Bounds on the Risk-Free Interest Rate in Incomplete Markets with and without Utility Functions Exhibiting Constant Absolute Risk Aversion*

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Abstract: In a model of a two-period exchange economy under uncertainty, we find an upper bound for the equilibrium risk-free interest rate when the expected aggregate endowment in the second period is no greater than the first-period aggregate endowment. We also find a lower bound when the agents' utility functions exhibit constant absolute risk aversion and the expected aggregate endowment in the second period is no smaller than the first-period counterpart. These bounds are independent of the degree of market incompleteness, and so these results show to what extent market incompleteness can explain the risk-free rate puzzle in this class of general equilibrium models with heterogeneous agents. A general method of finding lower bounds is also presented.

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1. Introduction

In this paper, we consider a model of an exchange economy under uncertainty with two consumption periods and one physical good, where consumption smoothing over time and uncertainty is done by asset transactions in financial markets. The preferences of each agent is represented by a time-independent, additively separable utility function and the discount factor is common across them. We are interested in how the risk-free interest rate depends on the primitives of the economy, in particular, the discount factor and the incompleteness of markets. This has been a focus of extensive research under the name, given by Weil (1992), of the risk-free rate puzzle. Kocherlakota (1996) provides an excellent survey on this topic.

The contribution of this paper is two fold. First, we show that the equilibrium price of the risk-free bond is no higher than the discount factor, provided the derivative of every agent's utility function is a convex function and the expected aggregate endowment in the second period is no larger than the first-period aggregate endowment. Note that there is no condition imposed on the utility functions other than the convexity of derivatives, and there is no assumption on the incompleteness of market; that is, we do not assume anything as to what kind of risky assets are available for trade. So just as Weil's (1992) original contribution, this result can be seen as a benchmark for the question of to what extent the market incompleteness can possibly explain the lower risk-free interest rate in general equilibrium models with heterogeneous agents.

Our second contribution is that if every agent's utility function exhibits constant absolute risk aversion (CARA for short) and the expected aggregate endowment in the second period is no smaller than the first-period aggregate endowment, then there is an upper bound for the bond price which only depends on the primitives of the economy. The bounds are succinctly related to the degree of risk aversion and the risk properties of initial endowments. This result complements the first one; no existing contribution has clarified a theoretical upper bound on the bond price (or lower bound on the interest rate) with incomplete markets. We emphasize that this result also assumes no condition on the incompleteness of markets. It therefore shows the range of the equilibrium risk-free interest rates explainable by the incompleteness of markets in the models with heterogeneous CARA agents. As a corollary to this result, we also show that the risk-free bond price can be made arbitrarily large with incomplete markets without changing the bond price with complete markets.

The reported bounds can be computed very simply. So this result also serves as a valuable tool for finding a rough estimate of equilibrium interest rates, since calculating an equilibrium price system can be a tedious exercise when markets are incomplete.

Levine and Zame (1998, 1999) considered an infinite-horizon economy under uncertainty with heterogeneous agents to investigate how the possibility of intertemporal income transfers weakens equilibrium implications of incomplete markets. A key step of their analysis is to find an upper bound on the interest rate. Our technique is inspired by theirs, though we do not need to make any a priori distinction between the cases with and without the aggregate risk as they did. The lower bound with CARA utility functions is obtained, on the other hand, as a corollary to the result of Hara (1998), which generalizes an earlier result shown in Elul (1997) on the risk-free rate puzzle.

The next section presents the general model of this paper. In Section 3 deals with general utility functions and shows that the risk-free bond price cannot be lower than the common discount factor. It also discuss a general method of finding bounds on the bond price. Section 4 takes up the case of CARA utility functions and find an upper bound for the bond price. Section 5 concludes and suggests a couple of directions of future research.

