

Production equilibria in vector lattices*

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Abstract

The general purpose of this paper is to prove quasiequilibrium existence theorems for production economies with general consumption sets in an infinite dimensional commodity space, without assuming any monotonicity of preferences or free-disposal in production.

The commodity space is a vector lattice commodity space whose topological dual is a sublattice of its order dual. We formulate two kinds of properness concepts for agents' preferences and production sets, which reduce to more classical ones when the commodity space is locally convex and the consumption sets coincide with the positive cone. Assuming properness allows for extension theorems of quasiequilibrium prices obtained for the economy restricted to some order ideal of the commodity space. As an application, the existence of quasiequilibrium in the whole economy is proved without any assumption of monotonicity of preferences or free-disposal in production.

Keywords and Phrases: linear vector lattices, competitive production equilibrium, quasiequilibrium, E -properness, F -properness.

JEL Classification Numbers: C 62, D 51

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1 Introduction

The purpose of this paper is to prove quasiequilibrium existence theorems for production economies with general consumption sets in an infinite dimensional commodity space, without assuming any monotonicity of preferences or free-disposal in production.

As indicated by the title of this paper, we consider economies defined on (infinite dimensional) vector lattice commodity spaces, a notion introduced by Aliprantis and Brown [2]. Here, as in Mas-Colell and Richard [21], Richard [25] and many others papers ([10], [23], [1], [26], [27], [13], [18]), we assume that the commodity space is a vector lattice whose topological dual is a sublattice of its order dual. As well-known, this setting, which covers most of the important infinite dimensional models, was introduced in order to include the models of commodity differentiation in Jones [17] and of intertemporal consumption in Huang and Kreps [16], not covered before by a number of equilibrium existence results (e.g., [19], [29], [30], [24], [4], [5], [15], [28]) requiring that the commodity space be a topological vector lattice. Even if it leaves out of its scope some commodity-price dualities of economic interest (a detailed discussion on relevant commodity-price dualities can be found in [3], [22]), such a setting is also the most general one used by now in equilibrium existence proofs, if one excepts a thought provoking paper by Aliprantis et al. [8] discarding the vector lattice property of the commodity space and its dual at the cost of an alternate theory of value with non-linear prices.

Our method of proof is to get quasiequilibria by decentralizing Edgeworth equilibria whose existence is guaranteed under relatively weak assumptions. When the preferred sets have an empty interior, the decentralization arguments use properness assumptions first introduced by Mas-Colell [19], then substantially weakened. Specifically, uniform properness of preferences was replaced by an assumption of pointwise properness at some particular allocations by Araujo and Monteiro [9], Duffie and Zame [14], in the particular case where the total endowment is a strictly positive element of the commodity space of an exchange economy with the positive cone as consumption sets of the agents. This result was extended by Podczeck [23] to the nonordered case and proved without any monotonicity assumption. For the more general case where the total endowment is not a quasi-interior element of the commodity space (specially, if this one has no quasi-interior element), Podczeck introduced a properness concept, called E -properness, stronger than pointwise properness but weaker than uniform properness.

It is this concept of E -properness that we mainly address in this paper. The economy under consideration is a production economy, a case not studied by Podczeck. We also consider more general consumption sets than the positive cone of the commodity space. We were stimulated to do it by two papers of

Tourky [26], [27] which perform, in a more general framework, an objective previously claimed by Back [10] and Boyd and McKenzie [12]. As a counterpart, the formulation of properness becomes then somewhat abstract. It uses, as well for preferences as for production sets, convex lattices which play the same role as the pretechnology sets introduced by Mas-Colell [20]. For preferences defined on the positive cone of a locally convex Riesz space, the E -properness defined in this paper exactly corresponds to Podczeck's E -properness, while for a production set in a locally convex Riesz commodity space, E -properness is quite similar to uniform properness as defined in Mas-Colell [20]. We introduce also a weaker properness that we call F -properness. As it will be seen, this concept, so useful for proving the existence of equilibrium in an exchange economy whose consumers have the positive cone as consumption set, is of a difficult use in a production economy or in an exchange economy with more general consumption sets.

As we look for an equilibrium existence theorem without assuming any monotonicity in preferences¹ or free-disposal in production, unlike Tourky's papers, we cannot make an argument directly in the whole commodity space. We first decentralize Edgeworth equilibria of an economy restricted to some well-chosen order ideal of the commodity space, an idea originated from Aliprantis et al. [4]. The extension of equilibrium prices to the whole commodity space is done using a technique borrowed from Podczeck [23] and adapted here to the case of a production economy.

The paper is organized as follows. In the next section, we define the model, set the main assumptions, discuss the properness definitions, state and prove theorems extending equilibrium prices of a restricted economy to continuous equilibrium prices for the initial economy. These theorems have their own interest. As a by-product, they show in particular that under E -properness (relative to the whole commodity space), a feasible allocation sustainable as a quasiequilibrium with discontinuous prices is also sustainable as a quasiequilibrium with continuous prices. Such a property, proved first by Yannelis and Zame [29] in the particular case of an exchange economy defined on a topological vector lattice having the total initial endowment as a strictly positive element, was re-proved later in several contexts ([3], [23],[26], [27]). It is obtained here in our general framework. The extension theorems are applied in Section 3 to establishing quasiequilibrium existence theorems. We then compare these results with similar ones obtained by Tourky [26], [27] in the same framework but under assumptions of strict monotonicity of preferences and free-disposal in production. The main proofs are given in the last section.

¹other than the extreme desirability assumption involved in the formulation of E - or F -properness

2 The model and extension results

2.1 The model

We consider a typical production economy in which the commodity space L is a partially ordered vector space equipped with a Hausdorff, linear topology τ . Let $I = \{1, \dots, I\}$ and $J = \{1, \dots, J\}$ be respectively the set of consumers and the set of firms. A consumer $i \in I$ is characterized by a consumption set $X_i \subset L$, an initial endowment $\omega_i \in L$ and a preference relation described by the point-to-set mapping $P_i : X_i \rightarrow X_i$, such that $P_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i -th agent to the bundle x_i . We will also use the notation $y_i \succ_i x_i$ which is equivalent to $y_i \in P_i(x_i)$. A producer (a firm) j is characterized by a production set $Y_j \subset L$. For every $j \in J$, each consumer i is also endowed with a share $\theta_i^j \geq 0$ of the profit of firm j , with $\sum_{i \in I} \theta_i^j = 1$. Let us set $\theta_i = (\theta_i^1, \dots, \theta_i^J)$. The model under study is a 5-tuple

$$\mathcal{E} = (I, J, (L, \tau), (X_i, P_i, \omega_i, \theta_i)_{i \in I}, (Y_j)_{j \in J}).$$

Let us denote by $\omega = \sum_{i \in I} \omega_i$ the total resources of the economy and let

$$\mathcal{A}(\mathcal{E}) = \{(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \mid \sum_{i \in I} x_i = \omega + \sum_{j \in J} y_j\}$$

be the set of all *feasible allocations*. In the following, $\mathcal{A}_X(\mathcal{E})$ will denote the projection of $\mathcal{A}(\mathcal{E})$ on $\prod_{i \in I} X_i$. We first recall some definitions.

