussed. If the conditions of Lemma 3.1 are met, then all the planner needs to do is choose an appropriate service rate in the market for vintage $N$. Note that, since all other vintages are in excess supply (so that the service rate is 1 in those markets), vintage-$N$ cars are unequivocally "worse" than all other cars: their quality is certainy worse, *and* demand is served at a lower rate.

Then, under the conditions of Lemma 3.1, a minimally modified leasing mechanism, whereby only demand vintage-$N$ cars is served at a rate less than 1, may be shown to achieve asymptotic efficiency.

However, if some intermediate market exhibits excess demand (i.e. if the conditions of Lemma 3.1 do not hold), then service rates must be chosen with some care. Specifically, setting the service rate equal to the ratio of supply and demand in such markets leads to the following problem.

It may be the case that, for some vintages $n$ and $m$ such that $n < m$ (so that cars of vintage $n$ are on average better than cars of vintage $m$), the ratio of supply to demand in market $n$ is lower than in market $m$. This may induce consumers in the $n$-th type bracket to temporarily rent vintage-$m$ cars, because the probability of receiving a unit is higher in that market, while they wait for supply of vintage-$n$ cars to build up. This induces *tainting* of vintage-$m$ cars, as in the previous section.

Hence, in order to achieve efficiency for all possible distributions of qualities, we explicitly ensure that *service rates be non-increasing in vintage.* This implies that, if $n < m$, then cars of vintage $n$ are unequivocally "better" than cars of vintage $m$: their average quality is higher, *and* demand is served at a (weakly) higher rate.

## 4.3 Formal Analysis and Main Result

We now focus on the formal details of our proposed mechanism.

To summarize the preceding discussion, as well as to introduce the required notation, at each time $t$ and for every vintage $n = 0, \dots, N$, the planner sets a *rental price* $r_n^t$ and a *buying price* $p_n^t$, as well as a *service rate* $e_n^t$.

The interpretation is that, if the vintage-$n$ market is open at date $t$, by paying the price $r_n^t$, an individual receives a car with probability $e_n^t$; in this case, she consumes it for one period, at the end of which she can decide whether to keep it or to return it. If she decides to keep it, then the following period she will be required to pay the price $p_n^t$, and will exit the market.

Observe two notational conventions: first, consumers pay a rental price even if they do not receive a car; hence, $r_n^t$ can be regarded as the price of a lottery ticket. Second, exercise prices are fixed at each time $t$, but paid at time $t+1$. Assuming that rental prices are only paid by consumers who are served, and that date-$t$ exercise prices are paid at the end of period $t$, is possible and of no consequence for the analysis, but notationally less convenient.

In the mechanism under consideration, prices are stationary after time $t = N$, so we shall only specify a time index when dealing with the first $N+1$ (numbered $0 \dots N$) periods.

At each time $t = 0, 1, \dots$, consumers observe all prices and service rates, and decide which car to rent. Vintage is observable and verifiable, whereas quality is neither.

### 4.3.1 Efficient Experimentation and the Trickle-Down Algorithm

The first order of business is to analyze the evolution of demand and supply in each market under the assumptions that:

1. Consumers follow the efficient experimentation policy;

2. The planner chooses non-increasing service rates so as to ensure that supply equals or exceeds effectively served demand.

We refer to the law of motion of demand and supply implied by the preceding two assumptions as the *trickle-down algorithm*.

Formally, denote by $S_n^t$ and $D_n^t$ the supply and demand of vintage-$n$ cars at the beginning of period $t$, for $t = 0, \dots$ and $n = \dots, N$. Let $S_0^0 = 1$, $S_n^0 = 0$ for $n > 0$, and $D_n^0 = \lambda_n$ for all $n$.[9]

[9] For notational convenience, we assume that demand and supply are also defined for $n = -1$ and $t = -1$; the respective values will be indicated as needed.

Define $R_n^t = \frac{S_n^t}{D_n^t}$, the fraction of demand for vintage-$n$ cars that can be served at time $t$ ($R_n^t > 1$ indicates that all demand can be served). Then, for every $t = 0, 1, \ldots$, let

$$e_0^t = \min(1, R_0^t) \quad \text{and} \quad \forall n = 1, \ldots, N: \; e_n^t = \min(e_{n-1}^t, R_n^t). \tag{5}$$

In words, the planner always attempts to serve as much demand as possible, given available supply, and given the constraint that service rates be non-increasing in vintage. Clearly, $e_n^t \leq R_n^t$; note also that service rates are zero in any market which is not (yet) open.

The quantities $D_n^t$ and $S_n^t$ can be defined inductively in terms of effective probabilities, as follows. First, at time $t-1$, $e_t^n$ consumers who request a vintage-$n$ car actually receive one; of these, $\frac{\lambda_n}{L_n}$ obtain a car of quality $q_n$, and hence leave the market. Thus,

$$D_n^t = D_n^{t-1}(1 - e_n^{t-1}\frac{\lambda_n}{L_n}) \tag{6}$$

where we let $D_n^{-1} = D_n^0$ and $e_n^{-1} = 0$ for convenience. Second, at time $t-1$ the supply of vintage-$n$ cars is diminished by $e_n^{t-1}D_n^{t-1}$, the number of successful consumers who request a vintage-$n$ car in that period; however, it is replenished in the amount $e_{n-1}^{t-1}\frac{L_n}{L_{n-1}}D_{n-1}^{t-1}$, corresponding to the number of consumers who successfully bid for a vintage-$(n-1)$ car in that period, but discover that the car they receive is of worse quality than $q_{n-1}$. Hence,

$$S_n^t = S_n^{t-1} - e_n^{t-1}D_n^{t-1} + e_{n-1}^{t-1}\frac{L_n}{L_{n-1}}D_{n-1}^{t-1} \tag{7}$$

where, again for notational convenience, we let $S_n^{-1} = S_n^0$, $e_n^{-1} = 0$ and $D_{-1}^t = 0$ for all $n$ and $t$ (including $n = -1$ and $t = -1$). Moreover, we also let $L_{-1} = 1$ and $e_{-1}^t = 0$.

It is also convenient to define

$$\eta_n^t = e_n^t\frac{\lambda_n}{L_n}, \tag{8}$$

the mass of consumers leaving market $n$ at time $t$.

Equations 5, 6, 7 and 8 define the trickle-down algorithm. The following result establishes the claim made at the beginning of this section, namely that, under the trickle-down algorithm, service rates are stationary. Moreover, the probability that a consumer leaves the market (i.e. rents a car and finds it to be of the best possible quality) is also decreasing in vintage, although the conditional probabilities $\frac{\lambda_n}{L_n}$ need not be ordered in any particular way. In other words, non-increasing service rates compensate for departures from the conditions in Lemma 3.1.

**Proposition 4.1** *For every $n = 0, \ldots, N$, $e_n^t = 0$ and $\eta_n^t = 0$ if $t < n$, and $e_n^t = e_n^n \equiv e_n$ and $\eta_n^t = \eta_n^n \equiv \eta_n$ if $t \geq n$; also, $e_0 = 1$. Moreover, $\eta_0 \geq \eta_1 \geq \ldots \geq \eta_N$.*

**Proof:** See Appendix.

Note that Proposition 4.1, together with the definition of $e_n^t$, implies that $e_n^t > 0$ and $\eta_n^t > 0$ for all $n$ and $t$ such that $t \geq n$.