2. The Model

There are two trading periods, and there is a single perishable good in each period. There is no uncertainty in the first period, when consumption good and assets are exchanged. At the beginning of the second period, the assets pay off, and then consumption takes place. The uncertainty in the second period is described by a probability measure space $(\Omega, \mathcal{B}, \mu)$. Denote by $\mathbf{1}$ (the μ-equivalent class of) the function from Ω to R that takes constant value one. The function $\mathbf{1}$ will be interpreted as the risk-free discount bond. Let X be a linear subspace in the set of all measurable functions from Ω to R such that $\mathbf{1} \in X$. We take the commodity space to be $R \times X$ and denote by $\mathbf{E}$ the expectation with respect to μ. A generic element of $R \times X$ will be denoted by $(x_0, \mathbf{x})$, where x_0 corresponds to consumption in the first period, and $\mathbf{x}$ is a random variable that corresponds to consumption in the second period. We shall later assume some minimal regularity condition on X, but our results cover interesting and commonly studied cases such as Ω is a finite set and $\mathcal{B}$ is the power set of X, or X is the linear subspace spanned by finitely many jointly normally distributed random variables.

There are H agents in the economy. Each *agent*, indexed $h \in \{1, \cdots, H\}$, is characterized by:

- Time invariant von Neumann Morgenstern utility function u^h. It is strictly concave and continuously differentiable. Its derivative Du^h is assumed to be a convex function.
- Initial endowment vector $(e_0^h, \mathbf{e}^h) \in R \times X$.

We assume that the agents have a common discount factor $\delta > 0$. Thus the preference relation of agent h is represented by the expected utility function $U^h : R \times X \to R$ defined by

$$U^h(x_0^h, \mathbf{x}^h) \equiv u^h\left(x_0^h\right) + \delta \int_\Omega u^h\left(\mathbf{x}^h(\omega)\right)\mu(d\omega) = u^h\left(x_0^h\right) + \delta \mathbf{E}\left(u^h\left(\mathbf{x}^h\right)\right).$$

The regularity assumption assumed throughout this paper is that this function is continuously differentiable and the first order necessary condition for utility maximization is well defined. This assumption is very mild and can be justified in various ways; for instance, a set of sufficient condition in terms of utility function and the underlying probability space can be found in Nielsen (1993, Proposition 1 and 5).

Although trade takes place sequentially, we define an agent's utility maximization problem and an equilibrium of the economy directly in terms of market spans and state price functions. This facilitates a simpler exposition, and it can be readily shown that it is equivalent to a model where asset trade is explicitly modelled. Namely, when the traded assets span a linear subspace M, called the *market span*, of the commodity space X, and their arbitrage-free prices coincides with a linear function $p : M \to R$, called the *state price function*, agent h's utility maximization problem is

$$\begin{array}{ll} \underset{(x_0^h, \mathbf{x}^h) \in R \times X}{\text{Max}} & U^h(x_0^h, \mathbf{x}^h) \\ \text{subject to:} & \mathbf{x}^h - \mathbf{e}^h \in M, \\ & (x_0^h - e_0^h) + p(\mathbf{x}^h - \mathbf{e}^h) \leq 0. \end{array}$$

The first constraint implies that the net trade vector $\mathbf{x}^h - \mathbf{e}^h$ can be achieved through asset trades, and the second constraint is the budget constraint. Note that the linearity of M and p means that there are no transaction costs; in particular there is no short sales constraint.

Note also that the first-period consumption is the numéraire, whose price equals one. The case of complete markets corresponds to the case where the market span M coincides with the commodity space X.

We say that a state price function p and a consumption allocation $((y_0^h, \mathbf{y}^h))_{h\in\{1,\cdots,H\}}$ constitute an *equilibrium* for the market span M if, for every h, $(y_0^h, \mathbf{y}^h)$ is a solution to the above maximization problem and $\sum_{h=1}^{H}(y_0^h, \mathbf{y}^h) = \sum_{h=1}^{H}(e_0^h, \mathbf{e}^h)$. It can then be shown that the asset markets clear automatically. So the equilibrium price of the risk-free discount bond is $p(\mathbf{1})$ and the equilibrium risk-free interest rate is $p(\mathbf{1})^{-1} - 1$. Hence a lower interest rate means an higher bond price, and vice versa.