A triple (x, y, p) is said to be a *quasiequilibrium* of \mathcal{E} iff $(x, y) \in \mathcal{A}(\mathcal{E})$, p is a linear functional, with $p \neq 0$, and

- (i) for every $i \in I$, $p \cdot x_i = p \cdot (\omega_i + \sum_{j \in J} \theta_i^j y_j)$ and $p \cdot x'_i \geq p \cdot x_i \forall x'_i \in P_i(x_i)$;
- (ii) for every $j \in J$, $p \cdot y'_j \leq p \cdot y_j \forall y'_j \in Y_j$.

This quasiequilibrium is said to be *nontrivial* if for some i_0 , $\inf p \cdot X_{i_0} < p \cdot x_{i_0}$

A quasiequilibrium such that $x'_i \in P_i(x_i)$ actually implies $p \cdot x'_i > p \cdot x_i$ is a *Walrasian equilibrium*.

On the other hand, $x \in \mathcal{A}_X(\mathcal{E})$ is said to be blocked by a nonempty coalition $B \subset I$ if there exists $x'_B \in \prod_{i \in B} X_i$ such that $\sum_{i \in B} (x'_{iB} - \omega_i) \in \sum_{i \in B} \sum_{j \in J} \theta_i^j Y_j$ and $x'_{iB} \in P_i(x_i) \forall i \in B$. The *core* of \mathcal{E} is the set of all $x \in \mathcal{A}_X(\mathcal{E})$ which are blocked by no (nonempty) coalition. Following Aliprantis et al. ([3], [4], [5]), $x \in \mathcal{A}_X(\mathcal{E})$ is said to be an *Edgeworth equilibrium* if, for every integer $r \geq 1$, the r -repetition of x belongs to the core of the r -replication of \mathcal{E} . Let $\mathcal{C}^e(\mathcal{E})$ denote the set of all Edgeworth equilibria of \mathcal{E} . As it is easily seen and proved in Florenzano [15], under convexity assumptions for consumption and production sets, the set of all Edgeworth equilibria $\mathcal{C}^e(\mathcal{E})$ contains the set $\mathcal{C}^f(\mathcal{E})$ of all $x \in \mathcal{A}_X(\mathcal{E})$ such that there exists no $t = (t_i) \in [0, 1]^I$, $t \neq 0$, and no $x'_t \in \prod_{t_i > 0} X_i$ satisfying

$$\sum_{i \in I} t_i (x'_{it} - \omega_i) \in \sum_{i \in I} t_i \sum_{j \in J} \theta_i^j Y_j$$

$$x'_{it} \in \text{co } P_i(x_i) \quad \forall i : t_i > 0.$$

The assumptions the economy will be required to satisfy are divided into several groups.

Structural assumptions (SA)

(i) L is a linear vector lattice (or Riesz space) endowed with a Hausdorff linear topology τ ;

(ii) L_+ is a closed cone in the τ -topology of L ;

(iii) L' is a sublattice of the order dual of L .

It is worth noticing that that we do not assume the topology τ to be locally solid. Note that if L were a locally solid Riesz space then requirements (ii), (iii) would be automatically valid. Since we avoid the solidness hypothesis, we need to require them explicitly. For more specific explanations and references, the reader is referred to [3], [6], [21].

The three following groups of assumptions are classical for existence of equilibrium and do not require special explanations. It should only be noticed that we do not make in (C) any local nonsatiation assumption. Local nonsatiation at every component of an attainable allocation will be a consequence of properness assumptions to be made later.

Consumption Assumptions (C)

For all $i \in I$,

(i) $X_i \subset L$ is convex, τ -closed, and $\omega_i \in X_i$;

(ii) $\forall x_i \in X_i$, the set $P_i^{-1}(x_i) = \{y_i \in X_i \mid x_i \in P_i(y_i)\}$ is $\sigma(L, L')^2$ -open in X_i ;

and for each $x = (x_i) \in \mathcal{A}_X(\mathcal{E})$,

(iii) (weak convexity and irreflexivity) $x_i \notin \text{co } P_i(x_i)$;

(iv) $P_i(x_i)$ is τ -open in X_i .

Production Assumption (P)

For all $j \in J$,

²In what follows, $\sigma(L, L')$ denotes the weak topology on L ; $\text{co } A$ denotes the convex hull of the set A and \overline{A} is its closure.

$Y_j \subset L$ is convex, τ -closed and $0 \in Y_j$.

Boundedness Assumption (B)

$\mathcal{A}(\mathcal{E})$ is $(\sigma(L, L'))^{I \times J}$ -compact.

As proved in Florenzano [15], under (C (i) – (iii)), (P), (B) for an economy defined on a Hausdorff linear topological vector space, $\mathcal{C}^e(\mathcal{E})$ is nonempty. If one assumes in addition (C (iv)), then $\mathcal{C}^f(\mathcal{E})$ is also nonempty. The same is true for the economy $\mathcal{E}|_K$ where K is any closed vector subspace of L containing all ω_i and $\mathcal{E}|_K$ is the production economy truncated to K whose consumption and production sets are respectively $X_i \cap K$ and $Y_j \cap K$.

2.2 Properness assumptions and extension of linear functionals

For decentralizing Edgeworth equilibria when the commodity space L is infinite dimensional, we now assume that (L, τ) satisfies (SA) and introduce properness assumptions.

Definition 2.1 Let K be some order ideal of L . A preference relation $P : X \rightarrow X$ is said to be F -proper relative to K at $x \in X$ if there exists a τ -open convex subset V_x of L , a convex lattice $Z_x \subset K$ verifying $Z_x + K_+ \subset Z_x$ and some subset A_x of L , radial³ at x , such that $x \in \bar{V}_x \cap Z_x$ and

$$\emptyset \neq V_x \cap Z_x \cap A_x \subset P(x) \quad (1)$$

If, moreover,

$$P(x) \cap A_x \subset \bar{V}_x \cap (Z_x + L_+) \quad (2)$$

then the preference relation is said to be E -proper at $x \in X$ relative to K .

To understand the previous definition, let us assume that (L, τ) is locally convex and that v_x , such that $x + \alpha v_x \in K_+$ for some $\alpha > 0$, and U_x are respectively a *properness vector* and a τ -open convex *properness 0-neighborhood*. Let Γ_x be the open convex cone with vertex 0 generated by $(\{v_x\} + U_x)$. Then $P : L_+ \rightarrow L_+$ is F -proper relative to K at $x \in K_+$ if there exists some $A_x \subset L$, radial at x , such that

$$(\{x\} + \Gamma_x) \cap K_+ \cap A_x \subset P(x)$$

³If A is a subspace of a vector space L , then A is called radial at a point $x \in A$ if for each $y \in L$ there exists a real number $\bar{\lambda}$, $0 < \bar{\lambda} \leq 1$ such that $(1 - \lambda)x + \lambda y \in A$ for every λ with $0 \leq \lambda \leq \bar{\lambda}$.

a definition originated from Yannelis and Zame [29] in the case $K = L$, while P is E -proper relative to K at $x \in K_+$ if there exists some $A_x \in L$, radial at x , such that

$$\begin{aligned} x + \beta v_x &\in P(x) \text{ for some } \beta > 0 \\ (P(x) + \Gamma_x) \cap K_+ \cap A_x &\subset P(x). \end{aligned}$$

In the first case, $V_x = \{x\} + \Gamma_x$ and $Z_x = K_+$ fit with F -properness as defined in Definition 2.1, while in the second case, $Z_x = K_+$ and $V_x = P(x) + \Gamma_x$ satisfy conditions 1 and 2 of the same definition, which corresponds to the definition of E -properness relative to K given by Podczeck in [23].