#### 4.3.2 Value Functions for Stationary Cutoff Policies ($t \geq N$)

At time $t = N$, the $N$-th market opens, so that, at $t = N$ and at all subsequent times, all markets are open. In particular, service rates are stationary. If prices are also stationary, the problem faced by the consumers is relatively easy to analyze.

Hence, we now focus on dates $t \geq N$, and indicate in the Appendix how to "jump-start" the economy, i.e. choose non-stationary prices at times $t = 0, \ldots, N-1$ so that consumers are willing to follow efficient experimentation and thereby implement the trickle-down algorithm in the initial $N$ periods as well.

In what follows, we assume that the planner has fixed constant rental and buying prices $r_n$ and $p_n$, as well as constant service rates $e_n$, for each vintage $n = 0, \ldots, N$. Thus, consumers face a stationary problem.

In order to establish the optimality of efficient experimentation, we need to analyze all possible deviations. However, since the problem faced by the consumer at dates $t \geq N$, it is sufficient to consider stationary policies. Moreover, if a consumer is willing to buy a car of quality $q_m$ at a price $p$, she is also willing to keep a car of quality $q > q_m$ at the same price. Hence, it is sufficient to consider policies whereby exercise of the option to buy is governed by a simple cutoff rule.

Thus, consider a consumer of type $\theta$ who adopts the following *stationary cutoff policy*: at each time $t$, she rents a car of vintage $n$, and keeps it iff it is of quality $q_m$ or better. Denote by $V_{n,m}(\theta)$ the value of such policy and, for notational convenience, let $\underline{q}_n = \mathbf{E}[q | q \leq q_n]$; then

$$V_{n,m}(\theta) = -r_n + e_n \left\{ \underline{q}_n \theta + \delta \left[ \sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n} \left( \frac{\theta q_\ell}{1-\delta} - p_n \right) + \frac{L_{m+1}}{L_n} V_{n,m}(\theta) \right] \right\} + (1-e_n)\delta V_{n,m}(\theta) \quad (9)$$

That is, in exchange for a rental price of $r_n$, the consumer enters a lottery in which, with probability $e_n$, she receives a car of vintage $n$, hence (in equilibrium) of expected quality $\underline{q}_n$. She obtains expected flow utility $\underline{q}_n \theta$ from consumption, and also has the opportunity to keep the car, if its quality is $q_m$ or better. If the car is of worse quality than $q_m$, or if she is unsuccessful in the lottery, she continues pursuing the same policy.

In Equation 9, $m < n$ implies that the sum in square brackets is over an empty set, and hence will be taken to equal zero. This indicates that the consumer rents a car of vintage $n$ in each period, and never keeps it. To avoid redundancy, we indicate the value of such a "pure consumption policy" by $V_{n,-1}(\theta)$:

$$V_{n,-1}(\theta) = \frac{e_n \underline{q}_n \theta - r_n}{1-\delta}. \tag{10}$$

Now Equation 9 may be rewritten as follows:

$$V_{n,m}(\theta) = w_{n,m} V_{n,-1}(\theta) + (1 - w_{n,m}) \sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n - L_{m+1}} \left( \frac{\theta q_\ell}{1-\delta} - p_n \right) \tag{11}$$

where

$$w_{n,m} = \frac{1-\delta}{(1-\delta) + \delta e_n \frac{L_n - L_{m+1}}{L_n}}, \tag{12}$$

which emphasizes that $V_{n,m}(\theta)$ is a weighted average of the expected payoff from repeated rental and the long-run expected payoff after purchasing the car. Also note:

**Remark 4.1** $\frac{dV_{N,N}(\theta)}{d\theta} = \frac{q_N}{1-\delta}$ *and, for* $0 \leq n < N$ *and* $m \geq n$,

$$\frac{dV_{n,-1}(\theta)}{d\theta} = \frac{e_n \underline{q}_n}{1-\delta} < \frac{dV_{n,m}(\theta)}{d\theta} = \frac{1}{1-\delta} \left[ w_{n,m} e_n \underline{q}_n + (1 - w_n) \sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n - L_{m+1}} q_\ell \right] < \frac{q_n}{1-\delta}.$$

**Remark 4.2** *For every* $n = 0, \ldots, N-1$, $w_{n,n}$ *can be rewritten as follows:*

$$w_{n,n} = \frac{1-\delta}{(1-\delta) + \delta \eta_n} \tag{13}$$

*Thus,* $w_{n,n} \leq w_{n+1,n+1}$, *and moreover* $\lim_{\delta \to 1} w_{n,n} = 0$.

#### 4.3.3 Prices

We now fix prices which induce efficient experimentation. First, as mentioned above, we ensure that, if an individual takes only per-period consumption into account, then she prefers to rent a car of the appropriate vintage. Hence, we set

$$r_N = e_N q_N \theta_N \quad \text{and} \quad r_n \text{ such that } e_n \underline{q}_n \theta_n - r_n = e_{n+1} \underline{q}_{n+1} \theta_n - r_{n+1}; \tag{14}$$

observe that rental prices are thus *independent of* $\delta$.

The following remark follows from standard arguments, together with the observation that service rates are non-increasing in vintage.

**Remark 4.3** *If rental prices are defined by Equation 14, for any type* $\theta \in [\theta_n, \theta_{n-1}]$, *the (unique, if* $\theta \neq \theta_n$*) best "pure consumption" policy involves renting vintage-n cars. Moreover,* $r_0 > r_1 > \ldots > r_N$.

Next, having defined rental prices $r_n$, $n = 0, \ldots, N$, we define buying prices via the equivalent of Equation 2 in Section 3:

$$p_N = \frac{\theta_N q_N}{1-\delta} \quad \text{and } \forall n = 0, \ldots, N-1, \; V_{n,n}(\theta_n) = V_{n+1,n+1}(\theta_n). \tag{15}$$

**Remark 4.4** *If buying prices are defined by Equation 15, then $p_0 > p_1 > \ldots > p_N$.*

**Proof:** Rewrite Equation 15 for $n < N$ as follows:

$$w_{n,n}V_{n,-1}(\theta_n) + (1 - w_{n,n})\left(\frac{\theta_n q_n}{1-\delta} - p_n\right) = w_{n+1,n+1}V_{n,-1}(\theta_n) + (1 - w_{n+1,n+1})\left(\frac{\theta_n q_{n+1}}{1-\delta} - p_{n+1}\right) \tag{16}$$

using Equation 14 to rewrite the pure-consumption parts.

Suppose that $p_n \leq p_{n+1}$. Since $q_n > q_{n+1}$, $\frac{\theta_n q_n}{1-\delta} - p_n > \frac{\theta_n q_{n+1}}{1-\delta} - p_{n+1}$. Thus, since $1 - w_{n,n} \geq 1 - w_{n+1,n+1}$ by Remark 4.2, we must necessarily have $V_{n,n}(\theta_n) > V_{n+1,n+1}(\theta_n)$, i.e. Equation 16 cannot hold.[10] ■

Now define the following rental prices:

$$r_N^L = \theta_N q_N \quad \text{and } \forall n = 0, \ldots, N-1, \; \theta_n q_n - r_n^L = \theta_n q_{n+1} - r_{n+1}^L. \tag{17}$$

These are the one-period rental prices that would achieve efficient sorting if qualities were observable. Our next result states that, if consumers are patient, buying prices approximate the net present value of rental prices under observable quality.