3. Bounds for the Risk-Free Interest Rate with General Utility Functions

In the first subsection, we provide an upper bound for the equilibrium risk-free interest rate or, equivalently, a lower bound for the risk-free bond prices. This is done for general utility functions, under the assumption that the expected aggregate endowment is non-increasing over time. In the second subsection, we discuss a general method of finding upper and lower bounds for the interest rate, which also serves as an introduction to the next section.

3.1. An Upper Bound for the Risk-Free Interest Rate

Under the regularity assumption, for *every* agent h, the following first order condition is met at a solution $\left(x_0^h, \mathbf{x}^h\right)$ of his utility maximization problem:

$$p(\mathbf{1})Du^h(x_0^h) = \delta\mathbf{E}\left(Du^h(\mathbf{x}^h)\right).$$

This can be rewritten as

$$p(\mathbf{1}) = \delta \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(x_0^h)} = \delta \frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)}, \tag{3.1}$$

where $\bar{x}^h = \mathbf{E}\left(\mathbf{x}^h\right)$. We write $\overline{e}^h = \mathbf{E}\left(\mathbf{e}^h\right)$, $\overline{e} = \sum_h \overline{e}^h$, and $e_0 = \sum_h e_0^h$. So $\overline{e}^h$ is agent h's expected endowment in the second period, $\overline{e}$ is the expected aggregate endowment in the second period, and e_0 the aggregate endowment in the first period.

Proposition 1. *Assume that $e_0 \geq \overline{e}$. Let M be a market span such that $\mathbf{1} \in M$. Also let a state price function $p : M \to R$ and a consumption allocation $\left((x_0^h, \mathbf{x}^h)\right)_{h\in\{1,\cdots,H\}}$ constitute an equilibrium for M. Then $p(\mathbf{1}) \geq \delta$.*

It is easy to show that if $\sum_h \mathbf{e}^h = e_0\mathbf{1}$, so that there is stationarity but no aggregate uncertainty, and the markets are complete, then $p(\mathbf{1}) = \delta$. This proposition thus implies that if the aggregate endowment is stationary, then the risk-free interest rate is equal or lower with incomplete markets than with complete markets.

Proof of Proposition 1. Since Du^h is convex for every h, we have

$$\mathbf{E}\left(Du^h(\mathbf{x}^h)\right) \geq Du^h\left(\bar{x}^h\right). \tag{3.2}$$

Since $\sum x_0^h = e_0 \geq \overline{e} = \sum_h \bar{x}^h$ by the assumption and the equilibrium condition, there is at least one $h = \bar{h}$ such that $x_0^{\bar{h}} \geq \bar{x}^{\bar{h}}$, and thus

$$Du^{\bar{h}}\left(\bar{x}^{\bar{h}}\right) \geq Du^{\bar{h}}\left(x_0^{\bar{h}}\right). \tag{3.3}$$

Plugging (3.2) and (3.3) into (3.1), we obtain $p(\mathbf{1}) \geq \delta$. ■

Notice that the equality (3.1) shows that if $e_0 \leq \overline{e}$ and the Du^h are *concave* in the relevant interval of wealth levels, then $p(\mathbf{1}) \leq \delta$ would follow. The more interesting case, however, is where the Du^h are *convex* as assumed earlier, because most frequently applied utility functions, such as those exhibiting constant absolute or relative risk aversions, have this property.

3.2. A General Approach to Finding Bounds

A bound in Proposition 1 is obtained under a very general assumption on utility functions, with no reference to the structure of market incompleteness. But it does not tie down sharply the relation between the agents' attitude toward risks and the equilibrium rate. In this subsection, we shall discuss how these can be more tightly related in general. The discussion will also clarify the intuition behind the result reported in the next section.

As can be seen from equality (3.1), the task of finding bounds for the risk-free interest rate, or, equivalently, for the risk-free bond price is one of finding bounds for $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(x_0^h)$ for *some* h. The right-hand side of (3.1) shows that it is the product of two factors. The first factor $Du^h(\overline{x}^h)/Du^h(x_0^h)$ is the intertemporal marginal rate of substitution. We have seen that if $e_0 \geq \overline{e}$, then this factor is no smaller than 1 for *some* h. Similarly, if $e_0 \leq \overline{e}$, then this factor is no larger than 1 for *some* h. By closely examining the risk attitude of this agent, we can find a better bound than 1.