From the two previous examples, it should be clear that properness assumptions on a preference relation $P : X \rightarrow X$, as stated in Definition 2.1, are as well assumptions on P as assumptions on X . As it is usually observed, P is E -proper at x if $P(x) \cap A_x \cap Z_x$ can be extended to a convex set $\tilde{P}(x)$ with a τ -interior point in $Z_x \cap A_x$ and such that $x \in \overline{\tilde{P}(x)}$. It is also worth noticing that condition 1 in Definition 2.1 implies that x is a point of local nonsatiation for P in K .

Definition 2.2 *Let K be some order ideal of L . A set $Y \subset L$ is said to be F -proper relative to K at $y \in Y$ if there exists a τ -open convex subset V_y of L , a convex lattice $Z_y \subset K$ verifying $Z_y - K_+ \subset Z_y$ and some subset A_y of L , radial at y , such that $y \in \overline{V_y} \cap Z_y$ and*

$$\emptyset \neq V_y \cap Z_y \cap A_y \subset Y \tag{3}$$

If, moreover,

$$Y \cap A_y \subset \overline{V_y} \cap (Z_y - L_+) \tag{4}$$

then the set is said to be E -proper at $y \in Y$ relative to K .

In order to understand the difference between F - and E -properness at $y \in Y \cap K$, let us assume once more that (L, τ) is locally convex and that $v_y \in K_+$ and U_y are respectively a *properness vector* and a τ -open convex *properness 0-neighborhood*. Let Γ_y be the open convex cone with vertex 0 generated by $(\{v_y\} + U_y)$. Then Y is F -proper relative to K at $y \in Y \cap K$ if there exists some $A_y \subset L$, radial at y , such that

$$(\{y\} - \Gamma_y) \cap \{z \in K \mid z^+ \leq y^+\} \cap A_y \subset Y$$

a condition similar if $K = L$ to the one imposed by Richard [24], while Y is E -proper relative to L at $y \in Y$ if, as in Mas-Colell [20], there exists some pretechnology convex lattice Z_y such that $Y \subset Z_y$, $Z_y - L_+ \subset Z_y$ and

$$(Y - \Gamma) \cap Z_y \subset Y.$$

Here $v \in L_+$ is a vector of uniform properness of Y , U is a τ -open convex 0-neighborhood and Γ is the open convex cone with vertex 0 generated by $(\{v\} + U)$.

Note that a production set $Y = \{0\}$ is neither F -proper nor E -proper. In counterpart, $Y = -L_+$ is F -proper and E -proper, relative to any order ideal K of L , at any point $y \in Y \cap K$.

Note also that, as well for preferences as for production, E -properness relative to L is quite similar to M -properness as defined in Tourky [26], [27].

Definition 2.3 *Let K be an order ideal of L containing all ω_i . An economy \mathcal{E} is said to be F -proper (resp. E -proper) relative to K at $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ if each preference relation and each production set is F -proper (resp. E -proper) relative to K at the corresponding component of (x, y) , with for each i , $\omega_i \in Z_{x_i}^i$ and for each j , $0 \in Z_{y_j}^j$ (where the sets $Z_{x_i}^i$ and $Z_{y_j}^j$ are taken from the definition of properness).*

If \mathcal{E} is a pure exchange economy, \mathcal{E} is F -proper (resp. E -proper) relative to K at $x \in \prod_{i \in I} X_i$ if each preference relation is F -proper (resp. E -proper) relative to K at the corresponding component of x , with for each i , $\omega_i \in Z_{x_i}^i$ (with $Z_{x_i}^i$ taken from the definition of properness).

Below considerations will be based on the following auxiliary result of convex analysis, the first part of which can be found in Podczeck [23].

Lemma 2.1 (MAIN AUXILIARY LEMMA) *Let (L, τ) be a Hausdorff topological vector space and let K be a vector subspace of L . Let Z be a convex subset of K and V be a convex τ -open subset of L such that $V \cap Z \neq \emptyset$. If p is a linear functional on K satisfying for some $z \in \overline{V} \cap Z$*

$$p \cdot z \leq p \cdot y \quad \forall y \in V \cap Z$$

then there exists a τ -continuous linear functional $\pi \in (L, \tau)'$ and a linear functional h on L such that $p = \pi|_K + h|_K$ and

$$\pi \cdot z \leq \pi \cdot y \quad \forall y \in V, \quad h \cdot z \leq h \cdot y \quad \forall y \in Z. \quad (5)$$

Let us assume in addition that (L, τ) is an ordered vector space and set $K_+ = L_+ \cap K$. If $Z + K_+ \subset Z$, then $h|_K \geq 0$, $\pi|_K \leq p$ and

$$p \cdot (z - y) = \pi \cdot (z - y) \quad \text{for each } y \leq z, \quad y \in Z. \quad (6)$$

Replacing the condition $Z + K_+ \subset Z$ by $Z - K_+ \subset Z$ and applying the previous lemma to the sets $-V$, $-Z$ and to the point $-z$, we get immediately the following corollary:

Corollary 2.1 *Let (L, τ) be a Hausdorff locally convex topological vector space and let K be a vector subspace of L . Let Z be a convex subset K and V be a*

convex τ -open subset of L such that $V \cap Z \neq \emptyset$. If p is a linear functional on K satisfying for some $z \in \overline{V} \cap Z$

$$p \cdot z \geq p \cdot y \quad \forall y \in V \cap Z$$

then there exists a τ -continuous linear functional $\pi \in (L, \tau)'$ and a linear functional h on L , such that $p = \pi|_K + h|_K$ and

$$\pi \cdot z \geq \pi \cdot y \quad \forall y \in V, \quad h \cdot z \geq h \cdot y \quad \forall y \in Z. \quad (7)$$

Let us assume in addition that (L, τ) is an ordered vector space and set $K_+ = L_+ \cap K$. If $Z - K_+ \subset Z$, then $h|_K \geq 0$, $\pi|_K \leq p$ and

$$p \cdot (z - y) = \pi \cdot (z - y) \quad \text{for each } y \geq z, \quad y \in Z. \quad (8)$$

Exploiting the previous lemma and its corollary, the following proposition describes the important properties of F -proper economies. Its proof makes extensive use of the Riesz decomposition property of L and the Riesz-Kantorovich formula applied to continuous linear functionals on L , both guaranteed by the structural assumptions (SA) on the commodity-price duality⁴.