**Lemma 4.2** *For every $n = 0, \ldots, N$, $\lim_{\delta \to 1}(1-\delta)p_n = r_n^L$ and therefore, for every $m \geq N$, $\lim_{\delta\to 1}(1-\delta)V_{n,m}(\theta) = \sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n - L_{m+1}} \theta q_n - r_n^L$.*

**Proof:** Note that $(1-\delta)p_N = r_N^L$ for any value of $\delta$. Now, from Equation 16,

$$p_n = \frac{\theta_n q_n}{1-\delta} - \frac{1 - w_{n+1,n+1}}{1 - w_{n,n}}\left(\frac{\theta_n q_{n+1}}{1-\delta} - p_{n+1}\right) + \frac{w_{n,n} - w_{n+1,n+1}}{1 - w_{n,n}} V_{n,-1}(\theta_n) \tag{18}$$

Arguing by induction, suppose that $\lim_{\delta\to 1}(1-\delta)p_{n+1} = r_{n+1}^L$; then Equation 18 shows that $\lim_{\delta\to 1}(1-\delta)p_n = \theta_n q_n - \theta_n q_{n+1} + r_{n+1}^L = r_n^L$, because $(1-\delta)V_{n,-1}(\theta_n) = e_n \underline{q}_n \theta_n - r_n$ (recall that $r_n$ is independent of $\delta$) and $w_{n,n}, w_{n+1,n+1} \to 0$ from Remark 4.2. The second claim follows directly from Equation 11. ■

[10] $V_{n,n}(\theta_n)$ places more weight on the long-run payoff than $V_{n+1,n+1}(\theta_n)$ does, and moreover the long-run payoff in the former is strictly higher than in the latter.

### 4.3.4 Optimality of Efficient Experimentation ($t \geq N$)

Apart from indicating that buying prices are approximately what they should be in an environment where efficiency is attainable, the preceding result implies that, if consumers are patient, they will be unwilling to follow an experimentation policy which prescribes buying a car other than the best possible given the vintage. In other words, the prices defined by Equations 14 and 15 make efficient experimentation *ex-post* incentive compatible.

**Remark 4.5** *There exists $\underline{\delta}_N < 1$ such that $\delta > \underline{\delta}_N$ implies that, for every type $\theta$, the best stationary cutoff policy involving vintage $n$ entails buying the car iff its quality is $q_n$.*

**Proof:** As $\delta \to 1$, $(1-\delta)V_{m,m}(\theta) \to \theta q_m - r_m^L < \theta q_m - r_n^L$ whenever $n > m$. Hence, we can find $\underline{\delta}_N < 1$ such that, for $\delta > \underline{\delta}_N$, $\frac{\overline{\theta} q_m}{1-\delta} - p_n < V_{m,m}(\overline{\theta})$, i.e. the maximal type $\overline{\theta}$ prefers to continue experimenting rather than keeping $q_m$ for a price $p_n$; but by Remark 4.1, lowering $\theta$ from $\overline{\theta}$ to any other $\theta \in [\theta_N, \overline{\theta})$ decreases the right-hand side by less than the left-hand side, so that strict inequality holds for all types. ■

The reason for the subscript "$N$" in $\underline{\delta}_N$ will become clear shortly. Coupled with Equation 15, Remarks 4.3 and 4.5 indicate that the optimal policy for a type $\theta_n \in [\theta_n, \theta_{n-1}]$ is either to rent vintage $n$ repeatedly for pure consumption purposes, or else to continue experimenting with vintage-$n$ cars until a car of quality $q_n$ is obtained. Thus, it remains to be shown that the latter policy is better.

**Proposition 4.3** *For $t \geq N$, there exists $\underline{\delta}_N < 1$ such that, for $\delta > \underline{\delta}_N$, the optimal (continuation) policy for type $\theta_n \in [\theta_n, \theta_{n-1}]$ involves experimenting with cars of vintage $n$, and buying a car iff it is of quality $q_n$.*

**Proof:** We first compare $V_{n,-1}(\theta_n)$ and the long-run payoff $\frac{\theta_n q_n}{1-\delta} - p_n$. Observe that, by construction, $V_{N,-1}(\theta_N) = 0 = \frac{\theta_N q_N}{1-\delta} - p_N$. Also, consumers in the lowest type bracket are indifferent between pure-consumption and experimentation. Next, suppose that

$$V_{n+1,-1}(\theta_{n+1}) = \frac{e_{n+1}\underline{q}_{n+1}\theta_{n+1} - r_{n+1}}{1-\delta} \leq \frac{\theta_{n+1} q_{n+1}}{1-\delta} - p_{n+1};$$

then, by raising $\theta$ from $\theta_{n+1}$ to $\theta_n$, the left-hand side increases by $\frac{e_{n+1}\underline{q}_{n+1}}{1-\delta}$, whereas the left-hand side increases by $\frac{q_{n+1}}{1-\delta}$, so we get

$$\frac{e_{n+1}\underline{q}_{n+1}\theta_n - r_{n+1}}{1-\delta} = V_{n,-1}(\theta_n) < \frac{\theta_n q_{n+1}}{1-\delta} - p_{n+1}$$

where the equality follows from Equation 14.[11] Now observe that $V_{n+1,n+1}(\theta_n)$ is a weighted average of $V_{n,-1}(\theta_n)$ and $\frac{\theta_n q_{n+1}}{1-\delta} - p_{n+1}$, with strictly positive weights; it follows that $V_{n+1,n+1}(\theta_n) > V_{n,-1}(\theta_n)$, which, since $V_{n+1,n+1}(\theta_n) = V_{n,n}(\theta_n)$ by our choice of buying prices, implies that also $V_{n,n}(\theta_n) > V_{n,-1}(\theta_n)$. Hence, *ex-ante*, the experimentation policy is better than the pure-consumption policy. Observe that, by Remark 4.1, the inequality is preserved for $\theta > \theta_n$.

Moreover, $V_{n,n}(\theta_n)$ is a weighted average of $V_{n,-1}(\theta_n)$ and $\frac{\theta_n q_n}{1-\delta} - p_n$, with strictly positive weights; hence, it must be the case that $\frac{\theta_n q_n}{1-\delta} - p_n > V_{n,-1}(\theta_n)$. That is, *ex-post*, after observing a car of quality $q_n$, type $\theta_n$ strictly prefers to keep it rather than continue experimenting. Again, the inequality is clearly preserved for $\theta > \theta_n$. This completes the inductive step. ■

Together with the results in the Appendix (see Subsection 6.3), we obtain the main result of this paper.

**Proposition 4.4** *There exists $\delta^* < 1$ such that, for $\delta > \delta^*$, the optimal policy for each consumer of type $\theta \in [\theta_n, \theta_{n-1}]$ involves experimenting with vintage $n$ at each time period, for $n = 0, \ldots, N$. Under this policy, for every $\epsilon > 0$ there exists $T(\epsilon) < \infty$, independent of $\delta$, such that a mass $1 - \epsilon$ of consumers receives a car of its designated quality by time $T(\epsilon)$.*

# 5 Comments

As promised in the Introduction, the modified leasing mechanism analyzed in Section 4 achieves asymptotic payoff efficiency, as well as efficient sorting, for any distribution of qualities.