Note first that if a von Neumann Morgenstern utility function u is more risk averse than another v, then, for every w_0 and w_1 with $w_0 \geq w_1$, we have

$$\frac{Du(w_1)}{Du(w_0)} \geq \frac{Dv(w_1)}{Dv(w_0)}.$$

Let $e_0 \geq \overline{e}$, then the agents must consume more in the first period in equilibrium, and so there is an agent who does so more than the average; that is, there is an $\overline{h}$ such that $x_0^{\bar{h}} - \overline{x}^{\bar{h}} \geq H^{-1}(e_0 - \overline{e}) \geq 0$. If agent $\overline{h}$ is more risk averse than an agent with constant absolute risk aversion with coefficient α on the interval $[\overline{x}^{\bar{h}}, x_0^{\bar{h}}]$ then using the inequality above for $v(w) = -\exp[-\alpha w]$, we have

$$\frac{Du^{\overline{h}}(\overline{x}^{\overline{h}})}{Du^{\overline{h}}(x_0^{\overline{h}})} \geq \frac{\exp\left[-\alpha\overline{x}^{\overline{h}}\right]}{\exp\left[-\alpha\overline{x}_0^{\overline{h}}\right]} \geq \exp\left(\alpha H^{-1}(e_0 - \overline{e})\right) \geq 1.$$

So the general recipe for finding a tighter lower bound for the term $Du^{\overline{h}}(\overline{x}^{\overline{h}})/Du^{\overline{h}}(x_0^{\overline{h}})$ when $e_0 \geq \overline{e}$ is to identify agent $\overline{h}$ who consumes more in the first period relative to the other agents and measure the minimum degree α of risk aversion on the interval $[\overline{x}^{\bar{h}}, x_0^{\bar{h}}]$. Then the lower bound for the bond price is improved form one to $\exp\left(\alpha H^{-1}(e_0 - \overline{e})\right)$.

For some utility functions, the coefficients of *relative* risk aversion vary in a much narrower range over relevant wealth levels than the coefficients of absolute risk aversions. We can then improve the above bound by using the intertemporal ratio $e_0/\overline{e}$ of aggregate endowments. Indeed, if the (generally non-constant) coefficients of *relative* risk aversions of $u^{\overline{h}}$ are larger than α, then we have

$$\frac{Du^h(\overline{x}^h)}{Du^h(x_0^h)} \geq \left(\frac{e_0}{\overline{e}}\right)^{\alpha} \geq 1.$$

Symmetric bounds can be obtained for the case of $e_0 \leq \overline{e}$.

The second factor $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$ shows how much, in ratio, the marginal utility from the bond is increased by the risk present in the second-period consumption. By Jensen's Inequality, this is no smaller than 1. It measures the degree of prudence of Kimball (1990). Indeed, if a von Neumann Morgenstern utility function u is more risk averse and more prudent than another v, then, for every $\mathbf{x} \in X$ with $\mathbf{E}(\mathbf{x}) = \overline{x}$, it can be shown that

$$\frac{\mathbf{E}\left(Du(\mathbf{x})\right)}{Du(\overline{x})} \geq \frac{\mathbf{E}\left(Dv(\mathbf{x})\right)}{Dv(\overline{x})}.$$

Once, for example, we know that the (generally non-constant) coefficients absolute risk aversions of *every* u^h lie in the interval $[\alpha, \beta]$, we can conclude that

$$\frac{\mathbf{E}\left(\exp(-\alpha\mathbf{x}^h)\right)}{\exp(-\alpha\overline{x}^h)} \leq \frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)} \leq \frac{\mathbf{E}\left(\exp(-\beta\mathbf{x}^h)\right)}{\exp(-\beta\overline{x}^h)}$$

for *every* h, which provides both the upper and lower bounds of $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$.