Proposition 2.1 *Let K be some order ideal of L containing all ω_i and let (x, y, p) be a quasiequilibrium of $\mathcal{E}|_K$ with a price $p \in K^*$, the algebraic dual of K . If \mathcal{E} is F -proper relative to K at (x, y) and if the sets $V_{x_i}^i, Z_{x_i}^i, V_{y_j}^j, Z_{y_j}^j$ are taken from the definition of properness, then there exist, for every $t \in I \cup J$, τ -continuous functionals $\pi_t \in (L, \tau)'$ such that $\pi_t|_K \leq p$, and*

$$\pi_i \cdot \overline{V_{x_i}^i} \geq \pi_i \cdot x_i, \quad \pi_j \cdot \overline{V_{y_j}^j} \leq \pi_j \cdot y_j, \quad i \in I, \quad j \in J. \quad (9)$$

Moreover, if $\pi = (\bigvee_{i \in I} \pi_i) \vee (\bigvee_{j \in J} \pi_j)$ then $\pi \in (L, \tau)'$ and

$$\forall i \in I, \quad \forall z_i \leq x_i, \quad z_i \in Z_{x_i}^i, \quad \pi_i \cdot (x_i - z_i) = \pi \cdot (x_i - z_i) = p \cdot (x_i - z_i) \quad (10)$$

$$\forall j \in J, \quad \forall z_j \geq y_j, \quad z_j \in Z_{y_j}^j, \quad \pi_j \cdot (z_j - y_j) = \pi \cdot (z_j - y_j) = p \cdot (z_j - y_j) \quad (11)$$

$$\pi \cdot (\omega - u) = p \cdot (\omega - u) \quad \text{for each } u \leq \omega, \quad u \in \sum_{i \in I} Z_{x_i}^i - \sum_{j \in J} Z_{y_j}^j \quad (12)$$

and, finally, for every $i \in I$

$$\pi \cdot x_i = \pi \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi \cdot y_j. \quad (13)$$

The following theorem is the first main result of this paper. It states that a feasible allocation being price supported in K can, because of E -properness relative to K , also be price supported in L . More precisely,

⁴For more on these two properties, see [7]

Theorem 2.1 *If, in the conditions of Proposition 2.1, \mathcal{E} is E -proper relative to K , then (x, y, π) is a quasiequilibrium of \mathcal{E} . Moreover if there is $i_0 \in I$ such that $p \cdot x'_{i_0} < p \cdot x_{i_0}$ for some $x'_{i_0} \in X_{i_0} \cap Z_{x_{i_0}}^{i_0}$, then (x, y, π) is a nontrivial quasiequilibrium.*

As a by-product, replacing K by L in the previous result, it should be noticed that if \mathcal{E} is E -proper relative to L , then any feasible allocation sustainable as a quasiequilibrium with a price vector not in $(L, \tau)'$ is also sustainable as a quasiequilibrium with continuous price. We extend here similar results of Podczeck [23], Tourky [26], [27] to the case of a production economy with general consumption sets without assuming any monotonicity of preferences or free-disposal in production.

Let us now come back to the case of F -proper economies. The following proposition gives sufficient conditions to get an analogue result under F -properness.

Proposition 2.2 *Assume, in the conditions of Proposition 2.1, that $\sum_i Z_{x_i}^i - \sum_j Z_{y_j}^j$ is a radial at ω subset of K and that each $P_i(x_i) \cap K$ and each $Y_j \cap K$ are τ -dense in (respectively) $P_i(x_i)$ and Y_j . Then $\pi|_K = p$ and (x, y, π) is a quasiequilibrium of \mathcal{E} .*

In view of applying Proposition 2.2, let us assume that the order ideal K is τ -dense in L . One can wonder how to guarantee that each $P_i(x_i) \cap K$ and each $Y_j \cap K$ are τ -dense in (respectively) $P_i(x_i)$ and Y_j . Let us first remark that under $(C(iv))$, for the τ -density of preferred sets it is enough to require the τ -density of $X_i \cap K$ in X_i . One important example of set having this property is the positive cone L_+ of L , as proved by Podczeck [23] in his Lemma 3, using the same structural assumptions as our $(SA(i) - (iii))^5$. It should be clear from this that any set of the form $\{k\} + L_+$ or $\{k\} - L_+$ for some $k \in K$ has the same property. Since K is assumed to be an order ideal of L , the same is true for any order interval with lower and upper bounds in K (note that, in view of $(SA)(ii)$, each such interval is also τ -closed). The following lemma and its obvious corollary give sufficient conditions for density which may have an economic interpretation.

Lemma 2.2 *Let some $Z \subset L$ be τ -closed, K be τ -dense in L and every $z \in Z$ satisfy*

– either there exists $a_z \in K$, $a_z \geq z$ and some τ -open, convex $U_z \subset L$ such that

$$\emptyset \neq U_z \cap (\{a_z\} - L_+) \subset Z, \quad \& \quad z \in \overline{U_z} \cap (\{a_z\} - L_+), \quad (14)$$

⁵The proof is given in Podczeck [23] in the case where K is the order ideal generated by a positive element of L , but it is easily checked that the same proof can be given for any other order ideal τ -dense in L .

– or there exists $a_z \in K$, $a_z \leq z$ and some τ -open, convex $U_z \subset L$ such that

$$\emptyset \neq U_z \cap (\{a_z\} + L_+) \subset Z, \quad \& \quad z \in \overline{U}_z \cap (\{a_z\} + L_+). \quad (15)$$

Then $\overline{K \cap Z} = Z$.

Corollary 2.2 *Let some closed $Z \subset L$ admit a representation of the form*

$$Z = \overline{U} \cap (B - L_+)$$

or of the form

$$Z = \overline{U} \cap (B + L_+),$$

for some τ -open convex set U and $B \subset K$ satisfying for every $b \in B$, $(\{b\} - L_+) \cap U \neq \emptyset$ in the first case and $(\{b\} + L_+) \cap U \neq \emptyset$ in the second case. Then $\overline{K} = L$ implies $\overline{K \cap Z} = Z$.

The conditions (14), (15) imposed in the lemma can be interpreted as a kind of “upper-properness” at the point z relative to K (“lower-properness” respectively), required for consumption sets and production sets in addition to the properness of preferences and production. In the properties postulated in Lemma 2.2, one restrictive point is the fact that every $z \in Z$ is supposed to have an upper (lower) bound in K . But being applied to consumption or production sets, this requirement may be economically interpreted. In fact, taking into account that $\omega_i \in K$ for each i , one may postulate that firms’ inputs (outputs) have to be chosen from $L(\overline{\omega}) \subset K$, the order ideal generated in K by $\overline{\omega} = \sum_i |\omega_i|$. Analogous hypothesis may be taken for admissible consumption plans of consumers⁶.

As shown in Corollary 2.2, consumption sets of the form $X_i = Z_i \cap (\{a_i\} + L_+)$, with $a_i \in K$, Z_i convex, τ -closed, with an interior point in $(\{a_i\} + L(\overline{\omega})_+)$, satisfy 15. This kind of assumption is evoked by Podczeck at the end of [23] as allowing, for an exchange economy, an extension of his equilibrium existence result under F -properness. A similar hypothesis for production sets seems more questionable.

3 Application to the quasiequilibrium existence problem

We now restrict ourselves to the case $K = L(\overline{\omega}) = \{x \in L \mid \exists \lambda > 0, |x| \leq \lambda \overline{\omega}\}$, the order ideal generated in L by $\overline{\omega} = \sum_i |\omega_i|$. Depending on the properties of L and of $\overline{\omega}$ relative to L , such a principal ideal may be τ -closed, τ -dense in L (obviously, it may have both properties). In most of cases, it is neither closed nor τ -dense in L .

⁶Note that such a kind of property automatically holds true if $X_i \subset L_+$ (since $0 \in K$).

In the first case, as already noticed, under Assumptions $(C (i) - (iv))$, (P) , (B) , we have $\mathcal{C}^f(\mathcal{E}_{|L(\bar{\omega})}) \neq \emptyset$. We thus can start from an element $(x, y) \in \mathcal{C}^f(\mathcal{E}_{|L(\bar{\omega})})$. The next proposition proves the existence of $p \in (L(\bar{\omega}), \leq)^\sim$ such that (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|L(\bar{\omega})}$.