We have argued that the mechanism we propose deviates from a simple resale mechanism precisely as mandated by incentive-compatibility issues: it incorporates just enough prices to ensure that efficient experimentation is optimal both *ex-ante* and *ex-post*, and deals with the possibility that intermediate vintages might be in short supply.

Moreover, the mechanism relies on anonymous information generated by payoff-relevant transactions, as is the case in the resale mechanism we consider in Section 3.

We also note that, apart from marginal types, consumers have *strict* incentives to follow the efficient experimentation policy. That is, the mechanism implements efficient sorting (and achieves asymptotic efficiency) as a strict (Nash or competitive) equilibrium.

This makes it *robust to perturbations in the assumptions* of the model. For instance, it can be shown that efficient experimentation remains optimal if simply driving a car for one period does

[11] The inequality is clearly strict for $n < N = 1$. Moreover, it must also be strict for $n = N$, because $e_N < 1$. To see that the latter inequality must hold, note that Equation 20 implies that $R_N^N = e_{N-1}\frac{\lambda_{N-1}}{L_{N-1}}\frac{L_N}{\lambda_N} = e_{N-1}\frac{\lambda_{N-1}}{L_{N-1}} < 1$, regardless of $e_{N-1}$.

not perfectly reveal its quality, and additional, costly effort must be expended to learn it.

More precisely, suppose that, at each date $t$, a car of quality $q_n$ yields per-period utility $(q+\epsilon_n^t)\theta$ to a consumer of type $\theta$, for some collection of mean-zero i.i.d. random variables $\{\epsilon_n^t\}$. Assume that, by driving the car for one period, a consumer observes $q_n + \epsilon_n^t$ but not $q_n$. Finally, assume that, at a small cost $c > 0$, the consumer can learn $q_n$ (perhaps by inspecting the engine, the tires, and so on).

Then it is possible to construct an equilibrium of the modified leasing mechanism in which consumers always expend the extra effort required to learn the quality of the car; the intuition is that buying prices can be fixed so as to reflect the actual quality of a car, whereas rental prices can be adjusted to compensate for the additional cost $c$. If a consumer does not exert the extra effort, and if the perceived quality $q_n + \epsilon_n^t$ of the car she rents is higher than the true quality $q_n$, she may end up buying a car at a price that is too high for its actual quality. Conversely, she may end up not buying a car that is actually of high quality (for its vintage) because its perceived quality is low.

Clearly, a positive cost $c$ of learning qualities generates an inefficiency; however, the argument shows that the equilibrium we construct in Section 4 is the limit of equilibria in environments with positive but vanishing costs of learning qualities, even if the distribution of the noise terms $\epsilon_n^t$ remains fixed as $c \to 0$.

It is possible to construct alternative mechanisms which achieve asymptotic efficiency in the setting under consideration. For instance, the planner might rent or simply lend a car to an arbitrary subset of consumers for one period, then ask them to report the quality of the car they have received. After receiving all reports, the planner can set the price of each car *as if* its reported quality was the actual one. In particular, the planner will sell a car of reported quality $q_n$ at the price $(1-\delta)r_n^L$ (see Equation 17).[12] Consumers then self-select based on prices.

If learning the quality of a car is costless, then it is a Nash equilibrium for consumers to report truthfully, because they will (almost) surely not receive the car they have rented in the first period.

This mechanism is simple to describe, but, loosely speaking, it relies on "soft information"—the reports elicited from the consumers. In particular, consumers would have no incentive to provide meaningful reports, if this was costly to them.

For instance, the mechanism just described is not robust to the kind of perturbation we have mentioned above. If learning the true quality is costly, consumers will at best report the perceived quality $q_n + \epsilon_n^0$. The planner may of course obtain more than one report, but this implies that efficient sorting will only obtain in the limit as infinitely many reports are collected.

[12] If the distribution of reports does not match the distribution of qualities, the planner repeats the procedure, until the distributions do match.

Thus, in the presence of costs of learning qualities, the alternative mechanism is arguably no simpler than the modified leasing mechanism we propose. Moreover, it still fails to provide strong incentives to report truthfully.

# 6 Appendix

## 6.1 Proof of key results in Section 3

### 6.1.1 Remark 3.1

Rewrite Equation 1 as follows:

$$\begin{aligned} p_n^t + V^t(n;\theta) &= \left(\frac{\lambda_n}{L_n}\frac{q_n}{1-\delta} + \frac{L_{n+1}}{L_n}\underline{q}_{n+1}\right)\theta + \delta\frac{L_{n+1}}{L_n}\left(p_{n+1}^{t+1} + V^{t+1}(n;\theta)\right) = \\ &= \left(\frac{\lambda_n}{L_n}\frac{q_n}{1-\delta} + \frac{L_{n+1}}{L_n}\underline{q}_{n+1}\right)\theta + \delta\frac{L_{n+1}}{L_n}\left(p_{n+1}^{t+1} - p_n^{t+1}\right) + \\ &\quad + \delta\frac{L_{n+1}}{L_n}\left(p_n^{t+1} + V^{t+1}(n;\theta)\right). \end{aligned}$$

Thus,

$$\begin{aligned} p_n^t + V^t(n;\theta) &= \sum_{s=t}^{\infty}\left(\delta\frac{L_{n+1}}{L_n}\right)^{s-t}\left(\frac{\lambda_n}{L_n}\frac{q_n}{1-\delta} + \frac{L_{n+1}}{L_n}\underline{q}_{n+1}\right)\theta + \\ &\quad + \delta\frac{L_{n+1}}{L_n}\sum_{s=t+1}^{\infty}\left(\delta\frac{L_{n+1}}{L_n}\right)^{s-(t+1)}\left(p_{n+1}^s - p_n^s\right) \end{aligned}$$

and Remark 3.1 follows.

### 6.1.2 Remark 3.2

Item (i) follows immediately from the first part of Equation 2. We prove (ii) by induction. First, for $n = N-1$, Equation 2 may be written as follows, using Remark 3.1:

$$B_{N-1}\theta_{N-1} - p_{N-1}^t - \delta\frac{L_N}{L_{N-1}}C_{N-1}^{t+1} = B_N\theta_{N-1} - p_N^t$$

where $B_N = q_N$. Rearranging terms, we get

$$(B_{N-1} - B_N)\theta_{N-1} = p_{N-1}^t - p_N^t + \delta\frac{L_N}{L_{N-1}}C_{N-1}^{t+1} = C_{N-1}^t$$

and the left-hand side is independent of $t$. Proceeding by induction, Equation 2 implies that

$$B_n\theta_n - p_n^t - \delta\frac{L_{n+1}}{L_n}C_n^{t+1} = B_{n+1}\theta_n - p_{n+1}^t - \delta\frac{L_{n+2}}{L_{n+1}}C_{n+1};$$

rearranging terms,

$$(B_n - B_{n+1})\theta_n + \delta\frac{L_{n+2}}{L_{n+1}}C_{n+1} = p_n^t - p_{n+1}^t + \delta\frac{L_{n+1}}{L_n}C_n^{t+1} = C_n^t$$

and (ii) follows. Now (iii) is immediate from the recursive decomposition of $C_n$.