We have thus obtained bounds for the second factor $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$ for *every* h and bounds for the first factor $Du^h(\overline{x}^h)/Du^h(x_0^h)$ for *some* h. By multiplication, we can obtain bounds for the bond price $p(\mathbf{1})$.

The difficulty in this approach to find the bounds for the bond price is that we need to identify the second-period equilibrium allocation $\left(\mathbf{x}^1, \cdots, \mathbf{x}^H\right)$. It can however be circumvented if all u^h exhibits constant absolute risk aversion.

4. A Lower Bound with CARA Utility Functions

From now on, we assume that utility function u^h has a constant coefficient $\alpha^h > 0$ of absolute risk aversion, so that $u^h(w) = -\alpha^h \exp\left(-\alpha^h w\right)$ for every h. Thus the utility function can now be written as

$$U^h(x_0^h, \mathbf{x}^h) = -\exp(-\alpha^h x_0^h) - \delta\mathbf{E}(\exp(-\alpha^h\mathbf{x}^h)).$$

Notice that Du^h is a convex function, and so the results of the previous section are applicable. For the rest of this section, we are concerned with the upper bound for the equilibrium price of the discount bond.

The following theorem was proved in Hara (1998), which generalized some of the results in Elul (1997) on the risk-free rate puzzle.

Proposition 2. *Let M and N be two market spans such that $\mathbf{1} \in M \subset N$. Let $p : M \to R$ be an equilibrium state price function for M and let a state price function $q : N \to R$ and a consumption allocation $((z_0^h, \mathbf{z}^h))_{h \in \{1,\cdots,H\}}$ constitute an equilibrium for N. Suppose that $\mathbf{z}^h - \mathbf{e}^h \notin M$ for some h. Then $p(\mathbf{1}) > q(\mathbf{1})$.*

This theorem implies that the less complete the markets M are, the lower the risk-free interest rate is. In particular, the equilibrium risk-free interest rate is lowest when M coincides with the line spanned by $\mathbf{1}$.

For each h, define

$$c^h = \frac{\mathbf{E}\left(\exp\left(-\alpha^h \mathbf{e}^h\right)\right)}{\exp\left(-\alpha^h \overline{e}^h\right)}.$$

This measures agent h's prudence evaluated at his initial endowment. It thus depends only on the primitives of the economy.

Another way to look at this number is to apply the second-order Taylor approximation $\exp(w) \approx 1 + w + 2^{-1}w^2$. Then $c^h \approx 1 + 2^{-1}\left(\alpha^h\right)^2 \mathrm{Var}(\mathbf{e}_h)$, or $\alpha^h \mathbf{S}(\mathbf{e}_h) \approx \left(2(c^h - 1)\right)^{1/2}$, where $\mathbf{S}$ denotes the standard deviation. The numbers c^h thus measure the variability of his second-period initial endowments weighted by the constant coefficients of absolute risk aversion. As shown by Duffie and Jackson (1990), Demange and Laroque (1995), Ohashi (1995), Rahi (1995), and others, they help provide a necessary and sufficient condition for the optimal asset structure when only a limited number of assets can be traded in markets.

We also define $\alpha = \max\{\alpha^1, \cdots, \alpha^H\}$ and

$$b = \exp\left(-\alpha H^{-1}\left(\overline{e} - e_0\right)\right),$$

which is no greater than one if $e^0 \leq \overline{e}$.

Proposition 3. *Assume that $e^0 \leq \overline{e}$. Let M be a market span such that $\mathbf{1} \in M$ and $p : M \to R$ be an equilibrium state price function for M. Then:*

1. $p(\mathbf{1}) \leq \delta b \max\{c^1, \cdots, c^H\}$.

2. If, furthermore, M coincides with the line spanned by $\mathbf{1}$, $e^0 = \overline{e}$, and $c^1 = \cdots = c^H$, then $p(\mathbf{1}) = \delta c^h$ for every h.