Proposition 3.1 *Let $(x, y) \in \mathcal{C}^f(\mathcal{E}_{|L(\bar{\omega})})$. Under Assumptions $(C (i), (iii), (iv))$ and (P) , if \mathcal{E} is F -proper relative to $L(\bar{\omega})$, then there exists $p \in (L(\bar{\omega}), \leq)^\sim$ such that (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|L(\bar{\omega})}$.*

Combining Proposition 3.1 with Theorem 2.1, we get immediately:

Theorem 3.1 *Under all assumptions made in Section 2.1 on \mathcal{E} , and if \mathcal{E} is E -proper, relative to the τ -closed order ideal $L(\bar{\omega})$, at every $(x, y) \in \mathcal{A}(\mathcal{E}_{|L(\bar{\omega})})$, then there exists a quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$.*

In the other cases, let us assume, as suggested above, that firms' inputs (outputs) have to be chosen from $L(\bar{\omega})$ while admissible consumption plans of consumers are to have their positive (negative) part in $L(\bar{\omega})$. As already noticed, such a condition is automatically satisfied in an exchange economy with consumption sets included in L_+ or having an inferior bound in $L(\bar{\omega})$. Then $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}_{|L(\bar{\omega})})$, $\mathcal{C}^f(\mathcal{E}) \subset \mathcal{C}^f(\mathcal{E}_{|L(\bar{\omega})})$ and we can yet apply Proposition 3.1 to get $p \in (L(\bar{\omega}), \leq)^\sim$ such that (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|L(\bar{\omega})}$. Combining this fact with Theorem 2.1, we get

Theorem 3.2 *Let us assume (SA) , (C) , (P) (B) and that $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}_{|L(\bar{\omega})})$. If E is E -proper, relative to $L(\bar{\omega})$, at every $(x, y) \in \mathcal{A}(\mathcal{E})$, then there exists a quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$.*

In case of F -properness, the following theorem summarizes results already obtained in Proposition 2.2, Lemma 2.2 and its corollary, combined again with Proposition 3.1.

Theorem 3.3 *Let us assume (SA) , (C) , (P) (B) and that $L(\bar{\omega})$ is τ -dense in L . Let us assume also either that*

(i) $\forall i, X_i = (\{a_i\} + L_+) \cap Z_i$, with $a_i \in L(\bar{\omega})$, Z_i convex, τ -closed, with an interior point in $(\{a_i\} + L(\bar{\omega})_+)$

(ii) and, in case of production, $\forall j, Y_j = (\{b_j\} - L_+) \cap Z_j$, with $b_j \in L(\bar{\omega})$, Z_j convex, τ -closed, with an interior point in $(\{b_j\} - L(\bar{\omega})_+)$

or that

(iii) $\forall i, X_i = (\{a_i\} - L_+) \cap Z_i$, with $a_i \in L(\bar{\omega})$, Z_i convex, τ -closed, with an interior point in $(\{a_i\} - L(\bar{\omega})_+)$

(iv) and, in case of production, $\forall j, Y_j = (\{b_j\} + L_+) \cap Z_j$, with $b_j \in L(\bar{\omega})$, Z_j convex, τ -closed, with an interior point in $(\{b_j\} + L(\bar{\omega})_+)$.

Then $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}|_{L(\bar{\omega})})$ and if \mathcal{E} is F -proper relative to $L(\bar{\omega})$ at every $(x, y) \in \mathcal{A}(\mathcal{E})$, \mathcal{E} has a quasiequilibrium $(\bar{x}, \bar{y}, \bar{\pi})$ with $\bar{\pi} \in (L, \tau)'$.

Our main concern in this paper was to obtain equilibrium existence results without assuming any monotonicity of preferences or free-disposal in production. The results stated in this section are to be compared with those obtained by Tourky [26], [27] in the same framework but with a technique of proof which heavily relies on both assumptions of strict monotonicity of preferences and free-disposal in production.

As already noticed, the E -properness used in this paper is quite similar to Tourky's M -properness, as far as E -properness is relative to L . Here, we require E -properness relative to the order-ideal $L(\bar{\omega})$, an assumption which is neither stronger nor weaker than M -properness. If one excepts the case (Theorem 3.1) where the order-ideal $L(\bar{\omega})$ is τ -closed in L , Theorem 3.2 requires the coincidence of $\mathcal{A}(\mathcal{E})$ with $\mathcal{A}(\mathcal{E}|_{L(\bar{\omega})})$, an assumption which is not made in Tourky. This assumption is automatically satisfied in an exchange economy with consumption sets equal to the positive cone. This assumption is still natural in an exchange economy with consumption sets not necessarily equal to the positive cone but bounded from below by some element of $L(\bar{\omega})$. It may seem rather artificial for a production economy to require the coincidence of $\mathcal{A}(\mathcal{E})$ with $\mathcal{A}(\mathcal{E}|_{L(\bar{\omega})})$, even if we give above sufficient conditions for this coincidence. For this reason, our results are a complement rather than a substitute to Tourky's results. In addition, when the order-ideal $L(\bar{\omega})$ is τ -dense in L , we give in Theorem 3.3, under F -properness relative to $L(\bar{\omega})$, a quasiequilibrium existence result subject to conditions which are again probably easier to interpret in the exchange case than in the production case.

A last comment is in order. Theorem 3.1 – 3.3 state only the existence of a quasiequilibrium. Additional conditions are necessary in order to guarantee the nontriviality of this quasiequilibrium, a first step towards the proof that this quasiequilibrium is actually an equilibrium. An example of such conditions is the following: Theorem 2.1 indicates how to deduce the nontriviality of quasiequilibrium in \mathcal{E} of the nontriviality of quasiequilibrium in $\mathcal{E}|_{L(\bar{\omega})}$. Using the same trick as in Proposition 4.1 of [13], sufficient conditions for the nontriviality of equilibrium in $\mathcal{E}|_{L(\bar{\omega})}$ can be stated using the condition of Corollary 2.2 for consumption sets included in the positive cone.

4 Proofs

Proof of Lemma 2.1. As already said, the first part of this lemma follows from Lemma 2 in Podczeck [23]⁷. Let us prove the second part.

Since $Z + K_+ \subset Z$, from (5) we conclude $h \cdot y \geq 0 \forall y \in K_+$, hence $h|_K \geq 0$ and $\pi|_K \leq p$. To prove (6), take any $y \in Z$, $y \leq z$. Now (5) gives $h \cdot y \geq h \cdot z$, but since $z - y \geq 0$, $z - y \in K$ and $h|_K \geq 0$, we have $h \cdot z \geq h \cdot y$ and conclude $h \cdot (z - y) = 0$. From $p = \pi|_K + h|_K$, we deduce

$$p \cdot (z - y) = \pi \cdot (z - y) + h \cdot (z - y) = \pi \cdot (z - y).$$

□

Proof of Proposition 2.1. Since (x, y, p) is a quasi-equilibrium of $\mathcal{E}|_K$, formulas (i)-(ii) from the definition of quasiequilibrium are fulfilled for $P_i(x) \cap K$ and $Y_j \cap K$ respectively. Now let us specify the cones

$$\Gamma_i = \{\lambda(v - x_i) \mid v \in V_{x_i}^i, \lambda > 0\}, \quad i \in I,$$

$$\Gamma_j = \{\lambda(v - y_j) \mid v \in V_{y_j}^j, \lambda > 0\}, \quad j \in J.$$