### 6.1.3 Remark 3.3

Note first that, for all $n = 0, \ldots, N-1$, $\lim_{\delta\to 1}(1-\delta)B_n = q_n$. The first claim is proved by induction, using Remark 3.2, Part (ii): for $n = N-1$, we have $\lim_{\delta\to 1}(1-\delta)C_{N-1} = \lim_{\delta\to 1}(1-\delta)(B_{N-1} - B_N)\theta_{N-1} = (q_{N-1} - q_N)\theta_{N-1} > 0$. For $n < N-1$, $\lim_{\delta\to 1}(1-\delta)C_n = (q_n - q_{n+1})\theta_n + \frac{L_{n+2}}{L_{n+1}}\lim_{\delta\to 1}(1-\delta)C_{n+1} > (q_n - q_{n+1})\theta_n$, and the induction is complete.

## 6.2 Proposition 4.1

**Proof:** First, note that, by Equation 6,

$$D_n^t = \lambda_n \prod_{s=n}^{t-1}(1 - \eta_n^s) \tag{19}$$

where the product of an empty set of factors (i.e. $t < n$) is taken to be equal to 1, as is customary.

We use a double induction argument. First, consider $n = 0$ and $t = 0$: $R_0^0 = \frac{1}{\lambda_0} > 1$, so $e_0^0 = 1$. Assume that the first claim is true for $s = 0, \ldots, t-1$ for $n = 0$. Then, from Equation 19, $D_0^t = \lambda_0(1-\eta_0)^t$. Note also that $\eta_0 = \lambda_0$. Now, from Equation 7, using the induction hypothesis for $n = 0$ and $s = 0, \ldots, t-1$,

$$S_n^t = 1 - \lambda_0 \sum_{s=0}^{t-1}(1-\lambda_0)^s = (1-\lambda_0)^t$$

so that $R_0^t = \frac{1}{\lambda_0} > 1$, and indeed $e_0^t = 1 = e_0$. Thus, the claim holds for $n = 0$ and every $t \geq 0$.

Now assume that the first and second claims are true for all $m = 0, \ldots, n-1$, and, for every such $m$, for every $s = 0, 1, \ldots$. Clearly, $R_n^t = 0$ for $t < n$; thus, consider $t = n$. From Equation 7,

$$S_n^n = e_{n-1}^{n-1}\frac{L_n}{L_{n-1}}D_{n-1}^{n-1} = e_{n-1}\frac{Ln}{L_{n-1}}\lambda_{n-1} = \eta_{n-1}L_n$$

whereas $D_n^n = \lambda_n$. Thus,

$$R_n^n = \frac{L_n}{\lambda_n}\eta_{n-1}. \tag{20}$$

Now, if $R_n^n > e_{n-1}$, or equivalently if $\eta_{n-1} > e_{n-1}\frac{\lambda_n}{L_n}$, then $e_n \equiv e_n^n = e_{n-1}^n = e_{n-1}$, where the last equality follows from the inductive hypothesis; in this case, $\eta_n = \frac{\lambda_n}{L_n}e_n = \frac{\lambda_n}{L_n}e_{n-1} < \eta_{n-1}$. If instead

$R_n^n \leq e_{n-1}$, then $e_n \equiv e_n^n = R_n^n \leq e_{n-1}^{n-1} = e_{n-1}$, and in this case $\eta_n = e_n \frac{\lambda_n}{L_n} = R_n^n \frac{\lambda_n}{L_n} = \eta_{n-1}$, where the last equality follows from Equation 20.

We conclude that $R_n^n > e_{n-1}$ implies $\eta_n < \eta_{n-1}$, and $R_n^n \leq e_n$ implies $\eta_n = \eta_{n-1}$; thus, the second claim is established.

We now claim that, for $t \geq n$,

$$R_n^t = \frac{R_n^n}{\eta_{n-1}} \left[ 1 - (1 - \eta_{n-1}) \left( \frac{1 - \eta_{n-1}}{1 - \eta_n} \right)^{t-n} \right]. \tag{21}$$

Observe that, for any $t \geq n$ for which the claim is true, $R_n^t = R_n^n$ if $R_n^n \leq e_{n-1}$, and $R_n^t \geq R_n^n$ (with equality only for $t = n$) if $R_n^n > e_{n-1}$; in either case, since by the induction hypothesis $e_{n-1}^t = e_{n-1}$, this implies that $e_n^t = e_n$ and hence $\eta_n^t = \eta_n$.

The claim is true for $t = n$. Thus, assume that it is true for $t - 1$. From Equations 7 and 19,

$$R_n^t = \frac{R_n^{t-1}}{1 - \eta_n^{t-1}} - \frac{\frac{L_n}{\lambda_n} \eta_n^{t-1}}{1 - \eta_n^{t-1}} + \frac{L_n \eta_{n-1} (1 - \eta_{n-1})^{t-n}}{\lambda_n \prod_{s=n}^{t-1} (1 - \eta_n^s)}$$

where we have used the induction assumption that $\eta_{n-1}^t = \eta_{n-1}$, and substituted for $e_n^{t-1}$ and $e_{n-1}$ using the definitions of $\eta_n^{t-1}$ and $\eta_{n-1}$. Now, assuming that the claim is true for $s = n, \ldots, t-1$, we can rewrite this as

$$R_n^t = \frac{R_n^{t-1}}{1 - \eta_n} - \frac{\frac{L_n}{\lambda_n} \eta_n}{1 - \eta_n} + \frac{L_n \eta_{n-1} (1 - \eta_{n-1})^{t-n}}{\lambda_n (1 - \eta_n)^{t-n}} = \frac{R_n^{t-1}}{1 - \eta_n} + \frac{R_n^n}{\eta_{n-1}} \left[ - \frac{\eta_n}{1 - \eta_n} + \eta_{n-1} \left( \frac{1 - \eta_{n-1}}{1 - \eta_n} \right)^{t-n} \right]$$

where the second equality uses Equation 20 to substitute for $\frac{L_n}{\lambda_n}$. We can now use the induction hypothesis to substitute for $R_n^{t-1}$:

$$\begin{aligned} R_n^t &= \frac{R_n^n}{\eta_{n-1}} \left[ \frac{1}{1 - \eta_n} - \frac{1 - \eta_{n-1}}{1 - \eta_n} \left( \frac{1 - \eta_{n-1}}{1 - \eta_n} \right)^{t-1-n} - \frac{\eta_n}{1 - \eta_n} + \eta_{n-1} \left( \frac{1 - \eta_{n-1}}{1 - \eta_n} \right)^{t-n} \right] = \\ &= \frac{R_n^n}{\eta_{n-1}} \left[ 1 - (1 - \eta_{n-1}) \left( \frac{1 - \eta_{n-1}}{1 - \eta_n} \right)^{t-n} \right] \end{aligned}$$

as required. ■

### 6.3 "Jump-Starting" the economy

At any time $t < N$, only markets $0 \ldots t$ are open, according to the trickle-down algorithm. If we maintained the same rental prices as above, consumers who are intended to experiment with vintages that are not yet available might have an incentive to rent better (lower) vintages for consumption purposes. To prevent this from happening, prices at time $t = 0, \ldots, N-1$ are determined according to

$$r_t^t = e_t \underline{q}_t \theta_t \quad \text{and } r_n \text{ such that} \quad e_n \underline{q}_n \theta_n - r_n^t = e_{n+1} \underline{q}_{n+1} \theta_n - r_{n+1}^t, \quad n = 0, \ldots, t-1; \tag{22}$$

that is, $n = t$ takes the place of $N$ in Equation 14. In particular, type $\theta_t$ is left with zero surplus, so lower types have no incentive to rent any one of the available vintages.