The second part of this proposition states that the upper bound of the bond price in the first part is indeed attained if the markets are least complete, the expected aggregate endowment is constant over time, and all agents have the same c^h. An important implication of this is that the equilibrium bond price in incomplete markets can be arbitrarily large while the bond price in the complete markets stays at the discount factor δ. To see this, first fix the initial endowments $(e_0^h, \mathbf{e}^h)$ and the coefficients α^h of absolute risk aversion so that $\sum_h \mathbf{e}^h = e_0\mathbf{1}$, $\alpha^1 = \cdots = \alpha^H$, and the $\mathbf{e}^h$ have the same distribution with positive variance. This implies that $c^1 = \cdots = c^H > 1$. For each $s > 0$, define $\mathbf{e}^h(s) = \mathbf{e}^h + s\left(\mathbf{e}^h - \overline{e}^h\mathbf{1}\right)$ and $c^h(s) = \mathbf{E}\left(\exp\left(-\alpha^h\mathbf{e}^h(s)\right)\right) / \exp\left(-\alpha^h\overline{e}^h\right)$. It can be shown that the equilibrium bond price in the complete markets is equal to δ regardless of the values of s, but the bond price in the least complete markets, $\delta c^h(s)$, becomes unboundedly large as $s \to \infty$.

Proof of Proposition 3. 1. By Proposition 2, we can assume that M is equal to the line spanned by $\mathbf{1}$. By equality (3.1), it suffices to show that

$$\frac{Du^{\overline{h}}(\overline{x}^{\overline{h}})}{Du^{\overline{h}}(x_0^{\overline{h}})} \leq b \tag{4.1}$$

for some $h = \overline{h}$; and

$$\frac{\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)}{Du^h(\overline{x}^h)} = c_h \tag{4.2}$$

for every h. The first inequality (4.1) follows since there exists an h such that $\overline{x}^h - x_0^h \geq H^{-1}(\overline{e} - e_0)$.

Since M is spanned by $\mathbf{1}$, we have $\mathbf{x}^h = \mathbf{e}^h + y^h\mathbf{1}$ for some $y^h \in R$. Thus $\bar{x}^h = \bar{e}^h + y^h$, and so we have $\mathbf{x}^h - \overline{x}^h\mathbf{1} = \mathbf{e}^h - \overline{e}^h\mathbf{1}$. Hence

$$\mathbf{E}\left(\exp(-\alpha^h(\mathbf{x}^h - \overline{x}^h\mathbf{1}))\right) = \mathbf{E}\left(\exp(-\alpha^h(\mathbf{e}^h - \overline{e}^h\mathbf{1}))\right) = c^h$$

every h.

2. The symmetric argument is applicable. Since $e^0 = \overline{e}$, we have $b = 1$ and

$$\frac{Du^{\overline{h}}(\overline{x}^{\overline{h}})}{Du^{\overline{h}}(x_0^{\overline{h}})} \geq 1$$

for some $h = \overline{h}$. By equality (4.2), $p(\mathbf{1}) \geq \delta c^{\overline{h}}$. Thus

$$p(\mathbf{1}) \geq \delta \min\{c_1, \cdots, c_H\}.$$

Since $\min\{c^1, \cdots, c^H\} = \max\{c^1, \cdots, c^H\}$, this and the first part establish the second part.

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5. Conclusion

In this paper, we have found the upper and lower bounds on the risk-free interest rates in a two-period model with incomplete asset markets. The upper bound was given for general utility functions, while the lower bound was only for CARA utility functions. We also discussed a general method of finding upper and lower bounds. These results will be useful in illustrating the risk-free rate puzzle in tractable general equilibrium models with incomplete asset markets.

As mentioned at the end of Section 3.2, an obstacle to obtaining a lower bound on the risk-free interest rate with general utility functions was that we cannot find bounds for the prudence $\mathbf{E}\left(Du^h(\mathbf{x}^h)\right)/Du^h(\overline{x}^h)$ at the second-period equilibrium consumption bundle $\mathbf{x}^h$. Finding these bounds is a good future research topic.

The crucial expression for the bond price was equality (3.1). It was useful because it identified the two determinants of the bond price, the intertemporal marginal rate of substitution and the prudence. This suggests the following extensions of our approach. The first step is to the case of time-separable, but time-dependent von Neumann Morgenstern utility functions. The next step is to the case of recursive utility functions.

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