These cones are obviously convex and open. In view of (1), (3), since A_{x_i} and A_{y_j} are assumed to be radial in L at the points x_i, y_j respectively, we see that

$$\emptyset \neq (\{x_i\} + \Gamma_i) \cap Z_{x_i}^i \subset \{x_i\} + \{\lambda(v - x_i) \mid v \in P_i(x_i) \cap K, \lambda > 0\}$$

and

$$\emptyset \neq (\{y_j\} + \Gamma_j) \cap Z_{y_j}^j \subset \{y_j\} + \{\lambda(v - y_j) \mid v \in Y_j \cap K, \lambda > 0\}.$$

Therefore, using (i), (ii) from the definition of quasiequilibrium, we may conclude that

$$p \cdot ((\{x_i\} + \Gamma_i) \cap Z_{x_i}^i) \geq p \cdot x_i, \quad i \in I,$$

$$p \cdot ((\{y_j\} + \Gamma_j) \cap Z_{y_j}^j) \leq p \cdot y_j, \quad j \in J.$$

From $x_i \in \overline{V_{x_i}^i}$ and $y_j \in \overline{V_{y_j}^j}$, we deduce $x_i \in \overline{(\{x_i\} + \Gamma_i) \cap Z_{x_i}^i}$ and $y_j \in \overline{(\{y_j\} + \Gamma_j) \cap Z_{y_j}^j}$. Now we can apply our *Main auxiliary lemma* (Lemma 2.1 and its corollary) to get the existence of τ -continuous functionals $\pi_i, \pi_j \in (L, \tau)'$ such that

$$\pi_i \cdot (\{x_i\} + \Gamma_i) \geq \pi_i \cdot x_i, \quad \pi_i \cdot (x_i - z_i) = p \cdot (x_i - z_i), \quad \forall z_i \leq x_i, z_i \in Z_{x_i}^i, \quad i \in I \quad (16)$$

and

$$\pi_j \cdot (\{y_j\} + \Gamma_j) \leq \pi_j \cdot y_j, \quad \pi_j \cdot (z_j - y_j) = p \cdot (z_j - y_j), \quad \forall z_j \geq y_j, z_j \in Z_{y_j}^j, \quad j \in J. \quad (17)$$

⁷Podczeck assumes in this lemma that (L, τ) is locally convex but does not use this extra assumption in his proof.

Clearly by the Γ_i and Γ_j specification and due to the continuity of π_i, π_j , the latter ones implies (9). Also, due to assumptions $Z_{x_i}^i + K_+ \subset Z_{x_i}^i$ and $Z_{y_j}^j - K_+ \subset Z_{y_j}^j$, these functionals satisfy

$$\pi_t|_K \leq p, \quad \forall t \in I \cup J.$$

Now we set $\pi = (\bigvee_{i \in I} \pi_i) \vee (\bigvee_{j \in J} \pi_j)$ and note that, in view of (SA)(iii), π is a τ -continuous linear functional. Also, since K was assumed to be an order ideal of L , we have

$$p \geq \pi|_K, \quad \pi \geq \pi_t, \quad t \in I \cup J. \quad (18)$$

This, in view of (16), gives for $z_i \leq x_i, z_i \in Z_{x_i}^i$

$$\pi \cdot (x_i - z_i) \geq \pi_i \cdot (x_i - z_i) = p \cdot (x_i - z_i)$$

and

$$p \cdot (x_i - z_i) \geq \pi \cdot (x_i - z_i),$$

that yields (10). Analogously (17) and (18) yields (11).

To prove (12), let us recall that (x, y) is feasible, i.e.

$$\sum_{i \in I} x_i = \sum_{i \in I} \omega_i + \sum_{j \in J} y_j.$$

Taking any $z_i \leq x_i, z_i \in Z_{x_i}^i$ and any $z_j \geq y_j, z_j \in Z_{y_j}^j$, we can write

$$\sum_{i \in I} (x_i - z_i) + \sum_{j \in J} (z_j - y_j) = \sum_{i \in I} \omega_i + \sum_{j \in J} z_j - \sum_{i \in I} z_i.$$

Applying p to the left and to the right sides of this equality and using (10), (11), we conclude

$$\begin{aligned} \pi \cdot (\omega + \sum_{j \in J} z_j - \sum_{i \in I} z_i) &= p \cdot (\omega + \sum_{j \in J} z_j - \sum_{i \in I} z_i), \\ \forall z_i \leq x_i, z_i \in Z_{x_i}^i \ \&\ \forall z_j \geq y_j, z_j \in Z_{y_j}^j. \end{aligned} \quad (19)$$

Now let $u = \sum_{i \in I} v_i - \sum_{j \in J} u_j, v_i \in Z_{x_i}^i, u_j \in Z_{y_j}^j$ and $u \leq \omega$. Let $z_i = x_i \wedge v_i, z_j = y_j \vee u_j$ for all i, j . Since Z_x^i and Z_y^j were assumed to be lattices, we have $z_i \in Z_{x_i}^i$ and $z_j \in Z_{y_j}^j$ for all i, j and also

$$u' \leq u \leq \omega, \quad u' = \sum_i z_i - \sum_j z_j,$$

and

$$\sum_i (x_i - z_i) + \sum_j (z_j - y_j) = \omega - u' \geq u - u' \geq 0.$$

Using Riesz's decomposition property (see for example th.1.2, p.3 in Aliprantis and Burkinshaw [6]), we may find $v'_i, i \in I$ and $u'_j, j \in J$ such that

$$0 \leq v'_i \leq x_i - z_i \ \&\ 0 \leq u'_j \leq z_j - y_j,$$

and $\sum v'_i + \sum u'_j = u - u'$. Remembering that K is an ideal and that $Z_{x_i}^i + K_+ \subset Z_{x_i}^i$, $Z_{y_j}^j - K_+ \subset Z_{y_j}^j$, one gets $v'_i, u'_j \in K$ and

$$x_i \geq v'_i + z_i \in Z_{x_i}^i, \quad y_j \leq z_j - u'_j \in Z_{y_j}^j, \quad \forall i, j. \quad (20)$$

Now

$$\begin{aligned} u &= u' + \sum_{i \in I} v'_i + \sum_{j \in J} u'_j = \sum_{i \in I} (z_i + v'_i) - \sum_{j \in J} (z_j - u'_j) \Rightarrow \\ \omega - u &= \sum_{i \in I} (x_i - (z_i + v'_i)) - \sum_{j \in J} (y_j - (z_j - u'_j)), \end{aligned}$$

that, in view of (19), (20), gives us the result.