We define buying prices as a function of rental prices as per Equation 22, under the assumptions that (1) the marginal type $\theta_t$ is indifferent between experimenting at time $t$ and waiting until time $t+1$ to begin experimentation, and (2) each type $\theta_n$, $n < t$, is indifferent between the experimentation policies involving vintage $n$ and vintage $n+1$.

Formally, as in Equation 9, we define the value at time $t$ of following the experimentation policy which entails renting a car of vintage $n$ and keeping it iff it is of quality $q_m$ or better, $V_{n,m}^t(\theta)$, by $V_{n,m}^N(\theta) = V_{n,m}(\theta)$ and, for $t < N$,

$$V_{n,m}^t(\theta) = e_n \underline{q}_n \theta - r_n^t + \delta e_n \frac{L_n - L_{m+1}}{L_m} \sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n - L_{m+1}} \left( \frac{\theta q_\ell}{1-\delta} - p_n^t \right) + \delta \left( 1 - e_n \frac{L_n - L_{m+1}}{L_n} \right) V_{n,m}^{t+1}(\theta) \tag{23}$$

In particular, according to the conventions maintained thus far, $V_{n,-1}^t(\theta) = e_n \underline{q}_n \theta - r_n^t + \delta V_{n,-1}^{t+1}(\theta)$. Observe that Equation 22 implies that $V_{n,-1}^t(\theta_n) = V_{n+1,-1}^t(\theta_n)$.

Since the first term in the right-hand side of Equation 23 equals $(1-\delta)\frac{e_n \underline{q}_n \theta - r_n^t}{1-\delta}$, $V_{n,m}^t(\theta)$ may also be viewed as a weighted average of a pure-consumption payoff, an expected consumption payoff after buying the car, and a continuation payoff. However, note that the pure-consumption payoff is *not* equal to $V_{n,-1}^t(\theta)$ (because rental prices change.)

**Remark 6.1** *For $t = 0, \ldots, N-1$, and for $0 \le n \le t$ and $m \ge n$,*

$$\frac{dV_{n,-1}(\theta)}{d\theta} = \frac{dV_{n,-1}^t(\theta)}{d\theta} = \frac{e_n \underline{q}_n}{1-\delta} < \frac{dV_{n,m}(\theta)}{d\theta} = \frac{dV_{n,m}^t(\theta)}{d\theta} < \frac{q_n}{1-\delta}.$$

**Proof:** Recall that, by Remark 4.1, $\frac{dV_{n,m}(\theta)}{d\theta} = (1-\delta)^{-1}[w_{n,m} e_n \underline{q}_n + (1-w_{n,m})\underline{q}_{n,m}] < (1-\delta)^{-1} q_n$; we write $\underline{q}_{n,m} = \mathbf{E}[q | q_m \le q \le q_n]$. On the other hand, $\frac{dV_{n,m}^{N-1}(\theta)}{d\theta} = (1-\delta)^{-1}[(1-\delta)e_n \underline{q}_n + \delta e_n \frac{L_n - L_{m+1}}{L_n} \underline{q}_{n,m} + \delta(1 - e_n \frac{L_n - L_{m+1}}{L_n}) w_{n,m} e_n \underline{q}_n + \delta(1 - e_n \frac{L_n - L_{m+1}}{L_n})(1 - w_{n,m})\underline{q}_{n,m}]$. From Equation 12, $(1-\delta) + \delta(1 - e_n \frac{L_n - L_{m+1}}{L_n}) w_{n,m} = w_{n,m}$, and $\delta e_n \frac{L_n - L_{m+1}}{L_n} + (1 - w_{n,m}) = 1 - w_{n,m}$, so the result holds for $t = N-1$. The proof is completed by induction. ■

We now define prices via the following conditions: for every $t = 0, \ldots, N-1$, and for every $n = 0, \ldots, t$,

$$V_{t,t}^t(\theta_t) = \delta V_{t,t}^{t+1}(\theta_t); \quad \forall n < t,\ V_{n,n}^t(\theta_n) = V_{n+1,n+1}^t(\theta_n). \tag{24}$$

Note that Equation 24, together with Remark 6.1, implies that $V^t_{t,t}(\theta) > \delta V^{t+1}_{t,t}(\theta)$ for $\theta > \theta_t$, and hence $V^t_{n,n}(\theta_n) > \delta V^{t+1}_{n,n}(\theta_n)$ for $n < t$. That is, every consumer type other than $\theta_t$ will strictly prefer to begin experimenting with her designated vintage immediately rather than in the next period.

It is convenient to also define $p^N_n = p_n$, $n = 0, \ldots, N$.

**Proposition 6.1** *If buying prices at times $t = 0, \ldots, N$ are defined as above, then there exists $\underline{\delta}$ such that, for $\delta > \underline{\delta}$, and for every $t = 0, \ldots, N$:*

*(1) $p^t_0 > \ldots > p^t_t$.*

*(2) $\frac{\theta_t q_t}{1-\delta} - p^t_t = V^{t+1}_{t,t}(\theta_t)$.*

*(3) If $t < N$, then for every $n = 0, \ldots, t$, $p^t_n > p^{t+1}_n$; thus, $V^t_{n,m}(\theta) < V^{t+1}_{n,m}(\theta)$ for all $\theta \in [\theta_t, \overline{\theta}]$ and $t \geq m \geq n$.*

*(4) For every $n = 0, \ldots, t$, and $\theta \in [\theta_n, \theta_{n-1}]$, $V^{t+1}_{n,n}(\theta_n) < \frac{\theta_n q_n}{1-\delta} - p^t_n$; also, $V^t_{n,n}(\theta_n) \geq V^t_{n,-1}(\theta_n)$, with equality only for $n = t = N$.*

*(5) For every $n = 0, \ldots, t$, $\theta \in [\theta_t, \overline{\theta}]$ and $m \in \{n, \ldots, t\}$, $\lim_{\delta \to 1}(1-\delta)p^t_n = r^L_n$ and $\lim_{\delta \to 1}(1-\delta)V^t_{n,m}(\theta) = \theta \sum_{\ell=n}^m \frac{\lambda_\ell}{L_n - L_{m+1}} q_\ell \theta - r^L_n$.*

*(6) For every $n = 0, \ldots, t-1$, $\theta \in [\theta_t, \overline{\theta}]$ and $m \in \{n+1, t\}$, $\frac{q_m \theta}{1-\delta} - p^t_n < V^{t+1}_{m,m}(\theta)$.*

**Proof:** For $t = N$, (1), (4), (5) and (6) restate results in the previous subsection; with the obvious convention that $V^{N+1}_{N,N} = V_{N,N}$, (2) holds because both sides of the equation are zero; finally, (3) does not apply. It will be crucial to note that prices and values become at and after time $t = N$.