To end the proof, we need to verify (13). Since we assumed $\omega_i \in Z_{x_i}^i$ for each $i \in I$, we can find $z_i \in Z_{x_i}^i$ such that $z_i \leq x_i \wedge \omega_i$. Now subtracting $p \cdot z_i$ from the left and the right sides of equalities $p \cdot x_i = p \cdot \omega_i + \sum_j \theta_i^j p \cdot y_j$, and using (10), we obtain for every $i \in I$

$$\pi \cdot (x_i - z_i) = p \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j p \cdot y_j.$$

Analogously, since $0 \in Z_{y_j}^j$, we have $y_j^+ \in Z_{y_j}^j$ for each $j \in J$. Now adding $\sum_j \theta_i^j p \cdot y_j^+$ to the right and the left sides of latter equalities and using (11), we get

$$\pi \cdot (x_i - z_i) + \sum_j \theta_i^j \pi \cdot (y_j^+ - y_j) = p \cdot (\omega_i - z_i) + \sum_j \theta_i^j p \cdot y_j^+, \quad \forall i \in I.$$

But $\omega_i \geq z_i$, $y_j^+ \geq 0$ and $p \geq \pi|_K$ imply

$$p \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j p \cdot y_j^+ \geq \pi \cdot (\omega_i - z_i) + \sum_{j \in J} \theta_i^j \pi \cdot y_j^+, \quad \forall i \in I$$

hence

$$\pi \cdot x_i \geq \pi \cdot \omega_i + \sum_{j \in J} \theta_i^j \pi \cdot y_j, \quad \forall i \in I.$$

That all these inequalities are actually equalities comes from the fact that (x, y) is feasible. \square

Proof of Theorem 2.1. For each $i \in I$, take and fix any $x'_i \in P_i(x_i) \cap A_{x_i}$, where the set A_{x_i} , radial at x_i in L , is chosen from (2). Due to (2), one can find $u_i \in Z_{x_i}^i$ such that $u_i \leq x'_i$ and define $v_i = u_i \wedge x_i$. In view of assumptions, we conclude $v_i \in Z_{x_i}^i$. Now applying (9), we obtain $\pi_i(x'_i - v_i) \geq \pi_i(x_i - v_i)$. This, (10) and $\pi \geq \pi_i$ allows us to write

$$\begin{aligned} \pi \cdot (x'_i - x_i) + p \cdot (x_i - v_i) &= \pi \cdot (x'_i - x_i) + \pi \cdot (x_i - v_i) = \pi \cdot (x'_i - v_i) \geq \\ &\pi_i \cdot (x'_i - v_i) \geq \pi_i \cdot (x_i - v_i) = p \cdot (x_i - v_i) \end{aligned}$$

which implies $\pi \cdot (x'_i - x_i) \geq 0$. We have thus proved for each $i \in I$

$$\pi \cdot (P_i(x_i) \cap A_{x_i}) \geq \pi \cdot x_i.$$

Since A_{x_i} is radial at x_i in L , we conclude

$$\pi \cdot P_i(x_i) \geq \pi \cdot x_i, \quad \forall i \in I,$$

that proves condition (i) of the definition of quasiequilibrium.

The proof of (ii) can be given symmetrically. Let j and $y'_j \in Y_j \cap A_{y_j}$ be fixed. Due to (4), one can find $u_j \in Z_{y_j}^j$ such that $u_j \geq y'_j$. Define $v_j = u_j \vee y_j$. Now (9) gives $\pi_j \cdot (y'_j - v_j) \leq \pi_j \cdot (y_j - v_j)$. This, (11) and $\pi \geq \pi_j$ allow us to write

$$\begin{aligned} \pi \cdot (y'_j - y_j) + p \cdot (y_j - v_j) &= \pi \cdot (y'_j - y_j) + \pi \cdot (y_j - v_j) = \pi \cdot (y'_j - v_j) \leq \\ &\pi_j \cdot (y'_j - v_j) \leq \pi_j \cdot (y_j - v_j) = p \cdot (y_j - v_j) \end{aligned}$$

which implies $\pi \cdot (y'_j - y_j) \leq 0$. We have thus proved for every $j \in J$,

$$\pi \cdot (Y_j \cap A_{y_j}) \leq \pi \cdot y_j.$$

Since each A_{y_j} is radial at y_j in L , we conclude

$$\pi \cdot Y_j \leq \pi \cdot y_j, \quad \forall j \in J.$$

To finish the proof, we show the non-triviality of (x, y, π) under the additional assumption that $p \cdot x'_{i_0} < p \cdot x_{i_0}$ for some $i_0 \in I$ and some $x'_{i_0} \in X_{i_0} \cap Z_{x_{i_0}}^{i_0}$. Let us set $u = x_{i_0} \wedge x'_{i_0}$, getting $u \in Z_{x_{i_0}}^{i_0}$. Using $x'_{i_0} - u \geq 0$, that by $p \geq \pi|_K$ implies $p \cdot (x'_{i_0} - u) \geq \pi \cdot (x'_{i_0} - u)$ and remembering (10), one can write

$$\begin{aligned} \pi \cdot (x'_{i_0} - x_{i_0}) &= \pi \cdot (x'_{i_0} - u) + \pi \cdot (u - x_{i_0}) \\ &\leq p \cdot (x'_{i_0} - u) + p \cdot (u - x_{i_0}) = p \cdot (x'_{i_0} - x_{i_0}) < 0. \end{aligned}$$

The proof is complete. □

Proof of Proposition 2.2. Let us apply Proposition 2.1 and consider $\pi = (\bigvee_{i \in I} \pi_i) \vee (\bigvee_{j \in J} \pi_j)$, such that π , π_i and π_j satisfy the relations (10)–(11). By (SA)(iii), we have $\pi \in (L, \tau)'$. We first show that $\pi|_K = p$.

Take any $y \in K$. Since K is an ideal of L we have $y^+, y^- \in K$. Now since $Z = \sum_i Z_{x_i}^i - \sum_j Z_{y_j}^j$ is a radial set at the point ω , we may find real $\lambda > 0$ such that $\omega - \lambda y^+, \omega - \lambda y^- \in Z$, that due to (12) gives

$$\pi \cdot (\omega - (\omega - \lambda y^+)) = p \cdot (\omega - (\omega - \lambda y^+)) \Rightarrow \lambda \pi \cdot y^+ = \lambda p \cdot y^+ \Rightarrow \pi \cdot y^+ = p \cdot y^+$$

and analogously $\pi \cdot y^- = p \cdot y^-$. Since $y = y^+ - y^-$, we conclude $\pi|_K = p$.

To finish the proof, let us remember the quasi-equilibrium properties of (x, y, p) and use the density of $P_i(x_i) \cap K$ and $Y_j \cap K$ respectively in $P_i(x_i)$ and Y_j , and the τ -continuity of π . \square

Proof of Lemma 2.2. Let us suppose (14) for some $z \in Z$, that is,

$$\emptyset \neq U_z \cap (\{a_z\} - L_+) \subset Z \text{ and } z \in \overline{U_z} \cap (\{a_z\} - L_+)$$

and prove that

$$z \in \overline{U_z \cap (\{a_z\} - L_+) \cap K} \subset \overline{Z \cap K}.$$

For all $z' \in U_z \cap (\{a_z\} - L_+)$, for all $\lambda : 0 < \lambda \leq 1$, using the fact that U_z is convex and τ -open, we get $z + \lambda(z' - z) \in U_z \cap (\{a_z\} - L_+)$. Thus $z \in \overline{U_z \cap (\{a_z\} - L_+)} \subset Z$. Now, as proved in Podczeck, $\overline{(\{a_z\} - L_+) \cap K} = \{a_z\} - L_+$. Since U_z is τ -open, one has also

$$\overline{U_z \cap (\{a_z\} - L_+) \cap K} = \overline{U_z \cap (\{a_z\} - L_+) \cap K},$$

hence $z \in \overline{U_z \cap (\{a_z\} - L_+) \cap K} \subset \overline{Z \cap K}$.

The case of (15) can be considered symmetrically.

Now since z is an arbitrary point of Z , we have $Z \subset \overline{Z \cap K}$. The reverse inclusion $\overline{Z \cap K} \subset Z$ follows from the assumption that Z is τ -closed. \square

Proof of Proposition 3.1. Let us recall that $L(\overline{\omega})$ can be endowed with the Riesz norm $\|x\|_{\overline{\omega}} = \inf\{\lambda > 0 \mid |x| \leq \lambda\omega\}$, so that the unit-ball is the order interval $[-\overline{\omega}, +\overline{\omega}]$. Moreover, the fact that on $L(\overline{\omega})$ the norm topology is finer than the topology induced by τ still holds true when assumptions (SA) replace the classical assumption that L is locally convex-solid. In what follows we will write $L(\overline{\omega})'$ for $(L(\overline{\omega}), \|\cdot\|_{\overline{\omega}})'$. Let $G \stackrel{\text{def}}{=} \text{co}(\bigcup_{i \in I} (\text{co}(P_i(x_i) \cap L(\overline{\omega}) - \sum_{j \in J} \theta_i^j Y_j \cap L(\overline{\omega}) - \omega_i))$.

From F -properness at x_i of preference P_i together with the assumptions on \mathcal{E} , we deduce that $G \neq \emptyset$ and that $x_i \in \overline{\text{co} P_i(x_i) \cap L(\overline{\omega})}^{\|\cdot\|_{\overline{\omega}}}$. It is easily verified that $(x, y) \in \mathcal{C}^f(\mathcal{E}_{|L(\overline{\omega})})$ implies $0 \notin G$. On the other hand, it follows also from the F -properness at x_i of P_i that G has a nonempty $\|\cdot\|$ -interior. The rest of the proof is routine. From the classical separation theorem, there exists $p \in L(\overline{\omega})$ such that $x'_i \in P_i(x_i) \cap L(\overline{\omega})$ implies $p \cdot x'_i \geq p \cdot x_i = p \cdot \omega_i + \sum_j \theta_i^j p \cdot y_j$ and $y'_j \in Y_j \cap L(\overline{\omega})$ implies $p \cdot y'_j \leq p \cdot y_j$. (x, y, p) is a quasiequilibrium of $\mathcal{E}_{|L(\overline{\omega})}$. Since $L(\overline{\omega})$, endowed with the Riesz norm, is a Banach lattice, $p \in (L(\overline{\omega}), \leq)^\sim$. \square

References

- [1] ALIPRANTIS, C.D., On the Mas-Colell – Richard Equilibrium theorem, *Journal of Economic Theory* **74**, 414-424 (1997)

- [2] ALIPRANTIS, C.D., BROWN, D.J., Equilibria in markets with a Riesz space of commodities, *Journal of Mathematical Economics* **11**, 189-207 (1983)
- [3] ALIPRANTIS, C.D., BROWN, D.J., BURKINSHAW, O., Existence and Optimality of Competitive Equilibria, Springer-Verlag, Berlin/New-York, 1989
- [4] ALIPRANTIS, C.D., BROWN, D.J., BURKINSHAW, O., Edgeworth equilibria, *Econometrica* **55** No.5, 1109-1137 (1987)
- [5] ALIPRANTIS, C.D., BROWN, D.J., BURKINSHAW, O., Edgeworth equilibria in production economies, *Journal of Economic Theory* **43**, 252-290 (1987)
- [6] ALIPRANTIS, C.D., BURKINSHAW, O., Locally solid Riesz spaces, Academic press, Inc., New-York, 1978
- [7] ALIPRANTIS, C.D., TOURKY, R., YANNELIS, N.C., The Riesz-Kantorovich formula and general equilibrium theory, to appear in *Journal of Mathematical Economics*
- [8] ALIPRANTIS, C.D., TOURKY, R., YANNELIS, N.C., A theory of value with auction prices, manuscript
- [9] ARAUJO A. AND P.K. MONTEIRO, Equilibrium without uniform conditions, *Journal of Economic Theory* **48**, 416-427 (1989)
- [10] BACK, K., Structure of consumption sets and existence of equilibria in infinite-dimensional spaces, *Journal of Mathematical Economics* **17**, 88-99 (1988)
- [11] BEWLEY, T., Existence of equilibria in economies with infinitely many commodities, *Journal of Economic Theory* **4**, 514-540 (1972)
- [12] BOYD, J.H., MCKENZIE, L.W. The existence of competitive equilibrium over an infinite horizon with production and general consumption sets, *International Economic review* **34**, 1-20 (1993)
- [13] DEGHDAK, M., FLORENZANO, M., Decentralizing Edgeworth equilibria in economies with many commodities, *Economic Theory* **14**, 297-310 (1999)
- [14] DUFFIE, D., ZAME, W.R., The consumption based capital asset pricing model, *Econometrica* **57**, 1274-1298 (1989)
- [15] FLORENZANO, M., Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences, *Journal of Mathematical Analysis and Applications*, **153**, 18-36 (1990)
- [16] HUANG, C., KREPS, D.M., Intertemporal preferences with a continuous time dimension, Sloan School Working Paper, 1986

- [17] JONES, L.E., A competitive model of commodity differentiation, *Econometrica* **52**, 507-530 (1984)
- [18] MARAKULIN, V.M., Production equilibria in vector lattices with unordered preferences: an approach using finite-dimensional approximations, *Couverture orange CEPREMAP* 9821, 1998
- [19] MAS-COLELL, A., The price equilibrium existence problem in topological vector lattices, *Econometrica* **54**, 1039-1053 (1986)
- [20] MAS-COLELL, A., Valuation equilibrium and Pareto optimum revisited, in W. Hildenbrand and A Mas Colell, eds, *Contribution to Mathematical Economics*, North Holland, New York, 1986
- [21] MAS-COLELL, A., RICHARD, S.F., A new approach to the existence of equilibria in vector lattices, *Journal of Economic Theory* **53**, 1-11 (1991)
- [22] MAS-COLELL, A., AND ZAME, W., Equilibrium theory in infinite dimensional spaces, In : Hildenbrand, W., Sonnenschein, H. (eds.): *Handbook of Mathematical Economics*, Vol. IV, North-Holland, Amsterdam, 1991.
- [23] PODCZEK, K., Equilibria in vector lattices without ordered preferences or uniform properness, *Journal of Mathematical Economics* **25**, 465-485 (1996)
- [24] RICHARD S.F., Competitive equilibria in Riesz spaces, mimeographed, G.S.I.A., Carnegie Mellon University, 1986
- [25] RICHARD, S.F., Production equilibria in vector lattices, *Journal of Mathematical Economics* **18**, 41-56 (1989)
- [26] TOURKY, R., A new approach to the limit theorem on the core of an economy in vector lattices, *Journal of Economic Theory* **78**, 321-328 (1998)
- [27] TOURKY, R., The limit theorem on the core of a production economy in vector lattices with unordered preferences, *Economic Theory* **14**, 219-226 (1999)
- [28] TOURKY, R., Production equilibria in locally proper economies with unbounded and unordered consumers, *Journal of Mathematical Economics* **32**, 303-315 (1999)
- [29] YANNELIS N.C., ZAME W.R., Equilibria in Banach lattices without ordered preferences, *Journal of Mathematical Economics* **15**, 85-110 (1986) and Preprint 71, Institute of Mathematics and its Applications, University of Minnesota, 1984
- [30] ZAME W.R., Competitive equilibria in production economies with an infinite dimensional commodity space, *Econometrica* **55**, 1075-1108 (1987).