Now suppose the result is true for some $t+1$, $t < N$. Note that $V^t_{n,n}(\theta_n) = V^t_{n+1,n+1}(\theta_n)$ is equivalent to

$$\eta_n \left( \frac{\theta_n q_n}{1-\delta} - p^t_n - V^{t+1}_{n,n}(\theta_n) \right) = \eta_{n+1} \left( \frac{\theta_n q_{n+1}}{1-\delta} - p^t_{n+1} - V^{t+1}_{n+1,n+1}(\theta_n) \right) \tag{25}$$

for $n < t$, where we used Equation 22 to eliminate the per-period consumption terms, and Equation 24 for time $t+1$ to reduce the weights multiplying the continuation value terms to $\delta\eta_n$ and $\delta_{n+1}$ respectively. As in the previous subsection, $V^{t+1}_{n,n}(\theta_n) = V^{t+1}_{n+1,n+1}(\theta_n)$ and $\eta_n \geq \eta_{n+1}$ imply that $p^t_n > p^t_{n+1}$, so (1) follows.

To prove (2), note that $V^t_{t,t}(\theta_t) = \delta V^{t+1}_{t,t}(\theta_t)$ reduces to

$$e_t \underline{q}_t \theta_t - r^t_t + \delta\eta_t \left( \frac{\theta_t q_t}{1-\delta} - p^t_t \right) + \delta(1-\eta_t) V^{t+1}_{t,t}(\theta_t) = \delta V^{t+1}_{t,t}(\theta_t).$$

Now $e_t \underline{q}_t - r^t_t = 0$ by construction, and we can subtract $\delta V^{t+1}_{t,t}(\theta_t)$ from both sides. Since $\eta_t > 0$, (2) follows.

We now prove (3) by induction on $n$. For $n = t$, note that, by (2), $\frac{\theta_t q_t}{1-\delta} - p_t^t = V_{t,t}^{t+1}(\theta_t)$; moreover, by the induction hypothesis on $t$, (3) and (4) hold at time $t+1$, so in particular $V_{t,t}^{t+1}(\theta_t) \leq V_{t,t}^{t+2}(\theta_t) < \frac{\theta_t q_t}{1-\delta} - p_t^{t+1}$ (the first inequality is weak only if $t = N-1$). This implies $p_t^t > p_t^{t+1}$.

Now suppose that $p_{n+1}^t > p_{n+1}^{t+1}$ for some $n < t$. Note that, at time $t+1$, an equation corresponding to 25 must hold. Hence, it must be the case that

$$\eta_n[(p_n^t - p_n^{t+1}) + (V_{n,n}^{t+1}(\theta_n) - V_{n,n}^{t+2}(\theta_n))] = \eta_{n+1}[(p_{n+1}^t - p_{n+1}^{t+1}) + (V_{n+1,n+1}^{t+1}(\theta_n) - V_{n+1,n+1}^{t+2}(\theta_n))]$$

so that, since $V_{n,n}^{\tau}(\theta_n) = V_{n+1,n+1}^{\tau}(\theta_n)$ for $\tau = t+1, t=2$ by Equation 24,

$$p_n^t - p_n^{t+1} = \frac{\eta_{n+1}}{\eta_n}(p_{n+1}^t - p_{n+1}^{t+1}) + \left(1 - \frac{\eta_{n+1}}{\eta_n}\right)[V_{n,n}^{t+2}(\theta_n) - V_{n,n}^{t+1}(\theta_n)];$$

also recall that $\eta_n \geq \eta_{n+1}$. Now, if $t = N-1$, the second term in the above weighted average is zero; otherwise, by the induction hypothesis, it is positive, because (3) holds at time $t+1$. Thus, in any case, $p_{n+1}^t > p_{n+1}^{t+1}$ implies $p_n^t > p_n^{t+1}$, so the induction on $n$ is completed. Finally, since, at time $t$, $r_n^t > r_n^{t+1}$ and $p_n^t > p_n^{t+1}$ for all $n = 0, \ldots, t$, Equation 23 and the induction hypothesis imply that $V_{n,m}^t(\theta) < V_{n,m}^{t+1}(\theta)$ for all types $\theta$ and $t \geq m \geq n$. Thus, (3) holds at time $t$.

To prove (4), we again apply induction on $n$. For $n = t$, we have $V_{t,t}^t(\theta_t) = \delta V_{t,t}^{t+1}(\theta_t)$ and, since $e_t \underline{q}_t \theta_t - r_t^t = 0$, $V_{t,-1}^t(\theta_t) = \delta V_{t,-1}^{t+1}(\theta_t)$. Hence, the induction hypothesis on $t$ implies that (4) holds at time $t+1$, which in turn yields $V_{t,t}^t(\theta_t) > V_{t,-1}^t(\theta_t)$.

Now assume that $V_{n+1,n+1}^t(\theta_{n+1}) > V_{n+1,-1}^t(\theta_{n+1})$ for some $n < t$. By Remark 6.1, this implies that $V_{n+1,n+1}^t(\theta_n) > V_{n+1,-1}^t(\theta_n) = V_{n,-1}^t(\theta_n)$, so also $V_{n,n}^t(\theta_n) > V_{n,-1}^t(\theta_n)$.

To prove the remaining inequalities, recall that, as was noted in the text, $V_{n,n}^t(\theta_n) > \delta V_{n,n}^{t+1}(\theta_n)$. This reduces to

$$e_n \underline{q}_n \theta_n - r_n^t + \delta\eta_n \left(\frac{\theta_n q_n}{1-\delta} - p_n^t\right) > \delta\eta_n V_{n,n}^{t+1}(\theta_n).$$

By the induction hypothesis on $t$, $V_{n,n}^{t+1}(\theta_n) > V_{n,-1}^{t+1}(\theta_n) > V_{n,-1}^t(\theta_n)$; thus, there exists $\underline{\delta}_1$ such that, for $\delta > \underline{\delta}_1$, $\delta\eta_n V_{n,n}^t(\theta_n) > e_n \underline{q}_n \theta_n - r_n^t$, and hence $\delta\eta_n V_{n,n}^{t+1}(\theta_n) > e_n \underline{q}_n \theta_n - r_n^t$. If, by contradiction, $V_{n,n}^{t+1}(\theta_n) \geq \frac{\theta_n q_n}{1-\delta} - p_n^t$, then for $\delta > \underline{\delta}_1$ the above inequality is violated. Hence, for $\delta > \underline{\delta}_1$, we must have $V_{n,n}^{t+1}(\theta_n) > \frac{\theta_n q_n}{1-\delta} - p_n^t$.

To prove (5), by (2) and the induction hypothesis on $t$, $\lim_{\delta\to 1}(1-\delta)p_t^t = \theta_t q_t - \lim_{\delta\to 1}(1-\delta)V_{t,t}^{t+1}(\theta_t) = \theta_t q_t - (\theta_t q_t - r_t^L) = r_t^L$, as needed. Now assume that the claim is true at time $t$ for $m = n+1, \ldots, t$; multiplying both sides of Equation 25 by $(1-\delta)$ and taking limits yields

$$\eta_n[\theta_n q_n - \lim_{\delta\to 1}(1-\delta)p_n^t - (\theta_n q_n - r_n^t)] = \eta_{n+1}[\theta_n q_{n+1} - r_{n+1}^L - (\theta_n q_{n+1} - r_n^L)] = 0$$

which, since $\eta_n > 0$, implies the required result. Now, from Equation 23, by the induction hypothesis on $t$,

$$\lim_{\delta\to 1} V^t_{n,m}(\theta) = e_n \frac{L_n - L_{m+1}}{L_m} \sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n - L_{m+1}} (\theta q_\ell - r^L_n) + \left(1 - e_n \frac{L_n - L_{m+1}}{L_n}\right) \left(\sum_{\ell=n}^{m} \frac{\lambda_\ell}{L_n - L_{m+1}} \theta q_\ell - r^L_n\right)$$

so the result follows.

Finally, to see that (6) holds, note that, by (5), $q_m\theta - \lim_{\delta\to 1}(1-\delta)p^t_n = q_m\theta - r^L_n < q_m\theta - r^L_m = \lim_{\delta\to 1}(1-\delta)V^{t+1}_{m,m}(\theta)$. Hence, it is possible to choose $\underline{\delta}_t < 1$ such that, for $\delta > \underline{\delta}_t$ implies that, for all $n$ and $m$ as in the claim, $\frac{q_m\overline{\theta}}{1-\delta} - p^t_n < V^{t+1}_{m,m}(\overline{\theta})$; the claim now follows from Remark 6.1.

To complete the proof, let $\underline{\delta} = \min\{\underline{\delta}_0, \ldots, \underline{\delta}_N\}$. ■

We now show, by backward induction on $t$, that the experimentation policy involving vintage $n$ is optimal for each type $\theta \in [\theta_n, \theta_{n-1}]$. The previous subsection shows that this is the case for $t = N$. Now suppose that it is also the case at time $t + 1 \leq N$, and consider the problem faced by a type $\theta \in [\theta_n, \theta_{n-1}]$ at time $t$.

By the induction hypothesis, if the agent does not leave the market at time $t$ (i.e. if she does not buy a car), her continuation strategy at time $t+1$ may be assumed to be the experimentation policy involving vintage $n$. By Proposition 6.1, (6), for $\delta > \underline{\delta}$ a consumer who has rented a vintage-$m$ car at time $t$ will only keep it if it is of quality $q_m$. Moreover, Equation 22 ensures that a consumer of type $\theta \in [\theta_n, \theta_{n-1}]$ who is only interested in time-$t$ consumption will rent a car of vintage $n$. Thus, the optimal continuation policy at time $t$ is necessarily one of the following: (P1) rent a car of vintage $n$ at $t$, do not buy it regardless of its quality, and experiment with vintage $n$ beginning with time $t+1$; (P2) rent a car of vintage $m \neq n$ at $t$, keep it iff it is of quality $q_m$, and experiment with vintage $n$ beginning with time $t + 1$; (P3) experiment with vintage $n$ beginning with time $t$.

Now Proposition 6.1, (4) shows that, for $\delta > \underline{\delta}$, if a consumer of type $\theta \in [\theta_n, \theta_{n-1}]$ rents a car of vintage $n$ which turns out to be of quality $q_n$, then she strictly prefers to buy it rather than experiment with vintage $n$ beginning with time $t + 1$; hence, (P1) above cannot be the optimal policy.

Next, suppose that a consumer of type $\theta \in [\theta_n, \theta_{n-1}]$ rents a car of vintage $m$ which turns out to be of quality $q_m$, with $m \neq n$. By Proposition 6.1, (5), $\theta q_m - \lim_{\delta\to 1} p^t_m = \theta q_m - r^L_m \leq \theta q_n - r^L_n = \lim_{\delta\to 1}(1 - \delta)V^{t+1}_{n,n}(\theta)$, with equality only for type $\theta_n$ and for $m = n - 1$. However, note that, since $V^{t+1}_{n+1,n+1}(\theta_n) = V^{t+1}_{n,n}(\theta_n)$, the value to type $\theta_n$ of experimenting with vintage $n+1$ at time $t$, keeping the car iff it is of quality $q_{n+1}$, and then experimenting with vintage $n$ forever after, is

$$e_{n+1}\underline{q}_{n+1}\theta_n + \delta\eta_{n+1}\left(\frac{q_{n+1}\theta_n}{1-\delta} - p^t_{n+1}\right) + \delta(1 - \eta_{n+1})V^{t+1}_{n,n}(\theta_n) =$$

$$= e_{n+1}\underline{q}_{n+1}\theta_n + \delta\eta_{n+1}\left(\frac{q_{n+1}\theta_n}{1-\delta} - p^t_{n+1}\right) + \delta(1-\eta_{n+1})V^{t+1}_{n+1,n+1}(\theta_n) =$$
$$= V^t_{n+1,n+1}(\theta_n) = V^t_{n,n}(\theta_n)$$

for type $\theta_n$, and strictly less than $V^t_{n,n}(\theta_n)$ for any type $\theta \in (\theta_n, \theta_{n-1}]$.[13] Hence, we can disregard this particular policy at time $t$.

For any other vintage $m \notin \{n, n+1\}$, we can find $\delta^*_{t,n,m}$ such that, for $\delta > \delta^*_{t,n,m}$:

(1) if $q_m > w_{n,n}e_n\underline{q}_n + (1-w_{n,n})q_n$, $\frac{q_m\theta_{n-1}}{1-\delta} - p^t_m < V^{t+1}_{n,n}(\theta_{n-1})$; by Remark 6.1, the same inequality will hold for all lower types $\theta \in [\theta_n, \theta_{n-1})$;

(2) if $q_m \leq w_{n,n}e_n\underline{q}_n + (1-w_{n,n})q_n$, $\frac{q_m\theta_n}{1-\delta} - p^t_m < V^{t+1}_{n,n}(\theta_n)$; again by Remark 6.1, the same inequality will hold for all higher types $\theta \in (\theta_n, \theta_{n-1}]$.

Thus, for $\delta > \delta^*_t = \max\{\delta^*_{t,n,m} : n = 0, \ldots, N,\ m \notin \{n, n+1\}\}$, no policy of type (P2) can be optimal at time $t$. Hence, the optimal policy at time $t$ involves experimenting with vintage $n$, and the induction is complete.

Taking $\delta^* = \max\{\underline{\delta}, \delta^*_0, \ldots, \delta^*_{N-1}\}$, we get Proposition 4.4.

# 7 References


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[13] To see this, note that $(1-\delta)$ times the derivative of the first expression with respect to $\theta$ is $(1-\delta)\underline{q}_{n+1} + \delta\eta_{n+1}q_{n+1} + \delta(1-\eta_{n+1})(1-\delta)\frac{dV^{t+1}_{n,n}(\theta_n)}{d\theta} = [1-\delta(1-\eta_{n+1})][w_{n+1,n+1}\underline{q}_{n+1} + (1-w_{n+1,n+1})q_{n+1}] + \delta(1-\eta_{n+1})[w_{n,n}\underline{q}_n + (1-w_{n,n})q_n]$. Since $w_{n,n} \leq w_{n+1,n+1}$, the first element of the weighted average is smaller than the second, which is $(1-\delta)\frac{dV^t_{n,n}(\theta_n)}{d\theta}$ (see the proof of Remark 6.1).